Journal of Applied Analysis and Computation
Volume 17, Number 1, February 2027, 27–38
Website:http://www.jaac-online.com
DOI:10.11948/20260009

IMAGE CHARACTERISTICS OF TWO-PARAMETER INVERSE GAUSSIAN DISTRIBUTION BASED ON COEFFICIENT OF VARIATION AND PARAMETER ESTIMATIONS OF CONSTANT STRESS ACCELERATED LIFE UNDER INVERSE POWER LAW MODEL

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Abstract This paper proposes a two-parameter inverse Gaussian distribution IGS(μ, m) based on the coefficient of variation, where one parameter μ is referred to as the scale parameter and the other m as the shape parameter. Its advantage lies in establishing a foundation for the accelerated life testing of the two-parameter inverse Gaussian distribution through a log-linear model of the scale parameter with stress. The main focus of the paper is to theoretically demonstrate the image characteristics of the density function, failure rate function, and mean residual life of the two-parameter inverse Gaussian distribution IGS(μ, m). For constant stress accelerated life testing under the inverse power law model, point estimation of parameters is provided. The paper concludes with two simulated case studies to illustrate the application of the method.

Keywords Two-parameter inverse Gaussian distribution, coefficient of variation, density function, failure rate function, mean residual life, constant stress accelerated life test, inverse power law model.

MSC(2010) 62N05.

1. Introduction

The Inverse Gaussian distribution (abbreviated as IG distribution) possesses numerous excellent probabilistic and statistical properties and has been widely applied in various fields including life testing in engineering, crack problems in fracture mechanics, fatigue estimation of metal material components, complex watershed confluence systems, and in many areas of management and biology. Numerous domestic and international literatures have touched upon the research and application of the two-parameter inverse Gaussian distribution, which are too extensive to list here but can be specifically found in references. Currently, research on the two-parameter inverse Gaussian distribution generally follows the definition IG(μ, λ) outlined in Definition 2.1. There are relatively few research findings on the accelerated life testing of products distributed

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according to the inverse Gaussian distribution. DeGroot & Goel, in Reference [3], studied the statistical analysis of the "step-stress test" of products distributed according to the inverse Gaussian under the Tampered Random Variable (TRV) model. Bhattacharyya & Fries, in Reference [1], investigated the maximum likelihood and least squares estimation of the inverse linear form of the regression model based on the inverse Gaussian distribution under accelerated life testing. Doksum & Hóyland, in Reference [4], considered a fatigue failure model subject to cumulative degradation by a continuous Gaussian process and continuous increasing stress. Under the TRV model, the failure time distribution is represented by the inverse Gaussian distribution function transformed over time, and the experimental model parameters with truncated data are estimated using the method of maximum likelihood. Onar & Padgett, in Reference [8], established a three-parameter family of inverse Gaussian-type distributions, which includes several models for the general system strength based on cumulative damage parameters. This model considers system size as a known accelerating variable, thus this family can be considered as an inverse Gaussian accelerated test distribution. Ye et al., in Reference [13], considered the design of accelerated degradation tests for inverse Gaussian processes without random effects.

2. Two-parameter inverse Gaussian distribution IG(μ, λ)

Definition 2.1. Let a non-negative continuous random variable X follows a two-parameter inverse Gaussian distribution with parameters μ, λ, denoted as $X \sim IG(\mu, \lambda)$. Its density function $f_X(x)$ and distribution function $F_X(x)$ are respectively defined as:

$$
\begin{aligned}
f_X(x) &= \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \\
F_X(x) &= \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right) + e^{\frac{2\lambda}{\mu}}\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right), \quad x > 0.
\end{aligned}
$$

In this context, the parameter $\mu > 0$ is referred to as the location parameter, which is also known in some literatures as the mean parameter, while $\lambda > 0$ is identified as the scale parameter. Furthermore, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$ respectively denote the density function and distribution function of the standard normal distribution.

Drawing from existing literatures, we can observe that the moments of the two-parameter inverse Gaussian distribution IG(μ, λ) exhibit the following properties.

Lemma 2.1. *For a non-negative continuous random variable $X \sim IG(\mu, \lambda)$, the following properties hold:*

(1) $E(X) = \mu$;

(2) for any real number k, $E\left(X^{k+1}\right) = \frac{(2k-1)\mu^2}{\lambda} E\left(X^k\right) + \mu^2 E\left(X^{k-1}\right)$;

(3) $E\left(X^{-1}\right) = \frac{1}{\mu} + \frac{1}{\lambda}$, $E\left(X^2\right) = \frac{\mu+\lambda}{\lambda}\mu^2$, $D(X) = \frac{\mu^3}{\lambda}$, $E\left[X - E(X)\right]^3 = \frac{3\mu^5}{\lambda^2}$, $E\left[X - E(X)\right]^4 = \frac{15\mu^7}{\lambda^3} + \frac{3\mu^6}{\lambda^2}$.

