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A NEW MORE ACCURATE HILBERT-TYPE INEQUALITY INVOLVING ONE PARTIAL SUM

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Abstract By using the weight coefficients, the Euler-Maclaurin summation formula and Abel summation by parts formula, a new more accurate Hilbert-type inequality with the power function as the internal variables involving one partial sum is given. The equivalent conditions of the best value related to a few parameters are obtained, and some particular inequalities are considered. We also provide the equivalent forms and the operator expressions.

Keywords Weight coefficient, Euler-Maclaurin summation formula, Hardy-Hilbert's inequality, internal variable, parameter, partial sum.

MSC(2010) 26D15.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy-Hilbert's inequality with the best value $\frac{\pi}{\sin(\pi/p)}$ (cf. [5], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1.1}$$

In 2006, by introducing parameters $\lambda_i \in (0, 2] (i = 1, 2), \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, an extension of (1.1) was provided by [15] as follows:

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \\ &< B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{1.2}$$

where, the constant factor $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B(u, v) = \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \, (u, v > 0)$$

is the beta function. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.2) reduces to (1.1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{1}{2}$, (1.2) reduces to Yang's inequality in [31].

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In 2019, by applying (1.2) and the technique of real analysis, Adiyasuren et al. [1] gave a new extended inequality of (1.2) with the kernel as $\frac{1}{(m+n)^{\lambda}}$ involving two partial sums as:

$$A_m = \sum_{i=1}^{m} a_i, B_n = \sum_{k=1}^{n} b_k \ (m, n \in \mathbf{N} = \{1.2.\cdots\}).$$

If $f(x), g(y) \geq 0, 0 < \int_0^{\infty} f^p(x)dx < \infty$, and $0 < \int_0^{\infty} g^q(y)dy < \infty$, then we still have the following Hardy-Hilbert's integral inequality (cf. [5], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.3}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.3) with their extensions are playing an important role in analysis and its applications (cf. [2–4, 7, 12, 16, 19, 27, 28, 30, 32, 37]). On the new study of Hilbert-type inequalities and its generalizations on time scales, please see the published papers [5, 23, 36].

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [5], Theorem 351): If $K(t)(t > 0)$ is decreasing, $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(t)t^{s-1}dt < \infty$, $a_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\int_0^{\infty} x^{p-1}\left(\sum_{n=1}^{\infty} K(nx)a_n\right)^p dx < \phi^p(\frac{1}{q})\sum_{n=1}^{\infty} a_n^p. \tag{1.4}$$

Some new extensions of (1.4) were provided by [20–22, 33, 34].

In 2016, by means of the technique of real analysis, Hong [11] considered some equivalent statements of the extensions of (1.1) with the best possible constant factor related to a few parameters. The other similar works about the extension of (1.3) were given by [8–10, 14, 26, 29, 35]. In 2024, Peng and Yang [18] provided an extension of Mulholland's inequality with the kernel as $\frac{1}{(\ln mn)^{\lambda}}$ $(\lambda > 0)$ involving one partial sum. Some similar results were published in [13, 24, 25]. The objective of this paper is to give a similar result of [18] in the more accurate kernel as

$$\frac{1}{[(m-\xi)^{\alpha} + (n-\eta)^{\beta}]^{\lambda}} \ (\lambda \in (0, 2], \alpha, \beta \in (0, 1], \xi, \eta \in [0, \frac{1}{2}]),$$

which is a generalization of [15].

In this paper, following the way of [1, 11, 15], by the means of the weight coefficients, the idea of introduced parameters, the Euler-Maclaurin summation formula and Abel summation by parts formula, a new more accurate Hilbert-type inequality with the power function as the internal variables involving one partial sum is given. The equivalent statements of the best possible constant factor related to a few parameters are provided, and some particular inequalities are deduced. We also obtain the equivalent forms and the operator expressions. The main results of this paper is an extended application of [1, 15], and the novelty of our work is a more perfect refinement of a few published results.

2. Some lemmas

In what follows, we suppose that

(A1). For $p > 1$ $(q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 2], \alpha, \beta \in (0, 1]$, $\xi, \eta \in [0, \frac{1}{4}]$, $\lambda_1 \in (0, \frac{3}{2\alpha}] \cap (0, \lambda)$, $\lambda_2 \in$

$(0, \frac{3}{2\beta} - 1] \cap (0, \lambda)$, $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. For $b_n \geq 0$, we define a partial sum as: $B_n := \sum_{k=1}^{n} b_k$ $(n \in \mathbf{N})$, such that $B_n = o(e^{t(n-\eta)^\beta})(t > 0; n \to \infty)$, $a_m \geq 0$, satisfying

$$0 < \sum_{m=1}^{\infty} (m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} (n-\eta)^{-q\beta\widehat{\lambda}_2-1} B_n^q < \infty. \tag{2.1}$$

Lemma 2.1. (cf. [32], Theorem 2.2.1) *(i) If* $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [m, \infty)$ $(m < n \in \mathbf{N})$ *with* $g^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, $P_i(t), B_i$ $(i \in \mathbf{N})$ *are the Bernoulli functions and the Bernoulli numbers of i-order, then*

$$\int_m^n P_{2q-1}(t) g(t) dt = \varepsilon_q \frac{B_{2q}}{2q} g(t)|_m^n (0 < \varepsilon_q < 1; q = 1, 2, \cdots). \tag{2.2}$$

In particular, for $q = 1, B_2 = \frac{1}{6}$, we have the following inequalities:

$$\frac{1}{12}(g(n) - g(m)) < \int_m^n P_1(t) g(t) dt < 0, \tag{2.3}$$

for $q = 2, B_4 = -\frac{1}{30}$, $n \to \infty$, it follows that

$$0 < \int_m^n P_3(t) g(t) dt < \frac{1}{120} g(m). \tag{2.4}$$

(ii) (cf. [32], Theorem 2.3.1) If $f(t)$ $(> 0) \in C^3[m, \infty), f^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, then we have the following Euler -Maclaurin summation formula:

$$\sum_{k=m}^{n} f(k) = \int_m^n f(t) dt + \frac{f(m) + f(n)}{2} + \int_m^n P_1(t) f'(t) dt, \tag{2.5}$$

$$\int_m^\infty P_3(t) f'(t) dt = -\frac{1}{12} f'(m) + \frac{1}{6} \int_m^\infty P_3(t) f'''(t) dt \ (m < n). \tag{2.6}$$

