Journal of Applied Analysis and Computation
Volume 17, Number 1, February 2027, 184–211
Website:http://www.jaac-online.com
DOI:10.11948/20250368

ON A CLASS OF CAUCHY-TYPE PROBLEMS ASSOCIATED WITH WEIGHTED FRACTIONAL DERIVATIVES

Soumia Bourchi[1], Yassine Adjabi[2], Hamid Boulares[3], Fahd Jarad[4,5], Sumati Kumari Panda[6], Abdelkader Moumen[7,†] and Mohamed Bouye[8]

Abstract This paper investigates the existence and uniqueness of mild solutions for a class of nonlocal Cauchy–type problems involving integro–differential equations with weighted ς–Caputo fractional derivatives. The equations incorporate generalized Volterra and Fredholm integral operators with singular kernels of order $\ell \in (0, 1)$, within the framework of an ordered Banach space. The main results are established by combining semigroup theory, monotone iterative techniques, upper and lower solution methods, Kuratowski's measure of noncompactness, generalized Gronwall inequalities, and fixed point theorems. Continuous dependence of mild solutions on initial data is also proved. Two concrete examples are provided to illustrate the applicability of the abstract results.

Keywords Weighted ς-Caputo fractional derivative, abstract Cauchy problem, Volterra integral operator, Fredholm integral operator, Kuratowski measure of noncompactness, C_0-semigroup, monotone iterative technique, upper and lower solutions, mild solution, existence and uniqueness, cone.

MSC(2010) 34A08, 34A12, 34B40, 26A33, 35R11, 35K90, 47D06.

1. Introduction

In recent years, fractional calculus has emerged as one of the most rapidly growing fields of research, capturing the interest of numerous scientists. This rise in attention can be attributed

†The corresponding author.
[1]Laboratory of Mathematics, Computer Science and Applications, Djilali Bounaama University, Khemis Miliana 44001, Algeria
[2]Department of Mathematics, Faculty of Sciences, University of M'hamed Bougara, Boumerdes 35000, and Dynamic Systems Laboratory, U.S.T.H.B, Algiers, Algeria
[3]Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, Guelma 24000, Algeria
[4]Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara 06790, Turkey
[5]Department of Engineering Mathematics and Artificial Intelligence, Azerbaijan Technical University, Baku, Azerbaijan
[6]Department of Mathematics, GMR Institute of Technology, Rajam-532127, Andhra Pradesh, India
[7]Department of Mathematics, College of Science, University of Ha'il, Ha'il 55473, Saudi Arabia
[8]Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia
Email: s.bourchi@univ-dbkm.dz(S. Bourchi), adjabiy@univ-boumerdes.dz(Y. Adjabi), boulareshamid@gmail.com(H. Boulares), fahd@cankaya.edu.tr(F. Jarad), sumatikumari.p@gmrit.edu.in(S. K. Panda), mo.abdelkader@uoh.edu.sa(A. Moumen), mbmahmad@kku.edu.sa(M. Bouye)

to the significant findings reported by researchers who have effectively employed tools from fractional calculus to model various phenomena [21,32]. The theoretical framework surrounding fractional differential equations has greatly benefited from these important contributions.

It is important to note, however, that the concept of the fractional derivative is not singular. Over the past decade, many researchers have proposed modifications and generalizations of classical fractional derivatives and integrals. For instance, in [2], the author introduced weighted fractional derivatives that rely on a kernel function. With carefully selected weights and kernel functions, these derivatives can reduce to classical fractional derivatives, including the Riemann–Liouville, Hadamard, and Caputo derivatives. The fractional derivatives (often abbreviated as FDs) of functions in relation to other functions depending on a kernel function are sometimes referred to as $\wp$-fractional operators, following the notation established by Almeida [4]. These derivatives represent a natural extension of the original operators found in classical fractional calculus. In a recent study, Jarad et al. [18] outlined the spaces in which the weighted fractional operators of a function with respect to another function are defined. They also presented a modified Laplace transform to effectively address these operators.

Integrodifferential equations are mathematical tools used in modeling complex phenomena across a wide range of fields, such as design, quantum theory, and neuroscience. These equations allow researchers to capture dynamic processes and relationships more effectively. As a result of their versatility and applicability, the investigation of integro–differential equations within the framework of Banach spaces, a type of complete normed vector space has garnered significant interest and attention from the scientific and mathematical communities in recent years [5, 17, 22, 29, 30, 34]. Undoubtedly, the fractional evolution equations have taken their share of attention. However, exact solutions to these types of equations remain elusive. Various methods have been developed to demonstrate the existence of solutions. Some of the most widely used approaches include the method of lower and upper solutions, fixed point theories, Mawhin's coincidence degree theory, and monotone iterative techniques. In [25] and [36], the authors extended the monotone iterative method to evolution equations in ordered Banach spaces. Zhang et al. [37] applied this method to evolution equations with classical initial conditions. Other researchers [13, 16, 31] have also used the same technique for different types of evolution equations. Their results were derived from the Sadovskii fixed point axiom, semigroup theory, and the upper-lower-solution methodology. By starting with a pair of ordered lower and upper solutions, it is possible to construct two monotone sequences that converge uniformly to the extremal solutions found between the upper and lower solutions for the problems being considered. Following this, the authors in [12, 20] made significant conceptual advancements in the monotone iterative approach. The primary finding of [12] was further extended by [34] to integro-differential equations, without accounting for impulses in the space $\mathscr{U}$.

$$\begin{cases} \digamma'(\Delta) = \chi(\Delta, \digamma(\Delta), \mathscr{G}\digamma(\Delta)), & \text{if } \Delta \in J, \\ \digamma(0) = \digamma_0. \end{cases} \tag{1.1}$$

The term $(\mathscr{G}\digamma)(\Delta)$ which could be interpreted as a control on the system is as outlined by $(\mathscr{G}\digamma)(\Delta) = \int_0^\Delta \mathcal{Y}(\Delta, \zeta, \digamma(\zeta))\mathrm{d}\zeta$ and $\chi(\Delta, \digamma(\Delta), \mathscr{G}\digamma(\Delta))$ is a given nonlinear function (see [22, 23]). Existence of solutions of fractional equations of Volterra type with the *Riemann-Liouville* derivatives was studied by Jankowski [17].

Notably, the fractional abstract Cauchy problem has received a lot of attention; for example, see [8]. Jaradat et al. [19] referred to as a mild solutions $\digamma(\Delta)$ of the initial value problem:

$$\begin{cases} \mathcal{D}^{\ell}\digamma(\Delta) = \mathscr{M}\digamma(\Delta) + \chi(\Delta, \digamma(\Delta), \mathscr{G}\digamma(\Delta), \mathscr{H}\digamma(\Delta)), & \text{if } \Delta \in J = (0, b], \\ \digamma(0) = \digamma_0, \end{cases} \tag{1.2}$$

such that the term $\mathcal{D}^{\ell}$ is a Caputo-type FD of order $0 < \ell \leq 1$ and $\mathscr{G}$, $\mathscr{H}$ are two linear operators defined by

$$(\mathscr{G}\digamma)(\Delta) = \int_0^{\Delta} \mathcal{Y}(\Delta, \zeta)\digamma(\zeta)\mathrm{d}\zeta \text{ and } (\mathscr{H}\digamma)(\Delta) = \int_0^{b} h(\Delta, \zeta)\digamma(\zeta)\mathrm{d}\zeta, \ b > 0, \tag{1.3}$$

such that $\mathscr{M}$ is the generator of a strongly continuous semigroup on a Banach space $\mathscr{U}$. This problem was studied in [5, 24] and the references therein. Similar problems (1.2) under some restrictive conditions appear in [10, 24, 30].

In recent years, significant progress has been made in the study of fractional and integro-differential equations involving generalized fractional derivatives and nonlocal conditions. Existence, uniqueness, and mild solution results have been established using fixed point techniques, semigroup theory, and related analytical tools; see [3, 9, 14, 26, 38].

Motivated by the interesting works [10, 18, 33], we investigate the existence and uniqueness of solutions of a class of fractional Cauchy problem for the following abstract evolution equation:

$$\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[\digamma] = \mathscr{M}\digamma(\Delta) + \chi(\Delta, \digamma, \mathscr{G}\digamma, \mathscr{H}\digamma), \text{if } \Delta \in J = (0, b], \tag{1.4}$$

together with Cauchy condition:

$$\hbar(0)\digamma(0) = \digamma_0, \hbar(0) \neq 0, \tag{1.5}$$

such that the term $\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)$ is the weighted ς–Caputo-type FD of order $0 < \ell \leq 1$ such that $\hbar$ is positive and continuous and $\mathscr{M}$, with domain $D(\mathscr{M}) \subset \mathscr{U} \subset \mathscr{U}$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{\mathscr{T}(\Delta)\}\,(\Delta \geq 0)$ on a Banach space $\mathscr{U}$. The nonlinearity $\chi : J \times \mathscr{U} \times \mathscr{U} \times \mathscr{U} \longrightarrow \mathscr{U}$ is a function that includes the Volterra generalized integral operator $\mathscr{G}$ and the Fredholm generalized integral operator $\mathscr{H}$ with singular kernels as given by

$$(\mathscr{G}\digamma)(\Delta) = \left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,\hbar]}\right)[\mathcal{Y}(\Delta, \cdot)\digamma(\cdot)]$$

and

$$(\mathscr{H}\digamma)(\Delta) = \left(\mathcal{J}_{0,b}^{\eta,[\varsigma,\hbar]}\right)[h(\Delta, \cdot)\digamma(\cdot)], 0 < \eta < \ell, \tag{1.6}$$

where $\mathcal{Y} \in C(\Lambda, \mathbb{R}^+)$, $\Lambda = \{(\Delta, \zeta) \in \mathbb{R}^2 \mid 0 \leq \zeta < \Delta \leq b\}$, $h \in C(\overline{\Lambda}, \mathbb{R}^+)$ and $\overline{\Lambda} = \{(\Delta, \zeta) \in \mathbb{R}^2 \mid 0 \leq \Delta, \zeta \leq b\}$.

The organization of the paper is as follows: Section 2 contains the preliminaries that will be used throughout the article. Section 3 discusses the existence and uniqueness of a mild solution to an abstract nonlocal Cauchy problem involving a class of fractional evolution equations with weighted ς-Caputo derivatives. This is achieved through a monotone adaptive approach, coupled with the concepts of upper and lower solutions. The results are derived using the notion of the measure of noncompactness, semigroup theory applied in cases where the C_0 semigroup is either compact or noncompact and a generalized Gronwall inequality for fractional differential equations, along with an analysis of the continuous dependence of the mild solutions. Finally, Section 4 provides examples that demonstrate the application of the derived results and highlights their significance.

2. Basic definitions and preliminaries

Notation

Throughout the paper, $\mathscr{U}$ denotes an ordered Banach space with norm $\|\cdot\|$ and positive cone $\mathcal{P}$. The symbol ς always denotes a strictly increasing differentiable function (the kernel), and $\hbar$ always denotes a positive continuous weight function with $\hbar(\Delta) \neq 0$. We write $\hbar^{-1}(\Delta) = 1/\hbar(\Delta)$ and $\tilde{\varsigma}(\Delta,\zeta) = \varsigma(\Delta) - \varsigma(\zeta)$. The operators $\mathscr{G}$ and $\mathscr{H}$ denote the generalized Volterra and Fredholm integral operators, respectively, as defined in (1.6). The operators $\mathscr{O}_\ell$, $\tilde{\mathbb{T}}_\ell$, $\mathbb{S}_\ell$, and $\tilde{\mathbb{S}}_\ell$ are solution operators associated with the weighted fractional Cauchy problem, defined in (2.7)–(2.8) and (3.12)–(3.13), respectively. We use $\mu(\cdot)$ for the Kuratowski measure of noncompactness, $\mathcal{E}_\ell$ for the Mittag-Leffler function, and p_ℓ, Ψ_ℓ for the Wright-type probability density functions. The notation $\|\cdot\|_\infty^\hbar$ denotes the weighted supremum norm. The shorthand $\chi(F) = \chi(\Delta, F(\Delta), (\mathscr{G}F)(\Delta), (\mathscr{H}F)(\Delta))$ is used throughout Section 3.

The following lemmas and definitions are necessary in order to support our main findings.

Let $\mathscr{U}$ be an ordered Banach space with the norm $\|.\|$, whose positive cone

$$\mathcal{P} = \{\mathscr{J} \in \mathscr{U} : \mathscr{J} \geqslant 0\}$$

is normal with normal constant $\mathrm{N} > 0$. Let $C(J, \mathscr{U})$ denote the Banach space of all continuous $\mathscr{U}$−valued functions on interval J, for all $\Delta \in J$.

Definition 2.1. The cone $\mathcal{P}$ is said to be:

- *Normal* whenever a constant $(\mathrm{N} > 0)$ exists so that $\|\mathscr{J}_1\| \leq \mathrm{N}\|\mathscr{J}_2\|$, if $0 \leq \mathscr{J}_1 \leq \mathscr{J}_2$, for all $\mathscr{J}_1, \mathscr{J}_2 \in \mathscr{U}$.
- (R) *regular* whenever every non-decreasing sequence that is bounded above is convergent. In other words, if $\{\mathscr{J}_n\}_{n\geqslant 1}$ is a sequence such that

$$\mathscr{J}_1 \leq \mathscr{J}_2 \leq \ldots \leq \mathscr{L},$$

for some $\mathscr{L} \in \mathscr{U}$, then there is $\mathscr{J} \in \mathscr{U}$ which implies $\lim_{n\longrightarrow\infty}\|\mathscr{J}_n - \mathscr{J}\| = 0$.

Definition 2.2. [2] The weighted fractional integral of order ℓ of F is defined by

$$\left(\mathcal{J}_{a,\Delta}^{\ell,[\varsigma,\hbar]}\right)[F] = \frac{\hbar^{-1}(\Delta)}{\Gamma(\ell)} \int_a^\Delta (\varsigma(\Delta) - \varsigma(\zeta))^{\ell-1}\, \hbar(\zeta)F(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta;\ \Delta \in [a,b], \tag{2.1}$$

where $\hbar(\Delta) \neq 0$ is a weighted function, $\hbar^{-1}(\Delta) = 1/\hbar(\Delta)$ and ς is a strictly increasing differentiable function. A dual form of the integrals (2.1) is

$$\left(\mathcal{J}_{\Delta,b}^{\ell,[\varsigma,\hbar]}\right)[F] = \frac{\hbar(\Delta)}{\Gamma(\ell)} \int_\Delta^b (\varsigma(\zeta) - \varsigma(\Delta))^{\ell-1}\, \hbar^{-1}(\zeta)F(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta, \tag{2.2}$$

where $\varsigma(\Delta)$ is assumed to be sufficiently smooth on $\Delta \in [a,b]$.

