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DIFFERENTIABILITY OF EIGENVALUES OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS IN POTENTIALS

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Abstract In this paper, the dependence of eigenvalues of one-dimensional Schrödinger operators with point interactions in potentials is studied. More precisely, the second-order Fréchet derivative of eigenvalues of one-dimensional Schrödinger operators with point interactions in potentials is presented.

Keywords One-dimensional Schrödinger operators, point interactions, eigenvalue, Fréchet derivatives.

MSC(2010) 34L05, 34L15.

1. Introduction

In this paper, we consider the following one-dimensional Schrödinger operators

$$-y''+q(x)y=\lambda y,\quad x\in[a,c)\cup(c,b]=I, \tag{1.1}$$

with the boundary conditions

$$y(a)\cos\alpha-y'(a)\sin\alpha=0, \tag{1.2}$$

$$y(b)\cos\beta-y'(b)\sin\beta=0, \tag{1.3}$$

and the point interactions

$$\begin{pmatrix} y(c+)\\ y'(c+)\end{pmatrix}=A\begin{pmatrix} y(c-)\\ y'(c-)\end{pmatrix}, \tag{1.4}$$

where $\alpha\in[0,\pi)$ and $\beta\in(0,\pi]$ are given parameters, A is 2×2 real matrix and $\det A=\rho>0$, $q\in L(I)$ and has finite limits at the left and right of discontinuous point c. It is well known that the problem (1.1)-(1.4) has a discrete spectrum consisting of an increasing infinite sequence of (real, simple) eigenvalues λ_n such that $\lambda_n\to+\infty$ as $n\to\infty$ and it can be ordered as

$$\lambda_1<\lambda_2<\cdots<\lambda_n\cdots\to\infty.$$

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The eigenvalues λ_n depend on the real-valued function q and the boundary data (α, β), respectively.

In the fields of mathematical physics and the theory of differential equations, one-dimensional Schrödinger operators have always occupied a core position. Such problems find extensive and in-depth applications in numerous areas including quantum mechanics, acoustics, and heat conduction. As a special form of the Sturm-Liouville (S-L) operator, the one-dimensional Schrödinger operator has received significant attention. When studying classical S-L problems, the dependence of eigenvalues on coefficients, boundary conditions, endpoints, and other elements involved in the problems [12,15–20,44] is a key research topic, such problems are mainly focus on how the eigenvalues dependent on these parameters, i.e., the continuity and differentiability of eigenvalues with respect to the parameters. This paper will focus on investigating the differentiability dependence of eigenvalues on potentials.

As an important component of differential equations, slight changes of potentials can have an impact on the eigenvalues and eigenfunctions of the entire system, the Fréchet derivatives provide a powerful tool for studying this dependence. V.A. Vinokurov has shown that eigenvalues and normalized eigenfunctions of S-L problems are analytic in potentials, which are considered as mappings from the Lebesgue space to the real-valued space and the Banach space of continuous functions, respectively (see [29–31]), one of their results in [29] is that the nonlinear mappings

$$\begin{cases} (\mathcal{L}^1(U), \|\cdot\|_1) \to \mathbb{R}, & q \mapsto \lambda_n(q), \\ (\mathcal{L}^1(U), \|\cdot\|_1) \to (C(U,\mathbb{R}), \|\cdot\|_\infty), & q \mapsto E_n(\cdot; q), \end{cases} \tag{1.5}$$

are proved to be analytic, where $U = [a, b]$. Moreover, Fréchet derivative can accurately characterize the rate of change of the functional (1.5) at a certain point, such as the following first-order derivative expression

$$\partial_q \lambda_n(q)(h) := \partial_s \lambda_n(q + sh)|_{s=0} \equiv \int_U (E_n(x; q))^2 h(x) \mathrm{d}x \tag{1.6}$$

for $h \in \mathcal{L}^1(U)$ [25]. In fact, when considering the kernel function in the dual space $(\mathcal{L}^1(U), \|\cdot\|_1)^* = L^\infty(U, \mathbb{R})$, expression (1.6) can also be written as

