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ON CONTROLLABILITY, STABILITY AND SOLVABILITY OF HILFER FRACTIONAL q-DYNAMICAL DELAY EQUATIONS ON TIME SCALES

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Abstract This article investigates a class of Hilfer-type fractional q-difference equations with time delay on time scales. Using Krasnoselskii's fixed point theorem and Banach contraction mapping principle, we obtain the existence and uniqueness of solutions for this equation. Meanwhile, we discuss the Hyers-Ulam stability of the solution for the equation by constructing a suitable perturbed equation, and obtain sufficient conditions of the controllability for the equation through the utilization of a non-zero bounded linear operator and a function that characterizes the control action. A numerical example is presented to validate the main results.

Keywords q-fractional dynamical equations, time scales, existence, Hyers-Ulam stability, controllability.

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1. Introduction

Quantum calculus, also known as q-calculus, is an important branch of mathematical analysis that establishes a class of calculus without the concept of limits. Jackson [17] first introduced the concepts of q-derivative and q-integral in the early 20th. Subsequent advancements in information technology have substantially expanded q-calculus applications, particularly in automatic control systems, power grid optimization and quantum-inspired mathematical physics models [21,31,44]. In the 1960s, Al-Salam [38] and Agarwal [2] made groundbreaking contributions by extending these concepts to non-integer order operators. Fractional quantum calculus, as a practical extension of integer-order quantum calculus, has stronger modeling capabilities, it can describe more complex dynamic behaviors, and has a wider range of application, especially in the applications of quantum mechanics and stochastic analysis [10,29].

In 1988, Hilger [13] first proposed the concept of time scales. His core aim was to unify differential equations in classical continuous analysis and difference equations in discrete analysis, and to break the limitations of single system modeling. Compared with traditional mathematical analysis tools, time scale theory has obvious advantages. It can realize unified analysis of continuous, discrete and hybrid dynamic systems through one set of theoretical framework avoiding repeated research. It also has strong modeling flexibility, which can adapt to various time scales such as non-uniform and piecewise ones, accurately fit the dynamic laws of actual systems, and has good theoretical expansibility, with its research conclusions being more universal. With these core advantages, time scale theory is now widely used in control science, biomathematics, quan-

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tum difference, economy and finance, electrical engineering, ecological environment and other disciplinary fields. It provides new research ideas and efficient tools for modeling, analyzing and solving various complex dynamic systems, and also lays a solid theoretical foundation for the research on time-scale fractional quantum difference equations in this paper. Over the past few decades, the solvability of fractional dynamical equations on time scales has been studied and some meaningful results have been produced, see [14, 16, 22, 24, 32, 33, 39, 41] and the references.

In recent years, a number of scholars have focused on the cross-integration of time-scale theory and quantum calculus. By introducing the "quantum parameter" into time scales, a unified description of quantum systems with continuous evolution and discrete mutations is achieved, providing a universal mathematical tool for the study of hybrid quantum systems and cross-scale quantum processes. In [7], the definitions and properties of fractional q-calculus on a specific time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n,\ n \text{ a nonnegative integer}\}$ were established, which provided some theoretical basis studies of subsequent problems. So far, the solvability of fractional q-dynamical equations (q-DEs) on time scales has been studied and some excellent results have been obtained, see [1, 4, 11, 25–28, 45] and its references. However, the solvability of fractional q-DEs on time scales has attracted extensive attention.

In 2021, Bakhtawar et al. [35] used Leray-Schauder and Banach fixed point theorems to prove the existence and uniqueness results, controllability and stability for a new class of mixed integral fractional delay dynamic systems with impulsive effects on time scales

$$\begin{cases} {}^{c,\mathbb{T}}D^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad\qquad + G\Big(\varsigma,\omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f} \mathcal{F}_1(\varsigma,s,\omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f} \mathcal{F}_2(\varsigma,s,\omega(s))\Delta s\Big) + \mathcal{H}\zeta(\varsigma), \\ \qquad\qquad \varsigma \in \mathbb{T}' = \mathbb{T} \setminus \{\varsigma_1, \varsigma_2, \dots, \varsigma_m\}, \sigma = (0,1), \\ \omega(\varsigma_k^+) - \omega(\varsigma_k^-) = \Xi_k\big(\omega(\varsigma_k^-)\big) + \Phi_k\big(\varsigma_k^-, \omega(\varsigma_k^-)\big), \quad k = 1, \dots, m, \\ \omega(\varsigma_0) = \omega_0, \end{cases}$$

and

$$\begin{cases} {}^{c,\mathbb{T}}D^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad\qquad + \mathcal{G}\Big(\varsigma,\omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f} \mathcal{F}_1(\varsigma,s,\omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f} \mathcal{F}_2(\varsigma,s,\omega(s))\Delta s\Big) + \mathcal{H}\zeta(\varsigma), \\ \qquad\qquad \varsigma \in (s_i, s_{i+1}] \cap \mathbb{T},\ i = 1, \dots, m,\ \sigma = (0,1), \\ \omega(\varsigma) = \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_{s_i}^{\varsigma} (\varsigma - s)^{\sigma-1} \hbar_i(s,\omega(s))\Delta s, \quad \varsigma \in (s_i, s_i] \cap \mathbb{T},\ i = 1, \dots, m, \\ \omega(\varsigma_0) = \omega_0, \end{cases}$$

where ${}^{c,\mathbb{T}}D^{\sigma}$ represents the classical Caputo derivative of fractional order σ on time scales $\mathbb{T}$. The regressive square matrix $A(\varsigma)$ is piecewise continuous, and $\mathcal{H} : \mathbb{T} \to \mathbb{T}$ is a bounded linear operator. By assuming $\mathbb{R}$ as the real number, $\zeta \in L^2(I, \mathbb{R})$ is a control map, $\mathbb{T}^0 := [\varsigma_0, \varsigma_f]_{\mathbb{T}}$, the pre-fixed numbers are $\varsigma_0 = s_0 < \varsigma_1 < s_1 < \varsigma_2 < \cdots < \varsigma_m < s_m < \varsigma_{m+1} = \varsigma_f$, and $\mathcal{F} : \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{F}_1, \mathcal{F}_2 : \mathbb{T}^0 \times \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{T}^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\hbar_i : (\varsigma_i, s_i] \cap \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, m$, $\mathbb{F} : (s_i, \varsigma_{i+1}] \cap \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, \dots, m$, $\mathcal{G} : (s_i, \varsigma_{i+1}] \cap \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, m$, $\Xi_k : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_k : \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings. In addition, we define the right limit and the left limit of $\omega(\varsigma)$ at ς_k as $\omega(\varsigma_k^+) = \lim_{\tau \to 0^+} \omega(\varsigma_k + \tau)$ and $\omega(\varsigma_k^-) = \lim_{\tau \to 0^-} \omega(\varsigma_k + \tau)$.

In 2022, Mouataz et al. [30] applied Krasnoselskii's fixed point theorem and Banach contraction mapping principle to establish the existence, uniqueness and stability of solutions for nonlinear neutral q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha(x(t)-g(t,x(\tau t)))=f(t,x(t),x(\tau t)), \quad t\in\mathbb{T}_a, \\ x(t)=\Psi(t), \quad t\in\mathbb{I}_\tau, \end{cases}$$

where $f:\mathbb{T}_a\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $g:\mathbb{T}_a\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo fractional q-difference equation of order $\alpha\in(0,1)$, $\mathbb{T}_a=\{aq^{-i}, i=0,1,2,\ldots\}$, $\tau=q^d\in\mathbb{T}_q$, $d\in N_0=\{0,1,2,\ldots\}$ and $\mathbb{I}_\tau=\{\tau a, q^{-1}\tau a, q^{-2}\tau a,\ldots,a\}$.

In 2023, Mahdi et al. [25] stated and proved the delta q-fractional Gronwall-type inequality on time scales

$$\begin{cases} {}^CD^\alpha_{\Delta_{q,\varsigma_0}}y(\varsigma)=g(\varsigma,y(\varsigma)), \\ y(\varsigma_0)=y_0, \end{cases}$$

where $0<\alpha\leq 1$, $\varsigma\geq\varsigma_0$, $\varsigma_0\in\mathbb{T}_q$, $g:\mathbb{T}_q\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous in y, and $g(\varsigma,0)=0$.