From Lemma 2.1, it can be seen that the coefficient of variation for the two-parameter inverse Gaussian distribution IG(μ, λ) is $\sqrt{\mu/\lambda}$, and the coefficient of variation is a widely applied parameter. Due to its ability to effectively reflect the dispersion level of variable values, it serves as an important indicator for measuring the stability of product quality. Considering that reliability engineering, especially in accelerated life testing and environmental factor analysis, requires the "invariance of failure mechanisms" in products, Reference [10] proved that a constant coefficient of variation is a necessary condition for the invariance of failure mechanisms in life distributions. References [12] and [15] delve deeply into the properties of acceleration factors, highlighting their essential connection with the invariance of failure mechanisms. They provided conditions for the invariance of failure mechanisms for various common life distributions and expressions for acceleration factors applicable to accelerated life tests. These studies also discussed time conversion formulas and the basic assumptions of accelerated life testing. However, the conditions for the invariance of failure mechanisms provided in References [12] and [15] essentially equate to a constant coefficient of variation. Thus, by introducing another parameter m (referred to as the shape parameter, which in reality is the square of the coefficient of variation of the two-parameter inverse Gaussian distribution IG(μ, λ)), another parametric expression of the inverse Gaussian distribution is obtained, denoted as IGS(μ, m) (where μ is the scale parameter, and m is the shape parameter). The introduction of this form of the inverse Gaussian distribution, assuming a log-linear model between the scale parameter and stress, lays the foundation for the accelerated life testing of the two-parameter inverse Gaussian distribution.

3. The two-parameter inverse Gaussian distribution IGS(μ, m) based on the coefficient of variation

Definition 3.1. Suppose a non-negative continuous random variable X follows a two-parameter inverse Gaussian distribution with a scale parameter μ and a shape parameter m, denoted as $X \sim$ IGS (μ, m). Its distribution function $F(x)$ and density function $f(x)$ are defined as:

$$F(x) = \Phi\left[\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}-1\right)\right] + e^{2m}\Phi\left[-\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}+1\right)\right],$$
$$f(x) = \sqrt{\frac{m\mu}{2\pi x^3}}\exp\left\{-\frac{m}{2\mu}\frac{(x-\mu)^2}{x}\right\}, \quad x>0,\ \mu>0,\ m>0.$$

It is noteworthy that if we assume a non-negative continuous random variable X follows a two-parameter inverse Gaussian distribution $X \sim$ IG(μ, λ) with parameters μ, λ, and denote m as $\frac{\lambda}{\mu}$, then its distribution function $F_X(x)$ can be expressed in the following form:

$$\begin{aligned} F_X(x) &= \Phi\left[\sqrt{\frac{\lambda/\mu}{x/\mu}}\left(\frac{x}{\mu}-1\right)\right] + e^{2\frac{\lambda}{\mu}}\Phi\left[-\sqrt{\frac{\lambda/\mu}{x/\mu}}\left(\frac{x}{\mu}+1\right)\right], \\ &= \Phi\left[\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}-1\right)\right] + e^{2m}\Phi\left[-\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}+1\right)\right]. \end{aligned}$$

From this, it can be seen that although the parameter μ is commonly referred to as the location parameter or mean parameter in the literature, it is more appropriate to consider it as the scale parameter. Meanwhile, the parameter m controls the shape of the density function $f_X(x)$ and failure rate function $s_X(x)$, making it more appropriate to term m as the shape parameter. To distinguish it from the original notation IG(μ, λ) of the inverse Gaussian distribution, after introducing the shape parameter m, this two-parameter inverse Gaussian distribution is denoted as IGS(μ, m).

If $X \sim$ IGS (μ, m), let $Y = \frac{x}{\mu}$, then $Y \sim$ IGS$(1, m)$, which means for $y > 0$, we have:

$$F_Y(y) = \Phi\Big[\sqrt{\frac{m}{y}}(y-1)\Big] + e^{2m}\Phi\Big[-\sqrt{\frac{m}{y}}(y+1)\Big],$$

$$f_Y(y) = \sqrt{\frac{m}{2\pi y^3}} \exp\Big[-\frac{m}{2y}(y-1)^2\Big] = e^m \sqrt{\frac{m}{2\pi}} y^{-3/2} \exp\Big[-\frac{m}{2}(y+y^{-1})\Big].$$

4. Parameter estimation for the two-parameter inverse Gaussian distribution IGS(μ, m) in the context of full sampling

Similar to IG(μ,λ), substituting $\lambda = m\mu$ into the equation easily yields some conclusions about the moments of the inverse Gaussian distribution IGS(μ, m).

Lemma 4.1. *For a non-negative continuous random variable $X \sim$IGS(μ, m), we have:*

(1) $E(X) = \mu$.