Definition 2.3. [2, 18] Let $\ell \geqslant 0$ and $n = [\ell] + 1$. The weighted Caputo FD of order ℓ of F is defined by

$$\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[F] = \frac{\hbar^{-1}(\Delta)}{\Gamma(n-\ell)} \int_0^\Delta (\varsigma(\Delta) - \varsigma(\zeta))^{n-\ell-1}\, F_{\varsigma,\hbar}^{[n]}(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta, \tag{2.3}$$

where

$$F_{\varsigma,\hbar}^{[k]}(\Delta) = \left[\frac{D_\Delta}{\varsigma'(\Delta)}\right]^k (\hbar(\Delta)F(\Delta)),\, k = 0, 1, 2, \ldots.$$

Lemma 2.1. [18]

$$\left(\mathcal{J}_{a,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left({}^{c}\mathcal{D}_{0,\zeta}^{\ell,[\varsigma,\hbar]}\right)[\mathcal{F}] = \mathcal{F}(\Delta) - \hbar^{-1}(\Delta)\sum_{k=0}^{n-1}\frac{(\varsigma(\Delta)-\varsigma(0))^{k}}{k!}\mathcal{F}_{k}(0), \tag{2.4}$$

where

$$\mathcal{F}_{k}(0) = \mathcal{F}_{\varsigma,\hbar}^{[k]}(\Delta)\Big|_{\Delta=0}.$$

Definition 2.4. [18] Let $\mathcal{F}$ and $\hbar$ be functions defined on the interval $[0,\infty)$ and $\mathcal{Y}$ be a strictly increasing function on $[0,\infty)$. Then the weighted Laplace transform of $\mathcal{F}$ is defined by

$$\mathcal{L}_{\mathcal{Y}}^{\hbar}\{\mathcal{F}\}(\zeta) = \int_{0}^{\infty} e^{-\zeta(\varsigma(\Delta)-\varsigma(0))}\hbar(\Delta)\mathcal{F}(\Delta)\varsigma'(\Delta)\mathrm{d}\Delta.$$

Definition 2.5. [18] The weighted convolution of functions $\mathcal{F}$ and h is defined by

$$\begin{aligned}\mathcal{F} *_{\varsigma}^{\hbar} h(\Delta) = \hbar^{-1}(\Delta)\int_{0}^{\Delta} & \hbar\left(\varsigma^{-1}\left(\varsigma(\Delta)+\varsigma(0)-\varsigma(\zeta)\right)\right)\\ & \times\mathcal{F}\left(\varsigma^{-1}\left(\varsigma(\Delta)+\varsigma(0)-\varsigma(\zeta)\right)\right)\hbar(\zeta)h(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta,\end{aligned}$$

where ς^{-1} is the inverse of ς.

In this paper, we consider semigroup for which $\mathscr{T}(\Delta)$ converges strongly to zero as $\Delta \longrightarrow \infty$. By the uniform boundedness principle, we have

$$\|\mathscr{T}(\Delta)\| \leq \mathrm{M}; \quad \text{where } \Delta \geq 0, \mathrm{M} \geq 1. \tag{2.5}$$

We note that if $\mathscr{M}$ is the infinitesimal generator of an analytic semigroup, then, for $\mathrm{C} \geq 0$ large enough, $(\mathscr{M} - \mathrm{C}I)$ is invertible and generates a bounded analytic semigroup

$$S(\Delta) = e^{-\mathrm{C}\Delta}\mathscr{T}(\Delta) \quad \text{on } \mathscr{U}; \quad \text{where } \Delta \geq 0. \tag{2.6}$$

We set

$$\mathscr{O}_{\ell}(\Delta,\zeta)\mathcal{F} = \int_{0}^{\infty} p_{\ell}(\flat)\,\mathscr{T}\left((\varsigma(\Delta)-\varsigma(\zeta))^{\ell}\flat\right)\mathcal{F}\mathrm{d}\flat \tag{2.7}$$

and

$$\tilde{\mathbb{T}}_{\ell}(\Delta,\zeta)\mathcal{F} = \ell\int_{0}^{\infty} \flat p_{\ell}(\flat)\,\mathscr{T}\left((\varsigma(\Delta)-\varsigma(\zeta))^{\ell}\flat\right)\mathcal{F}\mathrm{d}\flat, \mathcal{F} \in \mathscr{U}, \tag{2.8}$$

where $p_{\ell}(\Delta)$ is indeed a probability density function on $(0,\infty)$ so that

$$\int_{0}^{\infty} e_{\ell}^{-\Xi\varpi}\Psi_{\ell}(\varpi)\mathrm{d}\varpi = e^{-\Xi^{\ell}}, \text{ for } 0 < \ell < 1, \tag{2.9}$$

and

$$\Psi_{\ell}(\Delta) = \frac{\ell}{\Delta^{\ell+1}}p_{\ell}\left(\Delta^{-\ell}\right) \text{ or } p_{\ell}(\Delta) = \frac{1}{\ell}\Delta^{-1-\frac{1}{\ell}}\Psi\left(\Delta^{-\frac{1}{\ell}}\right) \geq 0, \tag{2.10}$$

with the function Ψ_{ℓ} defined by

$$\Psi_{\ell}(\varpi) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\varpi)^{-k\ell-1}\Gamma(\ell k+1)}{k!}\sin(k\pi\ell), \ \varpi \in (0,\infty).$$

Lemma 2.2. [39] *The operators $\mathscr{O}_\ell(\Delta,0)$ and $\tilde{\mathbb{T}}_\ell(\Delta,\zeta)$ possess the aforementioned characteristics:*
(i) *Fix $0\le\zeta\le\Delta$, $\mathscr{O}_\ell(\Delta,0)$ and $\tilde{\mathbb{T}}_\ell(\Delta,\zeta)$ are bounded linear operators with*

$$\|\mathscr{O}_\ell(\Delta,0)\|_\infty\le \mathrm{M} \text{ and } \|\mathscr{O}_\ell(\Delta,\zeta)\|_\infty\le\frac{\mathrm{M}}{\Gamma(\ell)}. \tag{2.11}$$

(ii) *For every $\digamma\in\mathscr{U}$, $\Delta\longrightarrow\mathscr{O}_\ell(\Delta,0)\digamma$ and $\Delta\longrightarrow\tilde{\mathbb{T}}_\ell(\Delta,\zeta)\digamma$ are continuous functions from $[0,\infty)$ into $\mathscr{U}$.*
(iii) *The operators $\mathscr{O}_\ell(\Delta,0)$ and $\tilde{\mathbb{T}}_\ell(\Delta,\zeta)$ are strongly continuous for all $0\le\zeta\le\Delta$, for every $\digamma\in\mathscr{U}$ and $0\le\zeta\le\Delta_1<\Delta_2\le b$ we have*

$$\|\mathscr{O}_\ell(\Delta_2,0)\digamma-\mathscr{O}_\ell(\Delta_1,0)\digamma\|_\infty\longrightarrow 0 \text{ and } \|\tilde{\mathbb{T}}_\ell(\Delta_2,\zeta)\digamma-\tilde{\mathbb{T}}_\ell(\Delta_1,\zeta)\digamma\|_\infty\longrightarrow 0 \text{ as } \Delta_1\longrightarrow\Delta_2. \tag{2.12}$$

(iv) *Whenever the semigroup $\mathscr{T}(\Delta)$ is continuous upon operator norm for each $\Delta>0$, then $\mathscr{O}_\ell(\Delta,0)$ and $\tilde{\mathbb{T}}_\ell(\Delta,\zeta)$ are continuous in $(0,+\infty)$ by the operator norm.*

Definition 2.6. [6,15] Suppose that $\mathscr{W}$ is a bounded set of the Banach space $\mathscr{U}$. The Kuratowski measure of noncompactness $\mu(.)$ is defined as

$$\mu(\mathscr{W}):=\inf\left\{\epsilon>0:\mathscr{W}\subset\cup_{i=1}^{n}\mathscr{W}_i,\operatorname{diam}(\mathscr{W}_i)\le\epsilon\right\},$$

where $\operatorname{diam}(\mathscr{W}_i)=\sup\{|\ \mathscr{L}-\mathscr{J}\ |:\ \mathscr{J},\ \mathscr{L}\in\mathscr{W}_i\}$ for $i=1,2,...n\in\mathbb{N}$.

We will use Lemmas 5.5 and 5.7 from [7]; Lemma 2.6 and Theorem 2.8 from [27]; Theorem 2.8 from [11]; and Propositions 1.6–1.7 from [28] respectively.

Lemma 2.3. [28] *Assume that $\{\digamma_n(\Delta)\}\,(n=1,2,...)$ is a sequence of Bochner integrable functions from J into $\mathscr{U}$ with $\|\digamma_n(\Delta)\|<\infty$, then the function $\mathscr{W}(\Delta)=\mu(\{\digamma_n(\Delta)\}_{n=1}^{\infty})$ is Lebesgue integrable on J and satisfies*

$$\mu\left(\left\{\int_0^\Delta \digamma_n(\zeta)\mathrm{d}\zeta\mid n\ge 1\right\}\right)\le 2\int_0^\Delta\mathscr{W}(\zeta)\mathrm{d}\zeta.$$

Definition 2.7. A function $\digamma\in C(J,\mathscr{U})$ satisfies

$$\begin{cases}\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[\underline{\digamma}]\le\mathscr{M}\underline{\digamma}(\Delta)+\chi(\Delta,\underline{\digamma}(\Delta),\mathscr{G}\underline{\digamma}(\Delta),\mathscr{H}\underline{\digamma}(\Delta)), & \text{if } 0<\Delta\le b,\\ \hbar(0)\,\underline{\digamma}(0)\le\digamma_0,\end{cases} \tag{2.13}$$

then $\underline{\digamma}$ is referred to as a lower solution of problem (1.4-1.5).

If every inequality in (2.13) is reversed, then $\overline{\digamma}$ is referred to as the upper solution to the problem 1.4-1.5).

Lemma 2.4. [1,35] *Let $\digamma$, v be two integrable functions and $\mathcal{Y}$ continuous, with domain $[a,b]$. Let $\varsigma\in C^1[a,b]$ an increasing function such that $\varsigma'(\Delta)\neq 0$, $\forall\Delta\in[a,b]$. Assume that*
(i) *$\digamma$ and v are nonnegative;*
(ii) *$\mathcal{Y}$ is nonnegative and nondecreasing. If*

$$\digamma(\Delta)\le v(\Delta)+\mathcal{Y}(\Delta)\int_a^\Delta(\varsigma(\Delta)-\varsigma(\zeta))^{\ell-1}\digamma(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta, \tag{2.14}$$

then

$$F(\Delta) \leq v(\Delta) + \int_a^\Delta \sum_{k=1}^{\infty} \frac{[\mathcal{Y}(\Delta)\Gamma(\ell)]^k}{\Gamma(\ell k)} \left(\varsigma(\Delta) - \varsigma(\zeta)\right)^{\ell k-1} v(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta \tag{2.15}$$

and

$$F(\Delta) \leq v(\Delta)\mathcal{E}_\ell\left(\mathcal{Y}(\Delta)\Gamma(\ell)\left(\varsigma(\Delta) - \varsigma(a)\right)^\ell\right), \ \Delta \in [a,b]. \tag{2.16}$$

In particular, when $v(\Delta) = 0$, then $F(\Delta) = 0$ for all $\Delta \in [a,b]$.

Theorem 2.1. [6] *Let D be a convex, closed, and bounded subset of a Banach space $\mathscr{U}$ and $Q : D \longrightarrow D$ is a μ- contraction mapping. Then the map Q has one fixed point in D.*

Let us list the following assumptions for convenience.

Remark 2.1. Assumptions $(\mathscr{M}_1)$–$(\mathscr{M}_6)$ are natural extensions of classical conditions used in the monotone iterative theory for evolution equations [33, 36]. Assumption $(\mathscr{M}_1)$ ensures the existence of ordered lower and upper bounds for the iteration. Assumptions $(\mathscr{M}_2)$ and (B_2) are one-sided Lipschitz conditions on χ that guarantee the monotonicity of the operator $\mathbb{K}$; they represent two alternative sign conventions for the nonlinearity. Assumption $(\mathscr{M}_3)$ (respectively (B_3)) provides bounds on the integral operators $\mathscr{G}$ and $\mathscr{H}$ consistent with the corresponding monotonicity condition. Assumptions $(\mathscr{M}_4)$ and $(\mathscr{M}_5)$ are standard Lipschitz-type growth conditions needed to control the contraction constant R_1 (respectively R_2). Assumption $(\mathscr{M}_6)$ is the measure-of-noncompactness condition required in the non-compact semigroup case (Theorems 3.3 and 3.5).

$(\mathscr{M}_1)$ There is an upper and a lower solution $\overline{F}_0, \underline{F}_0 \in C(J,\mathscr{U})$ of Cauchy problem (1.4-1.5) respectively, such that $\underline{F}_0 \leq \overline{F}_0$.

$(\mathscr{M}_2)$ χ is a function and there exist $\mathrm{C}_i\,(i=1,2,3) > 0$, such that

$$\begin{aligned}
&\hbar\chi(\Delta, \mathscr{J}_2, \mathscr{L}_2, \gamma_2) - \hbar\chi(\Delta, \mathscr{J}_1, \mathscr{L}_1, \gamma_1)\\
\geqslant& - \mathrm{C}_1\hbar(\Delta)\left(\mathscr{J}_2 - \mathscr{J}_1\right)(\Delta) - \mathrm{C}_2\hbar(\Delta)\left(\mathscr{L}_2 - \mathscr{L}_1\right)(\Delta)\\
&- \mathrm{C}_3\hbar(\Delta)\left(\gamma_2 - \gamma_1\right)(\Delta),
\end{aligned}$$

for any $\Delta \in J$, $\underline{F}_0(\Delta) \leq \mathscr{J}_1 \leq \mathscr{J}_2 \leq \overline{F}_0(\Delta)$, $\mathscr{G}\underline{F}_0(\Delta) \leq \mathscr{L}_1 \leq \mathscr{L}_2 \leq \mathscr{G}\overline{F}_0(\Delta)$ and $\mathscr{H}\underline{F}_0(\Delta) \leq \gamma_1 \leq \gamma_2 \leq \mathscr{H}\overline{F}_0(\Delta)$.

(B_2) χ is a function in $C([0,b] \times \mathscr{U} \times \mathscr{U} \times \mathscr{U}, \mathscr{U})$ and there exist constants $\mathrm{C}_1 \geq 0$ and $\mathrm{C}_i\,(i=2,3) > 0$, such that

$$\begin{aligned}
&\hbar\chi(\Delta, \mathscr{J}_2, \mathscr{L}_2, \gamma_2) - \hbar\chi(\Delta, \mathscr{J}_1, \mathscr{L}_1, \gamma_1)\\
\geqslant&\mathrm{C}_1\hbar(\Delta)\left(\mathscr{J}_2 - \mathscr{J}_1\right)(\Delta) + \mathrm{C}_2\hbar(\Delta)\left(\mathscr{L}_2 - \mathscr{L}_1\right)(\Delta)\\
&+ \mathrm{C}_3\hbar(\Delta)\left(\gamma_2 - \gamma_1\right)(\Delta),
\end{aligned}$$

for any $\Delta \in J$, $\underline{F}_0(\Delta) \leq \mathscr{J}_1 \leq \mathscr{J}_2 \leq \overline{F}_0(\Delta)$, $\mathscr{G}\underline{F}_0(\Delta) \leq \mathscr{L}_1 \leq \mathscr{L}_2 \leq \mathscr{G}\overline{F}_0(\Delta)$ and $\mathscr{H}\underline{F}_0(\Delta) \leq \gamma_1 \leq \gamma_2 \leq \mathscr{H}\overline{F}_0(\Delta)$.