In 2024, Ahmed et al. [3] considered the existence and uniqueness of solutions for boundary value fractional-order differential equations of the q-Hilfer and q-Caputo types with separated boundary conditions

$$\begin{cases} {}^HD_q^{\alpha,\beta}({}^CD_q^\delta z)(t)=f(t,z(t)), \quad t\in[0,T], \\ z(0)+\lambda_1\,{}^CD_q^{\gamma+\delta-1}z(0)=0, \quad x(T)+\lambda_2\,{}^CD_q^{\gamma+\delta-1}z(T)=0, \\ 0<\alpha,\delta,q<1, 0\leq\beta\leq 1, \lambda_1,\lambda_2\in\mathbb{R}, T>0, \end{cases}$$

where ${}^CD_q^\delta(\cdot)$ and ${}^HD_q^{\alpha,\beta}(\cdot)$ respectively denote the Caputo and Hilfer fractional q-derivatives of orders δ,α and type β such that $\gamma=\alpha+\beta(1-\alpha)$ with $\gamma+\delta>1$, and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

As is well known, in 1941, the mathematician Hyers [15] first proved that approximate solutions to linear functional equations could converge to exact solutions in Banach spaces. In 1978, Hyers and Rassias [36] extended this result to a more general case, known as Hyers-Ulam-Rassias stability. They mainly studied whether the solutions remained close to the unperturbed solutions when the systems and equations were subjected to perturbations, which was of significant value in theoretical and practical applications. Currently, research results on the Hyers-Ulam stability of fractional q-difference equations can be found in [6, 8, 43]. Similarly, mathematical control theory can be viewed as an interdisciplinary branch of engineering and mathematics. As a fundamental concept in mathematical control theory, it provides tools for determining whether a differential equation system can reach a desired state through control inputs. The need for control systems has already appeared in many phenomena, such as in the design of electrical circuits and communication systems. Moreover, control theory [12, 19, 20, 42] and Hyers-Ulam stability [5, 18, 37] for delay differential equations have been investigated in some research papers.

To the best of our knowledge, there are few studies on the solvability of fractional q-difference equations on time scales, especially the Hyers-Ulam stability and controllability of such problems. Therefore, in this paper, we investigate the solvability, Hyers-Ulam stability and controllability of the following boundary value problem for fractional q-DEs on time scales

$$\begin{cases} {}^H\Delta_q^{\alpha,\beta}({}^c\Delta_q^\delta(x(t)-g(t,x(\tau t))))=f(t,x(t),x(\tau t)), \quad t,\tau\in\mathbb{T}_q, \\ x(0)+\lambda_1\,{}^c\Delta_q^{\gamma+\delta-1}x(0)=0, \quad x(1)+\lambda_2\,{}^c\Delta_q^{\gamma+\delta-1}x(1)=0, \\ {}^c\Delta_q^{\gamma+\delta-1}g(0,x(0))=0, \quad {}^c\Delta_q^{\gamma+\delta-1}g(1,x(\tau))=0, \end{cases} \tag{1.1}$$

where ${}^{H}\Delta_q^{\alpha,\beta}(\cdot)$ and ${}^{c}\Delta_q^{\delta}(\cdot)$ are the Hilfer and Caputo fractional q-derivative on time scales of order α, δ. Respectively, $0<\alpha,\delta<1$, the type $0\le\beta\le1$ such that $\gamma=\alpha+\beta(1-\alpha)$ with $\gamma+\delta>1$, $f:\mathbb{T}_q\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{T}_q\times\mathbb{R}\to\mathbb{R}$ are continuous functions, and $\lambda_1,\lambda_2\in\mathbb{R}$. Moreover, we will add an non-zero bounded linear operator B and a control function u into problem (1.1), which results in

$$\begin{cases} {}^{H}\Delta_q^{\alpha,\beta}({}^{c}\Delta_q^{\delta}(x(t)-g(t,x(\tau t)))) = f(t,x(t),x(\tau t))+Bu(t), \quad t,\tau\in\mathbb{T}_q, \\ x(0)+\lambda_1\,{}^{c}\Delta_q^{\gamma+\delta-1}x(0)=0, \quad x(1)+\lambda_2\,{}^{c}\Delta_q^{\gamma+\delta-1}x(1)=0, \\ {}^{c}\Delta_q^{\gamma+\delta-1}g(0,x(0))=0, \quad {}^{c}\Delta_q^{\gamma+\delta-1}g(1,x(\tau))=0, \end{cases} \tag{1.2}$$

where $B:\mathbb{R}\to\mathbb{R}$ is assumed to be a bounded linear operator and the control function u belongs to $L^2(\mathbb{T}_q,\mathbb{R})$.

Here, the highlights of this paper are as follows:

Firstly, in [24], for the continuous time scale $\mathbb{T}=\mathbb{R}$, the Δ-derivative is defined by the forward jump operator $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$. At this time, $\sigma(t)=t$, the Δ-derivative reduces to the classical continuous derivative. For the discrete time scale $\mathbb{T}=\mathbb{Z}$, the forward jump operator satisfies $\sigma(t)=t+1$. Then the Δ-derivative simplifies to the forward difference, and the corresponding Δ-integral also has a unified expression. This property of time scale calculus has been widely used in control theory, signal processing, mathematical biology and quantum calculus. It provides a powerful tool for describing both continuous and discrete phenomena. Specifically, when $\mathbb{T}=\mathbb{R}$, the Δ-integral becomes the Riemann integral. It gives the measure between the function curve and the coordinate axes. When $\mathbb{T}=\mathbb{Z}$, the Δ-integral reduces to the summation operation, which corresponds to the sum of discrete values of the function. This paper studies the fractional q-dynamical equation defined on the time scale $\mathbb{T}_q$. By introducing the quantum parameter q, we get the forward jump operator $\sigma_q(t)=qt$. The corresponding Δ_q-derivative is the difference of the function at t and $\sigma_q(t)$. The Δ_q-integral is the summation on the scaling grid, which represents the total area of piecewise rectangles. As $q\to1^+$, the Δ_q-derivative and Δ_q-integral reduce to the classical derivative and integral, respectively. This unifies continuous and discrete theories, and strengthens the universality of the theory.

Secondly, we study a class of delayed Hilfer-type fractional quantum dynamic equations on time scales. We introduce the Hilfer Δ_q-derivative of order α and type β on time scales. When $\beta=0$, this derivative becomes the Riemann-Liouville Δ_q-derivative on time scales; when $\beta=1$, it becomes the Caputo Δ_q-derivative on time scales. This generalization expands the application scope of such fractional quantum dynamic equations, and has a wider application than reference [25]. Compared with the traditional Hilfer-type fractional quantum difference equations in reference [3], we define the Hilfer q-derivative on the time scale $\mathbb{T}_q$. The Δ_q-derivative on time scales can be used on any non-empty closed subset in the real number field $\mathbb{R}$. It can unify continuous, discrete and hybrid systems, and its modeling flexibility and application are much better than the traditional q-derivative.

Thirdly, based on the time scale framework, this paper studies the Hyers-Ulam stability of the solution for fractional quantum dynamical equations with time delay. Compared with traditional quantum difference equations [3], we combine time scale theory, fractional operators and time delay effects to establish a model that is more suitable for practical complex dynamical systems. When the quantum parameter $q\to1^+$, the model reduces to the corresponding stability results of fractional differential equations with time delay on the continuous time scale; when the time delay $\tau\to0$, it degenerates into the classical stability results of traditional quantum difference equations. Therefore, the unified analysis of continuous, discrete and quantum dynamical systems is realized. In addition, by Hyers-Ulam stability theory, the boundedness of errors under disturbances and approximate calculations is proved, which makes up for the lack of robustness analysis in traditional models and greatly improves the universality and engineering application value of the theory.

