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QUALITATIVE ANALYSIS OF A WAVE EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING AND LOGARITHMIC SOURCE TERMS*

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Abstract The main objective of this work is to investigate an initial–boundary value problem for a wave equation incorporating Balakrishnan–Taylor damping, weak damping, and a logarithmic source term. By employing the potential well method combined with an appropriate logarithmic Sobolev inequality, we establish the global existence and energy decay of solutions in the stable set of initial data. In addition, we show that when the initial data lie in the unstable set, the corresponding solutions blow up in infinite time in the cases $E(0) < d$ and $E(0) < 0$. These results provide a comprehensive description of the long-term behavior of solutions, highlighting the balance between dissipative effects and the logarithmic nonlinearity.

Keywords Wave equation, Balakrishnan-Taylor damping, logarithmic nonlinearity, global existence, general decay, blow up, nonlinear equations.

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1. Introduction

Wave equations with nonlinear damping and source terms have attracted considerable attention due to their wide range of applications in physics, engineering, and materials science. Such models arise in the study of viscoelastic materials, heat conduction with memory, acoustic wave propagation, and mechanical systems involving nonlocal or fractional damping effects. In particular, the inclusion of Balakrishnan–Taylor damping provides a more accurate description of dissipative mechanisms in media where classical damping fails to reflect memory or hereditary

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behavior. When combined with weak damping, the model captures multiple layers of energy dissipation that influence the long-term evolution of solutions; see [26, 29, 35].

Nonlinear source terms play a crucial role in determining whether the system exhibits global existence, decay, or blow-up. Among these, the logarithmic source term is of special interest because it grows more slowly than polynomial nonlinearities yet still introduces significant analytical challenges. The competition between damping terms and the logarithmic nonlinearity governs the asymptotic behavior of solutions, leading to rich dynamics that require delicate mathematical tools to analyze.

To understand the qualitative behavior of such systems, it is essential to investigate the global existence, decay rates, and blow-up scenarios of the solutions. Techniques such as the potential well method, energy estimates, and logarithmic Sobolev inequalities have proven effective in establishing thresholds that separate stable and unstable regimes. This work contributes to this line of research by providing a rigorous analysis of the interplay between Balakrishnan–Taylor damping, weak damping, and a logarithmic source term in the context of an initial–boundary value problem for a nonlinear wave equation; see [10, 13, 32].

In this paper, we study the initial boundary value problem:

$$\begin{cases} u_{tt} - \left(a + b\|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)\right) \Delta u + \omega u_t + \mu u = u \ln|u|, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. The constants a, b, σ, ω, and μ are fixed positive real numbers.

First, we recall some results related to problem (1.1) without logarithms. In the absence of the Balakrishnan-Taylor damping ($\sigma = 0$), when $b = 0$, the problem (1.1) reduces to a wave equation that has been extensively studied. When $b \neq 0$, the problem (1.1), which describes the nonlinear vibrations of an elastic string and is known as the Kirchhoff equation, is the well-known Kirchhoff equation introduced by Kirchhoff [21]. We refer interested readers to [22,27,36] and the references therein. Balakrishnan-Taylor damping ($\sigma \neq 0$) and $\omega = \mu = 0$ were initially proposed by Balakrishnan and Taylor [2], as well as by Bass and Zes [4]. It is related to the panel flutter equation and the spillover problem. So far, there have been some stability results for the problem involving Balakrishnan-Taylor damping; see, for instance, [11, 18, 34].

