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EXPANSION OF SOLUTION OF TIMOSHENKO BEAM WITH TIME DELAY IN INTERIOR DAMPING*

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Abstract In this paper, we study the solution expansion for Timoshenko beam with time delay in interior damping. We show that the set of its eigenvectors is complete in the state space, but the set of its eigenvectors does not form a Schauder basis for the state space. Besides, we also prove that the solution of this system still can be expressed by these eigenvectors in the form of infinite series under certain conditions.

Keywords Timoshenko beam, time delay, interior damping, solution expansion.

MSC(2010) 93C20, 35C30.

1. Introduction

Timoshenko beam mainly arises in the study of the mathematics control theory and engineering, since many applied problems in high-Tech can be reduced to such system (see [13, 16, 25]). Up to now, the researchers mainly devoted to study the stability and stabilization of Timoshenko beams. For the stability of Timoshenko beams, Krelifa et al. [15] gave the strong stability results for a Timoshenko system with thermoelasticity and two fractional damping terms. Liu and Zhang in [17] considered the stability of Timoshenko beam equation with locally distributed Kelvin-Voigt damping by using the frequency domain method and multiplier technique. Other literature can be seen in [4–7] and [9, 22, 24, 30]. Meanwhile, owing to the extensive applications of delayed systems, more and more researchers were devoted in investigating Timoshenko beams with delay. For instance, Han and Xu in [10] studied the exponential stability of a Timoshenko beam with delay times by spectral analysis. In [7, 11, 12, 14], the researchers gave the stability results for a kind of Timoshenko beams with delay using the Lyapunov functional method. Liu and Xu in [18, 19] designed dynamic feedback control strategy to stabilize a Timoshenko beam with distributed delay in the boundary control. A few years later, Xu et al. [31] stabilized a Timoshenko beam with internal input delays by designing dynamic feedback strategy. Recently, Xie and Chen discussed the rapid stabilization problem for Timoshenko beam system with the internal delay control [28]. Aounallah in [3] considered the Timoshenko beam system with a delay term in the internal fractional feedback in a bounded domain, and gave a stability result for this system. Other results can also be found in [1, 8, 21, 32] and the references therein.

Note that the results mentioned above mainly concerned the stability and stabilization of delayed Timoshenko beams. To our knowledge, few results have been obtained for such systems'

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solution structure, especially solution expansion, although some similar results have been investigated for delayed differential equations (DDEs) (see [26,34,35]). This is because the expansion of solution of delayed systems depends on their detailed spectral analysis and the properties of eigenvectors corresponding to its eigenvalues. Recently, the spectral analysis of the Timoshenko beam models without time delay has been widely investigated (see [2, 23]). However, the results of these systems with time delay are still scarce since the existence of time delay makes the calculation of the spectrum of such systems very complicated. In view of this, we tried to discuss the spectrum of a Timoshenko beam with time delay in interior damping in [27], and obtained that the spectrum of this system consists of all isolated eigenvalues of multiplicity one and distributes symmetrically with respect to the real axis. Meanwhile, we also obtained that a sequence of real eigenvalues diverges to negative infinity. Moreover, the solution expansion of delayed system also plays an important role in addressing the controllability problem. According to earlier works [33], it is known that the system without time delay is exactly controllable in finite time if and only if the observability inequality holds. For the delayed systems, a key question arises: When the systems are exactly controllable in finite time, whether the observability inequality still hold true. If this assertion holds, it is still needed to investigate the expansion of solution of systems before studying the exact controllability of these systems.

Considering the significance of solution expansion in the exact controllability of delayed systems, in this paper, we shall further investigate the expansion of solution of a Timoshenko beam with time delay in interior damping based on the results of [27]. This system is described by the partial differential equations:

$$\begin{cases} \rho w_{tt}(x,t) = K[w_{xx}(x,t) - \varphi_x(x,t)] - \alpha_1 w_t(x,t-\tau), \;\; x \in (0,1), \;\; t > 0, \\ I_\rho \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) + K[w_x(x,t) - \varphi(x,t)] - \alpha_2 \varphi_t(x,t-\tau), \;\; x \in (0,1), t > 0, \\ w(0,t) = \varphi(0,t) = 0, \; t > 0, \\ K[w_x - \varphi](1,t) = 0, \;\; EI\varphi_x(1,t) = 0, \; t > 0, \\ w(x,0) = w_0(x), \;\; w_t(x,0) = w_1(x), \quad x \in [0,1], \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \quad x \in [0,1], \\ w_t(x,s) = h_1(x,s), \quad x \in [0,1], \quad s \in [-\tau, 0], \\ \varphi_t(x,s) = h_2(x,s), \quad x \in [0,1], \quad s \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $w(x,t)$ is the deflection of the beam from its equilibrium and $\varphi(x,t)$ is the total rotatory angle of the beam respectively. t is the time variable, x is the spacial variable and $\tau > 0$ is the time delay; $h_1(x,s)$ and $h_2(x,s)$ are the history memory; α_1 and α_2 denote the damping coefficients; The coefficients ρ, I_ρ, E, I and K are the density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus respectively.

Similar to [27], we set

$$\begin{cases} \psi(x,y,t) = \sqrt{\rho} w_t(x,t-y\tau), \;\; x \in (0,1), \;\; y \in (0,1), \\ \eta(x,y,t) = \sqrt{I_\rho} \varphi_t(x,t-y\tau), \;\; x \in (0,1), \;\; y \in (0,1). \end{cases} \tag{1.2}$$

Substituting (1.2) into (1.1), then the system (1.1) can be rewritten as

$$
\begin{cases}
\rho w_{tt}(x,t) = K[w_{xx}(x,t) - \varphi_x(x,t)] - \dfrac{\alpha_1}{\sqrt{\rho}}\psi(x,1,t), \ \ x \in (0,1), \ \ t > 0,\\
I_\rho \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) + K[w_x(x,t) - \varphi(x,t)] - \dfrac{\alpha_2}{\sqrt{I_\rho}}\eta(x,1,t), \quad x \in (0,1), \ \ t > 0,\\
\tau\psi_t(x,y,t) + \psi_y(x,y,t) = 0, \quad x \in (0,1), \quad y \in (0,1), \quad t > 0,\\
\tau\eta_t(x,y,t) + \eta_y(x,y,t) = 0, \quad x \in (0,1), \quad y \in (0,1), \quad t > 0,\\
w(0,t) = \varphi(0,t) = 0, \quad t > 0,\\
K[w_x - \varphi](1,t) = 0, \ \ EI\varphi_x(1,t) = 0, \quad t > 0,\\
\psi(x,0,t) = \sqrt{\rho}w_t(x,t), \quad \eta(x,0,t) = \sqrt{I_\rho}\varphi_t(x,t), \ \ x \in (0,1), \ \ t > 0,\\
w(x,0) = w_0(x), \ \ w_t(x,0) = w_1(x), \quad x \in (0,1),\\
\varphi(x,0) = \varphi_0(x), \ \ \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,1),\\
\psi(x,y,0) = \sqrt{\rho}w_t(x,-y\tau) = \sqrt{\rho}h_1(x,-y\tau), \quad x \in (0,1), \ \ y \in (0,1),\\
\eta(x,y,0) = \sqrt{I_\rho}\varphi_t(x,-y\tau) = \sqrt{I_\rho}h_2(x,-y\tau), \quad x \in (0,1), \ \ y \in (0,1).
\end{cases}
\tag{1.3}
$$

Let $H^k[0,1]$ be the usual sobolev space of order k and $H_E^k[0,1] = \{f \in H^k[0,1] | f(0) = 0\}$. Take the space

$$
\mathcal{H} = H_E^1[0,1] \times H_E^1[0,1] \times L_\rho^2[0,1] \times L_{I_\rho}^2[0,1] \times L^2([0,1] \times [0,1]) \times L^2([0,1] \times [0,1])
$$

with the inner product

$$
\begin{aligned}
(W_1, W_2)_{\mathcal{H}} = &\int_0^1 K(w_1'(x) - \varphi_1(x))\overline{(w_2'(x) - \varphi_2(x))}dx + \int_0^1 EI\varphi_1'(x)\overline{\varphi_2'(x)}dx\\
&+ \int_0^1 \rho z_1(x)\overline{z_2(x)}dx + \int_0^1 I_\rho v_1(x)\overline{v_2(x)}dx\\
&+ \tau\int_0^1\int_0^1 \psi_1(x,y)\overline{\psi_2(x,y)}dydx + \tau\int_0^1\int_0^1 \eta_1(x,y)\overline{\eta_2(x,y)}dydx,\\
W_i = (w_i, &\varphi_i, z_i, v_i, \psi_i, \eta_i) \in \mathcal{H}, \ \ i = 1,2
\end{aligned}
$$

and the induced norm $\|W\|_{\mathcal{H}} = \sqrt{(W,W)_{\mathcal{H}}}$. Obviously, the space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space.

In $\mathcal{H}$, operator $\mathcal{A}$ is defined by

$$
\mathcal{A}\begin{pmatrix} w(x)\\ \varphi(x)\\ z(x)\\ v(x)\\ \psi(x,y)\\ \eta(x,y)\end{pmatrix} = \begin{pmatrix} z(x)\\ v(x)\\ \dfrac{K}{\rho}[w''(x) - \varphi'(x)] - \dfrac{\alpha_1}{\rho\sqrt{\rho}}\psi(x,1)\\ \dfrac{EI}{I_\rho}\varphi''(x) + \dfrac{K}{I_\rho}[w'(x) - \varphi(x)] - \dfrac{\alpha_2}{I_\rho\sqrt{I_\rho}}\eta(x,1)\\ -\dfrac{1}{\tau}\psi_y(x,y)\\ -\dfrac{1}{\tau}\eta_y(x,y)\end{pmatrix}
\tag{1.4}
$$

with domain

$$D(\mathcal{A}) = \left\{ (w,\varphi,z,v,\psi,\eta)^T \in \mathcal{H} \middle| \begin{array}{c} w,\varphi \in H_E^2[0,1], \quad z,v \in H_E^1[0,1] \\ \psi(x,y),\eta(x,y) \in H^1([0,1]\times[0,1]) \\ K[w'(1)-\varphi(1)]=0, \quad EI\varphi'(1)=0 \\ \psi(x,0)=\sqrt{\rho}z(x), \quad \eta(x,0)=\sqrt{I_\rho}v(x) \end{array} \right\}.$$

Then (1.3) can be written as an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dW(t)}{dt} = \mathcal{A}W(t), \quad t>0, \\ W(0)=W_0, \end{cases} \tag{1.5}$$

where $W(t) = (w(x,t),\varphi(x,t),w_t(x,t),\varphi_t(x,t),\psi(x,y,t),\eta(x,y,t))^T$ and
$W_0 = (w_0(x),\varphi_0(x),w_1(x),\varphi_1(x),\sqrt{\rho}h_1(x,-y\tau),\sqrt{I_\rho}h_2(x,-y\tau))^T \in \mathcal{H}$.

In [27], we have discussed the details of spectrum of system (1.5) and the corresponding eigenvectors on the assumption that $\frac{\alpha_1}{\rho} = \frac{\alpha_2}{I_\rho} = \alpha$. Moreover, we have also given the asymptotic expressions of the spectrum of this system. The corresponding results are stated in Section 2. In this paper, we continue to discuss the expansion of solution of (1.5).