Finally, we investigate the controllability of fractional q-dynamical equations with time delay on time scales. Compared with the traditional fractional calculus models on time scales in reference [35], the quantum time scale $\mathbb{T}_q$ adopted in this paper has a scaled non-uniform step structure, which is more consistent with the real dynamics of quantum systems and discrete scaling processes. After introducing the quantum parameter q, the model possesses a degeneration property: When $q \to 1^+$, it can be reduced to the fractional controllability problem on the classical continuous time scale. In addition, the model incorporates time delay simultaneously, which can describe more general lag behaviors, breaks through the limitations of traditional fractional models on time scales such as single structure, no time delay and single scale, and further expands the application scope and practical application value of controllability theory.

The content of this paper is organized as follows. In Sect.2, we recall some necessary preliminaries. By employing Krasnoselskii's fixed point theorem and Banach contraction mapping principle, the existence and uniqueness theorem of solutions to fractional q-DE (1.1) is established in Sect.3. In Sect.4, we derive the Hyers-Ulam stability of the solution for fractional q-DE (1.1). Further, in Sect.5, we study the controllability of solution for fractional q-DE (1.2). An example is given to verify the validity of results in Sect.6.

2. Preliminaries

In this section, we introduce some basic notations, definitions, lemmas and a proposition about fractional q-difference equations on time scales.

Let $\mathbb{T}_q$ be a time scale for $0 < q < 1$

$$\mathbb{T}_q = \{q^r : r \in \mathbb{Z}_+\} \cup \{0\},$$

where $\mathbb{Z}_+$ denotes the set of positive integers. A time scale is, in general, a closed subset of the real numbers.

For the function $\varpi : \mathbb{T}_q \to \mathbb{R}$, the delta q-derivative of $\varpi(\theta)$ is defined as

$$\Delta_q \varpi(\theta) = \frac{\varpi(q\theta) - \varpi(\theta)}{(q-1)\theta}, \quad \theta \in \mathbb{T}_q \setminus \{0\}.$$

In addition to a higher order of delta q-derivatives is also presented by

$$\Delta_q^0 \varpi(\theta) = \varpi(\theta), \quad \Delta_q^m \varpi(\theta) = \Delta_q(\Delta_q^{m-1} \varpi(\theta)) \quad (m = 1, 2, 3, \cdots).$$

The delta q-integral of $\varpi(\theta)$ is defined as

$$(I_{q,0}\varpi)(\theta) = \int_0^\theta \varpi(s)\Delta_q s = (1-q)\sum_{l=0}^{\infty} \theta q^l \varpi(\theta q^l),$$

and

$$(I_{q,a}\varpi)(\theta) = \int_a^\theta \varpi(s)\Delta_q s = \int_0^\theta \varpi(s)\Delta_q s - \int_0^a \varpi(s)\Delta_q s,$$

where $a, \theta \in \mathbb{T}_q$.

Definition 2.1. [25] The delta q-factorial function is as follows. If $l \in \mathbb{N}$, then

$$(\theta - s)_q^{(0)} = 1, \quad (\theta - s)_q^{(l)} = \prod_{j=0}^{l-1} (\theta - q^j s).$$

If α is a non positive integer, then

$$(\theta - s)_q^{(\alpha)} = \theta^\alpha \prod_{j=0}^{\infty} \frac{1 - \frac{\theta}{s} q^j}{1 - \frac{\theta}{s} q^{j+\alpha}}.$$

Lemma 2.1. [25] *Suppose $\kappa, \omega \in \mathbb{R}$, we have*

1. $(\theta - s)_q^{(\kappa+\omega)} = (\theta - s)_q^{(\kappa)} (\theta - q^\gamma s)_q^{(\omega)}$,
2. $(b\theta - bs)_q^{(\kappa)} = b^\kappa (\theta - s)_q^{(\kappa)}$,
3. $\Delta_q (\theta - s)_q^{(\kappa)} = \dfrac{1 - q^\kappa}{1 - q} (\theta - s)_q^{(\kappa - 1)}$,
4. $\Delta_q (\theta - s)_q^{(\kappa)} = -\dfrac{1 - q^\kappa}{1 - q} (\theta - qs)_q^{(\kappa - 1)}$.

Definition 2.2. [25] For $|q| < 1$, the q-Gamma function is defined as

$$\Gamma_q(\alpha) = \frac{(1-q)^{1-\alpha}}{(1-q)_q^{(1-\alpha)}}, \quad \alpha \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}_0\}.$$

Notice that the q-Gamma function satisfies

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha) = \frac{1-q^\alpha}{1-q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 0.$$

Lemma 2.2. [28] *If $\varpi(\theta)$ is defined and finite, then for $0 < \alpha < 1$*

$$\Delta_q^\alpha \varpi(\theta) = \Delta_q I_{\Delta_q}^{\alpha - 1} \varpi(\theta),$$

where $\theta \in [q^a, q^b] \subset \mathbb{T}_q$ with $a, b \in \mathbb{N} \cup \{0\}$, $b < a$.

Definition 2.3. [26] Let $\theta, \theta_0 \in \mathbb{T}_q$ and $0 < q < 1$. Then for the function $\varpi : \mathbb{T}_q \to \mathbb{R}$, the Riemann-Liouville delta q-fractional derivatives is given by

$$\Delta_{q,\theta_0}^\alpha \varpi(\theta) = \Delta_q^m I_{\Delta_q, \theta_0}^{m-\alpha} \varpi(\theta),$$

for α real value such that $\alpha \geq 0$ and $m = [\alpha] + 1$.

For $0 < \theta_0 < \theta$, $\alpha \in \mathbb{R}^+$. Then

$$\Delta_{q,\theta_0}^\alpha I_{\Delta_q,\theta_0}^\alpha \varpi(\theta) = \varpi(\theta).$$

Definition 2.4. [26] Let $\theta, \theta_0 \in \mathbb{T}_q$ and $0 < q < 1$. Then for the function $\varpi : \mathbb{T}_q \to \mathbb{R}$, the fractional delta q-integral of order $\alpha > 0$ is given by

$$\begin{aligned}
I_{\Delta_q,\theta_0}^0 \varpi(\theta) &= \varpi(\theta), \\
I_{\Delta_q,\theta_0}^\alpha \varpi(\theta) &= \frac{1}{\Gamma_q(\alpha)} \int_{\theta_0}^{\theta} (\theta - qs)_q^{(\alpha - 1)} \varpi(s) \Delta_q s,
\end{aligned}$$

and for $\alpha_1, \alpha_2 > 0$,

$$I_{\Delta_q,\theta_0}^{\alpha_1} I_{\Delta_q,\theta_0}^{\alpha_2} \varpi(\theta) = I_{\Delta_q,\theta_0}^{\alpha_1 + \alpha_2} \varpi(\theta).$$

For example, if we let $\varpi(\theta) = (\theta-\theta_0)_q^{(\mu)}$, then we obtain the q-analogue of integral formula for a power function

$$I^{\alpha}_{\Delta_q,\theta_0}(\theta-\theta_0)_q^{(\mu)} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\alpha+1)}(\theta-\theta_0)_q^{(\mu+\alpha)}, \quad 0<\theta_0<\theta,$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1,\infty)$.

Definition 2.5. [28] For $\varpi : \mathbb{T}_q \to \mathbb{R}$ at $\theta, \theta_0 \in \mathbb{T}_q$, the Caputo delta q-fractional derivatives of order $\alpha \geq 0$ is

$$\begin{aligned} {}^c\Delta^{\alpha}_{q,\theta_0}\varpi(\theta) &= I^{(m-\alpha)}_{\Delta_q,\theta_0}\Delta_q^m\varpi(\theta) \\ &= \frac{1}{\Gamma_q(m-\alpha)}\int_{\theta_0}^{\theta}(\theta-qs)^{(m-\alpha-1)}\Delta_q^m\varpi(s)\Delta_q s, \end{aligned}$$

where $m = [\alpha]+1$.