Lemma 2.2. *For* $s \in (0, 3], s_2 \in (1, \frac{3}{2\beta}] \cap (0, s).k_s(s_2) := B(s_2, s - s_2)$, *define the following weight coefficient:*

$$\varpi_s(s_2, m) := (m-\xi)^{\alpha(s-s_2)} \sum_{n=1}^{\infty} \frac{\beta(n-\eta)^{\beta s_2 - 1}}{[(m-\xi)^\alpha + (n-\eta)^\beta]^s} \ (m \in \mathbf{N}). \tag{2.7}$$

We have the following inequalities:

$$\begin{aligned} 0 &< k_s(s_2)(1 - O_1(\frac{1}{(m-\xi)^{\alpha s_2}})) \\ &< \varpi_s(s_2, m) \\ &< k_s(s_2) \ (m \in \mathbf{N}), \end{aligned} \tag{2.8}$$

where, $O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For fixed $m \in \mathbf{N}, a := (m-\xi)^{\alpha}$, we define a positive function $g(t)$ as follows: $g(t) := \frac{\beta(t-\eta)^{\beta s_2-1}}{[a+(t-\eta)^{\beta}]^s}(t > \eta)$. We divide two cases of $s_2 \in (0, \frac{1}{\beta}) \cap (0, s)$ and $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s)$ to obtain (2.8) as follows.

(i) For $s_2 \in (0, \frac{1}{\beta}) \cap (0, s)$, since $(-1)^{(i)}g(t) > 0$ $(t > \eta; i = 0, 1, 2)$, by Hermite-Hadamard's inequality (cf. [18]), setting $u = \frac{(t-\eta)^{\beta}}{a}$, we have

$$\begin{aligned}
\varpi_s(s_2, m) &= (m-\xi)^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(n) \\
&< (m-\xi)^{\alpha(s-s_2)} \int_{\frac{1}{2}}^{\infty} g(t)dt \\
&= \int_{\frac{(\frac{1}{2}-\eta)^{\beta}}{a}}^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} \\
&\leq \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} \\
&= k_s(s_2).
\end{aligned}$$

On the other-hand, in view of the decreasingness property of series, we obtain

$$\begin{aligned}
\varpi_s(s_2, m) &> (m-\xi)^{\alpha(s-s_2)} \int_1^{\infty} g(t)dt \\
&= \int_{\frac{(1-\eta)^{\beta}}{a}}^{\infty} \frac{u^{s_2-1}}{(1+u)^s} du \\
&= k_s(s_2) - \int_0^{\frac{(1-\eta)^{\beta}}{a}} \frac{u^{s_2-1}du}{(1+u)^s} \\
&= k_s(s_2)(1 - O_1(\frac{1}{(m-\xi)^{\alpha s_2}})) \\
&> 0,
\end{aligned}$$

where, $O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^{\beta}}{(m-\xi)^{\alpha}}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$, satisfying

$$0 < \int_0^{\frac{(1-\eta)^{\beta}}{(m-\xi)^{\alpha}}} \frac{u^{s_2-1}du}{(1+u)^s} \leq \int_0^{\frac{(1-\eta)^{\beta}}{(m-\xi)^{\alpha}}} u^{s_2-1}du = \frac{1}{s_2}\left[\frac{(1-\eta)^{\beta}}{(m-\xi)^{\alpha}}\right]^{s_2} \ (m \in \mathbf{N}).$$

Hence, we obtain (2.8).

(ii) For $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s)$, in view of (2.5) (for $n \to \infty$), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} g(n) = \int_1^{\infty} g(t)dt + \frac{g(1)}{2} + \int_1^{\infty} P_1(t)g'(t)dt = \int_{\eta}^{\infty} g(t)dt - h(a), \\
&h(a) := \int_{\eta}^{1} g(t)dt - \frac{g(1)}{2} - \int_1^{\infty} P_1(t)g'(t)dt.
\end{aligned}$$

We obtain $-\frac{g(1)}{2} = \frac{-\beta(1-\eta)^{\beta s_2-1}}{2[a+(1-\eta)^{\beta}]^s}$. By integration by parts, we find

$$\int_{\eta}^{1} g(t)dt = \beta \int_{\eta}^{1} \frac{(t-\eta)^{\beta s_2-1}dt}{[a+(t-\eta)^{\beta}]^s}$$

$$\begin{aligned}
&= \int_0^{(1-\eta)^\beta} \frac{u^{s_2-1}}{(a+u)^s} du \\
&= \frac{1}{s_2} \int_0^{(1-\eta)^\beta} \frac{du^{s_2}}{(a+u)^s} \\
&= \frac{u^{s_2}}{s_2(a+u)^s}|_0^{(1-\eta)^\beta} + \frac{s}{s_2} \int_0^{(1-\eta)^\beta} \frac{u^{s_2}}{(a+u)^{s+1}} du \\
&= \frac{(1-\eta)^{\beta s_2}}{s_2[a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \int_0^{(1-\eta)^\beta} \frac{du^{s_2+1}}{(a+u)^{s+1}} \\
&\geq \frac{(1-\eta)^{\beta s_2}}{s_2[a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \frac{u^{s_2+1}}{(a+u)^{s+1}}|_0^{(1-\eta)^\beta} \\
&\quad + \frac{s}{s_2(s_2+1)[a+(1-\eta)^\beta]^{s+1}} \int_0^{(1-\eta)^\beta} u^{s_2+1} du \\
&= \frac{(1-\eta)^{\beta s_2}}{s_2[a+(1-\eta)^\beta]^s} + \frac{s(1-\eta)^{\beta(s_2+1)}}{s_2(s_2+1)[a+(1-\eta)^\beta]^{s+1}} \\
&\quad + \frac{s(s+1)(1-\eta)^{\beta(s_2+2)}}{s_2(s_2+1)(s_2+2)[a+(1-\eta)^\beta]^{s+1}},
\end{aligned}$$

$$\begin{aligned}
-g'(t) &= -\frac{\beta(\beta s_2-1)(t-\eta)^{\beta s_2-2}}{[a+(1-\eta)^\beta]^s} + \frac{\beta^2 s(t-\eta)^{\beta+\beta s_2-2}}{[a+(1-\eta)^\beta]^{s+1}} \\
&= \frac{\beta(\beta s-\beta s_2+1)(t-\eta)^{\beta s_2-2}}{[a+(1-\eta)^\beta]^s} - \frac{\beta^2 sa(t-\eta)^{\beta s_2-2}}{[a+(1-\eta)^\beta]^{s+1}},
\end{aligned}$$

and for $\frac{1}{\beta} \leq s_2 \leq \frac{3}{2\beta}, 0 < \beta \leq 1, s_2 < s \leq 3$, it follows that

$$(-1)^i \frac{d^i}{dt^i} \frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+j}} > 0 \ (i=0,1,2,3; j=0,1).$$