The present paper is organized as follows. In Section 2, some results about the spectrum of (1.5) are listed. In Section 3, we obtain that the set of eigenvectors of Λ_n defined by (2.5) is complete in $\mathbb{X}$. In Section 4, we prove that this set of eigenvectors of Λ_n does not form a Schauder basis for the state space $\mathbb{X}$, although it is complete in $\mathbb{X}$. However, in Section 5, the semigroup generated by the operator Λ_n can be expanded as an infinite series by its eigenvectors and is differentiable. Moreover, in Section 6, it is proved that the semigroup generated by the operator $\mathcal{A}$ and the solution of (1.5) can be also expanded by its eigenvectors under certain conditions. Finally, we give a conclusion.

2. Preliminaries and some results

Let us introduce the following two useful lemmas from [32].

Lemma 2.1. *Let the differential operator $\mathcal{L}$ in $L_\rho^2[0,1]\times L_{I_\rho}^2[0,1]$ be defined by*

$$\mathcal{L}(w,\varphi)^T = (-\frac{K}{\rho}(w''(x)-\varphi'(x)), -\frac{EI}{I_\rho}\varphi''(x) - \frac{K}{I_\rho}(w'(x)-\varphi(x)))^T \tag{2.1}$$

with domain

$$D(\mathcal{L}) = \left\{ (w(x),\varphi(x)) \in H^2[0,1]\times H^2[0,1] \middle| \begin{array}{c} w(0)=\varphi(0)=0 \\ K(w'(1)-\varphi(1))=0 \\ EI\varphi'(1)=0 \end{array} \right\}. \tag{2.2}$$

Then $\mathcal{L}$ is a positive defined operator with compact resolvent in $L_\rho^2[0,1]\times L_{I_\rho}^2[0,1]$, its eigenvalues are

$$0<\mu_1<\mu_2<\cdots<\mu_n<\cdots \tag{2.3}$$

and the eigenfunctions $\Phi_n(x) = (w_n(x),\varphi_n(x))^T$ corresponding to μ_n are real functions and form a normalized orthogonal basis for $L_\rho^2[0,1]\times L_{I_\rho}^2[0,1]$.

Lemma 2.2. *Let $\Phi_n(x) = (w_n(x), \varphi_n(x))$ be the normalized eigenfunction corresponding to the eigenvalue μ_n of $\mathcal{L}$. Then it holds that*

$$\int_0^1 K|w_n'(x) - \varphi_n(x)|^2 dx + \int_0^1 EI|\varphi_n'(x)|^2 dx = \mu_n. \tag{2.4}$$

In the following discussion, we always assume $\frac{\alpha_1}{\rho} = \frac{\alpha_2}{I_\rho} = \alpha$. Let the space $\mathbb{X} = \mathbb{C}\times\mathbb{C}\times L^2[0,1]$. Define the unbounded linear operators $\Lambda_n : \mathbb{X} \to \mathbb{X}$ by

$$\Lambda_n = \begin{pmatrix} 0 & \sqrt{\mu_n} & 0 \\ -\sqrt{\mu_n} & 0 & -\alpha\delta_1 \\ 0 & 0 & -\frac{1}{\tau}\frac{d}{dy} \end{pmatrix} \tag{2.5}$$

with domain

$$D(\Lambda_n) = \{(a, b, z(y)) \in \mathbb{C} \times \mathbb{C} \times H^1[0,1] | z(0) = b\}, \tag{2.6}$$

where $\delta_1 z(y) := z(1)$.

In $\mathbb{X}$, define the inner product by

$$\begin{aligned} &\langle X_1, X_2\rangle_{\mathbb{X}} = a_1\overline{a_2} + b_1\overline{b_2} + \tau\int_0^1 z_1(y)\overline{z_2(y)}dy, \\ &X_i = (a_i, b_i, z_i(y)) \in D(\Lambda_n), \quad i = 1, 2 \end{aligned}$$

and the spaces $\mathcal{H}_n$ by

$$\mathcal{H}_n = \mathbb{X}^T \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix},$$

where "$\bullet$" denotes the Hadamard product of matrices, $G_1 = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & \sqrt{I_\rho} \end{pmatrix}$ and $G_2 = \begin{pmatrix} \frac{1}{\sqrt{\mu_n}} & 0 \\ 0 & \frac{1}{\sqrt{\mu_n}} \end{pmatrix}$.

The following results come from [27] are the spectral properties of (1.5).

Theorem 2.1. *Let $\mathcal{A}$ be defined as before. Then*

1) $\mathcal{A}$ is a closed and densely defined linear operator in $\mathcal{H}$.

2) $\sup\{\Re\lambda, \lambda \in \sigma(\mathcal{A})\} \le \frac{1}{2}\max\{(\rho + \frac{\alpha_1^2}{\rho}), (I_\rho + \frac{\alpha_2^2}{I_\rho})\}$.

3) $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$ and hence (1.5) is well-posed.

Theorem 2.2. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then*

1) $\mathcal{H}$ has an orthogonal decomposition $\mathcal{H} = \sum\limits_{n=1}^{\infty} \mathcal{H}_n$.

2) The decomposition is invariant under $\mathcal{A}$.

3) $\sigma(\mathcal{A}) = \overline{\bigcup_{n\ge1}\sigma(\Lambda_n)}$.

Theorem 2.3. *Let Λ_n be defined as above. Set*

$$\Delta(\lambda) = \lambda^2 + \mu_n + \alpha\lambda e^{-\tau\lambda}.$$

Then the following results are true:

(1) $\sigma(\Lambda_n) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\} = \{\lambda_{k,n}, \overline{\lambda_{k,n}} \mid \Im\lambda_{k,n} > 0, k \in \mathbb{N}\} \cup Q_0$, *where* Q_0 *is a set of possible real zeros of* $\Delta(\lambda)$*. Moreover,* $\lambda_{k,n}$ *has an asymptotic expression* $\lambda_{k,n} = \xi_k + \gamma_{k,n}$*.* ξ_k *and* $\gamma_{k,n}$ *have the following asymptotic expressions*

$$\xi_k = \frac{1}{\tau}(\ln|\alpha| - \ln w_k) + i(w_k - \frac{\ln w_k - \ln\tau|\alpha|}{\tau^2 w_k}) + O(\frac{1}{k}),$$
$$\gamma_{k,n} \simeq -\frac{\ln(1+\frac{\mu_n}{\xi_k^2})}{\tau + \frac{\xi_k^2-\mu_n}{\xi_k^2+\mu_n}\frac{1}{\xi_k}} + O(\frac{1}{k}),$$

where $w_k = \frac{1}{\tau}(2k + \frac{1}{2}\mathrm{sign}(\alpha))\pi$.

(2) If $\lambda \notin \sigma(\Lambda_n)$*, then for any* $(\widetilde{a}, \widetilde{b}, \widetilde{z}(y)) \in \mathbb{X}$*, the resolvent of* Λ_n *is defined as follows:*

$$R(\lambda,\Lambda_n)\begin{pmatrix}\widetilde{a}\\ \widetilde{b}\\ \widetilde{z}(y)\end{pmatrix} = \frac{1}{\Delta(\lambda)}\begin{pmatrix}(\lambda+\alpha e^{-\tau\lambda})\widetilde{a} + \sqrt{\mu_n}\widetilde{b} - \alpha\sqrt{\mu_n}\displaystyle\int_0^1 \tau e^{\tau\lambda(r-1)}\widetilde{z}(r)dr\\ \lambda\widetilde{b} - \sqrt{\mu_n}\,\widetilde{a} - \alpha\lambda\displaystyle\int_0^1 \tau e^{\tau\lambda(r-1)}\widetilde{z}(r)dr\\ (\lambda\widetilde{b} - \sqrt{\mu_n}\,\widetilde{a})e^{-\tau\lambda y} - \alpha\lambda e^{-\tau\lambda}e^{-\tau\lambda y}\displaystyle\int_y^1 \tau e^{\tau\lambda y}\widetilde{z}(r)dr\\ +(\mu_n+\lambda^2)\displaystyle\int_0^y \tau e^{\tau\lambda(r-y)}\widetilde{z}(r)dr\end{pmatrix}. \quad (2.7)$$

(3) If $\lambda \in \sigma(\Lambda_n)$*, then* λ *is a simple eigenvalue of* Λ_n *and the corresponding eigenvector is*

$$(a, b, z(y)) = b(\frac{\sqrt{\mu_n}}{\lambda}, 1, e^{-\tau\lambda y}). \quad (2.8)$$

The detailed proofs of above theorems can be found in [27].

Remark 2.1. In [27], from the proof of Theorem 2.2, we know that $\mathcal{A}$ has a diagonal decomposition under the decomposition $\{\mathcal{H}_n\}$ of $\mathcal{H}$, i.e., $\mathcal{A} = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n, \cdots)$. The results in [27] also show that the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of multiplicity one and distributes symmetrically with respect to the real axis. Moreover, there exist at most finite eigenvalues on the right-half complex plane for fixed Λ_n.

3. Completeness of the set of eigenvectors of Λ_n

In this section, we shall consider the completeness of set of eigenvectors of Λ_n. To begin with, we introduce the following definition and result comes from [29].

Definition 3.1. Suppose that L is an unbounded linear operator on Banach space $\mathbb{X}$, and $\sigma(L) = \sigma_p(L) = \{\lambda_k; k \in \mathbb{N}\}$ consists of all insolated eigenvalues of L. For each $\lambda_k \in \sigma(L)$, $E(\lambda_k; L)$ denotes the Riesz spectral projection corresponding to λ_k. Define the spectral subspace of L:

$$Sp(L) := \overline{\mathrm{span}\{E(\lambda_j; L)\mathbb{X} | \lambda_j \in \sigma(L)\}}.$$

If $Sp(L) = \mathbb{X}$, then the set of generalized eigenvectors of L is complete in $\mathbb{X}$. Otherwise, it is incomplete.

Lemma 3.1. *Suppose that L is an unbounded linear operator on $\mathbb{X}$. Then the set of generalized eigenvectors of L is complete in $\mathbb{X}$ if the following three conditions are fulfilled:*

1) for each $x \in \mathbb{X}$, $R(\lambda, L)x$ is a meromorphic function of finite exponential type, i.e.,

$$R(\lambda, L)x = \frac{Q(\lambda)x}{\Gamma(\lambda)}$$

with $Q(\lambda)x$ and $\Gamma(\lambda)$ are the entire functions satisfying $\|Q(\lambda)x\| \le M_1 e^{h_1|\lambda|}$ and $\|\Gamma(\lambda)\| \le M_2 e^{h_2|\lambda|}$;

2) there exists a ω such that

$$\|R(\lambda, L)\| \le \frac{M}{\Re\lambda - \omega}, \quad \Re\lambda > \omega.$$

3) there exists a linea γ with $\lambda \in \gamma$ and $\arg\lambda \in (\frac{\pi}{2}, \frac{3\pi}{2})$, for each $x \in \mathbb{X}$, it holds that

$$\|R(\lambda, L)x\| \le M, \quad \lambda \in \gamma.$$

In fact, Lemma 3.1 gives a sufficient conditions for the completeness of set of eigenvectors of operator L. In what follows, we shall use the conditions in Lemma 3.1 to prove the completeness of set of eigenvectors of Λ_n. Firstly, we shall give the estimations of the resolvent $R(\lambda, \Lambda_n)$ of Λ_n.