The logarithmic nonlinearity is of great interest in physics, as it appears naturally in inflationary cosmology, supersymmetric field theories, quantum mechanics, and nuclear physics [3, 12]. Birula and Mycielski [5,6] studied the following problem:

$$\begin{aligned} &u_{tt} - u_{xx} + u - \varepsilon u \ln|u|^2 = 0, \quad \text{in } [a,b] \times (0,T), \\ &u(a,t) = u(b,t) = 0, \quad \text{on } (0,T), \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \text{in } [a,b], \end{aligned} \tag{1.2}$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [16,31]. Cazenave and Haraux [9] considered

$$u_{tt} - \Delta u = u \ln|u|^k, \tag{1.3}$$

and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [15] used some compactness arguments and obtained the global existence of weak solutions for all $(u_0, u_1) \in H_0^1([a,b]) \times L^2([a,b])$ to the initial-boundary value problem (1.3) in the one-dimensional case. Lian et al. [23] studied (1.3), and by using the potential well combined with the logarithmic Sobolev inequality, they proved the global existence and infinite time blow up of solutions in cases $E(0) < d$ and $E(0) = d$. Furthermore, they proved the infinite time blow-up of solutions with $E(0) > 0$.

Hiramatsu et al. [20] introduced the following equation:

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|, \tag{1.4}$$

to study the dynamics of Q-balls in theoretical physics, a numerical study has been presented. However, there was no theoretical analysis of the problem (1.4). Han [19] proved the global existence of weak solutions for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ to the initial boundary value problem (1.4) in $\mathbb{R}^3$. By constructing a potential well and using the logarithmic Sobolev inequality, Zhang et al. [38] obtained the exponential decay estimates of energy with $E(0) < d$ for the equation (1.4). In another study, Yang et al. [33] studied the following Kirchhoff-type equation with a nonlinear logarithmic source term

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + |u_t|^{p-1} u_t = u^{k-1} \ln |u|. \tag{1.5}$$

By the potential well method, they proved the global existence, finite time blow up, and asymptotic behavior of solutions in the cases of $E(0) < d$ and $E(0) = 0$. They also proved the finite time blow up of solutions to the problem in the case $E(0) > 0$. For other related results, readers may see [8, 24, 30] and the references therein.

Motivated by the previous work, in this paper, We consider the asymptotic behavior of solutions to the initial boundary value problem of the logarithmic wave equation with Balakrishnan-Taylor and weak damping terms.

The paper is organized as follows. In Section 2, we provide some preliminary information and introduce potential wells. In Section 4, we prove the global existence of a solution. In Section 3, we establish the general decay result of the energy. In Section 5, we prove the infinite time blow up of solutions.

2. Preliminaries and potential wells

2.1. Preliminaries

In this subsection, we outline the variational framework for problem (1.1) and give some preliminary lemmas. We denote $\|\cdot\|_p$ and $\|\cdot\|_{H^1}$ as the usual $L^p(\Omega)$ norm and $H^1(\Omega)$ norm, respectively.

Lemma 2.1. ([1] Sobolev-Poincaré inequality) *Let q be a number with $2 \le q < \infty$ $(n = 1, 2)$ or $2 \le q < \frac{2n}{n-2}$ $(n \ge 3)$; then there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \le c_* \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega).$$

Lemma 2.2. ([17, 25] Logarithmic Sobolev inequality) *Let u be any function in $H_0^1(\Omega)$, and ρ be any positive real number. Then*

$$\int_\Omega u^2 \ln |u| dx \le \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{\rho^2}{2\pi}\|\nabla u\|_2^2 - n(1 + \ln \rho)\|u\|_2^2. \tag{2.1}$$

Now, we define the energy associated with problem (1.1) as follows:

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{\mu}{2}\|u\|_2^2 - \frac{1}{2}\int_\Omega u^2 \ln |u| dx + \frac{1}{4}\|u\|_2^2. \tag{2.2}$$

Lemma 2.3. *Let u be a solution to problem* (1.1)*. Then,*

$$E'(t) \le -\omega \|u_t\|_2^2 \le 0. \tag{2.3}$$

Now, we are ready to state the local existence of problem (1.1), whose proof can be found in [7, 14, 28]