Lemma 2.3. [28] *Let $\alpha>0$ and ϖ is defined in suitable domains. Then*

$$I^{\alpha}_{\Delta_q}\,{}^c\Delta_q^{\alpha}\varpi(\theta) = \varpi(\theta) - \sum_{l=0}^{\eta-1}\frac{(\theta-\theta_0)_q^{(l)}}{\Gamma_q(l+1)}\Delta_q^l\varpi(\theta_0),$$

and if $0<\alpha\leq 1$,

$$I^{\alpha}_{\Delta_q}\,{}^c\Delta_q^{\alpha}\varpi(\theta) = \varpi(\theta) - \varpi(\theta_0).$$

Definition 2.6. The Hilfer fractional delta q-derivative of order $0<\alpha<1$ and $0\leq\beta\leq 1$ for function $\varpi(\theta)$ is defined by

$${}^H\Delta_q^{\alpha,\beta}\varpi(\theta) = \Big(I^{\beta(1-\alpha)}_{\Delta_q}\Delta_q\big(I^{(1-\beta)(1-\alpha)}_{\Delta_q}\varpi\big)\Big)(\theta).$$

Remark 2.1. The Hilfer fractional delta q-derivative can be viewed as a generalization of the Riemann-Liouville and Caputo delta q-derivative:

(i) The operator ${}^H\Delta_q^{\alpha,\beta}$ also can be rewritten as

$${}^H\Delta_q^{\alpha,\beta}\varpi(\theta) = \Big(I^{(1-\alpha)\beta}_{\Delta_q}\Delta_q(I^{(1-\alpha)(1-\beta)}_{\Delta_q})\Big)\varpi(\theta) = I^{(1-\alpha)\beta}_{\Delta_q}\Delta_q^{\gamma}\varpi(\theta), \quad \gamma = \alpha+\beta-\alpha\beta.$$

(ii) Let $\beta=0$, the Riemann-Liouville fractional delta q-derivative can be presented as $\Delta_q^{\alpha} = {}^H\Delta_q^{\alpha,0}$.

(iii) Let $\beta=1$, the Caputo fractional delta q-derivative can be presented as ${}^c\Delta_q^{\alpha} = {}^H\Delta_q^{\alpha,1}$.

The following definition and lemmas are key tools in proving our results.

Definition 2.7. [30] A set D of sequences in l_∞ is uniformly Cauchy if for every $\varepsilon>0$, there exists an integer $\mathbb{N}^*$, such that $|x(t)-x(s)|<\varepsilon$ whenever $t,s\in\mathbb{N}^*$ for any $x=\{x(n)\}$ in D.

Lemma 2.4. [30] (Arzela-Ascoli theorem) *A bounded, uniformly Cauchy subset D of $l_\infty(\mathbb{T}_q)$ (all bounded real-valued sequences with domain $\mathbb{T}_q$) is relatively compact.*

Lemma 2.5. [30] (Banach fixed point theorem) *Let $(X,\|\cdot\|)$ be a Banach space and $A : X\to X$ be a contraction mapping, i.e.*

$$\|Ax-Ay\| \leq \lambda\|x-y\|,$$

for all $x,y\in X$, where $0<\lambda<1$. Then, A has a unique fixed point in X.

Lemma 2.6. [30] (Krasnoselskii's fixed point theorem) *Let D be a nonempty, closed, convex, and bounded subset of a Banach space $(X, \|\cdot\|)$. Suppose that $A_1 : D \to X$ and $A_2 : D \to X$ are two operators such that*

(i) A_1 is a contraction,

(ii) A_2 is continuous and $A_2(D)$ resides in a compact subset of X,

(iii) for any $x, y \in D$, $A_1x + A_2y \in D$.

Then, the operator equation $A_1x + A_2x = x$ has a solution $x \in D$.

Lemma 2.7. [34] (Sadovskii's fixed point theorem) *Let μ be a measures of noncompactness in a real Banach space B. Assume that Ω is a nonempty, closed, bounded and convex subset of B, a mapping $T : \Omega \to \Omega$ is continuous, and for any nonempty subset $X \subset \Omega$ such that $\mu(X) > 0$*

$$\mu(T(X)) < \mu(X).$$

Then T has at least one fixed point in Ω.

Next, we present an useful proposition regarding the measure of noncompactness, which will be used in the subsequent proofs.

Proposition 2.1. [23] *The Kuratowski's measure μ of non-compactness has the following properties:*

(i)Monotone: If the bounded subsets Q_1, Q_2 of X, $Q_1 \subseteq Q_2$ implies $\mu(Q_1) \le \mu(Q_2)$.

(ii)Nonsingular: If $\mu(\{a\} \cup \Omega) = \mu(\Omega)$ for every $a \in X$ and every nonempty subset $\Omega \subseteq X$.

(iii)Regular: If $\mu(\Omega) = 0$ is equivalent to the relative compactness of Ω.

(iv)Algebraic semi-additivity: $\mu(Q_1 + Q_2) = \mu(Q_1) + \mu(Q_2)$, where $Q_1 + Q_2 = \{x + y : x \in Q_1, y \in Q_2\}$.

(v)Semi-additivity: $\mu(Q_1 \cup Q_2) = max\{\mu(Q_1), \mu(Q_2)\}$.

$(vi)$$\mu(\lambda Q) = \|\lambda\|\mu(Q)$.

$(vii)$$\mu(Q) = \mu(\overline{Q})$.

Finally, present the following assumptions to illustrate our main results.

(H1) There exist positive continuous functions $L_f(\cdot), L_g(\cdot), J_i(\cdot) > 0, (i = 1, 2)$, such that

$$\begin{aligned}
&|f(t, x, z)| \le J_1(t), \quad |g(t, x)| \le J_2(t),\\
&|f(t, x, z) - f(t, y, w)| \le L_f(t)(\|x - y\| + \|z - w\|),\\
&|g(t, x) - g(t, y)| \le L_g(t)(\|x - y\|).
\end{aligned}$$

(H2) We define the linear operator $Wu : L^2(\mathbb{T}_q, \mathbb{R}) \to \mathbb{R}$ as

$$Wu = \frac{1}{\Gamma_q(\alpha+\delta)} \int_0^1 (1 - qs)^{(\alpha+\delta-1)} Bu(s)\Delta_q s \tag{2.1}$$

possesses a bounded invertible operator W^{-1}, this operator admits values in $L^2(\mathbb{T}_q, \mathbb{R})\backslash ker(W)$, and there exists a positive constant provided that $\|W^{-1}\| \le Mw$. Also, operator $B : \mathbb{R} \to \mathbb{R}$ is a non-zero bounded linear operator and have a positive constant M_B such that $\|B\| \le M_B$. Furthermore, $H(s, t) = \frac{(s-qt)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}$, and its norm is bounded by $\|H(s, t)\| \le \frac{1}{\Gamma_q(\alpha+\delta)}$.

Lemma 2.8. *If $B : \mathbb{R} \to \mathbb{R}$ is a non-zero bounded linear operator and $\int_0^1 (1-qs)^{(\alpha+\delta-1)}\Delta_q s > 0$, then the operator W has a bounded inverse operator W^{-1}.*

Proof. **Step 1.** We show that operator W is bounded. By the boundedness of B and Hölder's inequality, we have

$$|Wu| \leq C\|u\|_{L^2(\mathbb{T}_q,\mathbb{R})}$$

holds for some constant $C > 0$. Thus W is a bounded linear operator.