(2) For any real number k, $E\left(X^{k+1}\right) = (2k-1)\frac{\mu}{m} E\left(X^k\right) + \mu^2 E\left(X^{k-1}\right)$.

(3) $E\left(X^{-1}\right) = \frac{m+1}{m}\frac{1}{\mu}$, $E\left(X^2\right) = \frac{m+1}{m}\mu^2$, $D(X) = \frac{\mu^2}{m}$, $E\left[X-E(X)\right]^3 = \frac{3\mu^3}{m^2}$, $E\left[X-E(X)\right]^4 = \frac{15\mu^4}{m^3} + \frac{3\mu^4}{m^2}$.

(4) Coefficient of variation: $c_1 = \frac{\sqrt{D(X)}}{E(X)} = \frac{1}{\sqrt{m}}$.

(5) Skewness coefficient: $c_2 = \frac{E[X-E(X)]^3}{\sqrt{[D(X)]^3}} = \frac{3}{\sqrt{m}}$.

(6) Kurtosis coefficient: $c_3 = \frac{E[X-E(X)]^4}{[D(X)]^2} - 3 = \frac{15}{m}$.

By slightly modifying the conclusions in Reference [14], the following Lemma 4.2 can be derived.

Lemma 4.2. ([14]) *The inverse Gaussian distribution $X \sim$IGS(μ, m) has the following properties:*

1. *If $X \sim$IGS(μ, m), then for any $k > 0$, it satisfies $kX \sim IGS(k\mu, m)$.*
2. *If $X_i \sim IGS(\mu, m)$, $i = 1, 2, \ldots, n$ are mutually independent, then $\sum_{i=1}^{n} X_i \sim IGS(n\mu, nm)$.*
3. *If $X_i \sim IGS(\mu, m)$, $i = 1, 2, \ldots, n$ are mutually independent, then $\bar{X} \sim IGS(\mu, nm)$.*
4. *If $X_i \sim IGS(\mu_i,\ 2\mu_i)$, $i = 1, 2, \ldots, n$ are mutually independent, then $\sum_{i=1}^{n} X_i \sim IGS(\sum_{i=1}^{n}\mu_i,\ 2\sum_{i=1}^{n}\mu_i)$.*
5. *If $X \sim$IGS(μ, m), then when $m \to +\infty$, $\frac{X-\mu}{\sqrt{\mu^2/m}} \dot{\sim} N(0,1)$.*

Lemma 4.3. ([2, 7, 9, 11]) *Suppose a population $X \sim IGS(\mu, m)$, and let $X_1, X_2, \cdots, X_n$ be a sample of size n from the population X, then*

(1) $\frac{m}{\mu}\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{X_i} = \mu m \sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right) + \frac{nm}{\mu}\frac{(\overline{X}-\mu)^2}{\overline{X}}$;

(2) $\frac{m}{\mu}\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{X_i} \sim \chi^2(n)$, $\mu m \sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right) \sim \chi^2(n-1)$, $\frac{nm}{\mu}\frac{(\overline{X}-\mu)^2}{\overline{X}} = \frac{n\mu m}{\overline{X}}\left(\frac{\overline{X}}{\mu} - 1\right)^2 \sim \chi^2(1)$;

(3) $\mu m \sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)$ and $\frac{nm}{\mu}\frac{(\overline{X}-\mu)^2}{\overline{X}}$ are mutually independent.

From Lemma 4.3, it is evident that

$$\frac{n(n-1)\left(\frac{\overline{X}}{\mu}-1\right)^2}{\overline{X}\sum\limits_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\overline{X}}\right)}=\frac{\frac{nm}{\mu}\frac{(\overline{X}-\mu)^2}{\overline{X}}}{\frac{\mu m}{n-1}\sum\limits_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\overline{X}}\right)}\sim F(1,n-1)\,.$$

The following Theorem 4.1 provides point estimations of the parameters μ, m in the context of a full sample.

Theorem 4.1. *Let the population be $X \sim IGS(\mu, m)$, and let $X_1, X_2, \cdots, X_n$ be a sample of size n from the population X. Denote $\bar{X}=\frac{1}{n}\sum\limits_{i=1}^{n}X_i$, $\overline{X^{-1}}=\frac{1}{n}\sum\limits_{i=1}^{n}X_i^{-1}$, $\overline{X^2}=\frac{1}{n}\sum\limits_{i=1}^{n}X_i^2$,$S_n^2=\overline{X^2}-\bar{X}^2$, then:*

(1) If the moment equations $E(X), E(X^{-1})$ constructed using is utilized, the method of moment estimates of the parameters can be obtained as $\hat{\mu}=\bar{X}, \hat{m}=\frac{1}{\bar{X}\overline{X^{-1}}-1}$.