Theorem 2.1. (Local existence) *Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ hold. Then, there exists a unique local solution to the problem* (1.1) *in the class*

$$\begin{aligned} &u \in C\left([0,T]; H_0^1(\Omega)\right), \\ &u_t \in C\left([0,T]; H_0^1(\Omega)\right). \end{aligned}$$

2.2. Potential wells

In this subsection, we establish the corresponding method of potential wells, which is related to the logarithmic nonlinear term. For this goal, we define the functionals:

$$J(t) = J(u(t)) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{\mu}{2}\|u\|_2^2 - \frac{1}{2}\int_\Omega u^2 \ln |u| dx + \frac{1}{4}\|u\|_2^2, \tag{2.4}$$

and

$$I(t) = I(u(t)) = a\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + \mu\|u\|_2^2 - \int_\Omega u^2 \ln |u| dx. \tag{2.5}$$

Then, it is obvious that

$$J(t) = \frac{1}{2}I(t) + \frac{1}{4}\|u\|_2^2, \tag{2.6}$$

and

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t). \tag{2.7}$$

As in [37], we define the potential well depth

$$d = \inf\left\{\sup_{\lambda \ge 0} J(\lambda u); u \in H_0^1(\Omega), \|\nabla u\|_2^2 \ne 0\right\}, \tag{2.8}$$

and

$$0 < d = \inf_{u \in \mathcal{N}} J(u), \tag{2.9}$$

where $\mathcal{N}$ is the Nehari manifold, which is defined by

$$\mathcal{N} = \left\{u \in H_0^1(\Omega)\backslash\{0\} : I(u) = 0\right\}. \tag{2.10}$$

Then, the potential well (stable set) W can be defined by

$$W = \{u \in H_0^1(\Omega); I(u) > 0,\ J(u) < d\} \cup \{0\},$$

and the outer space of the potential well (unstable set) V, by

$$V = \{u \in H_0^1(\Omega); I(u) < 0,\ J(u) < d\}.$$

By a similar argument as [33], we can prove the following lemma.

Lemma 2.4. *For any $u \in H_0^1(\Omega)$, $\|\nabla u\|_2 \neq 0$. Then*
(1) $\lim_{\lambda\to 0} J(\lambda u) = 0$ *and* $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.
(2) *There is a unique $\lambda_1(u)$ such that* $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda_1} = 0$.
(3)

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) = \begin{cases} < 0,\ \lambda_1 < \lambda, \\ = 0,\ \lambda = \lambda_1, \\ > 0,\ 0 \leq \lambda < \lambda_1. \end{cases}$$

Lemma 2.5. *Let $u \in H_0^1(\Omega)$ and $r := (2\pi a)^{n/4}\, e^{n/2}$. Then, we have*
(i) If $0 < \|u\|_2 \leq r$, then $I(u) > 0$.
(ii) If $I(0) < 0$, then $\|u\|_2 > r$.
(iii) If $I(0) = 0$, and $\|u\|_2 \neq 0$, then $\|u\|_2 \geq r$.

Proof. Using the logarithmic Sobolev inequality in Lemma 2.3, for any $\rho > 0$, we have $I(t) \geq (a - \frac{\rho^2}{2\pi})||\nabla u||^2 + (n(1+\ln\rho) - \ln||u||)||u||^2$, so we need to demand $r = (2\pi a)^{\frac{n}{2}} e^n$.

$$\begin{aligned} I(t) &= a\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + \mu\|u\|_2^2 - \int_\Omega u^2 \ln|u| dx \\ &\geq a\|\nabla u\|_2^2 - \int_\Omega u^2 \left(\ln\frac{|u|}{\|u\|_2} + \ln\|u\|_2\right) dx \\ &\geq \left(a - \frac{\rho^2}{2\pi}\right)\|\nabla u\|_2^2 + \left(n(1+\ln\rho) - \ln\|u\|_2\right)\|u\|_2^2. \end{aligned} \tag{2.11}$$