Step 2. For any $y \in \mathbb{R}$, define

$$u(s) = \frac{y\Gamma_q(\alpha+\delta)}{B\int_0^1 (1-qs)^{(\alpha+\delta-1)}\Delta_q s}.$$

Substituting $u(s)$ into W, a direct calculation gives

$$Wu = \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1 (1-qs)^{(\alpha+\delta-1)}Bu(s)\,\Delta_q s = y,$$

hence W is surjective. Since $L^2(\mathbb{T}_q,\mathbb{R})$ and $\mathbb{R}$ are Banach spaces, and W is a bounded linear surjective operator, by the Banach inverse operator theorem, W admits a bounded inverse operator W^{-1}. □

3. Results of existence and uniqueness

Now, we investigate the existence and uniqueness of solutions for the fractional q-DE (1.1). Let $B(\mathbb{T}_q,\mathbb{R})$ be a Banach space endowed by the norm

$$\|u\| = \sup_{u\in\mathbb{T}_q}|u(t)| \tag{3.1}$$

and

$$\Omega = \{u \in B(\mathbb{T}_q,\mathbb{R}), \|u\| \leq \sigma\} \tag{3.2}$$

is a non-empty bounded closed and convex subset of $B(\mathbb{T}_q,\mathbb{R})$, where

$$\begin{aligned}
\sigma =& \max\{(\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)\frac{J_2^*}{1-\gamma_1}+\frac{J_1^*}{\Gamma_q(\alpha+\delta+1)(1-\gamma_1)}+\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|(1-\gamma_1)}|M|,\\
&\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|N|+\frac{M_BMu}{\Gamma_q(\alpha+\delta+1)}\},
\end{aligned}$$

$$\gamma_1 = (\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)L_g^*+\frac{2L_f^*}{\Gamma_q(\alpha+\delta+1)} < 1, \tag{3.3}$$

$$Mu = Mw(|x_1|+(1-\gamma_1)\sigma+\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|N|), \tag{3.4}$$

$$N = \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1 (1-qs)^{(\alpha+\delta-1)}Bu(s)\Delta_q s. \tag{3.5}$$

Lemma 3.1. *Suppose that $h : \mathbb{T}_q \to \mathbb{R}$ is a continuous function, and*

$$Q = (\lambda_2-\lambda_1)\Gamma_q(\gamma+\delta)+1 \neq 0, \tag{3.6}$$

then, the nonlinear problem

$$\begin{cases} {}^{H}\Delta_q^{\alpha,\beta}({}^{c}\Delta_q^{\delta}(x(t)-g(t,x(\tau t)))) = h(t), \quad t,\tau\in\mathbb{T}_q, \\ x(0)+\lambda_1^{c}\Delta_q^{\gamma+\delta-1}x(0)=0, \quad x(1)+\lambda_2^{c}\Delta_q^{\gamma+\delta-1}x(1)=0, \\ {}^{c}\Delta_q^{\gamma+\delta-1}g(0,x(0))=0, \quad {}^{c}\Delta_q^{\gamma+\delta-1}g(1,x(\tau))=0, \end{cases} \tag{3.7}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = &\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0))+g(t,x(\tau t)) \\ &+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s, \end{aligned}$$

where

$$M=\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}h(s)\Delta_q+\frac{\lambda_2}{\Gamma_q(\alpha-\gamma+1)}\int_0^1(1-qs)^{(\alpha-\gamma)}h(s)\Delta_q s.$$

Proof. Since

$${}^{H}\Delta_q^{\alpha,\beta}(\cdot)=I_{\Delta_q}^{\beta(1-\alpha)}\Delta_q I_{\Delta_q}^{(1-\beta)(1-\alpha)}(\cdot)=I_{\Delta_q}^{\beta(1-\alpha)}\Delta_q I_{\Delta_q}^{1-\gamma}(\cdot), \tag{3.8}$$

then, the first equation of (3.7) can be simplified as

$$I_{\Delta_q}^{\beta(1-\alpha)}\Delta_q I_{\Delta_q}^{(1-\beta)(1-\alpha)}({}^{c}\Delta_q^{\delta}(x(t)-g(t,x(\tau t))))=h(t). \tag{3.9}$$

Taking $I_{\Delta_q}^{\alpha}$ to both sides of Eq. (3.9), we yield

$${}^{c}\Delta_q^{\delta}(x(t)-g(t,x(\tau t))) = \frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-\gamma)}h(s)\Delta_q s+\frac{C_1}{\Gamma_q(\gamma)}t^{\gamma-1}. \tag{3.10}$$

Also, taking $I_{\Delta_q}^{\delta}$ to both sides of Eq. (3.10), we get

$$x(t) = \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s+g(t,x(\tau t))+\frac{C_1}{\Gamma_q(\gamma+\delta)}t^{\gamma+\delta-1}+C_2, \tag{3.11}$$

where $C_1, C_2\in\mathbb{R}$ are arbitrary constants.

Applying the operator ${}^{c}\Delta_q^{\gamma+\delta-1}$ to both sides of Eq. (3.11), we have

$${}^{c}\Delta_q^{\gamma+\delta-1}x(t) = \frac{1}{\Gamma_q(\alpha-\gamma+1)}\int_0^t(t-qs)^{(\alpha-\gamma)}h(s)\Delta_q s+C_1+{}^{c}\Delta_q^{\gamma+\delta-1}g(t,x(\tau t)). \tag{3.12}$$

Notice that $x(0)+\lambda_1^{c}\Delta_q^{\gamma+\delta-1}x(0)=0$ and ${}^{c}\Delta_q^{\gamma+\delta-1}g(0,x(0))=0$, then

$$x(0)+\lambda_1^{c}\Delta_q^{\gamma+\delta-1}x(0)=C_2+C_1\lambda_1+g(0,x(0))=0, \tag{3.13}$$

from Eqs. (3.11) and (3.12), we obtain

$$x(1)=\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s+g(1,x(\tau))+C_2+\frac{C_1}{\Gamma_q(\gamma+\delta)}, \tag{3.14}$$

$$ {}^{c}\Delta_q^{\gamma+\delta-1}x(1) = \frac{1}{\Gamma_q(\alpha-\gamma+1)}\int_0^1 (1-qs)^{(\alpha-\gamma)}h(s)\Delta_q s + C_1 + {}^{c}\Delta_q^{\gamma+\delta-1}g(1,x(\tau)). \tag{3.15} $$

From $x(1)+\lambda_2 {}^{c}\Delta_q^{\gamma+\delta-1}x(1)=0$ and ${}^{c}\Delta_q^{\gamma+\delta-1}g(1,x(\tau))=0$, we have

$$ \begin{aligned} C_1 &= -\frac{\Gamma_q(\gamma+\delta)}{Q}[g(1,x(\tau))-g(0,x(0))+M], \\ C_2 &= \frac{\lambda_1\Gamma_q(\gamma+\delta)}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0)). \end{aligned} $$

Hence,

$$ \begin{aligned} x(t) = {}& \frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0))+g(t,x(\tau t)) \\ &+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s, \end{aligned} \tag{3.16} $$

the proof is completed. □

A mapping A on Ω is given by

$$ \begin{aligned} Ax(t) = {}& \frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0)) \\ &+g(t,x(\tau t))+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s. \end{aligned} \tag{3.17} $$

For convenience, we denote the operator $A=A_1+A_2$, that is

$$ \begin{aligned} A_1x(t) &= \frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0))+g(t,x(\tau t)), \\ A_2x(t) &= \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s. \end{aligned} $$

Theorem 3.1. *Let Ω be defined by (3.2). Assume that ($H1$) holds, then there exists at least one solution of fractional q-DE (1.1).*

Proof. Consider the bounded closed interval $\mathbb{T}_q \subseteq \mathbb{R}$. By (H1), $L_f(\cdot), L_g(\cdot), J_i(\cdot)>0, (i=1,2)$ are positive continuous functions on $\mathbb{T}_q$. According to the Extreme Value Theorem, $J_1(t)$, $J_2(t)$, $L_f(\cdot)$ and $L_g(\cdot)$ attain their maximum values on $\mathbb{T}_q$, denoted by$J_i^* = \max_{t\in\mathbb{T}_q} J_i(t)$, $(i=1,2)$, $L_f^* = \max_{t\in\mathbb{T}_q} L_f(t)$, and $L_g^* = \max_{t\in\mathbb{T}_q} L_g(t)$. In what follows, we divide the proof into three steps.

Step 1. We show that A_1 is contractive on Ω. For any $x,y\in\Omega$, we have

$$ \begin{aligned} |A_1x(t)-A_1y(t)| = {}& |\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(1,y(\tau))-g(0,x(0)) \\ &+g(0,y(0))]+g(t,x(\tau t)-g(t,y(\tau t)) \\ &-\frac{1}{Q}(g(0,x(0))-g(0,y(0))))| \\ \le{}& |g(t,x(\tau t)-g(t,y(\tau t))|+\frac{1}{|Q|}|g(0,x(0))-g(0,y(0))| \end{aligned} $$

$$
\begin{aligned}
&+\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}[|g(1,x(\tau))-g(1,y(\tau))|\\
&+|g(0,x(0))-g(0,y(0))|]\\
\leq&(\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)\|x-y\|\\
<&\gamma_1\|x-y\|.
\end{aligned}
$$

Since $\gamma_1 < 1$, so A_1 is contractive on Ω.