$(\mathscr{M}_3)$ There exist constants $\mathrm{C}_i\,(i=4,5) > 0$ such that

$$\begin{aligned}
&\hbar(\Delta)\left(\mathscr{G}\mathscr{J}_2 - \mathscr{G}\mathscr{J}_1\right)(\Delta) \geq -\mathrm{C}_4\hbar(\Delta)\left(\mathscr{J}_2 - \mathscr{J}_1\right)(\Delta) \ \text{ and}\\
&\hbar(\Delta)\left(\mathscr{H}\mathscr{J}_2 - \mathscr{H}\mathscr{J}_1\right)(\Delta) \geq -\mathrm{C}_5\hbar(\Delta)\left(\mathscr{J}_2 - \mathscr{J}_1\right)(\Delta).
\end{aligned} \tag{2.17}$$

(B_3) There is a constant $0 \leq \mathrm{C}_4 \leq \frac{\mathrm{C}_1}{\mathrm{C}_2}$ and $0 \leq \mathrm{C}_5 \leq \frac{\mathrm{C}_1}{\mathrm{C}_3}$ such that (2.17) hold.

($\mathscr{M}_4$) There are nonnegative constants l_1, l_2, l_3 such that

$$\hbar(\Delta)\left(\chi(\Delta,\mathscr{J}_2,\mathscr{L}_2,\gamma_2)-\chi(\Delta,\mathscr{J}_1,\mathscr{L}_1,\gamma_1)\right)\le l_1\hbar(\Delta)(\mathscr{J}_2-\mathscr{J}_1)+l_2\hbar(\Delta)(\mathscr{L}_2-\mathscr{L}_1)+l_3\hbar(\Delta)(\gamma_2-\gamma_1),$$

for any $\Delta\in J$, $\underline{F}_0(\Delta)\le\mathscr{J}_1\le\mathscr{J}_2\le\overline{F}_0(\Delta)$, $\mathscr{G}\underline{F}_0(\Delta)\le\mathscr{L}_1\le\mathscr{L}_2\le\mathscr{G}\overline{F}_0(\Delta)$ and $\mathscr{H}\underline{F}_0(\Delta)\le\gamma_1\le\gamma_2\le\mathscr{H}\overline{F}_0(\Delta)$.

($\mathscr{M}_5$) There are nonnegative constants l_4, l_5 such that

$$\hbar(\Delta)(\mathscr{G}\mathscr{J}_2-\mathscr{G}\mathscr{J}_1)(\Delta)\le l_4\hbar(\Delta)(\mathscr{J}_2-\mathscr{J}_1)(\Delta)\ \text{and}$$
$$\hbar(\Delta)(\mathscr{H}\mathscr{J}_2-\mathscr{H}\mathscr{J}_1)(\Delta)\le l_5\hbar(\Delta)(\mathscr{J}_2-\mathscr{J}_1)(\Delta).$$

($\mathscr{M}_6$) There exist nonnegative constants l_{11}, l_{12} l_{13} such that for any bounded sets $\mathscr{W}_1,\mathscr{W}_2,\mathscr{W}_3\subset\mathscr{U}$,

$$\mu(\hbar(\Delta)\chi(\Delta,\mathscr{W}_1,\mathscr{W}_2,\mathscr{W}_3))\le l_{11}\mu(\hbar\mathscr{W}_1)+l_{12}\mu(\hbar\mathscr{W}_2)+l_{13}\mu(\hbar\mathscr{W}_3),\ \text{for}\ \Delta\in J.$$

For convenience, we write

$$\hbar:=\hbar(\Delta),\ \tilde{\varsigma}(\Delta,\zeta)=\varsigma(\Delta)-\varsigma(\zeta)\ \text{and}\ \ \chi(F)=\chi(\Delta,F(\Delta),(\mathscr{G}F)(\Delta),(\mathscr{H}F)(\Delta)).$$

3. Main results

In this section, we state and prove our main results.

3.1. Representation formula of mild solutions

To study the nonlinear problem (1.4-1.5), we first consider the associated linear problem:

$$\begin{cases}\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[F]=\mathscr{M}F(\Delta)+\chi(\Delta), & \Delta\in(0,b),\\ (\hbar F)(0)=F_0\in\mathscr{U}.\end{cases}\tag{3.1}$$

Lemma 3.1. *Assume $\chi\in L^1(J,\mathscr{U})$ and $0<\ell<1$. The mild solution of the linear Cauchy problem (3.1) is given by*

$$F(\Delta)=\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)\hbar(0)F(0)+\hbar^{-1}(\Delta)\int_0^\Delta(\varsigma(\Delta)-\varsigma(\zeta))^{\ell-1}\,\tilde{\mathbb{T}}_\ell(\Delta,\zeta)(\hbar\chi)(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\tag{3.2}$$

where χ is an abstract function defined on $[0,\infty)$ with values in $\mathscr{U}$ and the operators $\mathscr{O}_\ell(\Delta,0)$ and $\tilde{\mathbb{T}}_\ell(\Delta,\zeta)$ are defined by (2.7) and (2.8) respectively.

This shows that for $0<\ell<1$ qualitative behaviours of $\mathscr{T}(\Delta)$, $\mathscr{O}_\ell(\Delta,0)$ are crucial, in order to obtain information about $F(\Delta)$.

Proof. By making use of the weighted Laplace transform to (3.1), we get

$$\mathcal{L}_\varsigma^\hbar\{F\}(\Xi)=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}\hbar(0)F(0)+(\Xi^\ell-\mathscr{M})^{-1}\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi),\tag{3.3}$$

whenever $\Xi^\ell\in\rho(\mathscr{M})$. Therefore, we require a family of bounded and linear operators that can be transformed by Laplace and is strongly continuous, such as $\mathscr{O}_\ell(\Delta)$, so that

$$\hat{\mathbb{T}}_\ell(\Xi)=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}\ \text{for all}\ \Xi>0.\tag{3.4}$$

As an example, when $\ell = 1$, our investigation is precisely in line with the existence of a C_0-semigroup (or perhaps comparable, well posedness of $\mathcal{F}' = \mathscr{M}\mathcal{F}$), say $\mathscr{T}(\Delta)$.

$$\hat{T}(\Xi)\mathscr{J} = \int_0^\infty e^{-\Xi\zeta}\mathscr{T}(\zeta)\mathscr{J}\mathrm{d}\zeta = (\Xi - \mathscr{M})^{-1}\mathscr{J}, \text{ for all } \mathscr{J} \in \mathscr{U}. \tag{3.5}$$

We deduce that

$$\Xi^{\ell-1}(\Xi^\ell - \mathscr{M})^{-1}(\hbar\mathcal{F})(0) = \Xi^{\ell-1}\int_0^\infty e^{-\Xi^\ell\zeta}\mathscr{T}(\zeta)\mathcal{F}_0\mathrm{d}\zeta. \tag{3.6}$$

Putting $\zeta = (\varsigma(\Delta) - \varsigma(0))^\ell$ in (3.6) gives

$$\begin{aligned}
&= \int_0^\infty \ell\Xi^{\ell-1}(\varsigma(\Delta) - \varsigma(0))^{\ell-1} e^{-(\Xi(\varsigma(\Delta)-\varsigma(0)))^\ell}\mathscr{T}((\varsigma(\Delta) - \psi(0))^\ell)\mathcal{F}_0\varsigma'(\Delta)\mathrm{d}\Delta \\
&= \int_0^\infty -\frac{1}{\Xi}\frac{\mathrm{d}}{\mathrm{d}\Delta}\left(e^{-(\Xi(\varsigma(\Delta)-\varsigma(0)))^\ell}\right)\mathscr{T}((\varsigma(\Delta) - \varsigma(0))^\ell)\mathcal{F}_0\varsigma'(\Delta)\mathrm{d}\Delta \\
&= \int_0^\infty\int_0^\infty -\frac{1}{\Xi}\frac{\mathrm{d}}{\mathrm{d}\Delta}e^{-\Xi(\varsigma(\Delta)-\varsigma(0))\varpi}p_\ell(\varpi)\mathrm{d}\varpi\,\mathscr{T}((\varsigma(\Delta) - \varsigma(0))^\ell)\mathcal{F}_0\mathrm{d}\Delta \\
&= \int_0^\infty\int_0^\infty \varpi e^{-\Xi(\psi(\Delta)-\varsigma(0))\varpi}p_\ell(\varpi)\mathscr{T}((\varsigma(\Delta) - \varsigma(0))^\ell)\mathcal{F}_0\varsigma'(\Delta)\mathrm{d}\varpi\mathrm{d}\Delta.
\end{aligned} \tag{3.7}$$

Setting $\tilde{\varsigma}(\flat, 0) = (\varsigma(\Delta) - \varsigma(0))\varpi$ in (3.7) yields

$$\begin{aligned}
&= \int_0^\infty\int_0^\infty e^{-\Xi\tilde{\varsigma}(\flat,0)}p_\ell(\varpi)\mathscr{T}\left(\tfrac{\tilde{\varsigma}(\flat,0)\ell}{\varpi^\ell}\right)\mathcal{F}_0\varsigma'(\flat)\mathrm{d}\varpi\mathrm{d}\flat \\
&= \int_0^\infty e^{-\Xi\tilde{\varsigma}(\flat,0)}\left[\int_0^\infty \Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\mathcal{F}_0\mathrm{d}\varpi\right]\varsigma'(\flat)\mathrm{d}\flat \\
&= \int_0^\infty e^{-\Xi\tilde{\varsigma}(\flat,0)}\left[\hbar^{-1}(\flat)\int_0^\infty \Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\mathcal{F}_0\mathrm{d}\varpi\right]\hbar(\flat)\varsigma'(\flat)\mathrm{d}\flat
\end{aligned}$$

and, we have

$$\begin{aligned}
&(\Xi^\ell - \mathscr{M})^{-1}\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi) \\
&= \int_0^\infty e^{-\Xi^\ell\zeta}\mathscr{T}(\zeta)\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi)\mathrm{d}\zeta \\
&= \int_0^\infty\int_0^\infty \ell\left(\frac{\tilde{\varsigma}(\flat,0)}{\varpi}\right)^{\ell-1}e^{-\Xi\tilde{\varsigma}(\flat,0)}p_\ell(\varpi)\mathscr{T}\left(\left(\frac{\tilde{\varsigma}(\flat,0)}{\varpi}\right)^\ell\right)\frac{\varsigma'(\zeta)}{\varpi}\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi)\mathrm{d}\varpi\mathrm{d}\flat.
\end{aligned} \tag{3.8}$$

This implies that

$$\begin{aligned}
&= \int_0^\infty\int_0^\infty e^{-\Xi\tilde{\varsigma}(\flat,0)}\ell\varpi(\tilde{\varsigma}(\flat,0))^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\varsigma'(\flat)\mathrm{d}\varpi\mathrm{d}\flat\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi) \\
&= \int_0^\infty e^{-\Xi(\tilde{\varsigma}(\flat,0))}\left[\ell\hbar^{-1}(\flat)\int_0^\infty \varpi(\tilde{\varsigma}(\flat,0))^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\mathrm{d}\varpi\right]\hbar(\flat)\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi)\varsigma'(\flat)\mathrm{d}\flat \\
&= \mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\flat)\int_0^\infty \ell\varpi(\tilde{\varsigma}(\flat,0))^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\mathrm{d}\varpi\right\}(\Xi)\times\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi) \\
&= \mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\flat)\int_0^\infty \ell\varpi(\tilde{\varsigma}(\flat,0)))^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left((\tilde{\varsigma}(\flat,0))^\ell\varpi\right)\mathrm{d}\varpi *_\varsigma^\hbar \chi\right\}(\Xi).
\end{aligned}$$

Now, we get the solution using the inverse transform as

$$\hbar^{-1}(\flat)\int_0^\infty \ell\varpi\left(\tilde{\varsigma}(\flat,0)\right)^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left(\left(\tilde{\varsigma}(\flat,0)\right)^{\ell}\varpi\right)\mathrm{d}\varpi *_\varsigma^\hbar \chi(\flat) \tag{3.9}$$
$$=\hbar^{-1}(\Delta)\int_0^\Delta\int_0^\infty \ell\varpi\left(\tilde{\varsigma}(\Delta,\flat)\right)^{\ell-1}\Psi_\ell(\varpi)\mathscr{T}\left(\left(\tilde{\varsigma}(\Delta,\flat)\right)^{\ell}\varpi\right)\mathrm{d}\varpi\,(\hbar\chi)(\flat)\varsigma'(\flat)\mathrm{d}\flat.$$

On the other hand, we can confirm right away that the function $F\in C(J,\mathscr{U})$ given by (3.2) is a mild solution of problem (3.1). □

Lemma 3.2. *Suppose that $\mathscr{M}$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{\mathscr{T}(\Delta)\}$ $(\Delta\geqslant 0)$ in $\mathscr{U}$ for $0<\ell<1$, then*

$$\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left(\hbar^{-1}(.)\mathscr{O}_\ell(.,0)F_0\right)=\mathscr{M}\left(\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)F_0\right),$$
$$\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left(\hbar^{-1}(\Delta)\int_0^\Delta\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\ell-1}\tilde{\mathbb{T}}_\ell(\Delta,\zeta)\,(\hbar\chi)(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\right)$$
$$=\mathscr{M}\left(\hbar^{-1}(\Delta)\int_0^\Delta\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\ell-1}\tilde{\mathbb{T}}_\ell(\Delta,\zeta)\,(\hbar\chi)(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\right)+\chi(\Delta).$$

Proof. According to Lemma 2.4 in paper [33], (3.4) and Laplace transform, it is obvious to see that

$$\mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)F_0\right\}(\Xi)=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}F_0 \tag{3.10}$$

and

$$\begin{aligned}\mathcal{L}_\varsigma^\hbar\left\{\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left(\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)F_0\right)\right\}(\Xi)&=\Xi^\ell\mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)F_0\right\}(\Xi)-\Xi^{\ell-1}F_0\\&=\Xi^\ell\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}F_0-\Xi^{\ell-1}F_0\\&=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}\left[\Xi^\ell-(\Xi^\ell-\mathscr{M})\right]F_0\\&=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}\left[\Xi^\ell-\Xi^\ell+\mathscr{M}\right]F_0\\&=\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}\mathscr{M}F_0\\&=\mathscr{M}\Xi^{\ell-1}(\Xi^\ell-\mathscr{M})^{-1}F_0.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left(\hbar^{-1}(.)\mathscr{O}_\ell(.,0)F_0\right)=\mathscr{M}\left(\hbar^{-1}(\Delta)\mathscr{O}_\ell(\Delta,0)F_0\right).$$

Similarly, we have

$$\mathcal{L}_\varsigma^\hbar\left\{\left({}^c\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left[\hbar^{-1}(\Delta)\int_0^\Delta\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\ell-1}\tilde{\mathbb{T}}_\ell(\Delta,\zeta)\,(\hbar\chi)(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\right]\right\}(\Xi)$$
$$=\Xi^\ell\mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\Delta)\int_0^\Delta\left(\tilde{\varsigma}(\Delta,\zeta))\right)^{\ell-1}\tilde{\mathbb{T}}_\ell(\Delta,\zeta)\,(\hbar\chi)(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\right\}(\Xi)-\Xi^{\ell-1}.0$$
$$=\Xi^\ell\mathcal{L}_\varsigma^\hbar\left\{\hbar^{-1}(\Delta)\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\ell-1}\tilde{\mathbb{T}}_\ell(\Delta,\zeta)*_\varsigma^\hbar\chi\right\}(\Xi)$$
$$=\Xi^\ell\left(\Xi^\ell-\mathscr{M}\right)^{-1}\mathcal{L}_\varsigma^\hbar\{\chi\}(\Xi)$$

$$
\begin{aligned}
&= \left(\Xi^{\ell} - \mathscr{M} + \mathscr{M}\right)\left(\Xi^{\ell} - \mathscr{M}\right)^{-1} \mathcal{L}_{\varsigma}^{\hbar}\{\chi\}(\Xi) \\
&= \mathscr{M}\left(\Xi^{\ell} - \mathscr{M}\right)^{-1} \mathcal{L}_{\varsigma}^{\hbar}\{\chi\}(\Xi) + \mathcal{L}_{\varsigma}^{\hbar}\{\chi\}(\Xi).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left(\hbar^{-1}(\Delta)\int_{0}^{\Delta}(\tilde{\varsigma}(\Delta,\zeta))^{\ell-1}\,\tilde{\mathbb{T}}_{\ell}(\Delta,\zeta)\,(\hbar\chi)\,(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta\right) \\
&= \mathscr{M}\hbar^{-1}(\Delta)\int_{0}^{\Delta}(\tilde{\varsigma}(\Delta,\zeta))^{\ell-1}\,\tilde{\mathbb{T}}_{\ell}(\Delta,\zeta)\,(\hbar\chi)\,(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta + \chi(\Delta).
\end{aligned}
$$

□

Throughout this paper, we let $0 < \ell < 1$. Define two families $\{\mathbb{S}_{\ell}(\Delta,0)\}\,(\Delta \geqslant 0)$ and $\left\{\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\right\}(\Delta \geqslant 0)$ of linear operators by

$$
\mathbb{S}_{\ell}(\Delta,0)F = \int_{0}^{\infty} p_{\ell}(\flat)\,S\left((\varsigma(\Delta)-\varsigma(0))^{\ell}\flat\right)F(\flat)\,\mathrm{d}\flat \tag{3.12}
$$

and

$$
\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\,F = \ell\int_{0}^{\infty} \flat p_{\ell}(\flat)\,S\left((\varsigma(\Delta)-\varsigma(\zeta))^{\ell}\flat\right)F(\flat)\,\mathrm{d}\flat, F \in \mathscr{U}, \tag{3.13}
$$

for $0 \leq \Delta, \zeta \leq b$ and $F \in \mathscr{U}$, where p_{ℓ} is a probability density function defined on $(0,\infty)$ and S be defined as in (2.6).