By (2.3) and (2.5) (for $n \to \infty$), (2.1) and (2.6), we obtain

$$\begin{aligned}
&\beta(\beta s-\beta s_2+1) \int_1^\infty P_1(t) \frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^s} dt \\
&> -\frac{\beta(\beta s-\beta s_1+1)(1-\eta)^{\beta s_2-2}}{[a+(1-\eta)^\beta]^s} - \beta^2 sa \int_1^\infty P_1(t) \frac{(t-\eta)^{\beta s_2-2} dt}{[a+(t-\eta)^\beta]^{s+1}} \\
&= \frac{\beta^2 sa(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 sa}{6} \int_1^\infty P_3(t) [\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}]'' dt \\
&> \frac{\beta^2 sa(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 sa}{720} [\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}]''_{t=1} \\
&= \frac{\beta^2 s[a+(t-\eta)^\beta-(t-\eta)^\beta](1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 sa}{720} \left\{ \frac{(\beta s_2-2)(\beta s_2-3)(1-\eta)^{\beta s_2-4}}{[a+(1-\eta)^\beta]^{s+1}} \right. \\
&\quad \left. + \frac{\beta(s+1)(5-\beta-2\beta s_2)(1-\eta)^{\beta s_2+\beta-4}}{[a+(1-\eta)^\beta]^{s+2}} + \frac{\beta^2(s+1)(s+2)(1-\eta)^{\beta s_2+2\beta-4}}{[a+(1-\eta)^\beta]^{s+3}} \right\} \\
&= \frac{\beta^2 s(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^s} - \frac{\beta^2 s(1-\eta)^{\beta s_2+\gamma-2}}{12[a+(1-\eta)^\beta]^{s+1}}
\end{aligned}$$

$$-\frac{\beta^2 s}{720}\left\{\frac{(\beta s_2-2)(\beta s_2-3)}{[a+(1-\eta)^\beta]^s}+\frac{\beta(s+1)(5-\beta-2\beta s_1)}{[a+(1-\eta)^\beta]^{s+1}}(1-\eta)^\beta\right.$$
$$\left.+\frac{\beta^2(s+1)(s+2)}{[a+(1-\eta)^\beta]^{s+2}}(1-\eta)^{2\beta}\right\}(1-\eta)^{\beta s_2-4}.$$

Then we have

$$h(a)>\frac{(1-\eta)^{\beta s_2-4}}{[a+(1-\eta)^\beta]^s}h_1+\frac{s(1-\eta)^{\beta s_2+\beta-4}}{[a+(1-\eta)^\beta]^{s+1}}h_2+\frac{s(s+1)(1-\eta)^{\beta s_2+2\beta-4}}{[a+(1-\eta)^\beta]^{s+2}}h_3,$$

where,

$$\begin{aligned} h_1 &:= \frac{1}{s_2}(1-\eta)^4-\frac{\beta}{2}(1-\eta)^3-\frac{\beta-\beta^2 s_2}{12}(1-\eta)^2\\ &\quad -\frac{\beta^2 s(2-\beta s_2)(3-\beta s_2)}{720},\\ h_2 &:= \frac{(1-\eta)^4}{s_2(s_2+1)}-\frac{\beta^2(1-\eta)^2}{12}-\frac{\beta^3(s+1)(5-\beta-2\beta s_2)}{720},\ \text{and}\\ h_3 &:= \frac{(1-\eta)^4}{s_2(s_2+1)(s_2+2)}-\frac{\beta^4(s+2)}{720}. \end{aligned}$$

We find $h_1=\frac{g(s_2)}{720s_2}$, where,

$$\begin{aligned} g(s_2) &:= 720(1-\eta)^4-[360\beta(1-\eta)^3+60\beta(1-\eta)^2+6s\beta^2]s_2\\ &\quad +[60\beta^2(1-\eta)^2+5s\beta^2]s_2^2-s\beta^4 s_2^3\ (s_2\in[\frac{1}{\beta},\frac{3}{2\beta}]). \end{aligned}$$

We obtain that for $\beta\in(0,1], s\in(0,3]$,

$$\begin{aligned} g'(s_2) &:= -[360\beta(1-\eta)^3+60\beta(1-\eta)^2+6s\beta^2]\\ &\quad +2[60\beta^2(1-\eta)^2+5s\beta^3]s_2-3s\beta^4 s_2^2\\ &\le -360\beta(1-\eta)^3-60\beta(1-\eta)^2-6s\beta^2\\ &\quad +2[60\beta^2(1-\eta)^2+5s\beta^2]\frac{3}{2\beta}-3s\beta^2\\ &= 120\beta(1-\eta)^2(3\eta-2)+6s\beta^2\\ &\le 120\beta(1-\eta)^2\left[3\eta-2+\frac{1}{5(1-\eta)^2}\right]\\ &\le 120\beta(1-\eta)^2\left[\frac{3}{4}-2+\frac{1}{5(1-\frac{1}{4})^2}\right]\\ &= -2\beta(1-\eta)^2\frac{161}{3}\\ &< 0. \end{aligned}$$

Hence, we deduce to the following

$$h_1\ge\frac{1}{720s_2}g(\frac{3}{2\beta})$$

$$
\begin{aligned}
&= \frac{1}{720s_2}\left\{720(1-\eta)^4 - [360\beta(1-\eta)^3 + 60\beta(1-\eta)^2 + 6s\beta^2]\frac{3}{2\beta}\right. \\
&\quad \left. +[60\beta^2(1-\eta)^2 + 5s\beta^2]\frac{9}{4\beta^2} - s\beta^4\frac{27}{8\beta^3}\right\} \\
&> \frac{(1-\eta)^2}{16s_2}(16\eta^2 - 20\eta + 5) \\
&> 0 \ (\eta \in [0, \frac{1}{4}]).
\end{aligned}
$$