Theorem 3.1. *Let Λ_n be defined as before. Then for any $Y \in \mathbb{X}$, the resolvent of Λ_n satisfies*

$$\|R(\lambda, \Lambda_n)Y\| \le \frac{1}{\Re\lambda - \frac{1+\alpha^2}{2}}\|Y\|, \quad \forall \Re\lambda > \frac{1+\alpha^2}{2}. \tag{3.1}$$

Proof. For any $X = (a, b, z(y)) \in D(\Lambda_n)$, a simple calculation yields

$$\begin{aligned}
2\Re\langle \Lambda_n X, X\rangle_{\mathbb{X}} &= \langle \Lambda_n X, X\rangle_{\mathbb{X}} + \langle X, \Lambda_n X\rangle_{\mathbb{X}} \\
&= \sqrt{\mu_n} b\overline{a} + (-\sqrt{\mu_n} a - \alpha z(1))\overline{b} + \tau \int_0^1 (-\frac{1}{\tau} z'(y))\overline{z(y)} dy \\
&\quad + a\sqrt{\mu_n}\overline{b} + b(-\sqrt{\mu_n}\overline{a} - \alpha\overline{z(1)}) + \tau \int_0^1 z(y)\overline{(-\frac{1}{\tau} z'(y))} dy \\
&= -\alpha(z(1)\overline{b} + \overline{z(1)} b) + z(0)\overline{z(0)} - z(1)\overline{z(1)} \\
&= -\alpha(z(1)\overline{b} + \overline{z(1)} b) + |b|^2 - |z(1)|^2 \\
&\le 2|\alpha||b||z(1)| + |b|^2 - |z(1)|^2 \\
&\le (1+\alpha^2)\|X\|^2.
\end{aligned}$$

Thus when $\Re\lambda > \frac{1+\alpha^2}{2}$, we have

$$\begin{aligned}
\|(\lambda I - \Lambda_n)X\|\|X\| &\ge \|\langle(\lambda I - \Lambda_n)X, X\rangle\| = \|\langle \lambda X, X\rangle - \langle \Lambda_n X, X\rangle\| \\
&\ge \Re\lambda\langle X, X\rangle - \Re\langle \Lambda_n X, X\rangle \\
&\ge (\Re\lambda - \frac{1+\alpha^2}{2})\|X\|^2.
\end{aligned}$$

Let $Y = (\lambda I - \Lambda_n)X$, then $X = R(\lambda, \Lambda_n)Y$. When $\Re\lambda > \frac{1+\alpha^2}{2}$, from the above expression, we have

$$\|R(\lambda, \Lambda_n)Y\| \le \frac{1}{\Re\lambda - \frac{1+\alpha^2}{2}}\|Y\|.$$

The proof is complete. □

Theorem 3.2. *If $n \in \mathbb{N}$ is fixed, then for any $U = (\bar{a}, \bar{b}, \bar{z}(y)) \in \mathbb{X}$, the following estimation holds*

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} \|R(\lambda, \Lambda_n)U\|^2 = 0.$$

Furthermore, there exists a constant $\widehat{M} > 0$ such that

$$\limsup_{\lambda \in \mathbb{R}, \lambda \to -\infty} \|R(\lambda, \Lambda_n)U\|^2 < \frac{\widehat{M}}{\alpha^2} \int_0^1 |\bar{z}(r)|^2 dr.$$

Proof. We only need to consider the case of $\Delta(\lambda) \neq 0$.

For any $U = (\bar{a}, \bar{b}, \bar{z}(y))^T \in \mathbb{X}$, according to the resolvent equation $(\lambda I - \Lambda_n)(a, b, z(y))^T = U$ and (2.7), we have

$$\begin{aligned}
&|\Delta(\lambda)|^2 \|R(\lambda, \Lambda_n)U\|^2 \\
&= |\Delta(\lambda)|^2 (|a|^2 + |b|^2 + \tau \int_0^1 |z(y)|^2 dy) \\
&\leq 3\Big(|\lambda + \alpha e^{-\tau\lambda}|^2 + \mu_n + 2\tau\mu_n \frac{1 - e^{-2\tau\Re\lambda}}{2\tau\Re\lambda} \Big) |\bar{a}|^2 \\
&\quad + 3\Big(\mu_n + |\lambda|^2 + 2\tau|\lambda|^2 \frac{1 - e^{-2\tau\Re\lambda}}{2\tau\Re\lambda} \Big) |\bar{b}|^2 \\
&\quad + 3\Big(\frac{(\mu_n + |\lambda|^2 + |\mu_n + \lambda^2|^2)\alpha^2\tau}{2\Re\lambda} + \frac{\alpha^2|\lambda|^2 - |\mu_n + \lambda^2|^2}{(2\Re\lambda)^2} \Big) \tau \int_0^1 |\bar{z}(r)|^2 dr \\
&\quad + 3\Big(\frac{|\mu_n + \lambda^2|^2 - \alpha^2|\lambda|^2}{(2\Re\lambda)^2} - \frac{\alpha^2\tau(\mu_n + 2|\lambda|^2)}{2\Re\lambda} \Big) e^{-2\tau\Re\lambda} \tau \int_0^1 |\bar{z}(r)|^2 dr.
\end{aligned} \tag{3.2}$$

When $\lambda \in \mathbb{R}$ and $\lambda > 0$, we obtain that there exists a constant $\widetilde{M} > 0$ such that

$$\begin{aligned}
\|R(\lambda, \Lambda_n)U\|^2 \leq& \frac{\widetilde{M}}{|\Delta(\lambda)|^2} \Big[|\lambda|^2|\bar{a}|^2 + |\lambda|^2|\bar{b}|^2 + (|\lambda|^3 + |\lambda|^2 e^{-2\tau\lambda})\tau \int_0^1 |\bar{z}(y)|^2 dr \Big] \\
=& \frac{\widetilde{M}}{|\mu_n + \lambda^2 + \alpha\lambda e^{-\tau\lambda}|^2} \Big[|\lambda|^2|\bar{a}|^2 + |\lambda|^2|\bar{b}|^2 + (|\lambda|^3 + |\lambda|^2 e^{-2\tau\lambda})\tau \int_0^1 |\bar{z}(y)|^2 dr \Big].
\end{aligned}$$

Thus, we have

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} \|R(\lambda, \Lambda_n)U\|^2 = 0.$$

Moreover, if $\lambda \in \mathbb{R}$ and $\lambda < -1$, (3.2) implies that there exists a constant $\widehat{M} > 0$ such that

$$\|R(\lambda, \Lambda_n)U\|^2 \leq \frac{\widehat{M} e^{-2\lambda\tau}}{|\Delta(\lambda)|^2} \left(|\bar{a}|^2 + |\bar{b}|^2 + (|\lambda|^3 e^{2\lambda\tau} + |\lambda|^2)\tau \int_0^1 |\bar{z}(r)|^2 dr \right).$$

Thus, we have

$$\limsup_{\lambda \to -\infty} \|R(\lambda, \Lambda_n)U\|^2 \leq \frac{\widehat{M}}{\alpha^2} \int_0^1 |\bar{z}(r)|^2 dr.$$

□

The following corollary shows that the sequences $\{R(\lambda, \Lambda_n), n \geq 1\}$ are uniformly bounded.

Corollary 3.1. *Let $\lambda \in \bigcap_{n=1} \rho(\Lambda_n)$. For any $U = (\bar{a}, \bar{b}, \bar{z}(y)) \in \mathbb{X}$, there exists a constant $M(\lambda) > 0$ such that*

$$\limsup_{n\to\infty} \|R(\lambda, \Lambda_n)U\|^2 \leq M(\lambda)\|U\|^2.$$

Proof. If $\Delta(\lambda) \neq 0$, then there exists constant $M(\lambda) > 0$ such that

$$|\Delta(\lambda)|^2 \|R(\lambda, \Lambda_n)U\|^2 \leq M(\lambda)\left[\mu_n|\bar{a}|^2 + \mu_n|\bar{b}|^2 + \mu_n^2\tau\int_0^1 |\bar{z}(y)|^2 dr\right]$$

or equivalently,

$$\begin{aligned}\|R(\lambda, \Lambda_n)U\|^2 \leq & \frac{M(\lambda)}{|\Delta(\lambda)|^2}\left[\mu_n|\bar{a}|^2 + \mu_n|\bar{b}|^2 + \mu_n^2\tau\int_0^1 |\bar{z}(y)|^2 dr\right] \\ = & \frac{M(\lambda)}{|\mu_n + \lambda^2 + \alpha\lambda e^{-\tau\lambda}|^2}\left[\mu_n|\bar{a}|^2 + \mu_n|\bar{b}|^2 + \mu_n^2\tau\int_0^1 |\bar{z}(y)|^2 dr\right].\end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} \|R(\lambda, \Lambda_n)U\|^2 \leq M(\lambda)\tau\int_0^1 |\bar{z}(r)|dr \leq M(\lambda)\left(|\bar{a}|^2 + |\bar{b}|^2 + \tau\int_0^1 |\bar{z}(r)|dr\right) \leq M(\lambda)\|U\|^2.$$

$\square$

We have the following result.

Theorem 3.3. *Let Λ_n be defined as before. Then the set of eigenvectors of Λ_n is complete in $\mathbb{X}$.*

Proof. For any $Y = (a, b, z(y)) \in \mathbb{X}$, (2.7) shows that $R(\lambda, \Lambda_n)Y$ is a meromorphic function of finite exponential type. Hence the first condition in Lemma 3.1 holds. On the other hand, (3.1) implies that the second condition in Lemma 3.1 also holds.

Take $\gamma = \mathbb{R}_-$. For $\lambda \in \gamma$, according to Corollary 3.1, we have

$$\limsup_{\lambda\to\infty} \|R(\lambda, \Lambda_n)U\|^2 < \infty.$$

This means that the third condition in Lemma 3.1 holds. The desired result follows. $\square$

4. Non-basis property of eigenvectors of Λ_n

In this section, we shall verify that the set of eigenvectors of Λ_n can not form a Schauder basis for the state space $\mathbb{X}$, although it is complete in $\mathbb{X}$. To this end, we first discuss the spectrum of the adjoint operator of Λ_n. A direct calculation yields the adjoint operator of Λ_n, Λ_n^* is

$$\Lambda_n^*\begin{pmatrix}\widehat{a}\\ \widehat{b}\\ \widehat{z}(y)\end{pmatrix} = \begin{pmatrix}-\sqrt{\mu_n}\widehat{b}\\ \sqrt{\mu_n}\widehat{a} + \widehat{z}(0)\\ \frac{1}{\tau}\widehat{z}'(y)\end{pmatrix} \tag{4.1}$$

with the domain

$$D(\Lambda_n^*) = \{(\widehat{a}, \widehat{b}, \widehat{z}(y)) \in \mathbb{C}\times\mathbb{C}\times H^1[0,1] | \widehat{z}(1) = -\alpha\widehat{b}\}. \tag{4.2}$$

Theorem 4.1. *Let Λ_n^* be defined as above. then*

(1) $\sigma(\Lambda_n^*) = \sigma(\Lambda_n)$.