$$\partial_q \lambda_n(q) = (E_n(\cdot; q))^2. \tag{1.7}$$

At the same time, the result (1.6) for the Dirichlet eigenvalues and potentials was obtained in [25]. In particular, by solving the second-order Fréchet derivatives of the eigenvalues of S-L problems with respect to the potentials, one can gain a deeper understanding of how changes in the potentials affect changes in eigenvalues, which is of great significance for grasping the stability and vibrational properties of the system. In recent years, Guo et al obtained the second-order derivative of eigenvalues of regular S-L problems in potential function q and provided some applications in [13]. Based on the complete continuity of eigenvalues with respect to the potentials, in [32, 40], equation (1.7) can be used to study some typical optimization problems of the eigenvalues of S-L operators, which leads to a certain connection between linear and nonlinear stationary Schrödinger equations.

Generally speaking, for classical Sturm-Liouville problems or higher-order differential equations, the solutions or their quasi-derivatives are required to be absolutely continuous on the interval. However, numerous problems arising in mathematical physics and engineering applications fail to satisfy this property [21–23]. Such problems are referred to as problems with

interior discontinuities. Interior discontinuities mean that, at certain points inside the domain of the differential operator, the solution of the differential equation or its quasi-derivative may suffer a jump discontinuity. In recent years, discontinuous S-L problems have received increasing attention for its wide applications in physics and engineer, such problems are connected with discontinuous material properties, such as heat and mass transfer, vibrating string problems when the string loaded additionally with point masses, the heat transfer problems of the laminated plate of membrane, etc. To characterize the connection between the two sides of such points, some addition conditions need to be imposed on the discontinuous points, such conditions are usually called transmission conditions, interface conditions and point interactions etc, [1,5,8,9,24,34]. Compared with the classic S-L problems, the discontinuous S-L problems exhibit more complex mathematical structures and unique properties. Moreover, the distribution pattern and continuity of the eigenvalues and eigenfunctions have some differences. For example, discontinuity will cause changes in the number and distribution range of eigenvalues, and the smooth asymptotic behavior of the eigenfunctions will also be significantly affected. In recent years, numerous results for S-L problems with interface conditions were obtained, for example, oscillation theory, Hilbert space formulation and asymptotic estimates of eigenvalues [5–7], basis properties [21, 24], discrete spectrum and completeness of root functions [28], the dependence of eigenvalues [38, 42–44], inverse spectral problems [10, 35, 36], finite spectrum [2–4, 11], regular approximation [39], for more details, see the reference cited therein.

In the study of S-L problems, the dependence of eigenvalues on potentials is an important issue. In recent years, the first-order Fréchet derivative of eigenvalues of differential operators in potentials is well understood, for classical S-L problems, see [17]; for eigenparameter dependent S-L problems, see [38, 41]; for fourth-order differential operators, see [27], etc. In particular, Zhang and Wang obtained the first-order Fréchet derivative of eigenvalues with respect to the potentials for S-L problems with point interactions [42]. However, the results on the second-order Fréchet derivative of eigenvalues with respect to the potentials is relatively scarce. Existing studies mainly focus on the first-order derivative properties of eigenvalues of differential operators, and there are still many gaps and unsolved problems in the research on the second-order derivative in the discontinuous case. To fill this gap, this paper will investigate the second-order Fréchet derivative of eigenvalues with respect to the potentials for one-dimensional Schrödinger operators with point interactions.

This paper is organized as follows. In Section 2, we will present the second-order Fréchet derivative formula of the eigenvalues in potentials, namely Theorem 2.1. The formula for the second-order derivative of the eigenvalues in potentials when restrictions are imposed at the endpoints, that is Theorem 2.2. The proofs of Theorem 2.1 and Theorem 2.2 will be given in Section 3. In Section 4, we summarize the potential applications of the results obtained in this paper.