We find for $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}], s \in (0,3]$,

$$
\begin{aligned}
h_2 &\geq \frac{4\beta^2(1-\eta)^4}{15} - \frac{\beta^2(1-\eta)^2}{12} - \frac{\beta^2(s+1)}{144} \\
&\geq \beta^2\left[\frac{4(1-\frac{1}{4})^4}{15} - \frac{(1-\frac{1}{4})^2}{12} - \frac{1}{36}\right] \\
&= \frac{\beta^2}{360} \\
&> 0, \text{ and} \\
h_3 &\geq \beta^3\left[\frac{8}{105}(\frac{3}{4})^4 - \frac{1}{144}\right] > 0.
\end{aligned}
$$

Hence, we have $h(a) > 0$, and then by setting $u = \frac{(t-\eta)^\eta}{(m-\xi)^\alpha}$, we find

$$
\begin{aligned}
\varpi_s(s_2, m) &= (m-\xi)^{\alpha(s-s_2)}\sum_{n=1}^{\infty} g(n) \\
&< (m-\xi)^{\alpha(s-s_2)}\int_\eta^\infty g(t)dt \\
&= \int_0^\infty \frac{u^{s_2-1}du}{(u+1)^s} \\
&= B(s_2, s-s_2) \\
&= k_s(s_2).
\end{aligned}
$$

On the other-hand, by (2.5) (for $n \to \infty$), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} g(n) &= \int_1^\infty g(t)dt + \frac{g(1)}{2} + \int_1^\infty P_1(t)g'(t)dt \\
&= \int_1^\infty g(t)dt + H(a), \\
H(a) &:= \frac{g(1)}{2} + \int_1^\infty P_1(t)g'(t)dt.
\end{aligned}
$$

We have obtained that $\frac{g(1)}{2} = \frac{\beta(1-\eta)^{\beta s_2 - 1}}{2[a+(1-\eta)^\beta]^s}$, and

$$
g'(t) = -\frac{\beta(\beta s_2 - 1)(t-\eta)^{\beta s_2 - 2}}{[a+(1-\eta)^\beta]^s} + \frac{\beta^2 sa(t-\eta)^{\beta s_2 - 2}}{[a+(1-\eta)^\beta]^{s+1}}.
$$

For $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0,s), 0 < s \leq 3$, by (2.3), we find

$$- \beta(\beta s - \beta s_2 + 1) \int_1^\infty P_1(t) \frac{(t-\eta)^{\beta s_2 - 2} dt}{[a + (t-\eta)^\beta]^s} > 0, \text{ and}$$

$$\beta^2 sa \int_1^\infty P_1(t) \frac{(t-\eta)^{\beta s_2 - 2} dt}{[a + (t-\eta)^\beta]^{s+1}} > -\frac{\beta^2 sa(1-\eta)^{\beta s_2 - 2}}{12[a + (1-\eta)^\beta]^{s+1}} > -\frac{\beta(1-\eta)^{\beta s_2 - 2}}{4[a + (1-\eta)^\beta]^s}.$$

Hence, we have

$$H(a) > \frac{\beta(1-\eta)^{\beta s_2 - 1}}{2[a + (1-\eta)^\beta]^s} - \frac{\beta(1-\eta)^{\beta s_2 - 2}}{4[a + (1-\eta)^\beta]^s} > 0,$$

and then we obtain

$$\begin{aligned}
\omega_s(s_1, m) &= (m-\xi)^{\alpha(s-s_2)} \sum_{n=1}^\infty g(n) \\
&> (m-\xi)^{\alpha(s-s_2)} \int_1^\infty g(t) dt \\
&= (m-\xi)^{\alpha(s-s_2)} \left(\int_\eta^\infty g(t) dt - \int_\eta^1 g(t) dt \right) \\
&= k_s(s_2) \left[1 - \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2 - 1} du}{(u+1)^s} \right] \\
&> 0,
\end{aligned}$$

where, we set

$$O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0.$$

Therefore, inequalities (2.8) follow.

The lemma is proved. □

Lemma 2.3. *For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, s \in (0,3]$, $s_1 \in (0, \frac{3}{2\alpha}] \cap (0,s), s_2 \in (0, \frac{3}{2\beta}] \cap (0,s), k_s(s_1) = B(s_1, s - s_1)$, we have the following more accurate extended Hardy-Hilbert's inequality with the internal variables:*

$$\begin{aligned}
I &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{[(m-\xi)^\alpha + (n-\eta)^\beta]^s} \\
&\leq (\frac{k_s(s_2)}{\beta})^{\frac{1}{p}} (\frac{k_s(s_1)}{\alpha})^{\frac{1}{q}} \left\{ \sum_{m=1}^\infty (m-\xi)^{p[1-\alpha(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^\infty (n-\eta)^{q[1-\beta(\frac{s-s_1}{q} + \frac{s_1}{p})]-1} b_n^q \right\}^{\frac{1}{p}}. \qquad (2.9)
\end{aligned}$$

Proof. In the same way, we obtain the following inequalities for the next weight coefficient:

$$\begin{aligned}
0 &< k_s(s_1)(1 - O_2(\frac{1}{(n-\eta)^{\beta s_1}})) \\
&< \omega_s(s_1, n)
\end{aligned}$$

$$< k_s(s_1) \ (n \in \mathbf{N}), \tag{2.10}$$

where, $O_2(\frac{1}{(n-\eta)^{\beta s_1}}) = \frac{1}{k_s(s_1)} \int_0^{\frac{(1-\xi)^\alpha}{(n-\eta)^\beta}} \frac{u^{s_1-1}}{(1+u)^s} du > 0$.