(2) Assume that

$$\Phi_{\lambda_{k,n}} = (\frac{\sqrt{\mu_n}}{\lambda_{k,n}}, 1, e^{-\tau\lambda_{k,n}y})^T \tag{4.3}$$

is the eigenvector corresponding to $\lambda_{k,n} \in \sigma(\Lambda_n)$*, then*

$$\Psi_{\overline{\lambda}_{k,n}} = \xi_n^{-1}(\overline{\lambda}_{k,n})(-\frac{\sqrt{\mu_n}}{\overline{\lambda}_{k,n}} e^{\tau\overline{\lambda}_{k,n}}, e^{\tau\overline{\lambda}_{k,n}}, -\alpha e^{\tau\overline{\lambda}_{k,n}y})^T \tag{4.4}$$

is the eigenvector of Λ_n^* *corresponding to* $\overline{\lambda}_{k,n}$*, where*

$$\xi_n(\lambda_{k,n}) = -\frac{\mu_n}{(\lambda_{k,n})^2} e^{\tau\lambda_{k,n}} + e^{\tau\lambda_{k,n}} - \alpha\tau.$$

In addition,

$$\langle \Phi_{\lambda_{k,n}}, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} = 1 \ \ \text{and} \ \ \langle \Phi_{\lambda_{k,n}}, \Psi_{\overline{\lambda}_{m,n}} \rangle_{\mathbb{X}} = 0 \ \ \text{for} \ \ k \neq m. \tag{4.5}$$

Proof. For any $\lambda \in \sigma(\Lambda_n^*)$ and $Y = (\widehat{a}, \widehat{b}, \widehat{z}(y)) \in D(\Lambda_n^*)$, consider the eigenvalue problem $\Lambda_n^* Y = \lambda Y$, i.e.,

$$\begin{cases} -\sqrt{\mu_n}\widehat{b} = \lambda\widehat{a}, \\ \sqrt{\mu_n}\widehat{a} + \widehat{z}(0) = \lambda\widehat{b}, \\ \dfrac{1}{\tau}\widehat{z}'(y) = \lambda\widehat{z}(y), \quad y \in [0,1], \\ \widehat{z}(1) = -\alpha\widehat{b}. \end{cases} \tag{4.6}$$

Solving the third equation in (4.6), we have $\widehat{z}(y) = -\alpha\widehat{b}e^{-\tau\lambda(1-y)}$. Then substituting $\widehat{z}(y)$ into the second equation and combining with the first equation leads to $(\lambda^2 + \mu_n + \alpha\lambda e^{-\tau\lambda})\widehat{b} = 0$. Note that $\Delta(\lambda) = \lambda^2 + \mu_n + \alpha\lambda e^{-\tau\lambda}$, thus (4.6) has nonzero solution if and only if $\Delta(\lambda) = 0$. That is $\sigma(\Lambda_n^*) = \sigma(\Lambda_n)$. Further, we have the eigenvector corresponding to λ is of the form

$$\Psi_\lambda = \widehat{b}(-\frac{\sqrt{\mu_n}}{\lambda}, 1, -\alpha e^{-\tau\lambda(1-y)}), \quad \widehat{b} \neq 0.$$

On the other hand, suppose that $\lambda_{k,n} \in \sigma(\Lambda_n)$, $\zeta_{k,n} \in \sigma(\Lambda_n^*)$, and the corresponding eigenvectors are $\Phi_{\lambda_{k,n}}$ and $\Psi_{\zeta_{k,n}}$ respectively. A direct calculation gives

$$\begin{aligned} \lambda_{k,n}\langle \Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}} &= \langle \Lambda_n \Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}} \\ &= \langle \Phi_{\lambda_{k,n}}, \Lambda_n^* \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}} \\ &= \langle \Phi_{\lambda_{k,n}}, \zeta_{k,n} \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}} \\ &= \overline{\zeta}_{k,n} \langle \Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}}, \end{aligned}$$

which implies that $\langle \Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}} \rangle_{\mathbb{X}} = 0$ when $\lambda_{k,n} \neq \overline{\zeta}_{k,n}$.

In addition, let the eigenvectors

$$\Phi_{\lambda_{k,n}} = (\frac{\sqrt{\mu_n}}{\lambda_{k,n}}, 1, e^{-\tau\lambda_{k,n}y}), \quad \text{for} \ \ \lambda_{k,n} \in \sigma(\Lambda_n)$$

and

$$\Psi_{\zeta_{k,n}} = \xi_n^{-1}(\zeta_{k,n})(-\frac{\sqrt{\mu_n}}{\zeta_{k,n}}e^{\tau\zeta_{k,n}}, e^{\tau\zeta_{k,n}}, -\alpha e^{\tau\zeta_{k,n}y}) \quad \text{for} \quad \zeta_{k,n} \in \sigma(\Lambda_n^*).$$

Then when $\lambda_{k,n} = \overline{\zeta_{k,n}}$, we have

$$\begin{aligned}\langle\Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}}\rangle_{\mathbb{X}} =&\overline{\xi_n^{-1}(\zeta_{k,n})}\Big[\frac{\sqrt{\mu_n}}{\lambda_{k,n}}\overline{(-\frac{\sqrt{\mu_n}}{\zeta_{k,n}}e^{\tau\zeta_{k,n}})} + \overline{e^{\tau\zeta_{k,n}}} + \tau\int_0^1 e^{-\tau\lambda_{k,n}y}\overline{(-\alpha e^{\tau\zeta_{k,n}y})}dy\Big]\\ =&\overline{\xi_n^{-1}(\zeta_{k,n})}\Big[-\frac{\mu_n}{(\overline{\zeta_{k,n}})^2}e^{\tau\overline{\zeta_{k,n}}} + e^{\tau\overline{\zeta_{k,n}}} - \alpha\tau\Big]\\ =&1,\end{aligned}$$

thus we have

$$\langle\Phi_{\lambda_{k,n}}, \Psi_{\zeta_{k,n}}\rangle_{\mathbb{X}} = \begin{cases}1,\ \lambda_{k,n} = \overline{\zeta_{k,n}},\\ 0,\ \lambda_{k,n} \neq \overline{\zeta_{k,n}}.\end{cases}$$

The proof is complete. $\square$

Let $\lambda_{k,n} \in \sigma(\Lambda_n)$ be an eigenvalue of Λ_n and $E(\lambda_{k,n};\Lambda_n)$ be the corresponding Riesz spectral projection. Theorem 4.1 implies that for any $F \in \mathbb{X}$,

$$E(\lambda_{k,n};\Lambda_n)F = \langle F, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}}$$

and

$$\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}} = \|\Psi_{\overline{\lambda}_{k,n}}\|_{\mathbb{X}} \cdot \|\Phi_{\lambda_{k,n}}\|_{\mathbb{X}}.$$

Theorem 4.2. *The set of eigenvectors of Λ_n $\{\Phi_{\lambda_{k,n}}, k \in \mathbb{N}\}$ fails to be a Schauder basis for $\mathbb{X}$. Furthermore, we have*

$$\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}} = \|\Psi_{\overline{\lambda}_{k,n}}\|_{\mathbb{X}} \cdot \|\Phi_{\lambda_{k,n}}\|_{\mathbb{X}}\begin{cases}\leq \dfrac{M^{(n)}}{\sqrt{\Re\lambda_{k,n}}}e^{\tau\Re\lambda_{k,n}}, & \textit{for } \Re\lambda_{k,n} > 0,\\ \approx \dfrac{e^{\tau|\Re\lambda_{k,n}|}}{2\tau|\Re\lambda_{k,n}|}, & \textit{for } \Re\lambda_{k,n} < 0,\end{cases} \tag{4.7}$$

where $M^{(n)} > 0$ is a constant.

Proof. An elementary calculation reveals that

$$\begin{aligned}\|\Phi_{\lambda_{k,n}}\|_{\mathbb{X}}^2 &= \frac{\sqrt{\mu_n}}{\lambda_{k,n}}\overline{(\frac{\sqrt{\mu_n}}{\lambda_{k,n}})} + 1 + \tau\int_0^1 e^{-\tau\lambda_{k,n}y}e^{-\tau\overline{\lambda_{k,n}}y}dy\\ &= \frac{\mu_n}{|\lambda_{k,n}|^2} + 1 - \frac{1}{2\Re\lambda_{k,n}}[e^{-\tau2\Re\lambda_{k,n}} - 1]\end{aligned}$$

and

$$\begin{aligned}\|\Psi_{\overline{\lambda}_{k,n}}\|_{\mathbb{X}}^2 =& \frac{1}{-\frac{\mu_n}{|\overline{\lambda_{k,n}}|^2}e^{\tau\overline{\lambda_{k,n}}} + e^{\tau\overline{\lambda_{k,n}}} - \alpha\tau} \cdot \frac{1}{-\frac{\mu_n}{|\overline{\lambda_{k,n}}|^2}e^{\tau\lambda_{k,n}} + e^{\tau\lambda_{k,n}} - \alpha\tau}\\ &\times\Big[-\frac{\sqrt{\mu_n}}{\overline{\lambda_{k,n}}}e^{\tau\overline{\lambda_{k,n}}} \cdot \overline{(-\frac{\sqrt{\mu_n}}{\overline{\lambda_{k,n}}}e^{\tau\overline{\lambda_{k,n}}})} + e^{\tau\overline{\lambda_{k,n}}}\overline{e^{\tau\overline{\lambda_{k,n}}}}\\ &+\tau\int_0^1(-\alpha e^{\tau\overline{\lambda_{k,n}}y})\overline{(-\alpha e^{\tau\overline{\lambda_{k,n}}y})}dy\Big]\\ =& \frac{1}{|\xi_n(\overline{\lambda_{k,n}})|^2}\Big[\frac{\mu_n}{|\lambda_{k,n}|^2}e^{\tau2\Re\lambda_{k,n}} + e^{\tau2\Re\lambda_{k,n}} + \frac{\alpha^2}{2\Re\lambda_{k,n}}[e^{\tau2\Re\lambda_{k,n}} - 1]\Big].\end{aligned}$$

Hence, it is checked that

$$\begin{aligned}
\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}}^2 &= \|\Psi_{\overline{\lambda}_{k,n}}\|_{\mathbb{X}}^2 \cdot \|\Phi_{\lambda_{k,n}}\|_{\mathbb{X}}^2 \\
&= \frac{1}{|\xi_n(\overline{\lambda}_{k,n})|^2}\left[\frac{\mu_n}{|\lambda_{k,n}|^2}e^{\tau 2\Re\lambda_{k,n}} + e^{\tau 2\Re\lambda_{k,n}} + \frac{\alpha^2 e^{\tau 2\Re\lambda_{k,n}}}{2\Re\lambda_{k,n}} - \frac{\alpha^2}{2\Re\lambda_{k,n}}\right] \\
&\quad \times \left[\frac{\mu_n}{|\lambda_{k,n}|^2} + 1 - \frac{e^{-\tau 2\Re\lambda_{k,n}}}{2\Re\lambda_{k,n}} + \frac{1}{2\Re\lambda_{k,n}}\right] \\
&= \frac{1}{|\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}e^{\tau\overline{\lambda}_{k,n}} + e^{\tau\overline{\lambda}_{k,n}} - \alpha\tau|^2}\left[\frac{\mu_n}{|\lambda_{k,n}|^2}e^{\tau 2\Re\lambda_{k,n}} + e^{\tau 2\Re\lambda_{k,n}}\right. \\
&\quad \left. + \frac{\alpha^2 e^{\tau 2\Re\lambda_{k,n}}}{2\Re\lambda_{k,n}} - \frac{\alpha^2}{2\Re\lambda_{k,n}}\right] \cdot \left[\frac{\mu_n}{|\lambda_{k,n}|^2} + 1 - \frac{e^{-\tau 2\Re\lambda_{k,n}}}{2\Re\lambda_{k,n}} + \frac{1}{2\Re\lambda_{k,n}}\right].
\end{aligned}$$