Taking $\rho = \sqrt{2\pi a}$ in (2.11), we obtain

$$I(t) \geq \left(\frac{n(2+\ln(2\pi a))}{2} - \ln\|u\|_2\right)\|u\|_2^2. \tag{2.12}$$

(i) If $0 < \|u\|_2 \leq r$, then $I(u) \geq 0$ follows from (2.12).
(ii) If $I(u) < 0$, then by (2.12), we have

$$\left(\frac{n(2+\ln(2\pi a))}{2} - \ln\|u\|_2\right)\|u\|_2^2 < 0,$$

which means

$$\|u\|_2 > (2\pi a)^{n/2}\, e^n = r.$$

(iii) From (2.12) and $I(u) = 0$, we have

$$\left(\frac{n(2+\ln(2\pi a))}{2} - \ln\|u\|_2\right)\|u\|_2^2 \leq I(0) = 0,$$

which yields

$$\|u\|_2 \geq (2\pi a)^{n/2}\, e^n = r.$$

□

Lemma 2.6. *The depth of the potential well d in (2.9) satisfies*

$$d \geq \frac{1}{4}\,(2\pi a)^{n/2}\, e^n. \tag{2.13}$$

Proof. From the definition of d in (2.9), it follows that $u \in \mathcal{N}$. As a result, we obtain from (iii) in Lemma 2.5 that

$$0 = I(u) \geq \left(\frac{n(2+\ln(2\pi a))}{2} - \ln\|u\|_2\right)\|u\|_2^2, \tag{2.14}$$

which implies

$$\|u\|_2 \geq (2\pi a)^{n/2}\, e^n. \tag{2.15}$$

By virtue of (2.6), $I(u) = 0$, and (2.15), we obtain

$$J(u) = \frac{1}{2}I(t) + \frac{1}{4}\|u\|_2^2 = \frac{1}{4}\|u\|_2^2 \geq \frac{1}{4}\,(2\pi a)^{n/2}\, e^n, \tag{2.16}$$

which gives (2.10). □

Lemma 2.7. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. If $E(0) < d$ and $u_0 \in W$, then we have $u \in W$, and*

$$\|u\|_2^2 < 4d \ \ \textit{for all}\ \ t \in [0,T). \tag{2.17}$$

Proof. Let u be the weak solution of problem (1.1) with $E(0) < d$, $u_0 \in W$, and T being the maximum existence time of u. From Lemma 2.3 we have

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \leq E(0) < d, \quad \forall\, t \in [0,T). \tag{2.18}$$

Then we claim that $u \in W$ for all $t \in [0,T)$. If not, then there is a $t_0 \in (0,T)$ such that $I(u(t_0)) = 0$ and $I(u) > 0$ on $[0,t_0)$. From the definition of $\mathcal{N}$, we have $u(t_0) \in \mathcal{N}$. However, from (2.9) and the definition of $E(t)$, we have $E(t_0) \geq d$, which contradicts (2.18). Then we have $u \in W$ for all $t \in [0,T)$. From (2.6) and (2.18), it follows

$$d > E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) = \frac{1}{2}I(t) + \frac{1}{4}\|u\|_2^2 \geq \frac{1}{4}\|u\|_2^2. \tag{2.19}$$

This completes the proof. □

By a similar argument as in Lemma 2.7, we can prove the following lemma.

Lemma 2.8. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. If $E(0) < d$ and $u_0 \in V$, then we have $u \in V$.*

3. Global existence

In this section, we state and prove the global existence of solutions to (1.1) with $E(0) < d$.