Step 2. We show that A_2 is relatively compact on $\mathbb{T}_q$. For any $x \in \Omega$, take $t_1, t_2 \in \mathbb{T}_q$ with $0 < t_1 < t_2 \leq 1$. Let $d(t_1, t_2)$ denote the time-scale distance between t_1 and t_2 on $\mathbb{T}_q$. For any $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that whenever $d(t_1, t_2) < \delta_1$, we have

$$
\begin{aligned}
|A_2x(t_2)-A_2x(t_1)|=&|\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{t_2}(t_2-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s\\
&-\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{t_1}(t_1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s|\\
\leq&|\frac{1}{\Gamma_q(\alpha+\delta)}\int_{t_1}^{t_2}(t_2-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s|\\
&+|\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{t_1}[(t_2-qs)^{(\alpha+\delta-1)}-(t_1-qs)^{(\alpha+\delta-1)}]\\
&\times f(s,x(s),x(\tau s)\Delta_q s|\\
\leq&\frac{J_1^*}{\Gamma_q(\alpha+\delta)}\int_0^{t_1}|(t_2-qs)^{(\alpha+\delta-1)}-(t_1-qs)^{(\alpha+\delta-1)}|\Delta_q s\\
&+\frac{J_1^*}{\Gamma_q(\alpha+\delta)}\int_{t_1}^{t_2}(t_2-qs)^{(\alpha+\delta-1)}\Delta_q s\\
\leq&\frac{J_1^*}{\Gamma_q(\alpha+\delta+1)}(t_2-t_1)^{(\alpha+\delta)}\\
&+\frac{J_1^*}{\Gamma_q(\alpha+\delta+1)}[(t_2-t_1)^{(\alpha+\delta)}+t_2^{\alpha+\delta}-t_1^{\alpha+\delta}]\\
=&\frac{J_1^*}{\Gamma_q(\alpha+\delta+1)}[2(t_2-t_1)^{(\alpha+\delta)}+t_2^{\alpha+\delta}-t_1^{\alpha+\delta}]\\
\to&0.
\end{aligned}
$$

By Lemma 2.4, A_2 is relatively compact.

In order to show the continuity of A_2, let us consider any sequence $\{x_n\}$ which converges to x, we have

$$
\begin{aligned}
|A_2x_n(t)-A_2x(t)|=&|\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{t}(t-qs)^{(\alpha+\delta-1)}(f(s,x_n(s),x_n(\tau s))\\
&-f(s,x(s),x(\tau s)))\Delta_q s|\\
\leq&\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{1}(1-qs)^{(\alpha+\delta-1)}|f(s,x_n(s),x_n(\tau s))\\
&-f(s,x(s),x(\tau s))|\Delta_q s
\end{aligned}
$$

$$\leq \frac{2L_f^*}{\Gamma_q(\alpha+\delta+1)}\|x_n - x\|.$$

So, $A_2x_n \to A_2x$ as $x_n \to x$.

Step 3. We prove that for any $x, y \in \Omega$, $A_1x + A_2y \in \Omega$.

$$\begin{aligned}
|A_1x(t) + A_2y(t)| =& |\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1,x(\tau)) - g(0,x(0)) + M] \\
&- \frac{1}{Q}g(0,x(0)) + g(t,x(\tau t)) \\
&+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)} f(s,y(s),y(\tau s)\Delta_q s| \\
\leq& |\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1,x(\tau)) - g(1,0) - (g(0,x(0) \\
&- g(0,0)) + M)] - \frac{1}{Q}(g(0,x(0)) - g(0,0)) + (g(t,x(\tau t)) - g(t,0)) \\
&+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}(f(s,y(s),y(\tau s) - f(s,0,0))\Delta_q s \\
&+ \frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1,0) - g(0,0)] - \frac{1}{Q}g(0,0) + g(t,0) \\
&+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)} f(s,0,0)\Delta_q s| \\
\leq& (\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|} + 1)L_g^*\|x\| + \frac{2L_f^*}{\Gamma_q(\alpha+\delta+1)}\|y\| \\
&+ \frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|M| + (\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|} + 1)J_2^* \\
&+ \frac{J_1^*}{\Gamma_q(\alpha+\delta+1)} \\
\leq& \gamma_1\sigma + (1-\gamma_1)\sigma \\
=& \sigma.
\end{aligned}$$

According to Krasnoselskii's fixed point theorem, we know that fractional q-DE (1.1) has at least one fixed point. □

Theorem 3.2. *Assume that* $(H1)$ *holds, then there exists a unique solution of fractional q-DE* (1.1).

Proof. The operator A on Ω is defined as Eq. (3.17). To show that A maps Ω into itself, we get

$$\begin{aligned}
|Ax(t)| =& |\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1,x(\tau)) - g(0,x(0)) + M] - \frac{1}{Q}g(0,x(0)) \\
&+ g(t,x(\tau t)) + \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)} f(s,x(s),x(\tau s)\Delta_q s| \\
\leq& |\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1,x(\tau)) - g(1,0) - (g(0,x(0) - g(0,0)) + M)]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{Q}(g(0,x(0))-g(0,0))+(g(t,x(\tau t))-g(t,0))-\frac{1}{Q}g(0,0)+g(t,0)\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}(f(s,x(s),x(\tau s)-f(s,0,0))\Delta_q s\\
&+\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,0)-g(0,0)]\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}f(s,0,0)\Delta_q s|\\
\le&\gamma_1\sigma+(\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)J_2+\frac{J_1}{\Gamma_q(\alpha+\delta+1)}+\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|M|\\
\le&\gamma_1\sigma+(1-\gamma_1)\sigma\\
=&\sigma.
\end{aligned}
$$

Next, we show that A is a contraction mapping. Let $x, y\in\Omega$, we have

$$
\begin{aligned}
|A_2x(t)-A_2y(t)|=&|\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}(f(s,x(s),x(\tau s))-f(s,y(s),y(\tau s)))\Delta_q s|\\
\le&\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}|f(s,x(s),x(\tau s)-f(s,y(s),y(\tau s))|\Delta_q s\\
\le&\gamma_1\|x-y\|\\
<&\|x-y\|.
\end{aligned}
$$

From Lemma 2.5, indicating that A is contractive on Ω. We know that fractional q-DE (1.1) has a unique solution. □

4. Hyers-Ulam stability

We will study Hyers-Ulam stability for the solution of fractional q-DE (1.1), we give the necessary definition and theorem as follows.

Definition 4.1. [8] We call that fractional q-DE (1.1) is Hyers-Ulam stable if for each $\varepsilon>0$, each $\tilde{v}\in C(\mathbb{T}_q,\mathbb{R})$ satisfies the inequality

$$
|^H\Delta_q^{\alpha,\beta}(^c\Delta_q^{\delta}(\tilde{v}(t)-g(t,\tilde{v}(\tau t))))-f(t,\tilde{v}(t),\bar{v}(\tau t))|\le\varepsilon,\quad t\in\mathbb{T}_q, \tag{4.1}
$$

there exists a solution $x(t)$ of fractional q-DE (1.1) and $\eta>0$ such that

$$
|x(t)-\tilde{v}(t)|\le\eta\varepsilon. \tag{4.2}
$$

Theorem 4.1. *If the assumption* $(H1)$ *and the inequality* (4.2) *are satisfied, then fractional* q*-DE* (1.1) *is Hyers-Ulam stable.*