If $\Re\lambda_{k,n} > 0$, a direct calculation gives

$$\begin{aligned}
\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}}^2 &\le \frac{e^{\tau 2\Re\lambda_{k,n}}}{|\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}e^{\tau\overline{\lambda}_{k,n}} + e^{\tau\overline{\lambda}_{k,n}} - \alpha\tau|^2}\left[\frac{\mu_n}{|\lambda_{k,n}|^2} + 1 + \frac{\alpha^2}{2\Re\lambda_{k,n}}\right] \\
&\quad \times \left[\frac{\mu_n}{|\lambda_{k,n}|^2} + 1 + \frac{1}{2\Re\lambda_{k,n}}\right] \\
&= \frac{e^{\tau 2\Re\lambda_{k,n}}}{|\frac{1}{|\overline{\lambda}_{k,n}|^2}e^{\tau\overline{\lambda}_{k,n}} + \frac{e^{\tau\overline{\lambda}_{k,n}}}{\mu_n} - \frac{\alpha\tau}{\mu_n}|^2}\left[\frac{1}{|\lambda_{k,n}|^4} + \frac{2}{|\lambda_{k,n}|^2\mu_n}\right. \\
&\quad \left. + \frac{(\alpha^2+1)}{2\Re\lambda_{k,n}|\lambda_{k,n}|^2\mu_n} + \frac{1}{\mu_n^2} + \frac{\alpha^2+1}{2\Re\lambda_{k,n}\mu_n^2} + \frac{\alpha^2}{4(\Re\lambda_{k,n})^2\mu_n^2}\right].
\end{aligned}$$

Therefore, when $\Re\lambda_{k,n} > 0$, there exists a constant $M^{(n)} > 0$ such that

$$\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}}^2 = \|\Psi_{\overline{\lambda}_{k,n}}\|_{\mathbb{X}}^2 \cdot \|\Phi_{\lambda_{k,n}}\|_{\mathbb{X}}^2 \le \frac{(M^{(n)})^2}{\Re\lambda_{k,n}}e^{\tau 2\Re\lambda_{k,n}}.$$

If $\Re\lambda_{k,n} < 0$, then we have

$$\begin{aligned}
\|E(\lambda_{k,n};\Lambda_n)\|_{\mathbb{X}}^2 =& \frac{e^{\tau 2|\Re\lambda_{k,n}|}}{|\Re\lambda_{k,n}|^2} \cdot \frac{1}{|\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}e^{\tau\overline{\lambda}_{k,n}} + e^{\tau\overline{\lambda}_{k,n}} - \alpha\tau|^2} \\
& \times \left[\frac{\mu_n}{|\lambda_{k,n}|^2}|\Re\lambda_{k,n}|e^{-\tau 2|\Re\lambda_{k,n}|} + |\Re\lambda_{k,n}|e^{-\tau 2|\Re\lambda_{k,n}|}\right. \\
& \left. - \frac{\alpha^2}{2}e^{-\tau 2|\Re\lambda_{k,n}|} + \frac{\alpha^2}{2}\right] \cdot \left[\frac{\mu_n}{|\lambda_{k,n}|^2}|\Re\lambda_{k,n}|e^{-\tau 2|\Re\lambda_{k,n}|}\right. \\
& \left. + |\Re\lambda_{k,n}|e^{-\tau 2|\Re\lambda_{k,n}|} + \frac{1}{2} - \frac{1}{2}e^{-\tau 2|\Re\lambda_{k,n}|}\right] \\
=& \frac{e^{\tau 2|\Re\lambda_{k,n}|}}{|\Re\lambda_{k,n}|^2} \cdot \frac{1}{|-\alpha\tau + o(1)|^2} \cdot \left[\frac{\alpha^2}{2} + o(1)\right] \cdot \left[\frac{1}{2} + o(1)\right] \\
\approx& \frac{e^{\tau 2|\Re\lambda_{k,n}|}}{4\tau^2|\Re\lambda_{k,n}|^2}.
\end{aligned}$$

The expression (4.7) holds.

Moreover, (4.7) implies that $\sup_k \|E(\lambda_{k,n};\Lambda_n)\| = \infty$ as $\lambda_{k,n} \to -\infty$ and consequently the set of eigenvectors of Λ_n $\{\Phi_{\lambda_{k,n}}, \lambda_{k,n} \in \sigma(\Lambda_n)\}$ fails to be a schauder basis in $\mathbb{X}$. □

5. Expansion of semigroup associated with Λ_n

In this section, we shall discuss the expansion of semigroup associated with Λ_n. We have the following result.

Theorem 5.1. *Let Λ_n be defined as before. Then the following statements are true:*

(1) Λ_n generates a C_0 semigroup $e^{\Lambda_n t}$ on $\mathbb{X}$.

(2) For each $\lambda_{k,n} \in \sigma(\Lambda_n)$, let $\Phi_{\lambda_{k,n}}$ and $\Psi_{\overline{\lambda}_{k,n}}$ be defined as Theorem 4.1. Then for any $X \in \mathbb{X}$, when $t > 2\tau$, we have

$$\begin{aligned} e^{\Lambda_n t}X = &\sum_{j=1}^{N_n}\Big(e^{\lambda_{j,n}t}\langle X, \Psi_{\overline{\lambda}_{j,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t}\langle X, \Psi_{\lambda_{j,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{j,n}}\Big) \\ &+ \sum_{k=1}^{\infty}\Big(e^{\lambda_{k,n}t}\langle X, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n}t}\langle X, \Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}\Big), \end{aligned} \tag{5.1}$$

where $N_n > 0$ is the cardinality of the set of spectral of Λ_n with $\Re\lambda_{k,n} > 0$.

(3) When $t > 3\tau$, for any $X \in D(\Lambda_n)$, $\Lambda_n e^{\Lambda_n t}X$ can be expressed as the following infinite series

$$\begin{aligned} \Lambda_n e^{\Lambda_n t}X = &\sum_{j=1}^{N_n}\Big(\lambda_{j,n}e^{\lambda_{j,n}t}\langle X, \Psi_{\overline{\lambda}_{j,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{j,n}} + \overline{\lambda}_{j,n}e^{\overline{\lambda}_{j,n}t}\langle X, \Psi_{\lambda_{j,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{j,n}}\Big) \\ &+ \sum_{k=1}^{\infty}\Big(\lambda_{k,n}e^{\lambda_{k,n}t}\langle X, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}} + \overline{\lambda}_{k,n}e^{\overline{\lambda}_{k,n}t}\langle X, \Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}\Big). \end{aligned} \tag{5.2}$$

Proof. By Theorem 3.1, according to Hille-Yosida-Phillips Theorem (see [20]), we can easily obtain that Λ_n generates a C_0 semigroup $e^{\Lambda_n t}$ on $\mathbb{X}$.

For each $\lambda_{k,n} \in \sigma(\Lambda_n)$, since there exists at most finite eigenvalues with $\Re\lambda_{k,n} > 0$, then for any $X = (a_n, b_n, z_n(y))^T \in \mathbb{X}$, we can let

$$\begin{aligned} X_N = &\sum_{j=1}^{N_n}\Big(\langle X, \Psi_{\overline{\lambda}_{j,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{j,n}} + \langle X, \Psi_{\lambda_{j,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{j,n}}\Big) \\ &+ \sum_{k=1}^{N}\Big(\langle X, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}} + \langle X, \Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}\Big), \end{aligned}$$

where $\Re\lambda_{j,n} > 0, j = 1, 2, \ldots, N_n$. Then

$$\begin{aligned} e^{\Lambda_n t}X_N = &\sum_{j=1}^{N_n}\Big(e^{\lambda_{j,n}t}\langle X, \Psi_{\overline{\lambda}_{j,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t}\langle X, \Psi_{\lambda_{j,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{j,n}}\Big) \\ &+ \sum_{k=1}^{N}\Big(e^{\lambda_{k,n}t}\langle X, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n}t}\langle X, \Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}\Big) \end{aligned}$$

and

$$\Lambda_n e^{\Lambda_n t} X_N = \sum_{j=1}^{N_n} \left(\lambda_{j,n} e^{\lambda_{j,n} t} \langle X, \Psi_{\overline{\lambda}_{j,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{j,n}} + \overline{\lambda}_{j,n} e^{\overline{\lambda}_{j,n} t} \langle X, \Psi_{\lambda_{j,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{j,n}} \right)$$
$$+ \sum_{k=1}^{N} \left(\lambda_{k,n} e^{\lambda_{k,n} t} \langle X, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{k,n}} + \overline{\lambda}_{k,n} e^{\overline{\lambda}_{k,n} t} \langle X, \Psi_{\lambda_{k,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{k,n}} \right).$$

Set

$$M_1^{(n)} = \frac{2\tau |\Re \lambda_{k,n}|}{e^{-\tau \Re \lambda_{k,n}}} \sup_{k \geq 1} \|\Psi_{\overline{\lambda}_{k,n}}\| \cdot \|\Phi_{\lambda_{k,n}}\|.$$

From Theorem 2.3, we have $\Re\lambda_{k,n} \approx \frac{1}{\tau} \ln \frac{\tau|\alpha|}{(2k+\frac{1}{2}\mathrm{sign}(\alpha))\pi}$. Thus when $\Re\lambda_{k,n} < 0$, there exists a constant $M_1 > 0$ such that

$$\sum_{k=1}^{\infty} |e^{\lambda_{k,n} t}| \|\langle X, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{k,n}}\| + \sum_{k=1}^{\infty} |e^{\overline{\lambda}_{k,n} t}| \|\langle X, \Psi_{\lambda_{k,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{k,n}}\|$$
$$\leq \sum_{k=1}^{\infty} e^{\Re \lambda_{k,n} t} \|X\| \cdot \|\Psi_{\overline{\lambda}_{k,n}}\| \cdot \|\Phi_{\lambda_{k,n}}\| + \sum_{k=1}^{\infty} e^{\Re \overline{\lambda}_{k,n} t} \|X\| \cdot \|\Psi_{\lambda_{k,n}}\| \cdot \|\Phi_{\overline{\lambda}_{k,n}}\|$$
$$\leq 2\|X\| M_1^{(n)} \sum_{k=1}^{\infty} e^{\Re \lambda_{k,n} t} \frac{e^{-\tau \Re \lambda_{k,n}}}{2\tau |\Re \lambda_{k,n}|}$$
$$\leq M_1^{(n)} \|X\| \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{t-\tau}{\tau}} \sum_{k=1}^{\infty} \left[\frac{1}{\ln \frac{(2k+\frac{1}{2}\mathrm{sign}(\alpha))\pi}{\tau|\alpha|}} \cdot \left(\frac{1}{(k+\frac{1}{4}\mathrm{sign}(\alpha))} \right)^{\frac{t-\tau}{\tau}} \right]$$
$$\leq M_1^{(n)} M_1 \|X\| \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{t-\tau}{\tau}} \sum_{k=1}^{\infty} \left(\frac{1}{(k+\frac{1}{4}\mathrm{sign}(\alpha))} \right)^{\frac{t-\tau}{\tau}}.$$

Note that the series $\sum\limits_{k=1}^{\infty} \left(\frac{1}{(k+\frac{1}{4}\mathrm{sign}(\alpha))} \right)^{\frac{t-\tau}{\tau}}$ converges as $t > 2\tau$, we have the series

$$\sum_{k=1}^{\infty} e^{\lambda_{k,n} t} \langle X, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n} t} \langle X, \Psi_{\lambda_{k,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{k,n}}$$

converges absolutely.