(2) If the moment equations $E(X), E(X^2)$ constructed using is utilized, the method of moment estimates of the parameters can be obtained as $\hat{\mu}=\bar{X}, \hat{m}=\frac{\bar{X}^2}{S_n^2}$.

(3) The maximum likelihood estimates of the parameters are $\hat{\mu}=\bar{X}, \hat{m}=\frac{1}{\bar{X}\overline{X^{-1}}-1}$.

Note: It is evident that $\hat{\mu}=\bar{X}$ is the minimum variance unbiased estimate of the parameter μ.

5. The image characteristics of the two-parameter inverse Gaussian distribution IGS(μ, m)

The following text examines the image characteristics of the density function $f(x)$, failure rate function $s(x)$, and average remaining life $m(x)$ for the inverse Gaussian distribution IGS(μ, m).

Theorem 5.1. *For a non-negative continuous random variable $X \sim IGS\ (\mu, m)$, its density function $f(x)$ exhibits an "inverted bathtub" shape.*

Proof. Let the random variable $Y=\frac{X}{\mu}$. It is clear that the density function of Y is:

$$f_Y(y)=e^m\sqrt{\frac{m}{2\pi}}y^{-3/2}\exp\left[-\frac{m}{2}\left(y+y^{-1}\right)\right].$$

Since μ is a scale parameter, the image characteristics of $f(x)$ are the same as those of $f_Y(y)$. Thus, examining the image characteristics of $f_Y(y)$ suffices.

It is evident that: $\lim\limits_{y\to 0}f_Y(y)=0=\lim\limits_{y\to+\infty}f_Y(y)=0$, and

$$\begin{aligned}f_Y'(y)&=e^m\sqrt{\frac{m}{2\pi}}\Big\{-\frac{3}{2}y^{-5/2}\exp\Big[-\frac{m}{2}\left(y+y^{-1}\right)\Big]\Big\}\\&\quad-e^m\sqrt{\frac{m}{2\pi}}\Big\{\frac{m}{2}y^{-3/2}(1-y^{-2})\exp\Big[-\frac{m}{2}\left(y+y^{-1}\right)\Big]\Big\}\\&=\frac{e^m}{2}\sqrt{\frac{m}{2\pi}}y^{-7/2}\exp\Big[-\frac{m}{2}\left(y+y^{-1}\right)\Big](-my^2-3y+m).\end{aligned}$$

Let the function be $g(y)=-my^2-3y+m, y>0$.

If $y_0 = \frac{-3+\sqrt{4m^2+9}}{2m}$, then it follows that $g(y_0) = 0$.

When $0 < y < y_0$, $g(y) > 0$, $f'_Y(y) > 0$, $f_Y(y)$ is strictly increasing.

When $y > y_0$, $g(y) < 0$, $f'_Y(y) < 0$, $f_Y(y)$ is strictly decreasing.

Therefore, the inverse Gaussian distribution IGS(μ, m) has a mode at the point $x_0 = y_0\mu$, and its value is $f(x_0) = \frac{e^m}{\mu}\sqrt{\frac{m}{2\pi}}y_0^{-3/2}\exp\left[-\frac{m}{2}\left(y_0 + y_0^{-1}\right)\right]$. □

Lemma 5.1. ([5]) *Let T be a non-negative continuous random variable with a density function $f(t)$ that has a second-order derivative and a failure rate function $\lambda(t)$, denoted as $\eta(t) = -\frac{f'(t)}{f(t)}$. The following conclusions can be drawn:*

(1) If $\eta'(t) > 0$, i.e., $\eta(t)$ is "strictly increasing", then $\lambda(t)$ is also "strictly increasing".

(2) If $\eta'(t) < 0$, i.e., $\eta(t)$ is "strictly decreasing", then $\lambda(t)$ is also "strictly decreasing".

(3) If there exists a point t_0 where $t_0 > 0$, $\eta'(t_0) = 0$ and $\eta(t)$ shows a "strictly increasing then strictly decreasing" pattern, i.e., an "inverted bathtub" shape, then $\lambda(t)$ may also exhibit an "inverted bathtub" shape or may be "strictly decreasing".

(4) If there exists a point t_0 where $t_0 > 0$, $\eta'(t_0) = 0$ and $\eta(t)$ shows a "strictly decreasing then strictly increasing" pattern, i.e., a "bathtub" shape, then $\lambda(t)$ may also exhibit a "bathtub" shape or may be "strictly increasing".

If it is difficult to find such points in conditions (3) and (4) of Lemma 5.1, the following Lemma 5.2 may resolve this issue in certain cases.

Lemma 5.2. ([5]) *Let T be a non-negative continuous random variable with a density function $f(t)$ that has second-order derivative and a failure rate function $\lambda(t)$. Denote $\eta(t) = -\frac{f'(t)}{f(t)}$ and $\varepsilon = \lim_{t\to 0} f(t), \delta = \lim_{t\to 0}\frac{\eta(t)}{\lambda(t)}$. The following conclusions are drawn:*

(1) If $\eta(t)$ has a "bathtub" shape, then

(i) If $\varepsilon = 0$ or $\delta < 1$, $\lambda(t)$ is "strictly monotonically increasing".