2. Preliminaries and main results

In what follows, we fix (α,β) and consider $\lambda_n = \lambda_n(q)$ as nonlinear functionals of potentials $q \in \mathcal{L}^1(I)$. Firstly, we introduce a new Hilbert space $H = L^2[a,c) \oplus L^2(c,b]$, whose inner product is defined as

$$\langle f(x), g(x)\rangle = \rho \int_a^c f_1(x) g_1(x) \mathrm{d}x + \int_c^b f_2(x) g_2(x) \mathrm{d}x,$$

where

$$f(x)=\begin{cases} f_1(x), & x\in[a,c),\\ f_2(x), & x\in(c,b], \end{cases} \quad g(x)=\begin{cases} g_1(x), & x\in[a,c),\\ g_2(x), & x\in(c,b]. \end{cases}$$

The eigenfunction $E_n(x)=E_n(x;q)$ associated with λ_n can be normalized as

$$\|E_n\|_2=\left(\rho\int_a^c |E_n(x)|^2\mathrm{d}x+\int_c^b |E_n(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}=1. \tag{2.1}$$

Moreover, the first-order Fréchet derivative of the eigenvalue $\lambda_n(q)$ is given by

$$\begin{aligned}\partial_q\lambda_n(q)(h)&:=\partial_s\lambda_n(q+sh)|_{s=0}\\ &\equiv\rho\int_a^c |E_n(x;q)|^2h(x)\mathrm{d}x+\int_c^b |E_n(x;q)|^2h(x)\mathrm{d}x\end{aligned} \tag{2.2}$$

for $h\in\mathcal{L}^1(I)$, moreover, the problem (1.1)-(1.4) can be transferred to a self-adjoint operator in the Hilbert space H ([42]). Similarly, for $q,h\in\mathcal{L}^1(I)$, we define the second-order Fréchet derivative of $\lambda_n(q)$ by

$$\partial_{qq}\lambda_n(q)(h):=\partial_{ss}\lambda_n(q+sh)|_{s=0}\in\mathbb{R}. \tag{2.3}$$

For simplicity, denote

$$F:=\partial_q\lambda_n(q)(h)=\rho\int_a^c |E_n(x;q)|^2h(x)\mathrm{d}x+\int_c^b |E_n(x;q)|^2h(x)\mathrm{d}x\in\mathbb{R}. \tag{2.4}$$

The main results of this paper are as follows.

Theorem 2.1. *For any $q,h\in\mathcal{L}^1(I)$, let F be defined as in (2.4) and $U_n(x)=U_n(x;q,h)$ be the unique solution of the following inhomogeneous linear ordinary differential equation (ODE)*

$$-y''+(q(x)-\lambda_n(q))y=-E_n(x)(h(x)-F),\quad x\in I, \tag{2.5}$$

satisfying the initial conditions

$$y(a)=y'(a)=0, \tag{2.6}$$

and the point interactions

$$\begin{pmatrix} y(c+)\\ y'(c+)\end{pmatrix}=A\begin{pmatrix} y(c-)\\ y'(c-)\end{pmatrix}. \tag{2.7}$$

Then the second-order Fréchet derivative of the eigenvalues with respect to the potential function q of the problem (1.1)-(1.4) is given by

$$\begin{aligned}\partial_{qq}\lambda_n(q)(h)=&2\rho\int_a^c E_n(x)(h(x)-F)U_n(x)\mathrm{d}x\\ &+2\int_c^b E_n(x)(h(x)-F)U_n(x)\mathrm{d}x.\end{aligned} \tag{2.8}$$

Here $U_n(x;q,h)$ is given in formula (3.22).

Next, we still use the solution $U_n(x)$, but with some restriction on the boundary condition (1.3) at the right endpoint $x=b$.

Theorem 2.2. *For any $q, h \in \mathcal{L}^1(I)$, let $U_n(x)$ be defined as in Theorem 2.1. Then*

(i) $U_n(x)$ satisfies the boundary conditions (1.2)-(1.3).