By Holder's inequality (cf. [17]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{[(m-\xi)^\alpha + (n-\eta)^\beta]^s} \left\{ \frac{(m-\xi)^{\frac{\alpha(1-s_1)}{q}} [\beta(n-\eta)^{\beta-1}]^{\frac{1}{p}} a_m}{(n-\eta)^{\frac{\beta(1-s_2)}{p}} [\alpha(m-\xi)^{\alpha-1}]^{\frac{1}{q}}} \right\} \\ &\quad \times \left\{ \frac{(n-\eta)^{\beta(1-s_2)/p} [\alpha(m-\xi)^{\alpha-1}]^{1/q}}{(m-\xi)^{\alpha(1-s_1)/q} [\beta(n-\eta)^{\beta-1}]^{1/p}} b_n \right\} \\ &\le \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta \alpha^{1-p} a_m^p}{[(m-\xi)^\alpha + (n-\eta)^\beta]^s} \frac{(m-\xi)^{\alpha(1-s_1)(p-1)} (n-\eta)^{\beta-1}}{(n-\eta)^{\beta(1-s_2)} (m-\xi)^{(\alpha-1)(p-1)}} \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha \beta^{1-q} b_n^q}{[(m-\xi)^\alpha + (n-\eta)^\beta]^s} \frac{(n-\eta)^{\beta(1-s_2)(q-1)} (m-\xi)^{\alpha-1}}{(m-\xi)^{\alpha(1-s_1)} (n-\eta)^{(\beta-1)(q-1)}} \right\}^{\frac{1}{q}} \\ &= \frac{1}{\alpha^{1/q} \beta^{1/p}} \left\{ \sum_{m=1}^{\infty} \varpi_s(s_2, m) (m-\xi)^{p[1-\alpha(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \omega_s(s_1, n) (n-\eta)^{q[1-\beta(\frac{s-s_1}{q} + \frac{s_1}{p})]-1} b_n^q \right\}^{\frac{1}{p}}. \end{aligned}$$

Then by (2.8) and (2.10), we have (2.9).

The lemma is proved. □

Remark 2.1. In particular, for $s = \lambda + 1 \in (1, 3], \lambda \in (0, 2]$,

$$\begin{aligned} & s_1 = \lambda_1 \in (0, \frac{3}{2\alpha}] \cap (0, \lambda), \\ & s_2 = \lambda_2 + 1 \in (1, \frac{3}{2\alpha}] \cap (1, \lambda + 1), \lambda_2 \in (0, \frac{3}{2\alpha} - 1] \cap (0, \lambda), \end{aligned}$$

in (2.9), replacing b_n by $(n-\eta)^{\beta-1} B_n$, in view of (2.1), we have the following inequality:

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m (n-\eta)^{\beta-1} B_n}{[(m-\xi)^\alpha + (n-\eta)^\beta]^{\lambda+1}} \\ & < (\frac{k_{\lambda+1}(\lambda_2+1)}{\beta})^{\frac{1}{p}} (\frac{k_{\lambda+1}(\lambda_1)}{\alpha})^{\frac{1}{q}} \\ & \times \left\{ \sum_{m=1}^{\infty} (m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (n-\eta)^{-q\beta\widehat{\lambda}_2-1} B_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{2.11}$$

Lemma 2.4. *For $t > 0$, we have the following inequality:*

$$\sum_{n=1}^{\infty} e^{-(n-\eta)^\beta} b_n \le t\beta \sum_{n=1}^{\infty} e^{-t(n-\eta)^\beta} (n-\eta)^{\beta-1} B_n. \tag{2.12}$$

Proof. In view of $B_n e^{-t(n-\eta)^\beta} = o(1)$ $(n \to \infty)$, by Abel summation by parts formula, we find

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-(n-\eta)^\beta} b_n &= \lim_{n\to\infty} B_n e^{-t(n-\eta)^\beta} + \sum_{n=1}^{\infty}[e^{-t(n-\eta)^\beta} - e^{-t(n+1-\eta)^\beta}]B_n \\ &= \sum_{n=1}^{\infty}[e^{-t(n-\eta)^\beta} - e^{-t(n+1-\eta)^\beta}]B_n.\end{aligned}$$

We set function $g(x) := e^{-t(x-\eta)^\beta}, x \in (\eta, \infty)$. Then we find

$$g'(x) = -t\beta(x-\eta)^{\beta-1}e^{-t(x-\eta)^\beta} = -t\beta h(x),$$

where, for $\beta \in (0,1], h(x) := (x-\eta)^{\beta-1}e^{-t(x-\eta)^\beta}$ is decreasing in (η, ∞). By the differentiation mid-value theorem, we have

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-(n-\eta)^\beta} b_n &= -\sum_{n=1}^{\infty}(g(n+1) - g(n))B_n \\ &= -\sum_{n=1}^{\infty} g'(n+\theta_n)B_n \\ &= t\beta\sum_{n=1}^{\infty} h(n+\theta_n)B_n \le t\beta\sum_{n=1}^{\infty} h(n)B_n \\ &= t\beta\sum_{n=1}^{\infty}(n-\eta)^{\beta-1}e^{-t(n-\eta)^\beta}B_n \ (\theta_n \in (0,1)),\end{aligned}$$

namely, (2.12) follows.

The lemma is proved. □

3. Main results

Theorem 3.1. *With regards to the assumption (A1), we have the following more accurate Hilbert-type inequality involving one partial sum:*

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{[(m-\xi)^\alpha + (n-\eta)^\beta]^\lambda} \\ &< (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1) \\ &\times \left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{3.1}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have

$$0 < \sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1}a_m^p < \infty,\ 0 < \sum_{n=1}^{\infty}(n-\eta)^{-q\beta\widehat{\lambda}_2-1}B_n^q < \infty$$

and the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}}$$
$$<(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\lambda_2-1}B_n^q\right]^{\frac{1}{q}}. \tag{3.2}$$

Proof. In view of the following expression of the gamma function

$$\frac{1}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}e^{-[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]t}dt,$$

by (2.12), it follows that

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_m b_n\int_0^{\infty}t^{\lambda-1}e^{-[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]t}dt\\
&= \frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}\sum_{m=1}^{\infty}e^{-(m-\xi)^{\alpha}t}a_m\left[\sum_{n=1}^{\infty}b_n e^{-(n-\eta)^{\beta}t}\right]dt\\
&\leq \frac{\beta}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(n-\eta)^{\beta-1}a_m B_n\int_0^{\infty}t^{\lambda}e^{-[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]t}dt\\
&= \frac{\beta\Gamma(\lambda+1)}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m(n-\eta)^{\beta-1}B_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda+1}}.
\end{aligned}$$

Then by (2.11), for $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$, we have (3.1).