(ii) If $\varepsilon = \infty$ or $\delta > 1$, $\lambda(t)$ has a "bathtub" shape.

(2) If $\eta(t)$ has an "inverted bathtub" shape, then

(i) If $\varepsilon = 0$ or $\delta < 1$, $\lambda(t)$ retains an "inverted bathtub" shape.

(ii) If $\varepsilon = \infty$ or $\delta > 1$, $\lambda(t)$ is "strictly monotonically decreasing".

Theorem 5.2. *For a non-negative continuous random variable $X \sim$ IGS(μ, m), its failure rate function $s(x)$ has an "inverted bathtub" shape.*

Proof. The failure rate function is defined as:

$$s(x) = \frac{f(x)}{1-F(x)} = \frac{\sqrt{\frac{m\mu}{2\pi x^3}}\exp\left[-\frac{m}{2\mu}\frac{(x-\mu)^2}{x}\right]}{1-\Phi\left[\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}-1\right)\right] - e^{2m}\Phi\left[-\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}+1\right)\right]}.$$

Let the random variable be $Y = \frac{X}{\mu}$. It is evident that the failure rate function of Y is:

$$\begin{aligned} s_Y(y) &= \frac{f_Y(y)}{1-F_Y(y)} \\ &= \frac{\sqrt{\frac{m}{2\pi y^3}}\exp\left[-\frac{m(y-1)^2}{2y}\right]}{1-\Phi\left[\sqrt{\frac{m}{y}}(y-1)\right] - e^{2m}\Phi\left[-\sqrt{\frac{m}{y}}(y+1)\right]} \end{aligned}$$

$$= \frac{e^m\sqrt{\frac{m}{2\pi}}y^{-3/2}e^{-my/2}e^{-m/(2y)}}{1-\Phi\left[\sqrt{m}\left(y^{1/2}-y^{-1/2}\right)\right]-e^{2m}\Phi\left[-\sqrt{m}\left(y^{1/2}+y^{-1/2}\right)\right]}.$$

Since μ is a scale parameter, the graphical characteristics of $s(x)$ are the same as those of $s_Y(y)$. This means that it is sufficient to examine the graphical characteristics of $s_Y(y)$ alone.

When $y \to 0$,

$$s_Y(y) = \frac{e^m\sqrt{\frac{m}{2\pi}}e^{-my/2}}{1-\Phi\Bigg[\sqrt{m}\left(\sqrt{y}-\sqrt{y^{-1}}\right)\Bigg]-e^{2m}\Phi\Bigg[-\sqrt{m}\left(\sqrt{y}+\sqrt{y^{-1}}\right)\Bigg]}y^{-3/2}e^{-m/(2y)},$$

then $\lim\limits_{y\to 0} s_Y(y) = 0$.

When $y \to +\infty$,

$$s_Y(y) = \frac{y^{-3/2}e^{-my/2}}{1-\Phi\left[\sqrt{m}\left(\sqrt{y}-\sqrt{y^{-1}}\right)\right]-e^{2m}\Phi\left[-\sqrt{m}\left(\sqrt{y}+\sqrt{y^{-1}}\right)\right]}e^m\sqrt{\frac{m}{2\pi}}e^{-m/(2y)},$$

$$\begin{aligned}
&\lim_{y\to+\infty}\frac{y^{-3/2}e^{-my/2}}{1-\Phi\left[\sqrt{m}\left(\sqrt{y}-\sqrt{y^{-1}}\right)\right]-e^{2m}\Phi\left[-\sqrt{m}\left(\sqrt{y}+\sqrt{y^{-1}}\right)\right]}\\
&=\lim_{y\to+\infty}\frac{\left(-\frac{3}{2}y^{-5/2}e^{-my/2}-\frac{m}{2}y^{-3/2}e^{-my/2}\right)\frac{2}{\sqrt{m}}}{-\left(y^{-1/2}+y^{-3/2}\right)\varphi\left[\sqrt{m}\left(\sqrt{y}-\sqrt{y^{-1}}\right)\right]+e^{2m}\left(y^{-1/2}-y^{-3/2}\right)\varphi\left[-\sqrt{m}\left(\sqrt{y}+\sqrt{y^{-1}}\right)\right]}\\
&=\lim_{y\to+\infty}\frac{(3+my)e^{-my/2}\frac{\sqrt{2\pi}}{\sqrt{m}}}{(y^2+y)\exp\left[-\frac{m}{2}\left(\sqrt{y}-\sqrt{y^{-1}}\right)^2\right]-e^{2m}(y^2-y)\exp\left[-\frac{m}{2}\left(\sqrt{y}+\sqrt{y^{-1}}\right)^2\right]}\\
&=\frac{\sqrt{2\pi}}{\sqrt{m}}e^{-m}\lim_{y\to+\infty}\frac{3+my}{(y^2+y)e^{-m/(2y)}-(y^2-y)e^{-m/(2y)}}\\
&=\frac{\sqrt{2\pi}}{\sqrt{m}}e^{-m}\lim_{y\to+\infty}\frac{3+my}{2ye^{-m/(2y)}},
\end{aligned}$$

then $\lim\limits_{y\to+\infty} s_Y(y) = \frac{m}{2}$.