(ii) Assume that (1.3) *takes the following boundary condition*

$$y(b) = 0 \quad or \quad y'(b) = 0. \tag{2.9}$$

Then, in this case, the second-order Fréchet derivative of $\lambda_n(q)$ with respect to q can be expressed as

$$\begin{aligned} \partial_{qq}\lambda_n(q)(h) = &\rho U_n(c-)U_n^{'}(c-) - U_n(c+)U_n^{'}(c+) \\ &- 2\rho\int_a^c (U_n'^2 + (q(x) - \lambda_n(q))U_n^2)\mathrm{d}x \\ &- 2\int_c^b (U_n'^2 + (q(x) - \lambda_n(q))U_n^2)\mathrm{d}x. \end{aligned} \tag{2.10}$$

3. The proofs of main results

Let $q \in \mathcal{L}^1(I)$. For $h \in \mathcal{L}^1(I)$ and $s \in \mathbb{R}$, let us denote

$$Q(x,s) := q(x) + sh(x) - \lambda_n(q + sh) \in \mathcal{L}^1(I). \tag{3.1}$$

We consider the linear ODE

$$-y'' + Q(x,s)y = 0, \quad x \in I. \tag{3.2}$$

Here $' = \frac{d}{dx}$ and $'' = \frac{d^2}{dx^2}$; they can also be written as ∂_x and ∂_{xx} respectively. Let $y = \varphi(x,s)$ be the solution of equation Eq. (3.2) satisfying the initial condition

$$(y(a), y'(a)) = (\sin\alpha, \cos\alpha). \tag{3.3}$$

That is, for each s, one has

$$-\partial_{xx}\varphi(x,s) + Q(x,s)\varphi(x,s) = 0, \tag{3.4}$$

$$(\varphi(a,s), \partial_x\varphi(a,s)) = (\sin\alpha, \cos\alpha). \tag{3.5}$$

Moreover, for each s, from the definition of the eigenvalues, $\varphi(\cdot, s)$ also satisfies the boundary condition (1.3)

$$\varphi(b,s)\cos\beta - \partial_x\varphi(b,s)\sin\beta = 0, \tag{3.6}$$

and the point interactions (1.4)

$$\begin{pmatrix} \varphi(c+,s) \\ \partial_x\varphi(c+,s) \end{pmatrix} = A \begin{pmatrix} \varphi(c-,s) \\ \partial_x\varphi(c-,s) \end{pmatrix}. \tag{3.7}$$

Because $\lambda_n(q + sh)$ is analytic in s, as a mapping from $\mathbb{R}$ to $\mathcal{L}^1(I)$, $Q(\cdot, s)$ is analytic in s. In addition, the solution of the initial value problem of a linear ODE is analytic in coefficient potential, so $\varphi(x,s)$ and $\partial_x\varphi(x,s)$ are analytic in s. Therefore, by differentiating Eq. (3.4) with respect to s twice, we get the following inhomogeneous linear ODEs

$$-\partial_{xx}\partial_s\varphi(x,s) + Q(x,s)\partial_s\varphi(x,s) = -\partial_s Q(x,s)\varphi(x,s), \tag{3.8}$$

$$-\partial_{xx}\partial_{ss}\varphi(x,s)+Q(x,s)\partial_{ss}\varphi(x,s)=-\partial_{ss}Q(x,s)\varphi(x,s)-2\partial_sQ(x,s)\partial_s\varphi(x,s). \tag{3.9}$$

For simplicity, we denote these equations at $s=0$ by

$$\begin{cases} F:=\partial_s\lambda_n(q+sh)|_{s=0}=\partial_q\lambda_n(q)(h)\in\mathbb{R},\\ S:=\partial_{ss}\lambda_n(q+sh)|_{s=0}=\partial_{qq}\lambda_n(q)(h)\in\mathbb{R},\\ Q(x):=Q(x,0)=q(x)-\lambda_n(q),\\ \varphi(x):=\varphi(x,0),\\ \varphi_k(x):=\partial_{s^k}\varphi(x,s)|_{s=0},\quad k=1,2. \end{cases} \tag{3.10}$$