To prove the main theorems of this paper, an equivalent system given below is discussed.

Lemma 3.3. *Assume $\chi \in C(J,\mathscr{U})$ and $0 < \ell < 1$. The linear Cauchy problem's mild solution is:*

$$
\begin{cases}
\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[F] = \mathscr{M}F(\Delta) - \mathrm{C}F(\Delta) + \chi(\Delta), & \text{if } \Delta \in (0,1), \\
(\hbar F)(0) = F_0 \in \mathscr{U},
\end{cases} \tag{3.14}
$$

is given by

$$
F(\Delta) = \hbar^{-1}(\Delta)\mathbb{S}_{\ell}(\Delta,0)\hbar(0)F(0) + \hbar^{-1}(\Delta)\int_{0}^{\Delta}(\varsigma(\Delta)-\varsigma(\zeta))^{\ell-1}\,\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\,(\hbar\chi)\,(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta, \tag{3.15}
$$

where the constant C *is decided by the condition* $(\mathscr{M}_2)$ *or* (B_2)*, χ is an abstract function defined on $[0,\infty)$ and with values in $\mathscr{U}$ and the operators $\mathbb{S}_{\ell}$ and $\tilde{\mathbb{S}}_{\ell}$ are defined by (3.12) and (3.13), respectively.*

Evidently, the problem (1.4)-(1.5) is equivalent to that same problem listed below.

$$
\begin{cases}
\left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[F] = (\mathscr{M} - \mathrm{C}I)\,F(\Delta) + \chi(\Delta, F(\Delta), (\mathscr{G}F)(\Delta), (\mathscr{H}F)(\Delta)) \\
\qquad\qquad + \mathrm{C}F(\Delta), \text{if } \Delta \in (0,1), \\
(\hbar F)(0) = F_0.
\end{cases} \tag{3.16}
$$

Notice that the semigroup generated by $(\mathscr{M} - \mathrm{C}I)$ generated, it is uniformly bounded, positive and equicontinuous of families $\{\mathbb{S}_{\ell}(\Delta,0)\}\,(\Delta \geqslant 0)$ and $\left\{\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\right\}(\Delta,\zeta \geqslant 0)$. Furthermore, they possess the characteristics in Lemma 2.2, and they in particular satisfy

$$
\|\mathbb{S}_{\ell}(\Delta,0)F\|_{\infty} \leq \mathrm{M}\|F\|_{\infty} \text{ and } \|\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)F\|_{\infty} \leq \frac{\mathrm{M}}{\Gamma(\ell)}\|F\|_{\infty}. \tag{3.17}
$$

Definition 3.1. A function $\digamma \in C(J, \mathscr{U})$ is called a mild solution of (3.16) if it satisfies

$$\begin{aligned}
&\hbar(\Delta)\digamma(\Delta) \\
=&\mathbb{S}_{\ell}(\Delta,0)(\hbar\digamma)(0) \\
&+\int_0^{\Delta}(\tilde{\varsigma}(\Delta,\zeta))^{\ell-1}\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\left(\hbar(\zeta)\chi\left(\zeta,\digamma(\zeta),\mathscr{G}\digamma(\zeta),\mathscr{H}\digamma(\zeta)\right)+\mathrm{C}\hbar(\zeta)\digamma(\zeta)\right)\varsigma'(\zeta)\mathrm{d}\zeta,
\end{aligned}$$

where the operators $\mathbb{S}_{\ell}$ and $\tilde{\mathbb{S}}_{\ell}$ are defined by (3.12) and (3.13) respectively.

3.2. Existence and uniqueness results on mild solution

Let $D = [\underline{\digamma}_0, \overline{\digamma}_0] = \left\{\digamma \in C(J,\mathscr{U}) : \underline{\digamma}_0 \leq \digamma \leq \overline{\digamma}_0\right\}$ in addition, we establish an operator $\mathbb{K}$ mapping from D into $C(J,\mathscr{U})$ as

$$\hbar(\Delta)(\mathbb{K}\digamma)(\Delta) = \mathbb{S}_{\ell}(\Delta,0)(\hbar\digamma)(0) + \int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\left[\hbar\chi(\digamma) + \mathrm{C}(\hbar\digamma)(\zeta)\right]\mathrm{d}\zeta. \tag{3.18}$$

The problem (3.16) can be viewed as equivalent to the problem (3.18), where

$$\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta) = (\varsigma(\Delta)-\varsigma(\zeta))^{\ell-1}\tilde{\mathbb{S}}_{\ell}(\Delta,\zeta)\varsigma'(\zeta) \tag{3.19}$$

and

$$m_{(\ell,\varsigma)} = \max\left\{\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\mathrm{d}\zeta : \Delta \in [0,b]\right\} = \Gamma(\ell)\left\|\tilde{\mathbb{S}}_{\ell}\right\|_{\infty}^{\hbar}\left(\mathcal{J}_{0,b}^{\ell,[\varsigma,1]}\right)\left[\hbar^{-1}(.)\right], \tag{3.20}$$

where $\mathbb{S}_{\ell}$ and $\tilde{\mathbb{S}}_{\ell}$ are given by (3.12) and (3.13) respectively.

Theorem 3.1. *Suppose that $\mathscr{U}$ be an ordered Banach space, whose positive cone $\mathcal{P}$ is normal including normal constant* N. *Take $\mathscr{T}(\Delta)(\Delta \geqslant 0)$ is non-negative and the assumptions ($\mathscr{M}_1$)-($\mathscr{M}_5$) hold and*

$$\mathrm{R}_1 = m_{(\ell,\varsigma)}(l_1 + \mathrm{C} + l_2\max_{\Delta\in(0,b)}\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)] + l_3\max_{\Delta\in(0,b)}\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]) < 1, \tag{3.21}$$

where C *is given by (3.23).*

Therefore the Cauchy problem (3.16) has minimal and maximal mild solutions $\underline{\digamma}$ and $\overline{\digamma}$ between $\underline{\digamma}_0$ and $\overline{\digamma}_0$, which are attainable through the monotone iterative approach starting from $\underline{\digamma}_0$ and $\overline{\digamma}_0$ respectively.

Proof. It is clear that $\mathbb{K}$ is well defined. Then the mild solution of (3.16) is equivalent to the fixed point of $\mathbb{K}$. To determine the fixed point of $\mathbb{K}$, we employ the monotone iterative method of increasing operator.

The proof is divided into the following steps.

Step 1. In D, the map $\mathbb{K}$ is continuous.

Let $\{\digamma_n\}(n = 1,2,...)$ in $\mathscr{U}$ be a sequence such that $\{\digamma_n\}(n = 1,2,...) \longrightarrow \digamma \in \mathscr{U}$ as $n \longrightarrow \infty$. Since $\chi \in C(J \times \mathscr{U} \times \mathscr{U} \times \mathscr{U}, \mathscr{U})$ and $\hbar$ is continuous, the map $\digamma \mapsto \hbar\chi(\digamma)$ is continuous, so according to our supposition ($\mathscr{M}_2$), we have, for every $\Delta \in J$,

$$\hbar\digamma_n - \hbar\digamma \longrightarrow 0 \text{ and } \hbar\chi(\digamma_n) - \hbar\chi(\digamma)(\Delta) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and

$$\|\digamma_n - \digamma\|_\infty^\hbar \leq \frac{\epsilon}{2}, \|\chi(\digamma_n) - \chi(\digamma)\|_\infty^\hbar \leq \frac{\epsilon}{2}, \ \forall \epsilon > 0.$$

For almost every $\Delta \in J$, we get

$$\begin{aligned}
&|\hbar(\Delta)\left(\mathbb{K}\digamma_n\right)(\Delta) - \hbar(\Delta)\left(\mathbb{K}\digamma\right)(\Delta)| \\
=&\left|\int_0^\Delta \left[{}_\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar\chi(\digamma_n) - \hbar\chi(\digamma) + \mathrm{C}\left(\hbar\digamma_n - \hbar\digamma\right)(\zeta)\right]\mathrm{d}\zeta\right| \\
\leq& m_{(\ell,\varsigma)}\left(\|\hbar\chi(\digamma_n) - \hbar\chi(\digamma)\|_\infty + \mathrm{C}\left\|\hbar\digamma_n - \hbar\digamma\right\|_\infty\right) \\
\leq& m_{(\ell,\varsigma)}\left(\|\chi(\digamma_n) - \chi(\digamma)\|_\infty^\hbar + \mathrm{C}\left\|\digamma_n - \digamma\right\|_\infty^\hbar\right),
\end{aligned}$$

with Lebesgue dominated convergence result, which yields

$$\|(\mathbb{K}\digamma_n) - (\mathbb{K}\digamma)\|_\infty^\hbar \leq \epsilon. \tag{3.22}$$

$\|\left(\mathbb{K}\digamma_n\right) - (\mathbb{K}\digamma)\|_\infty^\hbar \longrightarrow 0$, as n tends to ∞. As a result, the map $\mathbb{K}$ is continuous in D.

Step 2. The map $\mathbb{K}$ is monotone increasing.

From $(\mathscr{M}_2)$ and $(\mathscr{M}_5)$, for $\digamma_1, \digamma_2 \in D$ with $\digamma_1 \leq \digamma_2$, we obtain that

$$\begin{aligned}
\hbar\chi(\digamma_2) - \hbar\chi(\digamma_1) \geq& -\mathrm{C}_1\left(\hbar(\Delta)\digamma_2 - \hbar(\Delta)\digamma_1\right) - \mathrm{C}_2\left(\hbar(\Delta)\mathscr{G}\digamma_2 - \hbar(\Delta)\mathscr{G}\digamma_1\right) \\
&-\mathrm{C}_3\left(\hbar(\Delta)\mathscr{H}\digamma_2 - \hbar(\Delta)\mathscr{H}\digamma_1\right) \\
\geq& -\left(\mathrm{C}_1 + \mathrm{C}_2 l_4 + \mathrm{C}_3 l_5\right)\left(\hbar(\Delta)\digamma_2 - \hbar(\Delta)\digamma_1\right).
\end{aligned}$$

We see that

$$\hbar\chi(\digamma_2) + \left(\mathrm{C}_1 + \mathrm{C}_2 l_4 + \mathrm{C}_3 l_5\right)\hbar(\Delta)\digamma_2 \geq \hbar\chi(\digamma_1) + \left(\mathrm{C}_1 + \mathrm{C}_2 l_4 + \mathrm{C}_3 l_5\right)\hbar(\Delta)\digamma_1.$$

If

$$\mathrm{C} = \left(\mathrm{C}_1 + \mathrm{C}_2 \mathrm{l}_4 + \mathrm{C}_3 \mathrm{l}_5\right) \geq 0, \tag{3.23}$$

then $\hbar\chi(\digamma) + \mathrm{C}\hbar(\digamma)$ is nondecreasing on $\digamma \in D$ for all $\Delta \in J$.

By the positivity of operators $\mathbb{S}_\ell(\Delta, 0)$ and $\tilde{\mathbb{S}}_\ell(\Delta,\zeta)$ for $0 \leq \zeta, \Delta \leq b$, we have

$$\begin{aligned}
&\hbar(\Delta)\left(\mathbb{K}\digamma_1\right)(\Delta) \\
=&\mathbb{S}_\ell(\Delta,0)\digamma_0 + \int_0^\Delta \left[{}_\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar\chi(\digamma_1) + \mathrm{C}\left(\hbar\digamma_1\right)(\zeta)\right]\mathrm{d}\zeta \\
\leq&\mathbb{S}_\ell(\Delta,0)\digamma_0 + \int_0^\Delta \left[{}_\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar\chi(\digamma_2) + \mathrm{C}\left(\hbar\digamma_2\right)(\zeta)\right]\mathrm{d}\zeta \\
=&\hbar(\Delta)\left(\mathbb{K}\digamma_2\right)(\Delta),
\end{aligned}$$

which implies

$$\left(\mathbb{K}\digamma_1\right)(\Delta) \leq \left(\mathbb{K}\digamma_2\right)(\Delta). \tag{3.24}$$

It's now sufficient to demonstrate $\underline{\digamma}_0 \leq \mathbb{K}\underline{\digamma}_0$ and $\mathbb{K}\overline{\digamma}_0 \leq \overline{\digamma}_0$.

In regard to this, suppose $\mathcal{F}(\Delta) = \left({}^\varsigma_\hbar\mathcal{D}^\ell_\Delta\right)\left[\underline{\digamma}_0\right] - \mathscr{M}\underline{\digamma}_0 + \mathrm{C}\underline{\digamma}_0$. Following the lines of Definition 2.7, we obtain $\mathcal{F}(\Delta) \leq \chi(\Delta, \underline{\digamma}_0, \mathscr{G}\underline{\digamma}_0, \mathrm{H}\underline{\digamma}_0) + \mathrm{C}\underline{\digamma}_0$ for $\Delta \in J$.