For $\lambda_1+\lambda_2=\lambda$, we have

$$k_{\lambda+1}(\lambda_2+1)=k_{\lambda+1}(\lambda_1)=B(\lambda_1,\lambda_2+1)=\frac{\lambda_2}{\lambda}B(\lambda_1,\lambda_2),$$

and then by (3.1), we have (3.2).

The theorem is proved. □

Theorem 3.2. *With regards to the assumption (A1), if $\lambda_1+\lambda_2=\lambda$, then the constant factor*

$$(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)$$

in (3.1) is the best possible; On the other-hand, if the same constant factor in (3.1) is the best possible, then for

$$\lambda-\lambda_1-\lambda_2\leq\min\{\frac{3}{2\alpha}-\lambda_1,\frac{3}{2\beta}-1-\lambda_1\},$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof. If $\lambda_1+\lambda_2=\lambda$, then (3.1) reduces to (3.2). For any $0<\varepsilon<\min\{p\lambda_1,q\lambda_2\}$, we set

$$\widetilde{a}_m:=m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1},\widetilde{b}_n:=n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}\ (m,n\in\mathbf{N}).$$

Since by (2.5) and (2.3), we have

$$\begin{aligned}
\widetilde{B}_n &:= \sum_{k=1}^{n} \widetilde{b}_k \\
&= \sum_{k=1}^{n} k^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} \\
&= \int_1^n t^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}dt + \frac{1}{2}[n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}+1] + \frac{\varepsilon_0}{12}[\beta(\lambda_2-\frac{\varepsilon}{q})-1][n^{\beta(\lambda_2-\frac{\varepsilon}{q})-2}-1] \\
&= \frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}[n^{\beta(\lambda_2-\frac{\varepsilon}{q})}+c_1+O_1(n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1})] \\
&\le \frac{n^{\beta(\lambda_2-\frac{\varepsilon}{q})}}{\beta(\lambda_2-\frac{\varepsilon}{q})}[1+|c_1|n^{-\beta(\lambda_2-\frac{\varepsilon}{q})}+|O_2(n^{-1})|](\varepsilon_0\in(0,1), n\in\mathbf{N}).
\end{aligned}$$

If there exists a positive constant $M \le (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)$, such that (3.2) is valid when we replace $(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)$ by M, then in particular, for $\xi=\eta=0$, we have

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(m^\alpha+n^\beta)^\lambda} \\
&< M\left[\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-q\beta\lambda_2-1}\widetilde{B}_n^q\right]^{\frac{1}{q}}. \qquad (3.3)
\end{aligned}$$

For derivable function $\alpha(x)\to 0$ $(x\to\infty)$, we obtain

$$\lim_{x\to\infty}\frac{(1+\alpha(x))^q-1}{\alpha(x)} = \lim_{x\to\infty}\frac{q(1+\alpha(x))^{q-1}\alpha'(x)}{\alpha'(x)} = q,$$

and then $(1+\alpha(x))^q = 1+O(\alpha(x))$ $(x\to\infty)$. So we have

$$[1+|c_1|n^{-\beta(\lambda_2-\frac{\varepsilon}{q})}+|O_2(n^{-1})|]^q = 1+O(|c_1|n^{-\beta(\lambda_2-\frac{\varepsilon}{q})}+|O_2(n^{-1})|)\ (n\in\mathbf{N}).$$

Hence, we find

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-q\beta\lambda_2-1}\widetilde{B}_n^q \\
&\le [\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}]^q\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}[1+|c_1|n^{-\beta(\lambda_2-\frac{\varepsilon}{q})}+|O_2(n^{-1})|]^q \\
&= [\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}]^q[\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}+\sum_{n=1}^{\infty}O(|c_1|n^{-\beta(\lambda_2-\frac{\varepsilon}{q})-1}+|O_2(n^{\alpha\beta\varepsilon-2})|)] \\
&= [\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}]^q(\sum_{n=2}^{\infty} n^{-\beta\varepsilon-1}+O_1(1)) \\
&< [\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}]^q(\int_1^{\infty} x^{-\beta\varepsilon-1}dx+O_1(1))
\end{aligned}$$

$$= [\frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}]^q (\frac{1}{\beta\varepsilon} + O_1(1)).$$

By (3.3) and the decreasingness property of series, we obtain

$$\begin{aligned}
\widetilde{I} &< \frac{M}{\beta(\lambda_2 - \frac{\varepsilon}{q})} \left(1 + \sum_{m=2}^{\infty} m^{-\alpha\varepsilon - 1}\right)^{\frac{1}{p}} (\frac{1}{\beta\varepsilon} + O_1(1))^{\frac{1}{q}} \\
&< \frac{M}{\beta(\lambda_2 - \frac{\varepsilon}{q})} \left(1 + \int_1^{\infty} x^{-\alpha\varepsilon - 1} dx\right)^{\frac{1}{p}} (\frac{1}{\beta\varepsilon} + O_1(1))^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon\beta(\lambda_2 - \frac{\varepsilon}{q})} \left(\varepsilon + \frac{1}{\alpha}\right)^{\frac{1}{p}} (\frac{1}{\beta} + \varepsilon O_1(1))^{\frac{1}{q}}.
\end{aligned}$$

By (2.10) (for $\xi = \eta = 0, s = \lambda \in (0, 2], s_1 = \widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \frac{3}{2\alpha}] \cap (0, \lambda)$, we find

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty} \left[n^{\beta(\lambda_2 + \frac{\varepsilon}{p})} \sum_{m=1}^{\infty} \frac{m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}) - 1}}{(m^\alpha + n^\beta)^\lambda} \right] n^{-\beta\varepsilon - 1} \\
&> \frac{1}{\alpha} k_\lambda(\widetilde{\lambda}_1) \sum_{n=1}^{\infty} (1 - O(\frac{1}{n^{\beta\widetilde{\lambda}_1}})) n^{-\beta\varepsilon - 1} \\
&= \frac{1}{\alpha} k_\lambda(\widetilde{\lambda}_1) \left(\sum_{n=1}^{\infty} n^{-\beta\varepsilon - 1} - \sum_{n=1}^{\infty} O(\frac{1}{n^{\beta(\lambda_1 + \frac{\varepsilon}{q}) + 1}}) \right) \\
&> \frac{1}{\alpha} k_\lambda(\widetilde{\lambda}_1) \left(\int_1^{\infty} x^{-\beta\varepsilon - 1} dx - O(1) \right) \\
&= \frac{1}{\varepsilon\alpha\beta} B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) \left(1 - \varepsilon\beta O(1)\right).
\end{aligned}$$