Since $f_Y(y) = e^m\sqrt{\frac{m}{2\pi}}y^{-3/2}\exp\left[-\frac{m}{2}\left(y+y^{-1}\right)\right]$,

$$f_Y'(y) = \frac{e^m}{2}\sqrt{\frac{m}{2\pi}}y^{-7/2}\exp\Bigg[-\frac{m}{2}\left(y+y^{-1}\right)\Bigg](-my^2-3y+m).$$

Let $\eta(y) = -\frac{f_Y'(y)}{f_Y(y)}$, then

$$\eta(y) = \frac{my^2+3y-m}{2y^2},\ \lim_{y\to 0}\eta(y) = -\infty,\ \lim_{y\to+\infty}\eta(y) = \frac{m}{2},\ \eta'(y) = -\frac{3}{2y}\left(y-\frac{2}{3}m\right).$$

Define $y_0 = \frac{2}{3}m$, it follows that $\eta'(y_0) = 0$, when $0 < y < y_0$, $\eta'(y) > 0$; when $y > y_0$, $\eta'(y) < 0$. Thus, $\eta(y)$ has an "inverted bathtub" shape, and $\varepsilon = \lim\limits_{y\to 0} f_Y(y) = 0$. According to Lemma 5.1 and Lemma 5.2, $s_Y(y)$ also exhibits an "inverted bathtub" shape. □

Next, we examine the graphical characteristics of the average residual life $m(x)$. The average residual life is

$$m(x) = \int_x^{+\infty}\frac{1-F(t)}{1-F(x)}dt = \frac{\int_x^{+\infty}\left\{1-\Phi\left[\sqrt{\frac{m}{t/\mu}}\left(\frac{t}{\mu}-1\right)\right]+e^{2m}\Phi\left[-\sqrt{\frac{m}{t/\mu}}\left(\frac{t}{\mu}+1\right)\right]\right\}dt}{1-\Phi\left[\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}-1\right)\right]-e^{2m}\Phi\left[-\sqrt{\frac{m}{x/\mu}}\left(\frac{x}{\mu}+1\right)\right]}.$$

Let the random variable be $Y = \frac{X}{\mu}$. It is clear that the mean residual life for Y is given by:

$$m_Y(y) = \int_y^{+\infty} \frac{1 - F_Y(t)}{1 - F_Y(y)}\, dt = \frac{\int_y^{+\infty} \left\{1 - \Phi\left[\sqrt{\frac{m}{t}}(t-1)\right] + e^{2m}\Phi\left[-\sqrt{\frac{m}{t}}(t+1)\right]\right\} dt}{1 - \Phi\left[\sqrt{\frac{m}{y}}(y-1)\right] - e^{2m}\Phi\left[-\sqrt{\frac{m}{y}}(y+1)\right]}.$$

Lemma 5.3. ([6]) *Consider a non-negative continuous random variable T with a failure rate function $\lambda(t)$ and a mean residual life function (abbreviated as mean residual life) $m(t)$. The following conclusions are derived:*

(1) If $\lambda(t)$ is "strictly monotonically increasing", then $m(t)$ is "strictly monotonically decreasing".

(2) If $\lambda(t)$ is "strictly monotonically decreasing", then $m(t)$ is "strictly monotonically increasing".

(3) If $\lambda(t)$ has a "bathtub" shape:

(i) If $\lim\limits_{t\to 0} m(t)\lambda(t) \le 1$, then $m(t)$ is "strictly monotonically decreasing".

(ii) If $\lim\limits_{t\to 0} m(t)\lambda(t) > 1$, then $m(t)$ takes on an "inverted bathtub" shape.

(4) If $\lambda(t)$ has an "inverted bathtub" shape:

(i) If $\lim\limits_{t\to 0} m(t)\lambda(t) \ge 1$, then $m(t)$ is "strictly monotonically increasing".

(ii) If $\lim\limits_{t\to 0} m(t)\lambda(t) < 1$, then $m(t)$ exhibits a "bathtub" shape.