Due to the positivity of operators and the Lemma 3.3, $\mathbb{S}_\ell(\Delta, 0)$ and $\tilde{\mathbb{S}}_\ell(\Delta,\zeta)$ for $0 \leq \zeta, \Delta \leq b$, we get

$$\hbar(\Delta)\underline{\digamma}_0 = \mathbb{S}_\ell(\Delta,0)\digamma_0 + \int_0^\Delta \left[{}_\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar(\zeta)\mathcal{F}(\zeta)\right]\mathrm{d}\zeta$$

$$\leq \mathbb{S}_\ell(\Delta,0)\digamma_0 + \int_0^\Delta \left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar(\zeta)\,\chi(\zeta,\underline{\digamma}_0,\mathscr{G}\underline{\digamma}_0,\mathscr{H}\underline{\digamma}_0)+\mathrm{C}\hbar(\zeta)\,\underline{\digamma}_0\right]\mathrm{d}\zeta$$
$$=\hbar(\Delta)\left(\mathbb{K}\underline{\digamma}_0\right).$$

Hence $\underline{\digamma}_0 \leq \mathbb{K}\underline{\digamma}_0$. Likewise, one can prove that $\mathbb{K}\overline{\digamma}_0 \leq \overline{\digamma}_0$. It further means that for $\digamma \in D$

$$\underline{\digamma}_0 \leq \mathbb{K}\underline{\digamma}_0 \leq \ldots \leq \mathbb{K}\digamma \leq \ldots \leq \mathbb{K}\overline{\digamma}_0 \leq \overline{\digamma}_0. \tag{3.25}$$

Hence, $\mathbb{K}$ is an increasing monotonic operator.

Step 3. $\mathscr{W}$ are bounded in $C(J,\mathscr{U})$.

The mentioned sequences $\left\{\underline{\digamma}_n\right\}$ and $\{\overline{\digamma}_n\}$ are defined. With the help of (3.25), by induction we obtain the equality. Also by continuity of $\left\{\underline{\digamma}_n\right\}$, $\{\overline{\digamma}_n\}$, which yields

$$\underline{\digamma}_n = \mathbb{K}\underline{\digamma}_{n-1} \text{ and } \overline{\digamma}_n = \mathbb{K}\overline{\digamma}_{n-1},\ n\in\mathbb{N}. \tag{3.26}$$

Then, by the monotonicity of $\mathbb{K}$, as a result of this

$$\underline{\digamma}_0 \leq \mathbb{K}\underline{\digamma}_0 = \underline{\digamma}_1 \leq \ldots \leq \underline{\digamma}_n \leq \ldots \leq \overline{\digamma}_n \leq \ldots \leq \mathbb{K}\overline{\digamma}_0 = \overline{\digamma}_1 \leq \overline{\digamma}_0. \tag{3.27}$$

Now, our aim is to get $\left\{\underline{\digamma}_n\right\}$ and $\{\overline{\digamma}_n\}$ are sequences in J that are uniformly convergent.

Let $\mathscr{W} = \left\{\underline{\digamma}_n \mid n\in\mathbb{N}^*\right\}$ and $\mathscr{W}_0 = \left\{\underline{\digamma}_{n-1} \mid n\in\mathbb{N}^*\right\}$. Then $\mathscr{W}_0 = \mathscr{W}\cup\left\{\underline{\digamma}_0\right\}$ and hence $\mu(\mathscr{W}(\Delta)) = \mu(\mathscr{W}_0(\Delta))$ for $\Delta\in J$.

In view of (3.27), since the positive cone $\mathcal{P}$ is normal, then $\mathscr{W}_0$ and $\mathscr{W}$ are bounded in $C(J,\mathscr{U})$.

Step 4. $\mathbb{K}(\mathscr{W})$ is equicontinuous on J.

From Step 2, for any $\digamma\in D$, we have

$$\begin{aligned}\hbar(\Delta)\,\chi(\Delta,\underline{\digamma}_0,\mathscr{G}\underline{\digamma}_0,\mathscr{H}\underline{\digamma}_0)+\mathrm{C}\hbar(\Delta)\,\underline{\digamma}_0 &\leq \hbar(\Delta)\,\chi(\Delta,\digamma,\mathscr{G}\digamma,\mathscr{H}\digamma)+\mathrm{C}\hbar(\Delta)\,\digamma(\Delta)\\ &\leq \hbar(\Delta)\,\chi(\Delta,\overline{\digamma}_0,\mathscr{G}\overline{\digamma}_0,\mathscr{H}\overline{\digamma}_0)+\mathrm{C}\hbar(\Delta)\,\overline{\digamma}_0.\end{aligned}$$

By the normality of the positive cone $\mathcal{P}$, there exists a constant $L>0$ such that

$$\|\chi(\Delta,\digamma,\mathscr{G}\digamma,\mathscr{H}\digamma)+\mathrm{C}\digamma\|_\infty^\hbar = \|\hbar\chi+\mathrm{C}\hbar\digamma\|_\infty \leq \left(\|\chi\|_\infty^\hbar + \mathrm{C}\,\|\digamma\|_\infty^\hbar\right) = L \text{ for } \digamma\in D. \tag{3.28}$$

For any $\underline{\digamma}_n\in\mathscr{W}$ and $0\leq\Delta_1<\Delta_2\leq b$, we have

$$\begin{aligned}&\left|\hbar(\Delta_2)\left(\mathbb{K}\underline{\digamma}_n\right)(\Delta_2)-\hbar(\Delta_1)\left(\mathbb{K}\underline{\digamma}_n\right)(\Delta_1)\right|\\
&\leq |\mathbb{S}_\ell(\Delta_2,0)\digamma_0-\mathbb{S}_\ell(\Delta_1,0)\digamma_0|\\
&\quad+\left|\int_0^{\Delta_2}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\chi(\underline{\digamma}_n)+\mathrm{C}\hbar\underline{\digamma}_n\right](\zeta)\mathrm{d}\zeta\right.\\
&\quad\left.-\int_0^{\Delta_1}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_1,\zeta)\left[\hbar\chi(\underline{\digamma}_n)+\mathrm{C}\hbar\underline{\digamma}_n\right](\zeta)\mathrm{d}\zeta\right|\\
&\leq |\mathbb{S}_\ell(\Delta_2,0)\digamma_0-\mathbb{S}_\ell(\Delta_1,0)\digamma_0|\\
&\quad+\left|\int_0^{\Delta_1}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\chi(\underline{\digamma}_n)+\mathrm{C}\hbar\underline{\digamma}_n\right](\zeta)\mathrm{d}\zeta\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Delta_1}^{\Delta_2}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta\\
&-\int_{0}^{\Delta_1}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_1,\zeta)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta\Bigg|\\
\le{}& |\mathbb{S}_\ell(\Delta_2,0)F_0-\mathbb{S}_\ell(\Delta_1,0)F_0|\\
&+\int_{0}^{\Delta_1}\left(\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)-\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_1,\zeta)\right)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta\\
&+\int_{\Delta_1}^{\Delta_2}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta.
\end{aligned}
$$

Denote

$$J_1=|\mathbb{S}_\ell(\Delta_2,0)F_0-\mathbb{S}_\ell(\Delta_1,0)F_0| \text{ and } J_2=\int_{\Delta_1}^{\Delta_2}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta,$$

$$J_3=\int_{0}^{\Delta_1}\left(\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)-\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_1,\zeta)\right)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta.$$

Next, we check if $\|J_i\|$ tends to 0 as $\Delta_2\longrightarrow\Delta_1$ $(i=1,2,3)$, which are not dependent on $F\in\left[\underline{F}_0,\overline{F}_0\right]$.

It is clear that $J_1\longrightarrow 0$ as $\Delta_2\longrightarrow\Delta_1$ By (3.17), we obtain

$$
\begin{aligned}
J_2&=\int_{\Delta_1}^{\Delta_2}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)\left[\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right](\zeta)\mathrm{d}\zeta\\
&\le\left\|\tilde{\mathbb{S}}_\ell\right\|_\infty\left\|\hbar\mathcal{X}(\underline{F}_n)+\mathrm{C}\hbar\underline{F}_n\right\|_\infty\int_{\Delta_1}^{\Delta_2}(\tilde{\varsigma}(\Delta_2,\zeta))^{\ell-1}\varsigma'(\zeta)\mathrm{d}\zeta\\
&\le\left\|\tilde{\mathbb{S}}_\ell\right\|_\infty^\hbar L\int_{\Delta_1}^{\Delta_2}(\tilde{\varsigma}(\Delta_2,\zeta))^{\ell-1}\varsigma'(\zeta)\mathrm{d}\zeta\\
&\le\frac{1}{\ell}\left\|\tilde{\mathbb{S}}_\ell\right\|_\infty^\hbar L\left(\tilde{\varsigma}(\Delta_2,\Delta_1)\right)^\ell.
\end{aligned}
$$

For J_3, we use

$$
\begin{aligned}
J_3=&\int_{0}^{\Delta_1}\left(\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_2,\zeta)-\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta_1,\zeta)\right)\left[\hbar\mathcal{X}(\underline{F})+\mathrm{C}\hbar\underline{F}\right](\zeta)\,\mathrm{d}\zeta\\
\le& L\int_{0}^{\Delta_1}\Big(\tilde{\varsigma}(\Delta_2,\zeta)\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)-\tilde{\varsigma}(\Delta_1,\zeta)\tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\\
&+\tilde{\varsigma}(\Delta_1,\zeta)\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)-\tilde{\varsigma}(\Delta_1,\zeta)\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)\Big)\varsigma'(\zeta)\,\mathrm{d}\zeta\\
\le& L\left(\int_{0}^{\Delta_1}(\tilde{\varsigma}(\Delta_2,\zeta)-\tilde{\varsigma}(\Delta_1,\zeta))\,\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)\varsigma'(\zeta)\,\mathrm{d}\zeta\right.\\
&\left.+\int_{0}^{\Delta_1}\left(\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)-\tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right)\tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\,\mathrm{d}\zeta\right).
\end{aligned}
$$

Consequently

$$J_{31}=\int_{0}^{\Delta_1}(\tilde{\varsigma}(\Delta_2,\zeta)-\tilde{\varsigma}(\Delta_1,\zeta))\,\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta)\varsigma'(\zeta)\ \mathrm{d}\zeta$$

$$\leq \left\|\tilde{\mathbb{S}}_\ell\right\|_\infty \int_0^{\Delta_1} \left(\tilde{\varsigma}(\Delta_2,\zeta) - \tilde{\varsigma}(\Delta_1,\zeta)\right) \varsigma'(\zeta)\, \mathrm{d}\zeta,$$

with

$$\tilde{\varsigma}(\Delta_2,\zeta) - \tilde{\varsigma}(\Delta_1,\zeta) = \left(\varsigma(\Delta_2) - \varsigma(\zeta)\right)^{\ell-1} - \left(\varsigma(\Delta_1) - \varsigma(\zeta)\right)^{\ell-1} \leq \left(\varsigma(\Delta_2) - \varsigma(\Delta_1)\right)^{\ell-1}$$

and hence $J_2 \longrightarrow 0$ and $J_{31} \longrightarrow 0$ as $\Delta_2 \longrightarrow \Delta_1$.

For $\Delta_1 = 0$ and $0 < \Delta_2 \leq b$, it is easy to see that

$$J_{32} = \int_0^{\Delta_1} \left(\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta) - \tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right) \tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\, \mathrm{d}\zeta \leq \frac{1}{\ell} \left\|\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta) - \tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right\|_\infty \tilde{\varsigma}(\Delta_1,0)^\ell.$$

Then $J_{32} = 0$, when $\Delta_1 \neq 0$ and for any $\epsilon \in (0,\Delta_1)$, we have

$$\begin{aligned}
J_{32} &= \int_0^{\Delta_1-\epsilon} \left(\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta) - \tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right) \tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\, \mathrm{d}\zeta \\
&\quad + \int_{\Delta_1-\epsilon}^{\Delta_1} \left(\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta) - \tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right) \tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\, \mathrm{d}\zeta \\
&\leq \sup_{0\leq\zeta<\Delta_1-\epsilon} \left|\tilde{\mathbb{S}}_\ell(\Delta_2,\zeta) - \tilde{\mathbb{S}}_\ell(\Delta_1,\zeta)\right| \int_0^{\Delta_1-\epsilon} \tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\, \mathrm{d}\zeta \\
&\quad + 2\left\|\tilde{\mathbb{S}}_\ell\right\|_\infty \int_{\Delta_1-\epsilon}^{\Delta_1} \tilde{\varsigma}(\Delta_1,\zeta)\varsigma'(\zeta)\, \mathrm{d}\zeta.
\end{aligned}$$

By (3.17), it follows that $J_{32} \longrightarrow 0$ as $\Delta_2 \longrightarrow \Delta_1$ and $\epsilon \longrightarrow 0$. Thus, we obtain

$$\|(\mathbb{K}F)(\Delta_2) - (\mathbb{K}F)(\Delta_1)\| \longrightarrow 0 \text{ independently of } F \in D \text{ as } \Delta_2 \longrightarrow \Delta_1, \tag{3.29}$$

which means that $\mathbb{K}(\mathscr{W})$ is equicontinuous on J.

Step 5. $\mathscr{W}(\Delta) = \{(\mathbb{K}F)(\Delta) : F \in D\}, \Delta \in J$, is relatively compact in $\mathscr{U}$.

By Lemma 2.3 and Proposition 1.6 in paper [28], we have

$$\begin{aligned}
\mu(\hbar\mathscr{G}\mathscr{W}_0(\Delta)) &= \mu\left(\left\{\hbar(\Delta)\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,\hbar]}\right)[\mathcal{Y}(\Delta,.)\,F_{n-1}(.)]\right\}_{n=1}^{\infty}\right), \\
\mu(\hbar\mathscr{H}\mathscr{W}_0(\Delta)) &= \mu\left(\left\{\left(\hbar(\Delta)\,\mathcal{J}_{0,b}^{\eta,[\varsigma,\hbar]}\right)[h(\Delta,.)\,F_{n-1}(.)]\right\}_{n=1}^{\infty}\right).
\end{aligned} \tag{3.30}$$

Setting

$$\hbar^{-1}(\Delta) = F_{n-1}(\Delta). \tag{3.31}$$

Then (3.30) is rewritten as

$$\hbar\mathscr{G}\mathscr{W}_0(\Delta) = \hbar(\Delta)\,F_{n-1}(\Delta)\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)] \tag{3.32}$$

and

$$\hbar\mathscr{H}\mathscr{W}_0(\Delta) = \hbar(\Delta)\,F_{n-1}(\Delta)\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]. \tag{3.33}$$

For any $\Delta \in J$, by $(\mathscr{M}_4)$ and Proposition 1.6 in paper [28] and Lemma 2.3 we have

$$\mu(\mathscr{W}(\Delta)) = \mu(\hbar\mathbb{K}\mathscr{W}_0(\Delta)) \tag{3.34}$$

$$
\begin{aligned}
&= \mu\left(\{\hbar F_{n-1}(\Delta)\}_{n-1}^{\infty}\right) \\
&= \mu\left(\left\{\mathbb{S}_{\ell}(\Delta,0)F_0 + \hbar(\Delta)\hbar^{-1}(\Delta)\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\right.\right. \\
&\qquad \times [\hbar\mathcal{X}(F_{n-1}) + \mathrm{C}(\hbar F_{n-1})](\zeta)\,\mathrm{d}\zeta\}_{n=1}^{\infty}) \\
&\le \mu\left(\left\{\hbar(\Delta)\hbar^{-1}(\Delta)\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\right.\right. \\
&\qquad \times (l_1\hbar F_{n-1} + l_2\hbar\mathscr{G}F_{n-1} + l_3\hbar\mathscr{H}F_{n-1} + \mathrm{C}\hbar F_{n-1})(\zeta)\,\mathrm{d}\zeta\}_{n=1}^{\infty}),
\end{aligned}
$$

with (3.20), (3.32), (3.33) and (3.34), we have

$$
\begin{aligned}
&\mu(\mathbb{K}\mathscr{W}_0(\Delta)) \\
&\le \mu\left(\left\{\hbar F_{n-1}(\Delta)\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\big(l_1 + l_2(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]})[\mathcal{Y}] + l_3(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]})[h] + \mathrm{C}\big)(\zeta)\,\mathrm{d}\zeta\right\}_{n=1}^{\infty}\right) \qquad (3.35) \\
&\le m_{(\ell,\varsigma)}\Big(l_1 + l_2(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]})[\mathcal{Y}(\Delta,.)] + l_3(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]})[h(\Delta,.)] + \mathrm{C}\Big)\sup_{\Delta\in J}\mu(\mathscr{W}_0(\Delta)) \\
&\le \mathrm{R}_1\sup_{\Delta\in J}\mu(\mathscr{W}_0(\Delta)).
\end{aligned}
$$

Since $\{\mathbb{K}F_n\}_{n=0}^{\infty}$ is equicontinuous on J, we get

$$\mu(\mathscr{W}) \le \mathrm{R}_1\mu(\mathscr{W}). \qquad (3.36)$$

Since $\mathrm{R}_1 < 1$, By using Proposition 1.7 in paper [28], we have $\mu(\mathscr{W}(\Delta)) = 0$ for $\Delta \in J$. As a result, $\mathscr{W}$ is relatively compact.