Then we have the following inequality:

$$\begin{aligned}
&\frac{1}{\alpha\beta} B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) \left(1 - \varepsilon\beta O(1)\right) \\
&< \varepsilon\widetilde{I} < \frac{M}{\beta(\lambda_2 - \frac{\varepsilon}{q})} \left(\varepsilon + \frac{1}{\alpha}\right)^{\frac{1}{p}} (\frac{1}{\beta} + \varepsilon O_1(1))^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, we find

$$(\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda_2 B(\lambda_1, \lambda_2) \leq M.$$

Hence, $M = (\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor of (3.2).

On the other-hand, for $\widehat{\lambda}_1 = \frac{\lambda - \lambda_1 - \lambda_2}{p} + \lambda_1, \widehat{\lambda}_2 = \frac{\lambda - \lambda_1 - \lambda_2}{q} + \lambda_2$ and the assumption, we find $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda, 0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \lambda, \widehat{\lambda}_2 B(\widehat{\lambda}_1, \widehat{\lambda}_2) \in \mathbf{R}_+$, $\widehat{\lambda}_1 \leq \frac{3}{2\alpha}$ and $\widehat{\lambda}_2 \leq \frac{3}{2\beta} - 1$. If the constant $(\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)$ in (3.1) is the best possible, then compare with the constants in (3.1) and (3.2) (for $\lambda_i = \widehat{\lambda}_i, i = 1, 2$), we have

$$(\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1) \leq (\frac{\beta}{\alpha})^{\frac{1}{q}} \widehat{\lambda}_2 B(\widehat{\lambda}_1, \widehat{\lambda}_2) \in \mathbf{R}_+.$$

It follows that

$$k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1) \le \frac{\widehat{\lambda}_2}{\lambda}B(\widehat{\lambda}_1,\widehat{\lambda}_2) = k_{\lambda+1}(\widehat{\lambda}_1). \tag{3.4}$$

By Holder's inequality, we obtain

$$\begin{aligned} k_{\lambda+1}(\widehat{\lambda}_1) &= k_{\lambda+1}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}) \\ &= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du}{(1+u)^{\lambda+1}} \\ &= \int_0^\infty \frac{1}{(1+u)^{\lambda+1}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\ &\le \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^{\lambda+1}}du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^{\lambda+1}}du\right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty \frac{v^{\lambda_2-1}dv}{(1+v)^{\lambda+1}}\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}du}{(1+u)^{\lambda+1}}\right]^{\frac{1}{q}} \\ &= k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1). \end{aligned} \tag{3.5}$$

In view of (3.4), (3.5) reduces to an equality. We observe that (3.5) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [17]) $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$. Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_1-\lambda_2} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and then $\lambda-\lambda_1-\lambda_2 = 0$, namely, $\lambda_1+\lambda_2 = \lambda$.

The theorem is proved. □

4. Equivalent forms and operator expressions

Theorem 4.1. *With regards to the assumption (A1), we have the following inequality equivalent to (3.1):*

$$\begin{aligned} J &:= \left\{\sum_{m=1}^{\infty}(m-\xi)^{q\alpha\widehat{\lambda}_1-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{[(m-\xi)^\alpha+(n-\eta)^\beta]^\lambda}\right]^q\right\}^{\frac{1}{q}} \\ &< (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{4.1}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, we have the following inequality equivalent to (3.2):

$$\begin{aligned} &\left\{\sum_{m=1}^{\infty}(m-\xi)^{q\alpha\lambda_1-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{[(m-\xi)^\alpha+(n-\eta)^\beta]^\lambda}\right]^q\right\}^{\frac{1}{q}} \\ &< (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1,\lambda_2)\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\lambda_2-1}B_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

Proof. Suppose that (3.6) is valid. By Holder's inequality, we have

$$I = \sum_{m=1}^{\infty}\left[(m-\xi)^{\frac{1}{q}-\alpha\widehat{\lambda}_1}a_m\right]\left[\sum_{n=1}^{\infty}\frac{(m-\xi)^{\alpha\widehat{\lambda}_1-\frac{1}{q}}b_n}{[(m-\xi)^\alpha+(n-\eta)^\beta]^\lambda}\right]$$

$$\leq \left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} J. \tag{4.3}$$

Then by (4.1), we have (3.1). On the other-hand, assuming that (3.1) is valid, we set

$$a_m := (m-\xi)^{q\alpha\widehat{\lambda}_1-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}}\right]^{q-1}, m \in \mathbf{N}.$$

Then, it follows that $J = \left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{q}}$.

If $J = 0$, then (4.1) is naturally valid; if $J = \infty$, then it is impossible that makes (4.1) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (3.1), we have

$$\begin{aligned}
0 &< \sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}a_m^p \\
&= J^q \\
&= I \\
&< (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)J^{q-1}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}, \\
J &< (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

namely, (4.1) follows, which is equivalent to (3.1).

The theorem is proved. □

Theorem 4.2. *With regards to the assumption (A1), if* $\lambda_1+\lambda_2 = \lambda$*, then the constant* $(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}} \times (\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)$ *in (4.1) is the best possible. On the other-hand, if the same constant in (4.1) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \leq \min\{\frac{3}{2\alpha}-\lambda_1, \frac{3}{2\beta}-1-\lambda_1\},$$

we have $\lambda_1+\lambda_2 = \lambda$.

Proof. If $\lambda_1+\lambda_2 = \lambda$, then by Theorem 3.2, the constant $(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2+1)k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)$ in (3.1) is the best value. Then by (4.3), we can show that the constant in (4.1) is the best value. Otherwise, we would reach a contradiction that the constant in (3.1) is not the best possible (for $\lambda_1+\lambda_2 = \lambda$).