Theorem 3.1. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Assume that $E(0) < d$ and $u_0 \in W$ hold. Then, the problem* (1.1) *admits a global weak solution*

$$u \in L^\infty(0, T; H_0^1(\Omega)),$$

and

$$u_t \in L^\infty(0, T; H_0^1(\Omega)).$$

Proof. It suffices to show that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2,$$

is bounded independent of t. From Lemma 2.3, we have

$$\|u_t\|_2^2 + a\|\nabla u\|_2^2 \le 2E(0) + \int_\Omega u^2 \ln |u| dx. \tag{3.1}$$

Using the logarithmic Sobolev inequality, we obtain

$$\|u_t\|_2^2 + \left(a - \frac{\rho^2}{2\pi}\right) \|\nabla u\|_2^2 + \frac{n(1+\ln \rho)}{2} \|u\|_2^2 \le 2E(0) + \|u\|_2^2 \ln \|u\|_2. \tag{3.2}$$

Choosing $0 < \rho \le \sqrt{2a\pi}$ so that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \le C_0 \left(1 + \|u\|_2^2 \ln \|u\|_2\right). \tag{3.3}$$

It follows from Lemma 2.7 that we have

$$\|u\|_2^2 \le 4d.$$

Then the inequality (3.3) becomes

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \le C_1, \tag{3.4}$$

where $C_1 > 0$ is a constant independent of t. This completes the proof. □

4. General decay

In this section, we prove the decay of solutions to problem (1.1). For this purpose, we set

$$G(t) = E(t) + \varepsilon \int_\Omega u_t u dx + \varepsilon \frac{\sigma}{4} \|\nabla u\|_2^4, \tag{4.1}$$

where $\varepsilon > 0$ will be chosen later. In order to show our stability result, we need the following lemma.

Lemma 4.1. *There exist two positive constants α_1 and α_2, such that*

$$\alpha_1 E(t) < G(t) < \alpha_2 E(t), \tag{4.2}$$

for ε sufficiently small.

Theorem 4.1. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Assume that $E(0) < d$ and $u_0 \in W$; there exist some positive constants K and k such that*

$$E(t) \leq K e^{-kt}, \ \ \forall\, t \geq 0. \tag{4.3}$$

Proof. Taking the derivative of (4.1) and using equation (1.1), we get

$$\begin{aligned} G'(t) &= E'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \int_\Omega u_{tt} u dx + \varepsilon\sigma \|\nabla u\|_2^2 \int_\Omega \nabla u \nabla u_t dx \\ &\leq -\left(\omega - \varepsilon\right) \|u_t\|_2^2 - \varepsilon a \|\nabla u\|_2^2 - \varepsilon b \|\nabla u\|_2^4 - \varepsilon\mu \|u\|_2^2 - \omega \int_\Omega u_t u dx \\ &+ \varepsilon \int_\Omega u^2 \ln |u| dx. \end{aligned} \tag{4.4}$$

In view of Young's inequality and Poincare's inequality, we have

$$\int_\Omega u_t u \leq \frac{1}{4\eta} \|u_t\|_2^2 + \eta \|u\|_2^2. \tag{4.5}$$

Combining (4.5) with (4.4) and using (2.2) for $N > 0$, we obtain

$$\begin{aligned} G'(t) &\leq -N\varepsilon E(t) + \left(\varepsilon \frac{N}{2} + \varepsilon + \omega(\frac{\varepsilon}{4\eta} - 1)\right) \|u_t\|_2^2 + \varepsilon b \left(\frac{N}{4} - 1\right) \|\nabla u\|_2^4 \\ &+ \varepsilon a \left(\frac{N}{2} - 1\right) \|\nabla u\|_2^2 + \varepsilon \left(\frac{N}{4} + \eta\omega - \mu\right) \|u\|_2^2 + \varepsilon \left(1 - \frac{N}{2}\right) \int_\Omega u^2 \ln |u| dx. \end{aligned} \tag{4.6}$$