Therefore, when $t > 2\tau$,

$$\lim_{N\to\infty} e^{\Lambda_n t} X_N = \sum_{j=1}^{N_n} \left(e^{\lambda_{j,n} t} \langle X, \Psi_{\overline{\lambda}_{j,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n} t} \langle X, \Psi_{\lambda_{j,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{j,n}} \right)$$
$$+ \sum_{k=1}^{\infty} \left(e^{\lambda_{k,n} t} \langle X, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n} t} \langle X, \Psi_{\lambda_{k,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{k,n}} \right),$$

i.e., the expression (5.1) holds.

Moreover, when $\lambda \in \sigma(\Lambda_n)$, $\Delta(\lambda) = \lambda^2 + \mu_n + \alpha\lambda e^{-\tau\lambda} = 0$ implies that $|\lambda| \leq |\alpha| e^{-\tau \Re \lambda}$. Then we have

$$\sum_{k=1}^{\infty} |\lambda_{k,n}| |e^{\lambda_{k,n} t}| \|\langle X, \Psi_{\overline{\lambda}_{k,n}} \rangle_{\mathbb{X}} \Phi_{\lambda_{k,n}}\| + \sum_{k=1}^{\infty} |\overline{\lambda}_{k,n}| |e^{\overline{\lambda}_{k,n} t}| \|\langle X, \Psi_{\lambda_{k,n}} \rangle_{\mathbb{X}} \Phi_{\overline{\lambda}_{k,n}}\|$$

$$
\begin{aligned}
&\le 2\|X\|M_1^{(n)}\sum_{k=1}^{\infty}|\lambda_{k,n}|\frac{e^{\Re\lambda_{k,n}(t-\tau)}}{2\tau|\Re\lambda_{k,n}|}\\
&\le 2\|X\|M_1^{(n)}\sum_{k=1}^{\infty}|\alpha|e^{-\tau\Re\lambda_{k,n}}\frac{e^{\Re\lambda_{k,n}(t-\tau)}}{2\tau|\Re\lambda_{k,n}|}\\
&\le 2|\alpha|\|X\|M_1^{(n)}\sum_{k=1}^{\infty}\frac{e^{\Re\lambda_{k,n}(t-2\tau)}}{2\tau|\Re\lambda_{k,n}|}\\
&\le |\alpha|M_1^{(n)}M_1\|X\|\left(\frac{\tau|\alpha|}{2\pi}\right)^{\frac{t-2\tau}{\tau}}\sum_{k=1}^{\infty}\left(\frac{1}{(k+\frac{1}{4}\mathrm{sign}(\alpha))}\right)^{\frac{t-2\tau}{\tau}}.
\end{aligned}
$$

Thus when $t > 3\tau$, the series

$$
\sum_{k=1}^{\infty}\lambda_{k,n}e^{\lambda_{k,n}t}\langle X,\Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}}+\overline{\lambda}_{k,n}e^{\overline{\lambda}_{k,n}t}\langle X,\Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}
$$

converges absolutely and it can easily get that the expression (5.2) holds as $t > 3\tau$. The desired results follow. □

For any $X = (a_n, b_n, z_n(y))^T \in \mathbb{X}$, we set

$$
\begin{aligned}
C_{k,n} &= \langle X,\Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\\
&= \xi_n^{-1}(\lambda_{k,n})\left[-a_n\frac{\sqrt{\mu_n}}{\lambda_{k,n}}e^{\tau\lambda_{k,n}}+b_ne^{\tau\lambda_{k,n}}-\alpha\tau\int_0^1 z_n(y)e^{\tau\lambda_{k,n}y}dy\right]
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
D_{k,n} &= \langle X,\Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\\
&= \xi_n^{-1}(\overline{\lambda}_{k,n})\left[-a_n\frac{\sqrt{\mu_n}}{\overline{\lambda}_{k,n}}e^{\tau\overline{\lambda}_{k,n}}+b_ne^{\tau\overline{\lambda}_{k,n}}-\alpha\tau\int_0^1 z_n(y)e^{\tau\overline{\lambda}_{k,n}y}dy\right].
\end{aligned}
\tag{5.4}
$$

Then (5.1) and (5.2) can be rewritten as

$$
\begin{aligned}
e^{\Lambda_n t}X &= \sum_{j=1}^{N_n}\left(e^{\lambda_{j,n}t}C_{j,n}\Phi_{\lambda_{j,n}}+e^{\overline{\lambda}_{j,n}t}D_{j,n}\Phi_{\overline{\lambda}_{j,n}}\right)\\
&\quad+\sum_{k=1}^{\infty}\left(e^{\lambda_{k,n}t}C_{k,n}\Phi_{\lambda_{k,n}}+e^{\overline{\lambda}_{k,n}t}D_{k,n}\Phi_{\overline{\lambda}_{k,n}}\right),\quad t>2\tau,
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
\Lambda_n e^{\Lambda_n t}X &= \sum_{j=1}^{N_n}\left(\lambda_{j,n}e^{\lambda_{j,n}t}C_{j,n}\Phi_{\lambda_{j,n}}+\overline{\lambda}_{j,n}e^{\overline{\lambda}_{j,n}t}D_{j,n}\Phi_{\overline{\lambda}_{j,n}}\right)\\
&\quad+\sum_{k=1}^{\infty}\left(\lambda_{k,n}e^{\lambda_{k,n}t}C_{k,n}\Phi_{\lambda_{k,n}}+\overline{\lambda}_{k,n}e^{\overline{\lambda}_{k,n}t}D_{k,n}\Phi_{\overline{\lambda}_{k,n}}\right),\quad t>3\tau,
\end{aligned}
\tag{5.6}
$$

respectively.

Remark 5.1. In Theorem 5.1, the conditions $t > 2\tau$ and $t > 3\tau$ are merely the sufficient conditions for the infinite series expansions of $e^{\Lambda_n t}$ and $\Lambda_n e^{\Lambda_n t}$, respectively. When $t < 2\tau$, $e^{\Lambda_n t}$ can not be expressed as the infinite series. Moreover, $t > 2\tau$ and $t > 3\tau$ are just the approximate estimations such that $e^{\Lambda_n t}$ and $\Lambda_n e^{\Lambda_n t}$ can be expressed as the infinite series respectively. The exact estimations for t can be obtained theoretically, but it is difficult to get them.

6. Expansion of solution of system (1.5)

In this section, we shall consider the expansion of solution of system (1.5) according to its eigenvectors. We begin with studying the expansion of semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$.

For any $W = ([w(x), \varphi(x)], [z(x), v(x)], [\psi(x, y), \eta(x, y)])^T \in \mathcal{H}$, by Theorem 2.2 we know that

$$W = \sum_{n=1}^{\infty} \begin{pmatrix} a_n \\ b_n \\ z_n(y) \end{pmatrix} \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix}$$

and

$$\mathcal{A}W = \sum_{n=1}^{\infty} \mathcal{A} \begin{pmatrix} a_n G_2\Phi_n(x) \\ b_n\Phi_n(x) \\ z_n(y)G_1\Phi_n(x) \end{pmatrix} = \sum_{n=1}^{\infty} \Lambda_n \begin{pmatrix} a_n \\ b_n \\ z_n(y) \end{pmatrix} \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix}.$$

The details can also be found in [27]. Next we shall verify that the semigroup $e^{\mathcal{A}t}$ can be expressed as the following convergent series

$$e^{\mathcal{A}t}W = \sum_{n=1}^{\infty} e^{\Lambda_n t} \begin{pmatrix} a_n G_2\Phi_n(x) \\ b_n\Phi_n(x) \\ z_n(y)G_1\Phi_n(x) \end{pmatrix} = \sum_{n=1}^{\infty} e^{\Lambda_n t} \begin{pmatrix} a_n \\ b_n \\ z_n(y) \end{pmatrix} \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix},$$

under certain conditions. i.e.,

$$\begin{aligned} e^{\mathcal{A}t}W = &\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} \left(e^{\lambda_{j,n}t}C_{j,n}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t}D_{j,n}\Phi_{\overline{\lambda}_{j,n}}\right) \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix} \\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(e^{\lambda_{k,n}t}C_{k,n}\Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n}t}D_{k,n}\Phi_{\overline{\lambda}_{k,n}}\right) \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix}. \end{aligned} \tag{6.1}$$

In fact, we have the following result.

Theorem 6.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then for any $W \in \mathcal{H}$, (6.1) is true as $t > \frac{9}{2}\tau$. Moreover, for any $W \in D(\mathcal{A})$, when $t > \frac{11}{2}\tau$, $\mathcal{A}e^{\mathcal{A}t}W$ can be also expressed as the form of an infinite convergent series*

$$\mathcal{A}e^{\mathcal{A}t}W = \sum_{n=1}^{\infty}\sum_{j=1}^{N_n} \left(\lambda_{j,n}e^{\lambda_{j,n}t}C_{j,n}\Phi_{\lambda_{j,n}} + \overline{\lambda}_{j,n}e^{\overline{\lambda}_{j,n}t}D_{j,n}\Phi_{\overline{\lambda}_{j,n}}\right) \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x) \end{pmatrix}$$

$$+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_{k,n}e^{\lambda_{k,n}t}C_{k,n}\Phi_{\lambda_{k,n}}+\overline{\lambda}_{k,n}e^{\overline{\lambda}_{k,n}t}D_{k,n}\Phi_{\overline{\lambda}_{k,n}}\right)\bullet\begin{pmatrix}G_2\Phi_n(x)\\ \Phi_n(x)\\ G_1\Phi_n(x)\end{pmatrix}.$$

Proof. According to (5.3) and (5.4), we give the estimations

$$\begin{aligned}|C_{k,n}|^2\leq&\frac{9\left[|a_n|^2\frac{\mu_n}{|\lambda_{k,n}|^2}e^{\tau2\Re\lambda_{k,n}}+|b_n|^2e^{\tau2\Re\lambda_{k,n}}+\alpha^2\tau^2\int_0^1|z_n(y)^2|dy\int_0^1e^{\tau2\Re\lambda_{k,n}y}dy\right]}{|-\frac{\mu_n}{(\lambda_{k,n})^2}e^{\tau\lambda_{k,n}}+e^{\tau\lambda_{k,n}}-\alpha\tau|^2}\\ =&\frac{9\left[|a_n|^2\frac{\mu_n}{|\lambda_{k,n}|^2}e^{\tau2\Re\lambda_{k,n}}+|b_n|^2e^{\tau2\Re\lambda_{k,n}}+\alpha^2\tau\frac{e^{\tau2\Re\lambda_{k,n}}-1}{2\Re\lambda_{k,n}}\int_0^1|z_n(y)^2|dy\right]}{|-\frac{\mu_n}{(\lambda_{k,n})^2}e^{\tau\lambda_{k,n}}+e^{\tau\lambda_{k,n}}-\alpha\tau|^2},\end{aligned}$$

and

$$\begin{aligned}|D_{k,n}|^2\leq&\frac{9\left[|a_n|^2\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}e^{\tau2\Re\overline{\lambda}_{k,n}}+|b_n|^2e^{\tau2\Re\overline{\lambda}_{k,n}}+\alpha^2\tau^2\int_0^1|z_n(y)^2|dy\int_0^1e^{\tau2\Re\overline{\lambda}_{k,n}y}dy\right]}{|-\frac{\mu_n}{(\overline{\lambda}_{k,n})^2}e^{\tau\overline{\lambda}_{k,n}}+e^{\tau\overline{\lambda}_{k,n}}-\alpha\tau|^2}\\ =&\frac{9\left[|a_n|^2\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}e^{\tau2\Re\overline{\lambda}_{k,n}}+|b_n|^2e^{\tau2\Re\overline{\lambda}_{k,n}}+\alpha^2\tau\frac{e^{\tau2\Re\overline{\lambda}_{k,n}}-1}{2\Re\overline{\lambda}_{k,n}}\int_0^1|z_n(y)^2|dy\right]}{|-\frac{\mu_n}{(\overline{\lambda}_{k,n})^2}e^{\tau\overline{\lambda}_{k,n}}+e^{\tau\overline{\lambda}_{k,n}}-\alpha\tau|^2}.\end{aligned}$$