On the other-hand, if the same constant in (4.1) is the best possible, then by the equivalency of (4.1) and (3.1), in view of $I = J^q$ (see the proof of Theorem 4.1), it follows that the same constant in (3.1) is the best possible. By Theorem 3.2, in view of the assumption, we have $\lambda_1+\lambda_2 = \lambda$.

The theorem is proved. □

Remark 4.1. For $\xi = \eta = 0$ in (3.2) and (4.2), we have the following equivalent inequalities with the best possible constant factor $(\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1, \lambda_2)$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} < (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1, \lambda_2) \times \left[\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} B_n^q\right)^{\frac{1}{q}}, \tag{4.4}$$

$$\left\{\sum_{m=1}^{\infty} m^{q\alpha\lambda_1 - 1}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m^\alpha + n^\beta)^\lambda}\right]^q\right\}^{\frac{1}{q}} < (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda_2 B(\lambda_1, \lambda_2)\left(\sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} B_n^q\right)^{\frac{1}{q}}. \tag{4.5}$$

Hence, (3.2) (resp. (4.2)) is a more accurate extension of (4.4) (resp. (4.5)).

Setting $\varphi(m) := (m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}, \psi(n) := (n-\eta)^{q(1-\beta\widehat{\lambda}_2)-1}$, $\Psi(n) := (n-\eta)^{-q\beta\widehat{\lambda}_2 - 1}$, wherefrom

$$\varphi^{1-q}(m) = (m-\xi)^{q\alpha\widehat{\lambda}_1 - 1} \ (m, n \in \mathbf{N}),$$

we define the following normed linear spaces:

$$l_{p,\varphi} := \{a = \{a_m\}_{m=1}^\infty; ||a||_{p,\varphi} := (\sum_{m=1}^{\infty}\varphi(m)|a_m|^p)^{\frac{1}{p}} < \infty\},$$

$$l_{q,\psi} := \{b = \{b_n\}_{n=1}^\infty; ||b||_{q,\psi} := (\sum_{n=1}^{\infty}\psi(n)|b_n|^q)^{\frac{1}{q}} < \infty\},$$

$$l_{q,\Psi} := \{B = \{B_n\}_{n=1}^\infty; ||B||_{q,\Psi} := (\sum_{n=1}^{\infty}\Psi(n)|B_n|^q)^{\frac{1}{q}} < \infty\},$$

$$l_{q,\varphi^{1-q}} := \{c = \{c_m\}_{m=1}^\infty; ||c||_{q,\varphi^{1-q}} := (\sum_{m=1}^{\infty}\varphi^{1-q}(m)|c_m|^q)^{\frac{1}{q}} < \infty\}.$$

For $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $B \in l_{q,\Psi}$, setting

$$c = \{c_m\}_{m=1}^\infty; c_m := \sum_{n=1}^{\infty}\frac{b_n}{[(m-\xi)^\alpha + (n-\eta)^\beta]^\lambda},$$

we can rewrite (4.1) as follows:

$$||c||_{q,\varphi^{1-q}} < (\frac{\beta}{\alpha})^{\frac{1}{q}}\lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)||B||_{q,\Psi} < \infty,$$

namely, $c \in l_{q,\varphi^{1-q}}$.

Definition 4.1. Define a Hilbert-type operator $T : l_{q,\psi} \to l_{q,\varphi^{1-q}}$ as follows: For any $b \in l_{q,\psi}$, there exists a unique representation $c = Tb \in l_{q,\varphi^{1-q}}$, such that for any $m \in \mathbf{N}$,$Tb(m) = c_m$. Define the formal inner product of Tb and $a \in l_{p,\varphi}$ and the norm of T as follows:

$$(Tb, a) := \sum_{m=1}^{\infty} a_m \sum_{n=1}^{\infty}\frac{b_n}{[(m-\xi)^\alpha + (n-\eta)^\beta]^\lambda} = I,$$

$$||T|| := \sup_{b(\neq 0)\in l_{q,\psi}}\frac{||Tb||_{q,\varphi^{1-q}}}{||B||_{q,\Psi}}.$$

By Theorem 4.1-Theorem 4.2, we have

Theorem 4.3. *With regards to the assumption (A1), if* $b(\geq 0) \in l_{q,\psi}$, $B \in l_{q,\Psi}, a(\geq 0) \in l_{p,\varphi}, ||b||_{q,\psi} > 0, ||B||_{q,\Psi} > 0, ||a||_{p,\varphi} > 0$, *then we have the following equivalent inequalities:*

$$(Tb, a) < (\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi} ||B||_{q,\Psi}, \tag{4.6}$$

$$||Tb||_{q,\varphi^{1-q}} < (\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1) ||B||_{q,\Psi}. \tag{4.7}$$

Moreover, If $\lambda_1 + \lambda_2 = \lambda$, *then the constant factor* $(\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda k_{\lambda+1}^{\frac{1}{p}}(\lambda_2 + 1) k_{\lambda+1}^{\frac{1}{q}}(\lambda_1)$ *in (4.6) and (4.7) is the best possible, namely* $||T|| = (\frac{\beta}{\alpha})^{\frac{1}{q}} \lambda_2 B(\lambda_1, \lambda_2)$. *On the other-hand, if the same constant factor in (4.6) (or (4.7)) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \leq \min\{\frac{3}{2\alpha} - \lambda_1, \frac{3}{2\beta} - 1 - \lambda_1\},$$

we have $\lambda_1 + \lambda_2 = \lambda$.

5. Conclusions

In this paper, by the use of the weight coefficients, the idea of introduced parameters, the Euler-Maclaurin summation formula and Abel summation by parts formula, a new more accurate Hilbert-type inequality with the power function as the internal variables involving one partial sum is obtained in Theorem 3.1. The equivalent conditions of the best value related to a few parameters are provided in Theorem 3.2, and some particular inequalities are given in Remark 4.1. The equivalent forms and the operator expressions are given in Theorem 4.1-4.3. The main results of this paper is an extended application of [1,15], and the novelty of our work is a more perfect refinement of a few published results. Compare with the published paper [18], this paper gives an extended Hilbert-type inequality with a more complex kernel involving one partial sum availably. The lemmas and theorems provide an extensive account of this type of inequalities.

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