Theorem 5.3. *For a non-negative continuous random variable $X \sim IGS(\mu, m)$, its mean residual life function $m(x)$ exhibits a "bathtub" shape.*

Proof. From Theorem 5.2, we know that the failure rate function $s(x)$ is in an "inverted bathtub" shape, and $\lim\limits_{x\to 0} s(x) = 0$. Consequently, $\lim\limits_{x\to 0} m(x)s(x) = 0 < 1$ is derived, and from Lemma 5.3, we know that $m(x)$ exhibits a "bathtub" shape. □

6. Parameter estimation for two-parameter inverse Gaussian distribution IGS(μ, m) under inverse power law model in constant-stress accelerated life testing

6.1. Assumptions and the inverse power law model

The statistical analysis of constant-stress accelerated life testing (abbreviated as constant-stress testing) for products following an inverse Gaussian distribution IGS(μ, m) under the inverse power law model is based on the following three basic assumptions:

Assumption 6.1. *It is assumed that at any stress level V, the lifetime X of the product follows a two-parameter inverse Gaussian distribution IGS(μ, m) with a shape parameter m and a scale parameter μ.*

Assumption 6.2. *The failure mechanism of the product is the same across all stress levels. That is, the shape parameter m of the product lifetime distribution remains the same for each stress level, while the scale parameter μ depends on the stress level.*

Assumption 6.3. *The scale parameter μ is related to the accelerated stress level V according to the inverse power law model.*

The inverse power law model refers to when voltage is used as the acceleration stress. Based on physical principles and empirical findings from experiments, it has been observed that for some products (such as insulating materials, capacitors, miniature motors, and certain electronic devices), the scale parameter μ (in hours) has the following inverse power law relationship with voltage (in volts): $\mu = \frac{1}{dV^c}$ where $d > 0, c > 0$ are constants. When the product is an electronic component, physical experiments have shown that c only depends on the type of component and is independent of its specifications.

After taking the logarithm of both sides of above equation, μ satisfies a log-linear relationship $\ln \mu = a + b\phi(V)$, where $a = -\ln d, b = -c$, and $\phi(V) = \ln V$ is a function of the stress V.

6.2. Point estimations of parameters

Assume that n products, each following a two-parameter inverse Gaussian distribution IGS(μ, m) with a shape parameter m and a scale parameter μ, are subjected to a constant-stress accelerated life test. Divide these n products into k groups, that is, $n_1, n_2, \cdots, n_k$, with the constant stress divided into k levels, that is $V_1 < V_2 < \cdots < V_k$. Under constant stress $V_i, i = 1, 2, \cdots, k$, the scale parameter follows the inverse power law model $\mu_i = \frac{1}{dV_i^c}$.

The density function at this point is

$$f_{V_i}(x) = \sqrt{\frac{m\mu_i}{2\pi x^3}} \exp\left[-\frac{m}{2\mu_i}\frac{(x-\mu_i)^2}{x}\right] = e^m \sqrt{\frac{m\mu_i}{2\pi x^3}} \exp\left[-\frac{m}{2}\left(\frac{x}{\mu_i} + \frac{\mu_i}{x}\right)\right].$$

Conduct life tests on these n_i products until all have failed, with the failure times of these products denoted as $x_{i1}, x_{i2}, \cdots, x_{in_i}$, and let $\bar{x}_i = \frac{1}{n_i}\sum\limits_{j=1}^{n_i} x_{ij}, \overline{x_i^{-1}} = \frac{1}{n_i}\sum\limits_{j=1}^{n} x_{ij}^{-1}, i = 1, 2, \cdots, k$.

It is evident that under stress $V_i, i = 1, 2, \cdots, k$, the point estimation of the scale parameter μ_i is $\hat{\mu}_i = \bar{x}_i$.

The likelihood function is

$$\begin{aligned} L &= \prod_{i=1}^{k}\prod_{j=1}^{n_i}\left\{e^m \sqrt{\frac{m\mu_i}{2\pi x_{ij}^3}} \exp\left[-\frac{m}{2}\left(\frac{x_{ij}}{\mu_i} + \frac{\mu_i}{x_{ij}}\right)\right]\right\} \\ &= (2\pi)^{-\sum\limits_{i=1}^{k} n_i}\left(\prod_{i=1}^{k}\prod_{j=1}^{n_i} x_{ij}\right)^{-3} e^{m\sum\limits_{i=1}^{k} n_i} m^{\frac{1}{2}\sum\limits_{i=1}^{k} n_i}\left(\prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}}\right)\exp\left[-\frac{m}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\mu_i} + \frac{\mu_i}{x_{ij}}\right)\right]. \end{aligned}$$