Step 6. Let $\underline{F}, \overline{F} \in C(J,\mathscr{U})$.

With the monotonicity (3.27), we can prove that $\{\underline{F}_n\}$ itself is convergent. For any $\Delta \in J$, we see that

$$
\begin{aligned}
\hbar(\Delta)\underline{F}_{n-1}(\Delta) &= \hbar(\Delta)\mathbb{K}\underline{F}_{n-1}(\Delta) \\
&= \mathbb{S}_{\ell}(\Delta,0)F_0 + \int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_{\ell}\right](\Delta,\zeta)\left[\left(\hbar\mathcal{X}(\underline{F}_{n-1}) + \mathrm{C}\hbar\underline{F}_{n-1}\right)\right](\zeta)\mathrm{d}\zeta.
\end{aligned}
$$

By taking n tends to ∞ in the preceding equation, and from Lebesgue dominated convergence result, we get

$$\underline{F} = \mathbb{K}\underline{F} \text{ with } \underline{F} \in C(J,\mathscr{U}).$$

By continuing same procedure as above, we get $\overline{F} = \mathbb{K}\overline{F}$. We notice that

$$\underline{F}_0 \le \underline{F} \le F^* \le \overline{F} \le \overline{F}_0. \qquad (3.37)$$

Now, our aim is to prove $\underline{F}$, $\overline{F}$ are the minimal and maximal fixed points of $\mathbb{K}$ on $[\underline{F}_0, \overline{F}_0]$, respectively. In any case $F^* \in D$ and F^* is a fixed point of $\mathbb{K}$, thus

$$\underline{F}_1 = \mathbb{K}\underline{F}_0 \le \mathbb{K}F^* = F^* \le \mathbb{K}\overline{F}_0 = \overline{F}_1. \qquad (3.38)$$

Induction provides us with $\underline{F}_n \le F^* \le \overline{F}_n$. By utilizing (3.27) and considering limit with n tends to ∞, we conclude that

$$\underline{F} \le F^* \le \overline{F}. \qquad (3.39)$$

Thus, $\underline{\digamma}$ and $\overline{\digamma}$ are minimal and maximal mild solutions of the Cauchy problem (3.16) on $[\underline{\digamma}_0, \overline{\digamma}_0]$, respectively and $\underline{\digamma}_0$, $\overline{\digamma}_0$ can indeed be attained through an iterative process (3.26) beginning with $\underline{\digamma}_0$ and $\overline{\digamma}_0$, respectively. □

Theorem 3.2. *Assume that $\mathscr{U}$ be an ordered Banach space, whose positive cone $\mathcal{P}$ is normal with normal constant* N. *Take $\mathscr{T}(\Delta)(\Delta \geqslant 0)$ is non-negative and the assumptions* $(\mathscr{M}_1)$, (B_2), (B_3) , $(\mathscr{M}_4)$ *and* $(\mathscr{M}_5)$ *hold and*

$$\mathrm{R}_2 = m_{(\ell,\varsigma)}(l_1 + \mathrm{C} + l_2 \max_{\Delta\in(0,b)} \left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)] + l_3 \max_{\Delta\in(0,b)} \left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]) < 1, \tag{3.40}$$

where C *is given by (3.41).*

Then the Cauchy problem (3.16) has minimal and maximal mild solutions $\underline{\digamma}$ and $\overline{\digamma}$ between $\underline{\digamma}_0$ and $\overline{\digamma}_0$.

Proof. Steps 1-2 and 6 are the same as before.

For Step 2: From (B_2) and (B_3), we obtain that

$$\begin{aligned} \hbar\mathcal{X}(\digamma_2) - \hbar\mathcal{X}(\digamma_1) &\geq \mathrm{C}_1 (\hbar(\Delta)\digamma_2 - \hbar(\Delta)\digamma_1) + \mathrm{C}_2 (\hbar(\Delta)\mathscr{G}\digamma_2 - \hbar(\Delta)\mathscr{G}\digamma_1) \\ &\quad + \mathrm{C}_3 (\hbar(\Delta)\mathscr{H}\digamma_2 - \hbar(\Delta)\mathscr{H}\digamma_1) \\ &\geq (\mathrm{C}_1 - \mathrm{C}_2\mathrm{C}_4 - \mathrm{C}_3\mathrm{C}_5)(\hbar(\Delta)\digamma_2 - \hbar(\Delta)\digamma_1) \\ &\geq 0. \end{aligned}$$

If

$$\mathrm{C} = (\mathrm{C}_1 - \mathrm{C}_2\mathrm{C}_4 - \mathrm{C}_3\mathrm{C}_5) \geq 0, \tag{3.41}$$

then $\hbar\mathcal{X}(\digamma)$ is nondecreasing on $\digamma \in D$ for all $\Delta \in J$. Namely,

$$\hbar\mathcal{X}(\digamma_1) \leq \hbar\mathcal{X}(\digamma_2) \text{ for all } \Delta \in J.$$

□

Theorem 3.3. *Assume that the conditions in Theorem 3.1 are met. Then the problem described by (3.16) has at least mild solution provided that*

$$\mathrm{R}_3 = 2m_{(\ell,\varsigma)}(l_{11} + \mathrm{C} + 2l_{12} \max_{\Delta\in(0,b)} \left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)] + 2l_{13} \max_{\Delta\in(0,b)} \left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]) < 1, \tag{3.42}$$

where C *is given by (3.23).*

Proof. Steps 1-4, and 6 are the same as before.

For Step 5: $\mathscr{W}(\Delta) = \{(\mathbb{K}\digamma)(\Delta) : \digamma \in D\}, \Delta \in J$, is relatively compact in $\mathscr{U}$.

For $\Delta \in J$, by $(\mathscr{M}_6)$ and from Lemma 3.1 in [33] and from Lemma 2.3, yields that

$$\begin{aligned} \mu(\hbar\mathscr{W}(\Delta)) &= \mu(\hbar\mathbb{K}\mathscr{W}_0(\Delta)) \\ &= \mu\left(\{\hbar\digamma_{n-1}\}_{n-1}^{\infty}(\Delta)\right) \\ &= \mu\left(\mathbb{S}_\ell(\Delta,0)\digamma_0 + \int_0^\Delta \left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar\mathcal{X}(\{\digamma_{n-1}\}) + \mathrm{C}(\{\hbar\digamma_{n-1}\})(\zeta)\right]\mathrm{d}\zeta\right) \\ &\leq \mu\left(\int_0^\Delta \left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar\mathcal{X}(\{\digamma_{n-1}\}) + \mathrm{C}(\{\hbar\digamma_{n-1}\})(\zeta)\right]\mathrm{d}\zeta\right) \end{aligned}$$

$$\leq 2\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\Big(l_{11}\mu\left(\{\hbar\digamma_{n-1}\}\right)+l_{12}\mu\left(\{\hbar\mathscr{G}\digamma_{n-1}\}\right)$$
$$+\,l_{13}\mu\left(\{\hbar\mathscr{H}\digamma_{n-1}\}\right)+\mathrm{C}\mu\left(\{\hbar\digamma_{n-1}\}\right)\Big)\mathrm{d}\zeta$$
$$\leq 2m_{(\ell,\varsigma)}\Big(l_{11}+\mathrm{C}+2l_{12}\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)]$$
$$+\,2l_{13}\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]\Big)\left[\mu(\hbar\mathscr{W}_0(\zeta))\right]$$
$$\leq \mathrm{R}_3\sup_{\Delta\in J}\mu(\mathscr{W}_0(\Delta)),\tag{3.43}$$

where $\mu(\hbar\mathscr{G}\mathscr{W}_0(\Delta))$ and $\mu(\hbar\mathscr{H}\mathscr{W}_0(\Delta))$ are given by

$$\mu(\hbar\mathscr{G}\mathscr{W}_0(\Delta))=\mu\left(\left\{\hbar(\Delta)\,\frac{\hbar^{-1}(\Delta)}{\Gamma(\eta)}\int_0^{\Delta}\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\eta-1}\mathcal{Y}(\Delta,\zeta)\hbar(\zeta)\digamma_{n-1}(\zeta)\varsigma'(\zeta)\,\mathrm{d}\zeta\right\}_{n=1}^{\infty}\right)\tag{3.44}$$
$$\leq 2\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)]\sup_{\Delta\in J}\mu\left\{\hbar\mathscr{W}_0(\Delta)\right\}$$

and

$$\mu(\hbar\mathscr{H}\mathscr{W}_0(\Delta))=\mu\left(\left\{\frac{\hbar(\Delta)\hbar^{-1}(\Delta)}{\Gamma(\eta)}\int_0^{b}\left(\tilde{\varsigma}(b,\zeta)\right)^{\eta-1}h(\Delta,\zeta)\hbar(\zeta)\digamma_{n-1}(\zeta)\varsigma'(\zeta)\,\mathrm{d}\zeta\right\}_{n=1}^{\infty}\right)\tag{3.45}$$
$$\leq 2\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]\sup_{\Delta\in J}\mu\left\{\hbar\mathscr{W}_0(\Delta)\right\},$$

respectively. Since $\{\mathbb{K}\digamma_n\}_{n=0}^{\infty}$ is equicontinuous on J, additionally by following the lines of Lemma 3.4 in [33], we get

$$\mu(\mathscr{W})\leq \mathrm{R}_3\mu(\mathscr{W}).\tag{3.46}$$

Since $\mathrm{R}_3<1$, following the lines of Lemma 2.12 in [33], we have $\mu(\mathscr{W})=0$. Therefore, the set $\mathscr{W}$ is relatively compact. □

Theorem 3.4. *From Theorem 3.2, then the problem (3.16) has at least mild solution gives that*

$$\mathrm{R}_4=2m_{(\ell,\varsigma)}(l_{11}+\mathrm{C}+2l_{12}\max_{\Delta\in(0,b)}\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)]+2l_{13}\max_{\Delta\in(0,b)}\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)])<1,\tag{3.47}$$

where C *is given by (3.41).*

In the following theorem, we will employ the monotone iterative procedure and Kuratowski to examine the uniqueness result of (3.16).

Theorem 3.5. *Assume that $\mathscr{U}$ is an ordered Banach space, whose positive cone $\mathcal{P}$ is normal with normal constant* N. *Take $\mathscr{T}(\Delta)(\Delta\geqslant 0)$ is non-negative and the assumptions $(\mathscr{M}_1)$-$(\mathscr{M}_6)$ hold. Then there exists a unique mild solution to the Cauchy problem (3.16) between $\underline{\digamma}_0$ and $\overline{\digamma}_0$ that might be built in an iterative manner using a monotone sequence beginning with $\underline{\digamma}_0$ and $\overline{\digamma}_0$, respectively.*

Proof. For $\Delta\in J$, let $\{\mathscr{J}_n\}\subset D$, $\{\mathscr{L}_n\}\subset[\mathscr{G}\underline{\digamma}_0,\mathscr{G}\overline{\digamma}_0]$ and $\{\gamma_n\}\subset[\mathscr{H}\underline{\digamma}_0,\mathscr{H}\overline{\digamma}_0]$ is a nondecreasing sequence. Besides $m,n=1,2,...$ including $m>n$, moreover, from $(\mathscr{M}_1)$-$(\mathscr{M}_2)$ and $(\mathscr{M}_4)$, we now have

$$0\leq\hbar(\Delta)\,\chi\left(\Delta,\mathscr{J}_m,\mathscr{L}_m,\gamma_m\right)-\hbar(\Delta)\,\chi\left(\Delta,\mathscr{J}_n,\mathscr{L}_n,\gamma_n\right)+\mathrm{C}\hbar(\Delta)\left(\mathscr{J}_m-\mathscr{J}_n\right)$$

$$\leq l_1\hbar(\Delta)(\mathscr{J}_m-\mathscr{J}_n)+l_2\hbar(\Delta)(\mathscr{L}_m-\mathscr{L}_n)+l_3\hbar(\Delta)(\gamma_m-\gamma_n)+\mathrm{C}\hbar(\Delta)(\mathscr{J}_m-\mathscr{J}_n).$$

By assumptions $(\mathscr{M}_4)$ and $(\mathscr{M}_5)$, we can obtain

$$\begin{aligned} 0 &\leq \hbar(\Delta)\chi(\Delta,\mathscr{J}_m,\mathscr{L}_m,\gamma_m)-\hbar(\Delta)\chi(\Delta,\mathscr{J}_n,\mathscr{L}_n,\gamma_n)+\mathrm{C}\hbar(\Delta)(\mathscr{J}_m-\mathscr{J}_n) \\ &\leq (l_1+\mathrm{C}+l_2l_4+l_3l_5)\hbar(\Delta)(\mathscr{J}_m-\mathscr{J}_n). \end{aligned} \tag{3.48}$$

From hypothesis, we have

$$\begin{aligned} &\|\chi(\Delta,\mathscr{J}_m,\mathscr{L}_m,\gamma_m)-\chi(\Delta,\mathscr{J}_n,\mathscr{L}_n,\gamma_n)+\mathrm{C}(\mathscr{J}_m-\mathscr{J}_n)\|_\infty^\hbar \\ \leq &\mathrm{N}(l_1+\mathrm{C}+l_2l_4+l_3l_5)\|\mathscr{J}_m-\mathscr{J}_n\|_\infty^\hbar. \end{aligned} \tag{3.49}$$

We see that

$$\begin{aligned} &\|\chi(\Delta,\mathscr{J}_m,\mathscr{L}_m,\gamma_m)-\chi(\Delta,\mathscr{J}_n,\mathscr{L}_n,\gamma_n)\|_\infty^\hbar \\ \leq &(\mathrm{C}+\mathrm{N}(l_1+\mathrm{C}+l_2l_4+l_3l_5))\|\mathscr{J}_m-\mathscr{J}_n\|_\infty^\hbar. \end{aligned} \tag{3.50}$$

Given this inequality and the overview of the noncompactness measure of noncompactness, it is evident that

$$\begin{aligned} \mu(\hbar(\Delta)\chi(\Delta,\digamma_{n-1},\mathscr{G}\digamma_{n-1},\mathscr{H}\digamma_{n-1})) &\leq (\mathrm{C}+\mathrm{N}(l_1+\mathrm{C}))\mu(\hbar\digamma_{n-1}) \\ &\quad+\mathrm{N}l_2\mu(\hbar\mathscr{G}\digamma_{n-1})+\mathrm{N}l_3\mu(\hbar\mathscr{H}\digamma_{n-1}), \end{aligned} \tag{3.51}$$

where $l_{11}=\mathrm{C}+\mathrm{N}(l_1+\mathrm{C})$, $l_{12}=\mathrm{N}l_2$ and $l_{13}=\mathrm{N}l_3$. Hence, $(\mathscr{M}_6)$ holds.