Proof. If x satisfies Eq. (4.2), then

$$
\begin{aligned}
&|\tilde{v}(t)-(\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]-\frac{1}{Q}g(0,x(0))+g(t,x(\tau t))\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s)|\le\varepsilon.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\tilde{v}(t)-(\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,\tilde{v}(\tau))-g(0,\tilde{v}(0))+M]-\frac{1}{Q}g(0,\tilde{v}(0))+g(t,\tilde{v}(\tau t))\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}f(s,\tilde{v}(s),\tilde{v}(\tau s)\Delta_q s)+(\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,\tilde{v}(\tau))\\
&-g(1,x(\tau))-g(0,\tilde{v}(0))+g(0,x(0))]-\frac{1}{Q}(g(0,\tilde{v}(0))-g(0,x(0)))+g(t,\tilde{v}(\tau t))-g(t,x(\tau t))\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}(f(s,\tilde{v}(s),\tilde{v}(\tau s))-f(s,x(s),x(\tau s)))\Delta_q s)\|\\
\le\,&\varepsilon+[(\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)L_g^*+\frac{2L_f^*}{\Gamma_q(\alpha+\delta+1)}]\|\tilde{v}-x\|\\
=\,&\varepsilon+\gamma_1\|\tilde{v}-x\|,
\end{aligned}
$$

and

$$\|\tilde{v}-x\|\le\frac{1}{1-\gamma_1}\varepsilon=\eta\varepsilon.$$

This completes the proof. □

5. Controllability

In this section, we investigate the controllability of fractional q-DE (1.2). Firstly, we present the concept of controllability and the integral equation that is equivalent to the problem discussed.

Definition 5.1. [40] We say that fractional q-DE (1.2) is controllable on $\mathbb{T}_q$, for any given initial state x_0 and any given final state x_1, there exists a continuous function $u\in L^2(\mathbb{T}_q,\mathbb{R})$ such that the solution x of fractional q-DE (1.2) satisfies $x(1)=x_1$.

Lemma 5.1. *A function $x\in\Omega$ is a solution of fractional q-DE* (1.2)*, if and only if, this function is a solution of the following integral equation*

$$
\begin{aligned}
x(t)=&\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+\hat{M}+\hat{N}]-\frac{1}{Q}g(0,x(0))+g(t,x(\tau t))\\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}h(s)\Delta_q s+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}Bu(s)\Delta_q s,
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{aligned}
\hat{M}=&\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s\\
&+\frac{\lambda_2}{\Gamma_q(\alpha-\gamma+1)}\int_0^1(1-qs)^{(\alpha-\gamma)}f(s,x(s),x(\tau s)\Delta_q s,\\
\hat{N}=&\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}Bu(s))\Delta_q s\\
&+\frac{\lambda_2}{\Gamma_q(\alpha-\gamma+1)}\int_0^1(1-qs)^{(\alpha-\gamma)}Bu(s)\Delta_q s.
\end{aligned}
$$

Proof. Let

$$z(t) = f(t, x(t), x(\tau t)) + Bu(t),$$

so fractional q-DE (1.2) can be written by

$${}^{H}\Delta_q^{\alpha,\beta}({}^{c}\Delta_q^{\delta}(x(t) - g(t, x(\tau t)))) = z(t).$$

Using the same method as Lemma 3.1, we can obtain

$$\begin{aligned} x(t) = &\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1, x(\tau)) - g(0, x(0))] + g(t, x(\tau t)) - \frac{1}{Q}g(0, x(0)) \\ &+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}z(s)\Delta_q s \\ &+ \frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1 (1-qs)^{(\alpha+\delta-1)}z(s)\Delta_q s \\ &+ \frac{\lambda_2}{\Gamma_q(\alpha-\gamma+1)}\int_0^1 (1-qs)^{(\alpha-\gamma)}z(s)\Delta_q s]. \end{aligned}$$

That is,

$$\begin{aligned} x(t) = &\frac{\lambda_1\Gamma_q(\gamma+\delta) - t^{\gamma+\delta-1}}{Q}[g(1, x(\tau)) - g(0, x(0)) + \hat{M} + \hat{N}] \\ &- \frac{1}{Q}g(0, x(0)) + g(t, x(\tau t)) \\ &+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}f(s, x(s), x(\tau s)\Delta_q s \\ &+ \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t (t-qs)^{(\alpha+\delta-1)}Bu(s)\Delta_q s. \end{aligned} \tag{5.2}$$

□

We apply Sadovskii's fixed point theorem to prove the exact controllability of fractional q-DE (1.2).

Theorem 5.1. *Assume that* $(H1)$ *and* $(H2)$ *are satisfied, and defined the control function as*

$$\begin{aligned} u(t) = &W^{-1}[x_1 - (\frac{\lambda_1\Gamma_q(\gamma+\delta) - 1}{Q}[g(1, x(\tau)) - g(0, x(0)) + \hat{M} + \hat{N}] \\ &- \frac{1}{Q}g(0, x(0)) + g(1, x(\tau)) + \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1 (1-qt)^{(\alpha+\delta-1)}f(t, x(t), x(\tau t)\Delta_q t)], \end{aligned} \tag{5.3}$$

then fractional q*-DE* (1.2) *is exactly controllable on* Ω.

Proof. From Lemma 5.1 and Eq. (5.3), we have

$$\begin{aligned} \|u(t)\| = &\|W^{-1}[x_1 - (\frac{\lambda_1\Gamma_q(\gamma+\delta) - 1}{Q}[g(1, x(\tau)) - g(0, x(0)) + \hat{M} + \hat{N}] \\ &- \frac{1}{Q}g(0, x(0)) + g(1, x(\tau)) + \frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1 (1-qt)^{(\alpha+\delta-1)}f(t, x(t), x(\tau t)\Delta_q t)]\| \\ \le &\|W^{-1}\|[|x_1| + |\frac{\lambda_1\Gamma_q(\gamma+\delta) - 1 + Q}{Q}||g(1, x(\tau))| \end{aligned}$$

$$
\begin{aligned}
&+|\frac{\lambda_1\Gamma_q(\gamma+\delta)-1}{Q}||\hat{M}+\hat{N}+\frac{1}{Q}g(0,x(0))| \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qt)^{(\alpha+\delta-1)}|f(t,x(t),x(\tau t)|\Delta_q t)] \\
\leq{}& Mw(|x_1|+(1-\gamma_1)\sigma+\frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|N|) \\
={}& Mu,
\end{aligned}
$$

and

$$
\begin{aligned}
x(1)={}&\frac{\lambda_1\Gamma_q(\gamma+\delta)+1}{Q}[g(1,x(\tau))-g(0,x(0))+\hat{M}+\hat{N}]-\frac{1}{Q}g(0,x(0))+g(1,x(\tau)) \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}Bu(s)\Delta_q s \\
={}&\frac{\lambda_1\Gamma_q(\gamma+\delta)+1}{Q}[g(1,x(\tau))-g(0,x(0))+\hat{M}+\hat{N}]-\frac{1}{Q}g(0,x(0))+g(1,x(\tau)) \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s \\
&+WW^{-1}\{[x_1-(\frac{\lambda_1\Gamma_q(\gamma+\delta)+1}{Q}[g(1,x(\tau))-g(0,x(0))+M+N] \\
&-\frac{1}{Q}g(0,x(0))+g(1,x(\tau))\Delta_q s+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qt)^{(\alpha+\delta-1)}f(t,x(t),x(\tau t)\Delta_q t)\} \\
={}&\frac{\lambda_1\Gamma_q(\gamma+\delta)+1}{Q}[g(1,x(\tau))-g(0,x(0))+\hat{M}+\hat{N}]-\frac{1}{Q}g(0,x(0))+g(1,x(\tau)) \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s \\
&+[x_1-(\frac{\lambda_1\Gamma_q(\gamma+\delta)+1}{Q}[g(1,x(\tau))-g(0,x(0))+\hat{M}+\hat{N}]+g(1,x(\tau))-\frac{1}{Q}g(0,x(0)) \\
&+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s)] \\
={}&x_1.
\end{aligned}
$$

Next, we will present the main proof of controllability for fractional q-DE (1.2). We define the operators $\hat{A}:\Omega\to\Omega$ and $\hat{A}(t)=\tilde{A}(t)+A(t)$, where

$$
\tilde{A}x(t)=\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}N+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}Bu(s)\Delta_q s, \tag{5.4}
$$

and $A(t)$ is the same as Eq. (3.17).