Its log-likelihood function is

$$\begin{aligned} \ln L = -(2\pi)\sum_{i=1}^{k} n_i - 3\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln x_{ij} + m\sum_{i=1}^{k} n_i + \frac{1}{2}(\ln m)\sum_{i=1}^{k} n_i \\ + \frac{1}{2}\sum_{i=1}^{k} n_i \ln \mu_i - \frac{m}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\mu_i} + \frac{\mu_i}{x_{ij}}\right), \end{aligned}$$

$$\frac{\partial \ln L}{\partial m} = \sum_{i=1}^{k} n_i + \frac{1}{2m}\sum_{i=1}^{k} n_i - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\mu_i} + \frac{\mu_i}{x_{ij}}\right).$$

Let $\frac{\partial \ln L}{\partial m} = 0$, then the following equation is obtained:

$$\sum_{i=1}^{k} n_i + \frac{1}{2m}\sum_{i=1}^{k} n_i - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{x_{ij}}{\mu_i} + \frac{\mu_i}{x_{ij}}\right) = 0.$$

After substituting its point estimate $\hat{\mu}_i = \overline{x}_i$ into the equation μ_i above, the point estimate of the parameter m is derived as follows:

$$\hat{m} = \frac{\sum\limits_{i=1}^{k} n_i}{\sum\limits_{i=1}^{k} \left[n_i \left(\overline{x}_i \overline{x_i^{-1}} - 1 \right) \right]}.$$

Let $\overline{\ln V} = \frac{1}{k} \sum\limits_{i=1}^{k} \ln V_i, \overline{\ln \mu} = \frac{1}{k} \sum\limits_{i=1}^{k} \ln \mu_i$, and denote

$$L_{VV} = \sum_{i=1}^{k} \left(\ln V_i - \overline{\ln V} \right)^2,$$

$$L_{V\mu} = \sum_{i=1}^{k} \left(\ln V_i - \overline{\ln V} \right) \left(\ln \mu_i - \overline{\ln \mu} \right).$$

Noting that $\ln \mu_i = -\ln d - c \ln V_i, i = 1, 2, \cdots, k$, then the point estimates of the parameters c, d are respectively:

$$\hat{c} = -\frac{L_{V\mu}}{L_{VV}}, \hat{d} = \exp \left\{ -\left(\overline{\ln \mu} + \hat{c} \overline{\ln V} \right) \right\}.$$

Example 6.1. Assume a constant stress level of $V_1 = 50, V_2 = 100$, with true parameter values $m = 0.1, c = 0.5, d = 0.01, \mu_1 = \frac{1}{dV_1^c}, \mu_2 = \frac{1}{dV_2^c}$. Through Monte-Carlo simulation, the simple random samples following a two-parameter inverse Gaussian distribution IGS(μ_i, m) under constant stress accelerated life testing are generated as shown in Table 1.

Table 1. Simulated data for Example 6.1.

Stress	Sample Data
$V_1 = 50$	31.2800, 10.4530, 16.1726, 6.5582, 52.3844, 1.6557, 4.9500, 1.2397, 0.5486, 0.3488
$V_2 = 100$	5.3978, 0.2901, 19.6648, 1.1804, 0.4029, 0.4203, 0.6470, 0.1941, 2.6402, 0.4931, 0.4941, 0.6365, 11.4666, 1.6411, 14.8186, 0.3106, 84.2045, 34.0538, 0.8002, 0.8573

Parameter estimates obtained by the method described in this paper are:

$$\hat{m} = 0.0964, \hat{c} = 0.4758, \hat{d} = 0.0124.$$

Example 6.2. Assume a constant stress level of $V_1 = 10, V_2 = 100$, with true parameter values $m = 0.2, c = 0.1, d = 0.05, \mu_1 = \frac{1}{dV_1^c}, \mu_2 = \frac{1}{dV_2^c}$. Through Monte-Carlo simulation, the simple random samples following a two-parameter inverse Gaussian distribution IGS(μ_i, m) under constant stress accelerated life testing are generated as shown in Table 2.

Parameter estimates obtained by the method described in this paper are:

$$\hat{m} = 0.2522, \hat{c} = 0.1078, \hat{d} = 0.0544.$$

Table 2. Simulated data for Example 6.2.

Stress	Sample Data
$V_1 = 10$	5.1116, 2.0686, 2.1421, 56.6267, 2.5191, 13.8157, 1.2024, 25.4537, 2.6490, 1.5713, 6.2317, 82.4323, 83.2373, 0.6818, 10.6844, 1.2920, 74.6820, 5.7713, 7.1316, 0.3783, 3.5293, 1.0485, 6.2358, 3.2617, 5.2445, 7.6840, 1.4043, 10.9091, 3.9746, 1.1091
$V_2 = 100$	5.4471, 2.8236, 44.5314, 1.0500, 3.4452, 40.6736, 1.3111, 12.8405, 2.2617, 3.0939, 3.5848, 6.9947, 5.0752, 17.0472, 2.8834, 15.3931, 4.9376, 1.3597, 31.4428, 17.5004

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Received January 2026; Accepted June 2026; Available online July 2026.