By (3.1), one has $\partial_sQ(x,s)|_{s=0}=h(x)-F$ and $\partial_{ss}Q(x,s)|_{s=0}=-S$. Therefore, Eq. (3.8) and Eq. (3.9) mean that $y=\varphi_k(x)(k=1,2)$ is a solution of the inhomogeneous linear ODE

$$-y''+Q(x)y=f_k(x),\quad x\in I,\quad k=1,2, \tag{3.11}$$

where

$$f_1(x)=(F-h(x))\varphi(x),\quad f_2(x)=S\varphi(x)-2(h(x)-F)\varphi_1(x). \tag{3.12}$$

Moreover, by taking the derivative of s twice for the initial condition (3.5), boundary condition (3.6) and point interactions (3.7) respectively, we get $y=\varphi_k(x)$, $k=1,2$, satisfying

$$(y(a),y'(a))=(0,0), \tag{3.13}$$

$$y(b)\cos\beta-y'(b)\sin\beta=0, \tag{3.14}$$

and

$$\begin{pmatrix} y(c+)\\ y'(c+)\end{pmatrix}=A\begin{pmatrix} y(c-)\\ y'(c-)\end{pmatrix}. \tag{3.15}$$

Lemma 3.1. *Let $q\in\mathcal{L}^1(I)$ be given. For any $h\in\mathcal{L}^1(I)$, one has*

(i) $\varphi_k(x)=\varphi_k(x;q,h)(k=1,2)$, are uniquely determined, and they are the solutions of the initial value problem (3.11)-(3.15).

(ii) $y=\varphi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ satisfying the boundary conditions (1.2)-(1.3) *and transmission conditions* (1.4).

Proof. (i) Obviously it holds true.

(ii) Since $\varphi(x)$ satisfies the initial value condition (3.3), then $\varphi(x)$ must satisfy the boundary condition (1.2). Moreover, due to the definition of the eigenvalues $\lambda_n(q)$, we get that $\varphi(x)$ satisfies (1.3) as well. For $k=1,2$, it is obvious from (3.13) and (3.14) that $y=\varphi_k(x)$ must satisfy the boundary conditions (1.2)-(1.3). □

Lemma 3.1 (ii) shows that $\varphi(x)$ is an eigenfunction associated with $\lambda_n(q)$, which is normalized as

$$E_n(x)=E_n(x;q)\equiv\frac{\varphi(x)}{\|\varphi(x)\|_2}, \tag{3.16}$$

where

$$\|\varphi\|_2^2=\rho\int_a^c|\varphi(x)|^2\mathrm{d}x+\int_c^b|\varphi(x)|^2\mathrm{d}x.$$

To derive the formulas for the Fréchet derivatives F and S, we can use the Fredholm principle.

Lemma 3.2. *Consider the following inhomogeneous linear ODE*

$$-y'' + Q(x)y = f(x), \quad x \in I, \tag{3.17}$$

where $f(x) \in \mathcal{L}^1(I)$. *If Eq.* (3.17) *admits a solution* $y(x)$ *satisfying the boundary conditions* (1.2)-(1.3) *and point interactions* (1.4), *it is necessary that*

$$\rho \int_a^c \varphi(x) f(x) \mathrm{d}x + \int_c^b \varphi(x) f(x) \mathrm{d}x = 0, \quad x \in I. \tag{3.18}$$

Proof. Let $\varphi(x)$ be as in (3.10). From Lemma 3.1 (ii), $y = \varphi(x)$ is a solution of the following boundary value problem

$$-y'' + Q(x)y = -y'' + (q(x) - \lambda_n(q))y = 0, \tag{3.19}$$

with conditions (1.2)-(1.4), where Eq. (3.19) is the corresponding homogeneous linear ODE of Eq. (3.17). Therefore, the solvability of problem (3.17), (1.2)-(1.4) is equivalent to condition (3.18). □

In the following we will give the proof of Theorem 2.1.