By the logarithmic Sobolev inequality, we have

$$\begin{aligned} G'(t) &\leq -N\varepsilon E(t) + \left\{\varepsilon \frac{N}{2} + \varepsilon + \omega \left(\frac{\varepsilon}{4\eta} - 1\right)\right\} \|u_t\|_2^2 \\ &+ \varepsilon \left\{\frac{N}{4} + \eta\omega - \mu + \left(1 - \frac{N}{2}\right) \left(\ln \|u\|_2 - \frac{n}{2}(1 + \ln \rho)\right)\right\} \|u\|_2^2 \\ &+ \varepsilon \left(1 - \frac{N}{2}\right) \left(\frac{\rho^2}{2\pi} - a\right) \|\nabla u\|_2^2 + \varepsilon b \left(\frac{N}{4} - 1\right) \|\nabla u\|_2^4. \end{aligned} \tag{4.7}$$

It follows from (2.2), (2.6), and

$$E(t) \leq E(0) < d,$$

that we get

$$\ln \|u\|_2^2 \leq \ln(4J(t)) \leq \ln(4E(0)) < \ln(4d). \tag{4.8}$$

Consequently, we have

$$\begin{aligned} G'(t) &\leq -N\varepsilon E(t) + \left\{\varepsilon \left(\frac{N}{2} + 1 + \omega \frac{1}{4\eta}\right) - \omega\right\} \|u_t\|_2^2 \\ &+ \varepsilon \left\{\frac{N}{4} + \eta\omega - \mu + \frac{1}{2} \left(1 - \frac{N}{2}\right) (\ln(4d) - n(1 + \ln \rho))\right\} \|u\|_2^2 \\ &+ \varepsilon \left(1 - \frac{N}{2}\right) \left(\frac{\rho^2}{2\pi} - a\right) \|\nabla u\|_2^2 + \varepsilon b \left(\frac{N}{4} - 1\right) \|\nabla u\|_2^4. \end{aligned} \tag{4.9}$$

At this point, we choose $0 < N < 2$ and η such that

$$\frac{N}{4} - 1 < 0 \quad \text{and} \quad \eta\omega - \mu > 0.$$

When N and η are fixed, we choose ε to be small enough such that

$$\varepsilon\left(\frac{N}{2} + 1 + \omega\frac{1}{4\eta}\right) - \omega < 0.$$

Once N, η and ε are fixed, we choose

$$e^{\frac{2\alpha}{n}} \leq \rho^2 \leq 2a\pi,$$

where

$$\alpha = \left(\frac{2}{2-N}\right)\left(\frac{N}{4} + \eta\omega - \mu\right) + \ln(4d) - n,$$

and taking ρ small enough such that

$$\frac{\rho^2}{2\pi} - a < 0 \quad \text{and} \quad \frac{N}{4} + \eta\omega - \mu + \frac{1}{2}\left(1 - \frac{N}{2}\right)(\ln(4d) - n(1 + \ln\rho)) < 0.$$

Then the inequality (4.9) becomes

$$G'(t) \leq -N\varepsilon E(t), \ \ \forall\, t > 0. \tag{4.10}$$

Further, by (4.2), we have

$$G'(t) \leq -\frac{N}{\alpha_2} G(t), \ \forall\, t > 0. \tag{4.11}$$

A simple integration of (4.11) over $(0, t)$ yields

$$G(t) \leq -Ce^{-kt}, \ \forall\, t > 0. \tag{4.12}$$

Consequently, by (4.2) once again, we obtain

$$E(t) \leq -Ke^{-kt}, \ \forall\, t > 0. \tag{4.13}$$

This completes the proof. □

5. Blow up

In this section, we state and prove the infinite time blow up of solutions to problem (1.1) in the cases $E(0) < d$ and $E(0) < 0$.