Therefore, we have

$$|C_{k,n}|^2\leq\begin{cases}\frac{9\left[\frac{\mu_n}{|\lambda_{k,n}|^2}+1+\alpha^2\frac{1}{2\Re\lambda_{k,n}}\right]}{|-\frac{\mu_n}{(\lambda_{k,n})^2}+1-\alpha\tau e^{-\tau\lambda_{k,n}}|^2}\cdot\left[|a_n|^2+|b_n|^2+\tau\displaystyle\int_0^1|z_n(y)|^2dy\right],\ \Re\lambda_{k,n}>0,\\ \frac{9\left[\frac{\mu_n}{|\lambda_{k,n}|^2}+1+\alpha^2\frac{e^{-2\tau\Re\lambda_{k,n}}}{2|\Re\lambda_{k,n}|}\right]}{|-\frac{\mu_n}{(\lambda_{k,n})^2}+1-\alpha\tau e^{-\tau\lambda_{k,n}}|^2}\cdot\left[|a_n|^2+|b_n|^2+\tau\displaystyle\int_0^1|z_n(y)|^2dy\right],\Re\lambda_{k,n}<0\end{cases}$$

and

$$|D_{k,n}|^2\leq\begin{cases}\frac{9\left[\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}+1+\alpha^2\frac{1}{2\Re\overline{\lambda}_{k,n}}\right]}{|-\frac{\mu_n}{(\overline{\lambda}_{k,n})^2}+1-\alpha\tau e^{-\tau\overline{\lambda}_{k,n}}|^2}\cdot\left[|a_n|^2+|b_n|^2+\tau\displaystyle\int_0^1|z_n(y)|^2dy\right],\ \Re\lambda_{k,n}>0,\\ \frac{9\left[\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}+1+\alpha^2\frac{e^{-2\tau\Re\overline{\lambda}_{k,n}}}{2|\Re\overline{\lambda}_{k,n}|}\right]}{|-\frac{\mu_n}{(\overline{\lambda}_{k,n})^2}+1-\alpha\tau e^{-\tau\overline{\lambda}_{k,n}}|^2}\cdot\left[|a_n|^2+|b_n|^2+\tau\displaystyle\int_0^1|z_n(y)|^2dy\right],\Re\lambda_{k,n}<0.\end{cases}$$

Moreover, note that $\Phi_{\lambda_{k,n}}$ is defined by (4.3), then a simple calculation leads to

$$\left\|\Phi_{\lambda_{k,n}}\bullet\begin{pmatrix}G_2\Phi_n(x)\\ \Phi_n(x)\\ G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2=\left\|\begin{pmatrix}\frac{\sqrt{\mu_n}}{\lambda_{k,n}}G_2\Phi_n(x)\\ \Phi_n(x)\\ e^{-\tau\lambda_{k,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2$$

$$
\begin{aligned}
&= \int_0^1 K\left|\frac{1}{\lambda_{k,n}} w_n'(x) - \frac{1}{\lambda_{k,n}}\varphi_n(x)\right|^2 dx + \int_0^1 EI\left|\frac{1}{\lambda_{k,n}}\varphi_n'(x)\right|^2 dx \\
&\quad + \int_0^1 \rho|w_n(x)|^2 dx + \int_0^1 I_\rho|\varphi_n(x)|^2 dx \\
&\quad + \tau\int_0^1\int_0^1 |e^{-\tau\lambda_{k,n}y}\sqrt{\rho}w_n(x)|^2 dxdy \\
&\quad + \tau\int_0^1\int_0^1 |e^{-\tau\lambda_{k,n}y}\sqrt{I_\rho}\varphi_n(x)|^2 dxdy \\
&\le \frac{\mu_n}{|\lambda_{k,n}|^2} + 1 + \frac{1-e^{-\tau 2\Re\lambda_{k,n}}}{2\Re\lambda_{k,n}}.
\end{aligned}
$$

Similarly, we obtain

$$
\left\|\Phi_{\overline{\lambda}_{k,n}} \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2 = \left\|\begin{pmatrix} \frac{\sqrt{\mu_n}}{\overline{\lambda}_{k,n}} G_2\Phi_n(x) \\ \Phi_n(x) \\ e^{-\tau\overline{\lambda}_{k,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2 \le \frac{\mu_n}{|\overline{\lambda}_{k,n}|^2} + 1 + \frac{1-e^{-\tau 2\Re\overline{\lambda}_{k,n}}}{2\Re\overline{\lambda}_{k,n}}.
$$

Denote

$$
\begin{aligned}
\mathcal{S} = &\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}\left(e^{\lambda_{j,n}t}C_{j,n}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t}D_{j,n}\Phi_{\overline{\lambda}_{j,n}}\right) \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix} \\
&+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(e^{\lambda_{k,n}t}C_{k,n}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{k,n}t}D_{k,n}\Phi_{\overline{\lambda}_{j,n}}\right) \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix}.
\end{aligned}
$$

Combining with the above estimations and $\Delta(\lambda_{k,n}) = 0$, we show that there exist constants $M_2, M_3 > 0$ such that

$$
\begin{aligned}
\|\mathcal{S}\|_{\mathcal{H}}^2 \le &16\left\|\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} e^{\lambda_{j,n}t}C_{j,n}\begin{pmatrix} \frac{\sqrt{\mu_n}}{\lambda_{j,n}} G_2\Phi_n(x) \\ \Phi_n(x) \\ e^{-\tau\lambda_{j,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2 \\
&+ 16\left\|\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} e^{\overline{\lambda}_{j,n}t}D_{j,n}\begin{pmatrix} \frac{\sqrt{\mu_n}}{\overline{\lambda}_{j,n}} G_2\Phi_n(x) \\ \Phi_n(x) \\ e^{-\tau\overline{\lambda}_{j,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2 \\
&+ 16\left\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} e^{\lambda_{k,n}t}C_{k,n}\begin{pmatrix} \frac{\sqrt{\mu_n}}{\lambda_{k,n}} G_2\Phi_n(x) \\ \Phi_n(x) \\ e^{-\tau\lambda_{k,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2
\end{aligned}
$$

$$
\begin{aligned}
&+16\left\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{\overline{\lambda}_{k,n}t}D_{k,n}\begin{pmatrix}\frac{\sqrt{\mu_n}}{\overline{\lambda}_{k,n}}G_2\Phi_n(x)\\ \Phi_n(x)\\ e^{-\tau\overline{\lambda}_{k,n}y}G_1\Phi_n(x)\end{pmatrix}\right\|_{\mathcal{H}}^2\\
\le\,&16\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}|C_{j,n}|^2\left[\frac{\mu_n}{|\lambda_{j,n}|^2}+1+\frac{1}{2\Re\lambda_{j,n}}\right]\\
&+16\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\overline{\lambda}_{j,n}t}|D_{j,n}|^2\left[\frac{\mu_n}{|\overline{\lambda}_{j,n}|^2}+1+\frac{1}{2\Re\overline{\lambda}_{j,n}}\right]\\
&+16\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}|C_{k,n}|^2\left[\frac{\mu_n}{|\lambda_{k,n}|^2}+1+\frac{e^{-2\tau\Re\lambda_{k,n}}}{2|\Re\lambda_{k,n}|}\right]\\
&+16\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\overline{\lambda}_{k,n}t}|D_{k,n}|^2\left[\frac{\mu_n}{|\overline{\lambda}_{k,n}|^2}+1+\frac{e^{-2\tau\Re\overline{\lambda}_{k,n}}}{2|\Re\overline{\lambda}_{k,n}|}\right]\\
\le\,&32\times 9\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}\frac{[\frac{\mu_n}{|\lambda_{j,n}|^2}+1+\alpha^2\frac{1}{2\Re\lambda_{j,n}}]}{|-\frac{\mu_n}{(\lambda_{j,n})^2}+1-\alpha\tau e^{-\tau\lambda_{j,n}}|^2}\left[\frac{\mu_n}{|\lambda_{j,n}|^2}+1+\frac{1}{2\Re\lambda_{j,n}}\right]\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+32\times 9\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}\frac{\left[\frac{\mu_n}{|\lambda_{k,n}|^2}+1+\alpha^2\frac{e^{-2\tau\Re\lambda_{k,n}}}{2|\Re\lambda_{k,n}|}\right]}{|-\frac{\mu_n}{(\lambda_{k,n})^2}+1-\alpha\tau e^{-\tau\lambda_{k,n}}|^2}\left[\frac{\mu_n}{|\lambda_{k,n}|^2}+1+\frac{e^{-2\tau\Re\lambda_{k,n}}}{2|\Re\lambda_{k,n}|}\right]\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&72\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}\frac{(\mu_n+|\lambda_{j,n}|^2)^2[4(\Re\lambda_{j,n})^2+2(1+\alpha^2)\Re\lambda_{j,n}+\alpha^2]}{(\Re\lambda_{j,n})^2|(\tau\lambda_{j,n}-1)\mu_n+\lambda_{j,n}^2+\tau\lambda_{j,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+\frac{72}{\alpha^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}\frac{[4\alpha^2|\Re\lambda_{k,n}|^2+(2\alpha^2+2)|\Re\lambda_{k,n}|(\mu_n+|\lambda_{k,n}|^2)+(|\lambda_{k,n}|^2+\mu_n)^2]}{|\Re\lambda_{k,n}|^2|(\tau\lambda_{k,n}-1)\mu_n+(\lambda_{k,n})^2+\tau\lambda_{k,n}^3|^2}\\
&\times(\mu_n+|\lambda_{k,n}|^2)^2\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&288\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}\frac{|\lambda_{j,n}^2+\mu_n|^2[4+\frac{2(1+\alpha^2)}{\Re\lambda_{j,n}}+\frac{\alpha^2}{(\Re\lambda_{j,n})^2}]}{|(\tau\lambda_{j,n}-1)\mu_n+\lambda_{j,n}^2+\tau\lambda_{j,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+\frac{288}{\alpha^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}\frac{|\lambda_{k,n}^2+\mu_n|^2[4\alpha^2+2(\alpha^2+1)\frac{|\lambda_{k,n}|^2+\mu_n}{|\Re\lambda_{k,n}|}+\frac{(|\lambda_{k,n}|^2+\mu_n)^2}{|\Re\lambda_{k,n}|^2}]}{|(\tau\lambda_{k,n}-1)\mu_n+\lambda_{k,n}^2+\tau\lambda_{k,n}^3|^2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&288\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}\frac{|\lambda_{j,n}^2+\mu_n|^2[4+\frac{2(1+\alpha^2)}{\Re\lambda_{j,n}}+\frac{\alpha^2}{(\Re\lambda_{j,n})^2}]}{|(\tau\lambda_{j,n}-1)\mu_n+\lambda_{j,n}^2+\tau\lambda_{j,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+\frac{288}{\alpha^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}\frac{|\lambda_{k,n}^2+\mu_n|^4[\frac{4\alpha^2}{|\lambda_{k,n}^2+\mu_n|^2}+\frac{4(\alpha^2+1)}{|\Re\lambda_{k,n}||\lambda_{k,n}^2+\mu_n|}+\frac{4}{|\Re\lambda_{k,n}|^2}]}{|(\tau\lambda_{k,n}-1)\mu_n+\lambda_{k,n}^2+\tau\lambda_{k,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&288M_2\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}t}\frac{|\lambda_{j,n}^2+\mu_n|^2}{|(\tau\lambda_{j,n}-1)\mu_n+\lambda_{j,n}^2+\tau\lambda_{j,n}^3|^2}\cdot\left[|a_n|^2+|b_n|^2\right.\\
&\left.+\tau\int_0^1|z_n(y)|^2dy\right]+\frac{288}{\alpha^2}M_3\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}t}\frac{|\lambda_{k,n}^2+\mu_n|^4}{|(\tau\lambda_{k,n}-1)\mu_n+\lambda_{k,n}^2+\tau\lambda_{k,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&288\alpha^4M_2\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}(t-2\tau)}\frac{1}{|(\tau\lambda_{j,n}-1)\mu_n+\lambda_{j,n}^2+\tau\lambda_{j,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+288\alpha^6M_3\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}(t-4\tau)}\frac{1}{|(\tau\lambda_{k,n}-1)\mu_n+\lambda_{k,n}^2+\tau\lambda_{k,n}^3|^2}\\
&\times\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right].
\end{aligned}
$$