The proof of Theorem 2.1. Applying Lemma 3.2 to Eq. (3.11), we get

$$\rho \int_a^c \varphi(x) f_k(x) \mathrm{d}x + \int_c^b \varphi(x) f_k(x) \mathrm{d}x = 0, \quad k = 1, 2. \tag{3.20}$$

For $k = 1$, it follows from (3.12) that Eq. (3.20) becomes

$$\rho \int_a^c (F - h(x))\varphi^2(x) \mathrm{d}x + \int_c^b (F - h(x))\varphi^2(x) \mathrm{d}x = 0,$$

i.e.,

$$\begin{aligned} F =& \frac{\rho \int_a^c \varphi^2(x) h(x) \mathrm{d}x + \int_c^b \varphi^2(x) h(x) \mathrm{d}x}{\rho \int_a^c \varphi^2(x) \mathrm{d}x + \int_c^b \varphi^2(x) \mathrm{d}x} \\ =& \frac{\rho \int_a^c \varphi^2(x) h(x) \mathrm{d}x + \int_c^b \varphi^2(x) h(x) \mathrm{d}x}{(\|\varphi\|_2)^2} \\ =& \rho \int_a^c \left(\frac{\varphi(x)}{\|\varphi\|_2}\right)^2 h(x) \mathrm{d}x + \int_c^b \left(\frac{\varphi(x)}{\|\varphi\|_2}\right)^2 h(x) \mathrm{d}x \\ =& \rho \int_a^c |E_n(x)|^2 h(x) \mathrm{d}x + \int_c^b |E_n(x)|^2 h(x) \mathrm{d}x \end{aligned} \tag{3.21}$$

(see (3.16)). This gives another deduction of formula (2.2).

Let F be defined as in (3.21). For $k = 2$, it follows from (3.12) that we have Eq. (3.20)

$$\rho \int_a^c (S\varphi^2(x) - 2\varphi(x)(h(x) - F)\varphi_1(x)) \mathrm{d}x + \int_c^b (S\varphi^2(x) - 2\varphi(x)(h(x) - F)\varphi_1(x)) \mathrm{d}x = 0.$$

By (3.16), we obtain

$$\begin{aligned} S =& \frac{\rho \int_a^c 2\varphi(x)(h(x) - F)\varphi_1(x) \mathrm{d}x + \int_c^b 2\varphi(x)(h(x) - F)\varphi_1(x) \mathrm{d}x}{\rho \int_a^c \varphi^2(x) \mathrm{d}x + \int_c^b \varphi^2(x) \mathrm{d}x} \\ =& 2\rho \int_a^c \left(\frac{\varphi(x)}{\|\varphi\|_2}\right)(h(x) - F)\frac{\varphi_1(x)}{\|\varphi\|_2} \mathrm{d}x + 2 \int_c^b \left(\frac{\varphi(x)}{\|\varphi\|_2}\right)(h(x) - F)\frac{\varphi_1(x)}{\|\varphi\|_2} \mathrm{d}x \\ =& 2\rho \int_a^c E_n(x)(h(x) - F)\frac{\varphi_1(x)}{\|\varphi\|_2} \mathrm{d}x + 2 \int_c^b E_n(x)(h(x) - F)\frac{\varphi_1(x)}{\|\varphi\|_2} \mathrm{d}x. \end{aligned}$$

Thus we get (2.5), where

$$U_n(x) = U_n(x;q,h) := \frac{\varphi_1(x)}{\|\varphi\|_2}. \tag{3.22}$$

Dividing Eq. (3.11) by the factor $\|\varphi\|_2$ and using the conditions (3.12) and (3.13), one sees that $U_n(x)$ is the solution of the initial value problem (2.5)-(2.7). The proof of Theorem 2.1 is finished. □

The following is the proof of Theorem 2.2.

The proof of Theorem 2.2. (i) It is obvious from Lemma 3.1 (ii) that the solution $U_n(x)$ of (2.5) satisfies the boundary conditions (1.2)-(1.3) and point interactions (1.4).