Thus, by Theorem 3.1, the Cauchy problem (3.16) has the minimal mild solution $\underline{\digamma}$ and the maximal mild solution $\overline{\digamma}$ on D.

Now, we show that $\underline{\digamma}=\overline{\digamma}$. For $\Delta\in J$, by the positivity of operator $\tilde{\mathbb{S}}_\ell$, we have

$$\begin{aligned} 0 &\leq \hbar(\Delta)(\overline{\digamma}-\underline{\digamma})(\Delta) \\ &= \hbar(\Delta)(\mathbb{K}\overline{\digamma}-\mathbb{K}\underline{\digamma})(\Delta) \\ &\leq \int_0^\Delta\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)(\hbar\chi(\bar{\digamma})-\hbar\chi(\underline{\digamma})+\mathrm{C}(\hbar\overline{\digamma}-\hbar\underline{\digamma}))(\zeta)\mathrm{d}\zeta \\ &\leq \int_0^\Delta\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)((l_1+\mathrm{C})|\hbar\overline{\digamma}-\hbar\underline{\digamma}|+l_2|\hbar\mathscr{G}\overline{\digamma}-\hbar\mathscr{G}\underline{\digamma}|+l_3|\hbar\mathscr{H}\overline{\digamma}-\hbar\mathscr{H}\underline{\digamma}|)(\zeta)\mathrm{d}\zeta, \\ (\overline{\digamma}-\underline{\digamma})(\Delta) &\leq \hbar^{-1}(\Delta)\left(l_1+l_2\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,.)]+l_3\left(\mathcal{J}_{0,b}^{\eta,[\varsigma,1]}\right)[h(\Delta,.)]+\mathrm{C}\right) \\ &\qquad\times\left\|\tilde{\mathbb{S}}_\ell\right\|_\infty^\hbar\int_0^\Delta(\tilde{\varsigma}(\Delta,\zeta))^{\ell-1}(\overline{\digamma}-\underline{\digamma})(\zeta)\varsigma'(\zeta)\mathrm{d}\zeta. \end{aligned} \tag{3.52}$$

By Lemma 2.4 and (3.52), we obtain that $\underline{\digamma}=\overline{\digamma}$ on J. Hence, $\underline{\digamma}=\overline{\digamma}$ is the unique mild solution of the Cauchy problem (3.16) on D. Following the lines of Theorem 3.1, by employing a monotone iterative procedure beginning with either the $\underline{\digamma}_0$ or $\overline{\digamma}_0$, the solution can be determined. □

Theorem 3.6. *Assume that $\mathscr{U}$ is an ordered Banach space, whose positive cone $\mathcal{P}$ is normal with normal constant* N*. Take $\mathscr{T}(\Delta)(\Delta\geqslant 0)$ is non-negative. The assumptions $(\mathscr{M}_1)$, (B_2), (B_3), $(\mathscr{M}_5)$ and $(\mathscr{M}_6)$ hold. Then the Cauchy problem (3.16) has the unique mild solution between $\underline{\digamma}_0$ and $\overline{\digamma}_0$ which can be iteratively constructed by monotone sequence beginning with $\underline{\digamma}_0$ and $\overline{\digamma}_0$, respectively.*

3.3. Continuous dependence on initial conditions

Here, we characterise the problem's (3.16) continuous dependence on its initial conditions.

Theorem 3.7. *Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be mild solutions to the following problems (3.16) and*

$$\begin{cases} \left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)\left[\tilde{\mathcal{F}}\right] = (\mathscr{M} - \mathrm{C})\,\tilde{\mathcal{F}} + \chi(\Delta, \tilde{\mathcal{F}}, \mathscr{G}\tilde{\mathcal{F}}, \mathscr{H}\tilde{\mathcal{F}}) + \mathrm{C}\tilde{\mathcal{F}}, \quad \text{if } \Delta \in J, \\ \left(\hbar\tilde{\mathcal{F}}\right)(0) = \tilde{\mathcal{F}}_0 \in \mathscr{U}, \end{cases} \tag{3.53}$$

respectively where C *is given by (3.23). Then, for all $\Delta \in J$, we have*

$$\|\mathcal{F} - \tilde{\mathcal{F}}\|_{\infty}^{\hbar} \le \mathrm{N}\,\|\mathbb{S}_\ell\|_{\infty}\left\|\mathcal{F}_0 - \tilde{\mathcal{F}}_0\right\|_{\infty}\mathcal{E}_\ell\left(\left[\left\|\tilde{\mathbb{S}}_\ell\right\|_{\infty}(l_1 + \mathrm{C} + l_2 l_4 + l_3 l_5)\right]\Gamma(\ell)\left(\tilde{\varsigma}(b,0)\right)^{\ell}\right). \tag{3.54}$$

Proof. By Lemma 3.3 and Definition 3.1, we have

$$\begin{aligned} \tilde{\mathcal{F}}(\Delta) &= \hbar^{-1}(\Delta)\mathbb{S}_\ell(\Delta,0)\tilde{\mathcal{F}}_0 \\ &\quad + \hbar^{-1}(\Delta)\int_0^{\Delta}\left[\varsigma\tilde{\mathbb{S}}_\ell\right](\Delta,\zeta)\left[\hbar(\zeta)\chi\big(\zeta, \tilde{\mathcal{F}}(\zeta), \mathscr{G}\tilde{\mathcal{F}}(\zeta), \mathrm{H}\tilde{\mathcal{F}}(\zeta)\big) + \mathrm{C}\hbar(\zeta)\tilde{\mathcal{F}}(\zeta)\right]\mathrm{d}\zeta. \end{aligned} \tag{3.55}$$

Following the lines of Theorem 3.1, we get

$$\begin{aligned} &\left|\hbar\mathcal{F}(\Delta) - \hbar\tilde{\mathcal{F}}(\Delta)\right| \\ &\le \|\mathbb{S}_\ell\|_{\infty}\left|\mathcal{F}_0 - \tilde{\mathcal{F}}_0\right| \\ &\quad + \left\|\tilde{\mathbb{S}}_\ell\right\|_{\infty}((l_1 + \mathrm{C} + l_2 l_4 + l_3 l_5))\int_0^{\Delta}\left(\tilde{\varsigma}(\Delta,\zeta)\right)^{\ell-1}\left(\left|\hbar\mathcal{F}(\zeta) - \hbar\tilde{\mathcal{F}}(\zeta)\right|\right)\varsigma'(\zeta)\,\mathrm{d}\zeta. \end{aligned} \tag{3.56}$$

Applying the generalized Gronwall inequality (Lemma 2.4) to (3.56), with $v(\Delta) = \|\mathbb{S}_\ell\|_{\infty}|\mathcal{F}_0 - \tilde{\mathcal{F}}_0|$ and $\mathcal{Y}(\Delta) = \|\tilde{\mathbb{S}}_\ell\|_{\infty}(l_1 + \mathrm{C} + l_2 l_4 + l_3 l_5)$, which is a nonnegative nondecreasing constant, we get

$$\left|\hbar\left(\mathcal{F} - \tilde{\mathcal{F}}\right)(\Delta)\right| \le \|\mathbb{S}_\ell\|_{\infty}\left|\mathcal{F}_0 - \tilde{\mathcal{F}}_0\right|\mathcal{E}_\ell\left(\left[\left\|\tilde{\mathbb{S}}_\ell\right\|_{\infty}(l_1 + \mathrm{C} + l_2 l_4 + l_3 l_5)\right]\Gamma(\ell)\left(\tilde{\varsigma}(b,0)\right)^{\ell}\right). \tag{3.57}$$

The normality of the positive cone $\mathcal{P}$ allows us to obtain (3.54). □

3.4. Examples

In this section, we provide two concrete examples that show how our abstract concepts can be generalized to real-world issues without attempting to be general.

Example 3.1. We adopt the nonlinear mixed generalized Volterra- Fredholm fractional temporal derivative partial integro-differential equation in $\mathscr{U} = L^2([0,\pi])$,

$$\begin{cases} \left({}^{c}\mathcal{D}_{0,\Delta}^{\ell,[\varsigma,\hbar]}\right)[\mathcal{F}(.,\mathscr{J})] = \mathscr{M}\mathcal{F}(\Delta,\mathscr{J}) + \chi(\Delta,\mathscr{J},\mathcal{F}(\Delta,\mathscr{J}),\mathscr{G}\mathcal{F}(\Delta,\mathscr{J}),\mathscr{H}\mathcal{F}(\Delta,\mathscr{J})), \\ \text{if } \Delta \in (0,1), \ \mathscr{J} \in [0,\pi], \end{cases} \tag{3.58}$$

with initial and boundary conditions

$$\begin{cases} \hbar(0)\,\mathcal{F}(0,\mathscr{J}) = \mathcal{F}_0(\mathscr{J}) = \dfrac{1}{2}\sin(\mathscr{J}). & \mathscr{J} \in [0,\pi], \\ \mathcal{F}(\Delta,0) = \mathcal{F}(\Delta,\pi) = 0; & \Delta \in [0,1]. \end{cases} \tag{3.59}$$

Let $\mathcal{P} = \{\mathscr{L} \in \mathscr{U} \mid \mathscr{L}(\mathscr{J}) \geqslant 0 \text{ a.e. } \mathscr{J} \in [0,\pi]\}$. Consequently, $\mathcal{P}$ is normal cone in Banach space $\mathscr{U}$. The operator $\mathscr{M} : D(\mathscr{M}) \subset \mathscr{U} \longrightarrow \mathscr{U}$ as follows:

$$\mathscr{M}\digamma = \mathbf{\Delta}\digamma, \tag{3.60}$$

where $\mathbf{\Delta}$ denotes the Laplacian operator, with the domain

$$D(\mathscr{M}) = \left\{\digamma \in \mathscr{U} \mid \digamma, \frac{\partial \digamma}{\partial \mathscr{J}} \text{ are absolutely continuous}, \frac{\partial^2 \digamma}{\partial \mathscr{J}^2} \in \mathscr{U}, \digamma(0) = \digamma(\pi) = 0\right\}.$$

It is well known that $\mathscr{M}$ is the infinitesimal generator of an analytic semigroup. $\mathscr{T}(\Delta)\,(\Delta \geq 0)$ on $\mathscr{U}$. Additionally, $\mathscr{M}$ has a discrete spectrum and the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $\mathscr{L}_n(\mathscr{J}) = \sqrt{2/\pi}\sin(n\mathscr{J})$. In addition, $\{\mathscr{L}_n : n \in \mathbb{N}\}$is an orthonormal basis for $\mathscr{U}$.

Moreover

$$\mathscr{T}(\Delta)\digamma = \sum_{n=1}^{\infty} e^{-n^2\Delta}\,(\digamma, \mathscr{L}_n)\,\mathscr{L}_n,\ \|\mathscr{T}(\Delta)\| \leq e^{-\Delta} \leq 1 \text{ for all } \Delta \geq 0\ , \digamma \in \mathscr{U} \tag{3.61}$$

and

$$\mathscr{M}\digamma = \sum_{n=1}^{\infty} n^2\,(\digamma, \mathscr{L}_n)\,\mathscr{L}_n,\ \digamma \in D(\mathscr{M}). \tag{3.62}$$

It follows that $\{\mathscr{T}(\Delta)\}\,(\Delta \geqslant 0)$ is a uniformly bounded compact semigroup.

Therefore, we have that for $\digamma \in \mathscr{U}$, the two characterize operators $\mathcal{S}_\ell(\Delta, 0)$ and $\tilde{\mathcal{S}}_\ell(\Delta, \zeta)$ can be written as

$$\begin{aligned} \mathbb{S}_\ell(\Delta, 0)\digamma &= \sum_{n=1}^{\infty} \mathcal{E}_\ell\left(-\left(n^2 + \mathrm{C}\right)\Delta^\ell\right)(\digamma, \mathscr{L}_n)\,\mathscr{L}_n, \\ \text{and}\quad \tilde{\mathbb{S}}_\ell(\Delta, \zeta)\digamma &= \sum_{n=1}^{\infty} \mathcal{E}_{\ell,\ell}\left(-\left(n^2 + \mathrm{C}\right)\Delta^\ell\right)(\digamma, \mathscr{L}_n)\,\mathscr{L}_n. \end{aligned} \tag{3.63}$$

Clearly,

$$\|\mathbb{S}_\ell(\Delta, 0)\digamma\|_\infty \leq \|\digamma\|_\infty \text{ and } \|\tilde{\mathbb{S}}_\ell(\Delta, \zeta)\digamma\|_\infty \leq \frac{1}{\Gamma(\ell)}\|\digamma\|_\infty. \tag{3.64}$$

Now define $\digamma(\Delta, .) = \digamma(\Delta)(.)$, we can reconstruct the fractional partial differential problem (3.58) as an abstract fractional functional differential equation (1.4).

Further, for any $\Delta \in [0, 1]$, we define

$$\chi(\Delta, \digamma(\Delta), \mathscr{G}\digamma(\Delta), \mathscr{H}\digamma(\Delta)) = a(\Delta)\,\digamma(\Delta) + b(\Delta)\,\mathscr{G}\digamma(\Delta) + c(\Delta)\,\mathscr{H}\digamma(\Delta), \tag{3.65}$$

where

$$a(\Delta) = \frac{1}{\Gamma(\eta)}\left(\Delta^\eta + (1-\Delta)^\eta\right),\ b(\Delta) = \frac{\eta}{\rho} \text{ and } c(\Delta) = \frac{\eta}{\rho},\ \ell + \eta = 1.$$

Letting $\ell = \frac{4}{5}$, $\eta = \frac{1}{5}$ and $\rho = 70$, we have

$$a^* = 0.37924,\ b^* = 0.28571 \times 10^{-2} \text{ and } c^* = b^*.$$

We set

$$\mathcal{Y}(\Delta,\zeta)=\left(\frac{1-\Delta}{\zeta}\right)^{\eta}\ \text{for } 0<\zeta,\Delta<1$$

and

$$h(\Delta,\zeta)=\begin{cases}\left(\dfrac{1-\zeta}{\Delta-\zeta}\right)^{\ell} & \text{for } 0<\zeta<\Delta<1,\\ \left(\dfrac{1-\zeta}{\zeta-\Delta}\right)^{\ell} & \text{for } 0<\Delta<\zeta<1.\end{cases}$$

By choosing $\varsigma(\Delta)=\Delta$, $\underline{F}_0=0$ and $\overline{F}_0=\hbar^{-1}(\Delta)\sin(\mathscr{J})$ for $(\mathscr{J},\Delta)\in[0,\pi]\times[0,1]$. It is easy to see that $\underline{F}_0$ and $\overline{F}_0$ are lower and upper solutions, respectively and $\underline{F}_0\leq\overline{F}_0$ and

$$\chi(\Delta,\underline{F}_0,\mathscr{G}\underline{F}_0,\mathscr{H}\underline{F}_0)\geqslant 0,\ \text{and } \chi(\Delta,\overline{F}_0,\mathscr{G}\overline{F}_0,\mathscr{H}\overline{F}_0)\geqslant 0,$$

that demonstrates that the assertion $(\mathscr{M}_1)$ of Theorem 3.1 holds good.