5.1. Blow up for $E(0) < d$

Theorem 5.1. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ hold. Assume that $0 < E(0) < \frac{1}{4}r^2 < d$, $u_0 \in V$, and the constant μ satisfy*

$$\mu > \ln(r + 1) > 0, \tag{5.1}$$

where r is defined as in Lemma 2.5. Then, the solution to problem (1.1) *blows up in an infinite amount of time.*

Proof. Set

$$H(t) = E_1 - E(t), \tag{5.2}$$

where

$$E(0) < E_1 < \frac{1}{4}r^2.$$

It follows from (2.3) and (5.2) that

$$H'(t) = -E'(t) \geq \omega \|u_t\|_2^2 \geq 0, \tag{5.3}$$

and

$$0 < H(0) = E_1 - E(0) \leq H(t) \leq \frac{1}{2}\int_\Omega u^2 \ln |u| dx. \tag{5.4}$$

Next, we define

$$\Psi(t) = H(t) + \varepsilon \int_\Omega u_t u dx + \varepsilon \frac{\sigma}{4}\|\nabla u\|_2^4 + \varepsilon \frac{\omega}{2}\|u\|_2^2, \tag{5.5}$$

where ε is a positive constant to be specified later.

Taking the derivative of $\Psi(t)$ and using (1.1), we have

$$\Psi'(t) = H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon a\|\nabla u\|_2^2 - \varepsilon b\|\nabla u\|_2^4 - \varepsilon\mu\|u\|_2^2 + \varepsilon\int_\Omega u^2 \ln |u| dx. \tag{5.6}$$

It follows from (2.2) and (5.2) that, for a constant $M > 0$, we see that

$$\begin{aligned}\Psi'(t) \geq\ & H'(t) + \varepsilon\left(1 + \frac{M}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{M}{2} - 1\right)\|\nabla u\|_2^2 + \varepsilon b\left(\frac{M}{4} - 1\right)\|\nabla u\|_2^4 \\ & + \varepsilon\mu\left(\frac{M}{2} - 1\right)\|u\|_2^2 - \varepsilon\left(\frac{M}{2} - 1\right)\int_\Omega u^2 \ln |u| dx + \varepsilon M H(t) \\ & + \varepsilon M\left(\frac{1}{4}\|u\|_2^2 - E_1\right).\end{aligned} \tag{5.7}$$

Since $u_0 \in V$ and Lemma 2.8, we have $u \in V$ and $I(u) < 0$; then $\|u\|_2 > r$. Note that

$$E_1 < \frac{r^2}{4} < \frac{1}{4}\|u\|_2^2,$$

when combined with (5.7), and by the logarithmic Sobolev inequality, we have

$$\begin{aligned}\Psi'(t) \geq\ & H'(t) + \varepsilon\left(1 + \frac{M}{2}\right)\|u_t\|_2^2 + \varepsilon a\left(\frac{M}{2} - 1\right)\|\nabla u\|_2^2 + \varepsilon b\left(\frac{M}{4} - 1\right)\|\nabla u\|_2^4 \\ & + \varepsilon\mu\left(\frac{M}{2} - 1\right)\|u\|_2^2 - \varepsilon\left(\frac{M}{2} - 1\right)\int_\Omega u^2 \ln |u| dx + \varepsilon M H(t) \\ \geq\ & H'(t) + \varepsilon\left(1 + \frac{M}{2}\right)\|u_t\|_2^2 + \varepsilon\left(\frac{M}{2} - 1\right)\left(a - \frac{\rho^2}{2\pi}\right)\|\nabla u\|_2^2 \\ & + \varepsilon b\left(\frac{M}{4} - 1\right)\|\nabla u\|_2^4 + \varepsilon\left(\frac{M}{2} - 1\right)\left\{\mu + \left(\frac{n}{2}(1 + \ln\rho) - \ln\|u\|_2\right)\right\}\|u\|_2^2.\end{aligned} \tag{5.8}$$