(ii) Since $U_n(x)$ is the solution of (2.5), i.e.,

$$-U_n'' + Q(x)U_n = -E_n(x)(h(x) - F). \tag{3.23}$$

Since we now consider the boundary conditions (1.2) and (2.9), we know that $U_n(x) = \varphi_1(x)/\|\varphi\|_2$ satisfies

$$U_n(a) = U_n'(a) = 0, \quad U_n(b)U_n'(b) = 0. \tag{3.24}$$

Taking the inner product of $U_n(x)$ with equation (3.23), we obtain

$$\begin{aligned}
&\rho\int_a^c U_n(x)E_n(x)(h(x)-F)\mathrm{d}x + \int_c^b U_n(x)E_n(x)(h(x)-F)\mathrm{d}x\\
=&-\rho\int_a^c U_n(x)(-U_n''(x)+Q(x)U_n(x))\mathrm{d}x - \int_c^b U_n(x)(-U_n''(x)+Q(x)U_n(x))\mathrm{d}x\\
=&\,\rho\int_a^c U_n''(x)U_n(x)\mathrm{d}x - \rho\int_a^c Q(x)U_n^2(x)\mathrm{d}x + \int_c^b U_n''(x)U_n(x)\mathrm{d}x - \int_c^b Q(x)U_n^2(x)\mathrm{d}x\\
=&\,\rho\Big[U_n(x)U_n'(x)|_a^c - \int_a^c U_n'(x)\mathrm{d}U_n(x) - \int_a^c Q(x)U_n^2(x)\mathrm{d}x\Big]\\
&+U_n(x)U_n'(x)|_c^b - \int_c^b U_n'(x)\mathrm{d}U_n(x) - \int_c^b Q(x)U_n^2(x)\mathrm{d}x\\
=&\,\rho U_n(c-)U_n^{'}(c-) - U_n(c+)U_n^{'}(c+)\\
&-\rho\int_a^c (U_n'^2 + Q(x)U_n^2)\mathrm{d}x - \int_c^b (U_n'^2 + Q(x)U_n^2)\mathrm{d}x.
\end{aligned}$$

Hence formula (2.10) can be deduced from (2.8) and the above equality. □

4. Conclusion

In this paper, we mainly obtained second-order Fréchet derivative for one-dimensional Schrödinger operators with point interactions in potentials. The main motivation of this paper stems, on the one hand, from the wide applications of Schrödinger operators with point interactions in various fields, and on the other hand, from their potential applications in mathematics, physics and related disciplines. We will explain their potential applications from both mathematical and physical perspectives, respectively. Mathematically, the second-order Fréchet derivative of the eigenvalues of a Schrödinger operator with respect to the potential function characterizes the curvature and quadratic form structure of eigenvalues as functionals of the

potential in the potential function space. It can be used to determine the convexity and concavity of spectral functionals, analyze the second-order asymptotic behavior of eigenvalues under small perturbations, solve inverse problems such as reconstructing potentials from spectral data, and describe the nonlinear coupling effects of potential perturbations at different positions via integral kernels, providing a fundamental analytical tool for spectral theory and perturbation analysis of differential operators. Physically, this second-order derivative corresponds to the second-order perturbation correction of energy levels with respect to the potential field in the stationary Schrödinger equation in quantum mechanics, reflecting the contribution of virtual intermediate-state transitions to energy levels. It also characterizes the stability of bound states such as the ground state, and is applied in systems including quantum dots, impurity potentials, and lattice defects to analyze the sensitivity of energy levels to potential deformations. It can be further extended to vibrations, acoustics, and elastic structures to describe the second-order response of natural frequencies to distributions of material stiffness, density, and other parameters, offering physical support for structural optimization, potential design, and inverse problem inversion. In the future work, we will work on general Sturm-Liouvile problems with point interactions, and explore the second-order Fréchet derivatives of eigenvalues with respect to the coefficient function, potential function and weight function. We further investigate qualitative properties including monotonicity and concavity of eigenvalues with respect to these functions.

Conflict of interest. The authors have no conflicts to disclose.

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