Since $\Re\lambda_{k,n}\approx\frac{1}{\tau}\ln\frac{\tau|\alpha|}{(2k+\frac{1}{2}\mathrm{sign}(\alpha))\pi}$ for fixed Λ_n, therefore there exist $M_4,M_5>0$ such that

$$
\begin{aligned}
\|\mathcal{S}\|^2\le\,&288\alpha^4M_2M_4\sum_{n=1}^{\infty}\sum_{j=1}^{N_n}e^{2\Re\lambda_{j,n}(t-2\tau)}\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+288\alpha^6M_3M_5\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\Re\lambda_{k,n}(t-4\tau)}\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
\le\,&288\alpha^4M_2M_4\left(\frac{\tau|\alpha|}{2\pi}\right)^{\frac{2(t-2\tau)}{\tau}}\sum_{j=1}^{N_n}\left(\frac{1}{(j+\frac{1}{4}\mathrm{sign}(\alpha))}\right)^{\frac{2(t-2\tau)}{\tau}}\\
&\times\sum_{n=1}^{\infty}\left[|a_n|^2+|b_n|^2+\tau\int_0^1|z_n(y)|^2dy\right]\\
&+288\alpha^6M_3M_5\left(\frac{\tau|\alpha|}{2\pi}\right)^{\frac{2(t-4\tau)}{\tau}}\sum_{k=1}^{\infty}\left(\frac{1}{(k+\frac{1}{4}\mathrm{sign}(\alpha))}\right)^{\frac{2(t-4\tau)}{\tau}}
\end{aligned}
$$

$$\times \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 + \tau \int_0^1 |z_n(y)|^2 dy \right].$$

Moreover, since the series $\sum\limits_{k=1}^{\infty} \left(\frac{1}{k+\frac{1}{4}\mathrm{sign}(\alpha)} \right)^{\frac{2(t-4\tau)}{\tau}}$ converges as $t > \frac{9}{2}\tau$, then there exists $M_6, M_7 > 0$ such that

$$\begin{aligned}
\|\mathcal{S}\|^2 \leq & \, 288\alpha^4 M_2 M_4 M_6 \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-2\tau)}{\tau}} \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 + \tau \int_0^1 |z_n(y)|^2 dy \right] \\
& + 288\alpha^6 M_3 M_5 M_7 \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-4\tau)}{\tau}} \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 + \tau \int_0^1 |z_n(y)|^2 dy \right] \\
\leq & \, M \left[\left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-2\tau)}{\tau}} + \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-4\tau)}{\tau}} \right] \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 + \tau \int_0^1 |z_n(y)|^2 dy \right] \\
= & \, M \left[\left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-2\tau)}{\tau}} + \left(\frac{\tau|\alpha|}{2\pi} \right)^{\frac{2(t-4\tau)}{\tau}} \right] \|W\|^2,
\end{aligned}$$

where $M = \max\{288\alpha^4 M_2 M_4 M_6, 288\alpha^6 M_3 M_5 M_7\}$.

Therefore, for $t > \frac{9}{2}\tau$, the series $\mathcal{S}$ converges abosolutely. Furthermore, for any $W \in \mathcal{H}$, when $t > \frac{9}{2}\tau$, the semigroup $e^{\mathcal{A}t}$ can be expanded as the following infinite convergent series

$$\begin{aligned}
e^{\mathcal{A}t}W = & \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} \left(e^{\lambda_{j,n}t} C_{j,n} \Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t} D_{j,n} \Phi_{\overline{\lambda}_{j,n}} \right) \bullet \begin{pmatrix} G_2 \Phi_n(x) \\ \Phi_n(x) \\ G_1 \Phi_n(x) \end{pmatrix} \\
& + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(e^{\lambda_{k,n}t} C_{k,n} \Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n}t} D_{k,n} \Phi_{\overline{\lambda}_{k,n}} \right) \bullet \begin{pmatrix} G_2 \Phi_n(x) \\ \Phi_n(x) \\ G_1 \Phi_n(x) \end{pmatrix}.
\end{aligned}$$

Similarly, for any $W \in D(\mathcal{A})$, when $t > \frac{11}{2}\tau$, we obtain that $\mathcal{A}e^{\mathcal{A}t}W$ can be expressed as the form of the following infinite convergent series

$$\begin{aligned}
\mathcal{A}e^{\mathcal{A}t}W = & \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} \left(\lambda_{j,n} e^{\lambda_{j,n}t} C_{j,n} \Phi_{\lambda_{j,n}} + \overline{\lambda}_{j,n} e^{\overline{\lambda}_{j,n}t} D_{j,n} \Phi_{\overline{\lambda}_{j,n}} \right) \bullet \begin{pmatrix} G_2 \Phi_n(x) \\ \Phi_n(x) \\ G_1 \Phi_n(x) \end{pmatrix} \\
& + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_{k,n} e^{\lambda_{k,n}t} C_{k,n} \Phi_{\lambda_{k,n}} + \overline{\lambda}_{k,n} e^{\overline{\lambda}_{k,n}t} D_{k,n} \Phi_{\overline{\lambda}_{k,n}} \right) \bullet \begin{pmatrix} G_2 \Phi_n(x) \\ \Phi_n(x) \\ G_1 \Phi_n(x) \end{pmatrix}.
\end{aligned}$$

The desired result follows. □

For system (1.5), we have the following result.

Theorem 6.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then when $t > \frac{11}{2}\tau$, the solution to the system (1.5) can be expanded according to its eigenvectors as follows:*

$$\begin{aligned} W(t) &= e^{\mathcal{A}t}W_0 \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{N_n}\left(e^{\lambda_{j,n}t}\langle X_0, \Psi_{\overline{\lambda}_{j,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{j,n}} + e^{\overline{\lambda}_{j,n}t}\langle X_0, \Psi_{\lambda_{j,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{j,n}}\right)\bullet\begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(e^{\lambda_{k,n}t}\langle X_0, \Psi_{\overline{\lambda}_{k,n}}\rangle_{\mathbb{X}}\Phi_{\lambda_{k,n}} + e^{\overline{\lambda}_{k,n}t}\langle X_0, \Psi_{\lambda_{k,n}}\rangle_{\mathbb{X}}\Phi_{\overline{\lambda}_{k,n}}\right)\bullet\begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix}, \end{aligned} \tag{6.2}$$

where N_n is the cardinality of the set of spectral of Λ_n with $\Re\lambda_{k,n} > 0$; Φ_λ and Ψ_λ are defined in Theorem 5.1; X_0 satisfies $W_0 = \sum\limits_{n=1}^{\infty} X_0 \bullet \begin{pmatrix} G_2\Phi_n(x) \\ \Phi_n(x) \\ G_1\Phi_n(x)\end{pmatrix}$.

7. Conclusion

In this paper, we considered the expansion of the solution of Timoshenko beam with time delay in interior damping. We prove that the set of eigenvectors is complete, but it does not form a Schauder basis for the state space. When $t > \frac{9}{2}\tau$, we obtained that the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ can be expanded in the form of (6.1) according to its eigenvectors. Besides, when $t > \frac{11}{2}\tau$, the solution of system (1.5) can be also expanded in the form of the infinite series. In fact, this method in this paper can be applied to other models described by PDEs.

Appendix A Proof of Lemma 3.1

Proof. Suppose that there is no spectrum of L on these lineas. Since $R(\lambda, L)$ is uniformly bounded on the linea $\arg z = \theta_i$, taking the quotient space $\widetilde{\mathbb{X}} = \mathbb{X}/Sp(L)$ and denoting the induced resolvent of $R(\lambda, L)$ on $\widetilde{\mathbb{X}}$ by $R(\lambda, \widetilde{L})$, then it is easily obtained that $R(\lambda, \widetilde{L})$ is an entire function on $\widetilde{\mathbb{X}}$, and have the same properties with $R(\lambda, L)$. For any $x \in \mathbb{X}$, $[x] \in \widetilde{\mathbb{X}}$, we take any $f \in \widetilde{\mathbb{X}}^*$ and define the function

$$F(z) = \langle f, R(z, \widetilde{L})[x]\rangle, \quad \forall z \in \mathbb{C}.$$

Then $F(z)$ is an entire function and bounded on the linea $\arg z = \theta_i (i = 1, 2, \cdots, 2n+2)$. Denote the angular domain G_i by

$$G_i = \{x \in \mathbb{C} | \theta_{i-1} \le \arg z \le \theta_i\}, \quad i = 1, 2, \cdots, 2n+2,$$

and the flare angle of G_i by $\frac{\pi}{\alpha_i}$. Since $\frac{\pi}{\alpha_i} < \frac{\pi}{n}$ and the resolvent $R(\lambda, L)$ is of order μ, therefore, $F(z)$ is an analytic function in G_i, $F(z) \le M$ on ∂G_i and when $z \in G_i$, $|z| = r \to +\infty$,

$F(z) = O(e^{r^\mu})$. Moreover, note that $\mu \leq n < \alpha_i (i = 1, 2, \cdots, 2n+2)$, according to Phragmen-Lindelöf theorem, it holds that $F(z)$ is uniformly bounded in $\overline{G_i} (i = 1, 2, \cdots, n)$. So $F(z)$ is uniformly bounded in the complex plane. Furthermore, it follows that $F(z)$ is a constant. By the arbitrariness of $f \in \widetilde{\mathbb{X}}^*$, it holds that $R(z, \widetilde{T})$ is a constant. So $R(z, \widetilde{T})[x] = 0$. In addition, from the the arbitrariness of $x \in \mathbb{X}$, it shows that $Sp(L) = \mathbb{X}$.

Since $R(\lambda, L)x$ is meromorphic function of finite exponential type for each $x \in \mathbb{X}$, so $\mu(L) = 1$. From the above discussion, we easily conclude that the result of Lemma 3.1 holds. □

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