By Lemma 2.8, we take ρ satisfying $0 < \rho^2 < 2a\pi$, and then

$$\frac{n}{2}(1+\ln \rho) - \ln \|u\|_2 < 0,$$

since $\|u\|_2 > r$. Exploiting (5.1), we obtain

$$\mu + \left(\frac{n}{2}(1+\ln \rho) - \ln \|u\|_2\right) > 0.$$

Now, choosing $M > 4$, the inequality (5.8) becomes

$$\Psi'(t) \geq \gamma \left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|u\|_2^2 + H(t)\right), \tag{5.9}$$

where γ is a positive constant for some fixed $\varepsilon > 0$. Consequently, we have

$$\Psi(t) \geq \Psi(0) > 0.$$

On the other hand, we can easily see that

$$\begin{aligned}\Psi(t) &\leq H(t) + \frac{\varepsilon}{2}\|u_t\|_2^2 + \varepsilon\frac{(1+\omega)}{2}\|u\|_2^2 + \varepsilon\frac{\sigma}{2}\|\nabla u\|_2^4 \\ &\leq \kappa\left(H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_2^4\right).\end{aligned} \tag{5.10}$$

Combining (5.9) with (5.10), we find that

$$\Psi'(t) \geq \xi\Psi(t), \tag{5.11}$$

namely,

$$\Psi(t) \geq Ke^{\xi t}, \tag{5.12}$$

i.e.,

$$\lim_{t\to+\infty} \Psi(t) = +\infty. \tag{5.13}$$

This shows that the weak solution u to problem (1.1) blows up at infinity. □

5.2. Blow up for $E(0) < 0$

Theorem 5.2. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ hold. Assume that $E(0) < 0$, $u_0 \in V$, and the constant μ satisfy*

$$\mu > \ln(r+1) > 0, \tag{5.14}$$

where r is defined as in Lemma 2.5. Then, the solution to problem (1.1) *blows up in infinite time.*

Proof. Set

$$H(t) = -E(t), \tag{5.15}$$

then, $E(0) < 0$, (2.3), and (5.15) give

$$H'(t) = -E'(t) \geq \omega \|u_t\|_2^2 \geq 0, \tag{5.16}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{2} \int_\Omega u^2 \ln |u| dx. \tag{5.17}$$

Next, we define

$$\Psi(t) = H(t) + \varepsilon \int_\Omega u_t u dx + \varepsilon \frac{\sigma}{4} \|\nabla u\|_2^4 + \varepsilon \frac{\omega}{2} \|u\|_2^2, \tag{5.18}$$

where ε is a positive constant. Taking the derivative of $\Psi(t)$ and using (1.1), we have

$$\Psi'(t) = H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon a \|\nabla u\|_2^2 - \varepsilon b \|\nabla u\|_2^4 - \varepsilon \mu \|u\|_2^2 + \varepsilon \int_\Omega u^2 \ln |u| dx. \tag{5.19}$$

It follows from (2.2) and (5.15) that, for a constant $M > 4$, we see that

$$\begin{aligned} \Psi'(t) \geq H'(t) + \varepsilon \left(1 + \frac{M}{2}\right) \|u_t\|_2^2 + \varepsilon a \left(\frac{M}{2} - 1\right) \|\nabla u\|_2^2 + \varepsilon b \left(\frac{M}{4} - 1\right) \|\nabla u\|_2^4 \\ + \varepsilon \mu \left(\frac{M}{2} - 1\right) \|u\|_2^2 + \varepsilon \frac{M}{4} \|u\|_2^2 - \varepsilon \left(\frac{M}{2} - 1\right) \int_\Omega u^2 \ln |u| dx + \varepsilon M H(t). \end{aligned} \tag{5.20}$$

The remainder of the proof is the same as that of Theorem 5.1; by estimating the growth rate, we can conclude the results. This completes the proof. □

6. Conclusion

In this work, we analyzed an initial–boundary value problem for a nonlinear wave equation equipped with Balakrishnan–Taylor damping, weak damping, and a logarithmic source term. By employing the potential well method and a suitable logarithmic Sobolev inequality, we established conditions that ensure the global existence of solutions and derived energy decay estimates in the stable regime of initial data. Moreover, we proved that solutions corresponding to initial data lying in the unstable set blow up in infinite time, which provides a sharp classification of the long-term dynamics based on the initial energy level.

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