We assume that the nonlinearity χ is globally Lipschitz continuous $\chi\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ \mathbb{R}^+)$,and satisfies Lipschitz condition

$$|\chi(F_2)-\chi(F_1)|\leq l_1\left|(F_2-F_1)(\Delta)\right|+l_2\left|\mathscr{G}(F_2-F_1)(\Delta)\right|+l_3\left|\mathscr{H}(F_2-F_1)(\Delta)\right|. \tag{3.66}$$

From (3.66), we obtain that

$$\mathrm{C}_1=l_1=a^*,\ \mathrm{C}_2=l_2=b^*,\ \mathrm{C}_3=l_3=c^*, \tag{3.67}$$

that is, $(\mathscr{M}_2)$ and $(\mathscr{M}_5)$ hold. With

$$\mathbf{B}(\ell,\eta)=\int_0^1\Delta^{\ell-1}(1-\Delta)^{\eta-1}\,\mathrm{d}\Delta,$$

and (2.1), we have

$$\left(\mathcal{J}_{0,1}^{\eta,[\varsigma,1]}\right)[h(\Delta,\zeta)]=\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[1]+\left(\mathcal{J}_{\Delta,1}^{\eta,[\varsigma,1]}\right)[1]=\frac{1}{\Gamma(1+\eta)}\left(\Delta^{\eta}+(1-\Delta)^{\eta}\right). \tag{3.68}$$

Hence (3.68) follows from Definition 4.1 in paper [18], and

$$\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[\mathcal{Y}(\Delta,\zeta)]=(1-\Delta)^{\eta}\left(\mathcal{J}_{0,\Delta}^{\eta,[\varsigma,1]}\right)[1]=\frac{(1-\Delta)^{\eta}\Delta^{\eta}}{\Gamma(1+\eta)}. \tag{3.69}$$

Due to

$$\begin{aligned}\mathrm{C}_4=l_4&=\max\left\{\frac{(1-\Delta)^{\eta}\Delta^{\eta}}{\Gamma(1+\eta)}\right\}=\frac{\left(\frac{1}{2}\right)^{2\eta}}{\Gamma(\eta+1)}=0.82535\ \text{ and}\\ \mathrm{C}_5=l_5&=\max\left\{\frac{1}{\Gamma(1+\eta)}\left(\Delta^{\eta}+(1-\Delta)^{\eta}\right)\right\}=2\frac{\left(\frac{1}{2}\right)^{\eta}}{\Gamma(\eta+1)}=1.8962.\end{aligned}$$

The last assumption (3.21) allows us to determine the range of R_1 which is obviously

$$\mathrm{R}_1=0.83108<1, \tag{3.70}$$

with $\mathrm{M}=1$, $m_{(\ell,\varsigma)}=1.0737$ and

$$0<\mathrm{C}\equiv \mathrm{C}_1+\mathrm{C}_2 l_4+\mathrm{C}_3 l_5=0.38702.$$

As a result of Theorem 3.1, we acquire that the problem's (3.58-3.59) minimal and maximal mild solutions are between the lower solution $\underline{\mathcal{F}}_0$ and upper solution $\overline{\mathcal{F}}_0$.

The conditions (B_2-B_3) are satisfied with

$$\begin{aligned}&\mathrm{C}_1=\frac{\eta}{\Gamma(\eta)},\ \mathrm{C}_2=\eta l_2,\ \mathrm{C}_3=\eta l_3, \\ &0<\mathrm{C}\equiv(\mathrm{C}_1-\mathrm{C}_2 l_4-\mathrm{C}_3 l_5)=0.42009\times 10^{-1} \text{ and } \mathrm{R}_2=0.46065.\end{aligned} \tag{3.71}$$

As a result, all of the mentioned conditions of Theorem 3.3 are met. In accordance with the Theorem 3.3, a unique mild solution exists between the problem's lower and upper solutions (3.58-3.59).

Moreover, the mild solution $\mathcal{F}$ of the problem (3.58-3.59) is continuously dependent on the initial data (3.58-3.59) with

$$\hbar^{-1}(\Delta)=e^{-\Delta},\ \mathcal{F}_0=\frac{1}{2}\sin(\mathscr{J}) \text{ and } \tilde{\mathcal{F}}_0=\frac{1}{4}\sin(\mathscr{J})$$

and

$$\left|\left(\mathcal{F}-\tilde{\mathcal{F}}\right)(\Delta)\right|\le 0.67958\times\mathcal{E}_\ell(0.38702)=1.8784,$$

where

$$\mathcal{E}_\ell(\mathscr{Z})\simeq\frac{1}{\ell}\exp(\mathscr{Z}^{\frac{1}{\ell}}),\ |\arg(\mathscr{Z})|\le\frac{1}{2}\ell\pi.$$

Example 3.2. We take into account the below initial-boundary value problem of time fractional partial differential equation

$$\begin{cases}\left({}^{c}\mathcal{D}^{\ell,[\varsigma,\hbar]}_{0,\Delta}\right)[\mathcal{F}(.,\mathscr{J})]=\mathscr{M}\mathcal{F}(\Delta,\mathscr{J})+\chi(\Delta,\mathscr{J},\mathcal{F}(\Delta,\mathscr{J}),\mathscr{G}\mathcal{F}(\Delta,\mathscr{J}),\mathscr{H}\mathcal{F}(\Delta,\mathscr{J})),\\ \text{if } \Delta\in(0,1),\ \mathscr{J}\in[0,\pi],\end{cases} \tag{3.72}$$

with initial and boundary conditions (3.59).

Take $\ell=0.7$, $\eta=0.3$, $\varsigma(\Delta)=\Delta$, $\hbar^{-1}(\Delta)=1$, with

$$\begin{aligned}&\chi(\Delta,\mathcal{F}(\Delta),\mathscr{G}\mathcal{F}(\Delta),\mathscr{H}\mathcal{F}(\Delta))\\ &=\left(a_1-a_0\Delta^2\right)\mathcal{F}(\Delta)+b_0\Delta^{\ell}\mathscr{G}\mathcal{F}(\Delta)+c_0\Delta^2\mathscr{H}\mathcal{F}(\Delta),\quad \ell+\eta=1,\end{aligned} \tag{3.73}$$

where

$$a_1=\frac{1}{\rho} \text{ and } a_0=\frac{1}{\rho\Gamma(2+\eta)},\ b_0=\frac{1}{\rho}\hbar(\Delta),\ c_0=\frac{1}{\rho}\hbar(\Delta),$$

which implies that $(\mathscr{M}_2)$ and $(\mathscr{M}_5)$ are satisfied with

$$\rho=144,\ \mathrm{C}_1=l_1=0.6946\times\times 10^{-2},\ \mathrm{C}_2=l_2=\frac{1}{\rho},\ \mathrm{C}_3=l_3=\frac{\Gamma(2+\eta)}{\rho} \tag{3.74}$$

and

$$\left(\mathcal{J}^{\eta,[\varsigma,1]}_{0,\Delta}\right)[1]=\frac{\Delta^{1+\eta}}{\Gamma(2+\eta)} \text{ and } \left(\mathcal{J}^{\eta,[\varsigma,1]}_{0,1}\right)[1]=\frac{1}{\Gamma(2+\eta)},$$

$$\mathrm{C}_4 = \mathrm{C}_5 = \frac{1}{\Gamma(2+\eta)}.$$

By setting $\digamma(\Delta)(.) = \digamma(\Delta, .)$, the fractional partial differential equation (3.72) with (3.59) can be modeled as an abstract Cauchy problem of the form (1.4-1.5). Letting

$$\mathcal{Y}(\Delta, \zeta) = \zeta \hbar^{-1}(\Delta), \text{ for } 0 \leq \zeta < \Delta \leq 1$$

and

$$h(\Delta, \zeta) = \zeta \hbar^{-1}(\Delta), \text{ for } 0 \leq \zeta, \Delta \leq 1.$$

Now, take $\underline{\digamma}_0 = 0$ and $\overline{\digamma}_0 = \hbar^{-1}(\Delta) \sin(\mathscr{J})$ for $(\mathscr{J}, \Delta) \in [0, \pi] \times [0, 1]$. It is quite complicated to ensure that $\underline{\digamma}_0$, $\overline{\digamma}_0$ are lower and upper solutions of (3.72-3.59).

The condition

$$\mathrm{R}_3 = 0.8690 < 1, \tag{3.75}$$

is satisfied with $\mathrm{M} = 2$ and $\mathrm{N} = 3$ where $l_{11} = (\mathrm{N}(l_1 + \mathrm{C}))$, $l_{12} = \mathrm{N} l_2$ and $l_{13} = \mathrm{N} l_3$, $m = 2.201$,

Moreover, the condition $(\mathscr{M}_6)$ is satisfied with

$$0 < \mathrm{C} = 0.1984 \times 10^{-1}. \tag{3.76}$$

As a result, Theorem 3.3's requirements are all met. Due to the Theorem 3.3, the problem (3.72-3.59) has a unique mild solution among the lower and upper solutions.

4. Conclusions

In this paper, we establish sufficient conditions for demonstrating the existence and uniqueness of mild solutions for a class of weighted ς-Caputo fractional abstract Cauchy-type problems involving integro-differential equations. These problems incorporate both Volterra and Fredholm generalized integral operators. Additionally, we prove the continuous dependence on initial conditions. A weight function, commonly found in the literature as a special case for various fractional derivatives, serves as the foundation for defining the generalized fractional derivative. Consequently, the boundary value problems examined in this paper are broader in scope. The results presented are formulated in an abstract manner, allowing for their application in future research involving evolution equations with perturbations, delays, and nonlocal terms.

Acknowledgments. The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through large group research project under Grant Number RGP2/559/47.

Author contributions. All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Conflict of interest. All authors declare no conflicts of interest in this paper.

Use of AI tools declaration. The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Funding. This research received no specific funding.

References

[1] Y. Adjabi, F. Jarad and T. Abdeljawad, *On generalized fractional operators and a Gronwall type inequality with applications*, Filomat, 2017, 31, 5457–5473.

[2] O. Agrawal, *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Calc. Anal. Appl., 2012, 15(4), 700–711.

[3] K. Ali, A. El-Sayed and M. Abdelrahman, *Existence and uniqueness of solutions for a fractional q-integro-differential equation involving the Caputo–Fabrizio fractional derivative with nonlocal conditions*, J. Math. Comput. Sci., 2024, 29, 1–18.

[4] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., 2017, 44, 460–481.

[5] A. Anguraj, P. Karthikeyan and J. J. Trujillo, *Existence of solutions to fractional mixed integro-differential equations with nonlocal initial condition*, Adv. Differ. Equ., 2011, Article ID 690653, 12 pp.

[6] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*, Commentationes Mathematicae Universitatis Carolinae, 1980, 21(1).

[7] J. Banas and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.

[8] E. Bazhlekova, *Evolution Equations in Fractional Banach Spaces*, PhD thesis, Eindhoven University of Technology, 2001.

[9] A. Ben Brahim and D. F. M. Torres, *Existence and uniqueness of mild solutions for a class of ψ-Caputo time-fractional systems of order from one to two*, Fract. Calc. Appl. Anal., 2025, 28(1), 1–27.

[10] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., 1991, 162, 494–505.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[12] S. W. Du and V. Lakshmikantham, *Monotone iterative technique for differential equations in Banach spaces*, J. Math. Anal. Appl., 87, (1982), 454–459.

[13] H. Gou and Y. Li, *The method of lower and upper solutions for impulsive fractional evolution equations*, Ann. Funct. Anal., 11, (2020), 350.

[14] S. Hasan, A. Jleli and B. Samet, *Existence and uniqueness results for a class of fractional integro-stochastic differential equations*, Fractal and Fractional, 2025, 9(1), Article 42.

[15] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector valued functions*, Nonlinear Anal., 1983, 7(12), 1351–1371.

[16] R. W. Ibrahim and S. Momani, *Upper and lower bounds of solutions for fractional integral equations*, Surveys in Mathematics and its Applications, 2007, 2, 145–156.

[17] T. Jankowski, *Fractional equations of Volterra type involving a Riemann-Liouville derivative*, Appl. Math. Lett., 2013, 26, 344–350.

[18] F. Jarad, T. Abdeljawad and K. Shah, *On the weighted fractional operators of a function with respect to another function*, Fractals, 2020, 28(8), 2040011.

[19] O. K. Jaradat, A. Al-Omari and S. Momani, *Existence of the mild solution for fractional semilinear initial value problems*, Nonlinear Anal., 2008, 69, 3153–3159.

[20] H. Jian, B. Liu and S. Xie, *Monotone iterative solutions for nonlinear fractional differential systems with deviating arguments*, Appl. Math. Comput., 2015, 262, 1–14.

[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[22] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., 2008, 69(8), 2677–2682.

[23] B. Li and H. Gou, *Monotone iterative method for the periodic boundary value problems of impulsive evolution equations in Banach spaces*, Chaos, Solitons and Fractals, 2018, 110, 209–215.

[24] K. Li and J. Jia, *Existence and uniqueness of mild solutions for abstract delay fractional differential equations*, Comput. Math. Appl., 2011, 62, 1398–1404.

[25] Y. Li, *The positive solutions of abstract semilinear evolution equations and their applications*, Acta Math. Sin., 1996, 39(5), 666–672.

[26] H. Lmou, A. Boutarfa and S. Muthaiah, *Existence theory on the Caputo-type fractional differential Langevin hybrid inclusion with variable coefficient*, Bound. Value Probl., 2024, Article ID 48.

[27] E. Malkowsky and V. Rakocevic, *An introduction into the theory of sequence spaces and measures of noncompactness*, Zb. rad. Beogr., 2000, 17, 143–234.

[28] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal., TMA, 1980, 4, 985–999.

[29] S. K. Panda, A. Atangana and T. Abdeljawad, *Existence results and numerical study on novel coronavirus 2019-Ncov/Sars-Cov-2 model using differential operators based on the generalized Mittag-Leffler kernel and fixed points*, Fractals, 2022, 30(8), 2240214.

[30] D. N. Pandey, A. Ujlayan and D. Bahuguna, *On a solution to fractional order integro-differential equations with analytic semigroups*, Nonlinear Anal., 2009, 71, 3690–3698.

[31] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, Berlin, 1983.

[32] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon & Breach Science Publishers, Yverdon, 1993.

[33] A. Suechoei and P. Sa Ngiamsunthorn, *Extremal solutions of φ-Caputo fractional evolution equations involving integral kernels*, AIMS Mathematics, 2021, 6(5), 4734–4757.

[34] J. X. Sun and Z. Q. Zhao, *Extremal solutions initial value problem for integro-differential of mixed type in Banach spaces*, Ann. Diff. Eqs., 1992, 8, 469–475.

[35] J. Vanterler da, C. Sousa and E. Capelas de Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ -Hilfer operator*, Differ. Equ. Appl., 2019, 11, 87–106.

[36] G. Wang, R. P. Agarwal and A. Cabada, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, Appl. Math. Lett., 2012, 25, 1019–1024.

[37] X. Zhang and Y. Li, *Monotone iterative technique for fractional partial differential equations with impulses*, J. Comput. Anal. Appl., 2018, 25(3).

[38] M. Zhou and Y. Zhou, *Existence of mild solutions for fractional integro-differential equations with Hilfer derivatives*, Mathematics, 2024, 12(18), Article 2876.

[39] Y. Zhou and F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal., RWA, 2010, 11, 4465–4475.

Received December 2025; Accepted July 2026; Available online July 2026.