Start the following steps to complete the proof.

Step 1. The operator $\tilde{A}$ maps Ω into itself. For each $t\in\mathbb{T}_q$ and $x\in\Omega$, it follows from hypothesis (H2) and Eq. (5.4) that

$$\begin{aligned}\|\tilde{A}x(t)\| =& |\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}N+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qt)^{(\alpha+\delta-1)}Bu(t)\Delta_q t\|\\ \leq& \frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|N|+|\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^1(1-qt)^{(\alpha+\delta-1)}Bu(t)\Delta_q t|\\ \leq& \frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}|N|+\frac{M_BMu}{\Gamma_q(\alpha+\delta+1)}\\ \leq& \sigma,\end{aligned}$$

which implies that $\tilde{A}$ maps Ω into itself.

Step 2. Let x_n be a sequence in Ω satisfying $x_n \to x$ as $n \to \infty$. Then, for each $t \in \mathbb{T}_q$, one has

$$\begin{aligned}\|Ax_n(t)-Ax(t)\| =& \|\frac{\lambda_1\Gamma_q(\gamma+\delta)-t^{\gamma+\delta-1}}{Q}[g(1,x_n(\tau))-g(1,x(\tau))-g(0,x_n(0))\\ &+g(0,x(0))]-\frac{1}{Q}(g(0,x_n(0))-g(0,x(0))+g(t,x_n(\tau t))\\ &-g(t,x(\tau t))+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}f(s,x_n(s),x_n(\tau s)\\ &-f(s,x(s),x(\tau s)\Delta_q s\|\\ \leq& \frac{|\lambda_1|\Gamma_q(\gamma+\delta)+1}{|Q|}(\|g(1,x_n(\tau))-g(1,x(\tau))\|\\ &+\|g(0,x_n(0))-g(0,x(0))\|)\\ &+\frac{1}{|Q|}\|g(0,x_n(0))-g(0,x(0))\|+\|g(t,x_n(\tau t))-g(t,x(\tau t))\|\\ &+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^t(t-qs)^{(\alpha+\delta-1)}\|f(s,x_n(s),x_n(\tau s)\\ &-f(s,x(s),x(\tau s)\|\Delta_q s\\ \leq& [(\frac{2|\lambda_1|\Gamma_q(\gamma+\delta)+3}{|Q|}+1)L_g^*+\frac{2L_f^*}{\Gamma_q(\alpha+\delta+1)}]\|x-x_n\|\\ =& \gamma_1\|x-x_n\|.\end{aligned}$$

We know that $\|Ax_n - Ax\| \to 0$ as $n \to \infty$. This means that A is continuous.

Step 3. Now we show that $A(\Omega)$ is bounded and equicontinuous. From step 3 of Theorem 3.1, it is clear that $A(\Omega)$ is bounded. It remains to show that $A(\Omega)$ is equicontinuous. Indeed, we have

$$\begin{aligned}\|Ax(t_2)-Ax(t_1)\| =& \|\frac{\lambda_1\Gamma_q(\gamma+\delta)-t_2^{\gamma+\delta-1}}{Q}[g(1,x(\tau))-g(0,x(0))+M]\\ &-\frac{1}{Q}g(0,x(0))+g(t_2,x(\tau t_2))\\ &+\frac{1}{\Gamma_q(\alpha+\delta)}\int_0^{t_2}(t_2-qs)^{(\alpha+\delta-1)}f(s,x(s),x(\tau s)\Delta_q s\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_1 \Gamma_q(\gamma+\delta) - t_1^{\gamma+\delta-1}}{Q}[g(1, x(\tau)) - g(0, x_(0)) + M] \\
&+ \frac{1}{Q} g(0, x(0)) - g(t_1, x(\tau t_1)) \\
&- \frac{1}{\Gamma_q(\alpha+\delta)} \int_0^{t_1} (t_1 - qs)^{(\alpha+\delta-1)} f(s, x(s), x(\tau s) \Delta_q s \\
\leq & \frac{2J_1^*}{|Q|} \| t_1^{\gamma+\delta-1} - t_2^{\gamma+\delta-1} \| + \frac{|M|}{|Q|} \| t_1^{\gamma+\delta-1} - t_2^{\gamma+\delta-1} \| \\
&+ \| g(t_2, x(\tau t_2)) - g(t_1, x(\tau t_1)) \| \\
&+ \frac{J_1^*}{\Gamma_q(\alpha+\delta)} \int_0^{t_1} ((t_2 - qs)^{(\alpha+\delta-1)} - (t_1 - qs)^{(\alpha+\delta-1)}) \Delta_q s \\
&+ \frac{J_1^*}{\Gamma_q(\alpha+\delta)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha+\delta-1)} \Delta_q s \\
\leq & \frac{2J_1^* + |M|}{|Q|} \| t_1^{\gamma+\delta-1} - t_2^{\gamma+\delta-1} \| + \| g(t_2, x(\tau t_2)) - g(t_1, x(\tau t_1)) \| \\
&+ \frac{2(t_1 - t_2)^{(\alpha+\delta)} + t_2^{\alpha+\delta} - t_1^{\alpha+\delta}}{\Gamma_q(\alpha+\delta+1)} J_1^* \\
\to & \, 0,
\end{aligned}
$$

the same as Theorem 3.1 in step 2. Hence, $A(\Omega)$ is equicontinuous. As a consequence of Lemma 3.2 and Arzelà-Ascoli theorem, it follows that the operator A is relatively compact, the operator $\hat{A} = \tilde{A} + A$ is continuous and maps bounded sets into bounded sets. Also, it is relatively compact with respect to the operator A which implies $v(A(\Omega)) = 0$. It follows from the inclusion $\tilde{A}(\Omega) \subset \Omega$ and the equality $v(A(\Omega)) = 0$ that

$$
v(\hat{A}(\Omega)) \leq v(\tilde{A}(\Omega)) + v(A(\Omega)) \leq v(\Omega).
$$

Since $\hat{A}(\Omega) \subset \Omega$ is a convex, closed and bounded set, that is, all conditions of Sadovskii fixed point theorem are satisfied, we conclude that the operator $\hat{A}$ has a fixed point $x \in \hat{A}(\Omega)$ that is a solution of fractional q-DE (1.2) with $x(1) = x_1$. Therefore, fractional q-DE (1.2) is controllable on Ω. □

6. Applications

Consider the sequential fractional q-DE involving q-Hilfer and q-Caputo fractional derivatives

$$
\begin{cases}
{}^H\Delta_{3/4}^{3/4,1/4}({}^c\Delta_{3/4}^{1/4}(x(t) - g(t, x(\tau t)))) = f(t, x(t), x(\tau t)), \quad t, \tau \in \mathbb{T}_q, \\
x(0) + \lambda_1 \, {}^c\Delta_{3/4}^{1/16} x(0) = 0, \quad x(1) + \lambda_2 \, {}^c\Delta_{3/4}^{1/16} x(1) = 0, \\
{}^c\Delta_{3/4}^{1/16} g(0, x(0)) = 0, \quad {}^c\Delta_{3/4}^{1/16} g(1, x(\tau)) = 0,
\end{cases} \tag{6.1}
$$

where $\alpha = 3/4$, $\beta = 1/4$, $\gamma = 13/16$, $\delta = 1/4$, $q = 3/4$, $\lambda_1 = 11/13$, $\lambda_2 = 7/11$, and

$$
f(t, x(t), x(\tau t)) = t^2 + \frac{|x(t)|}{|x(t)| + 23} + \frac{|x(\tau t)|}{|x(\tau t)| + 20},
$$

$$g(t,x(\tau t)) = t^4 + \frac{|x(\tau t)|}{|x(\tau t)|+11}.$$

By calculations, we have $\gamma+\delta-1 = 1/16$. These information can be used to find that $J_1^* = 3$, $J_2^* = 2$, $L_g^* = 0.1$ and $L_f^* = 0.05$, so

$$\begin{aligned}&|f(t,x(t),x(\tau t)) - f(t,x_1(t),x_1(\tau t))| \le (1/23+1/20)|x-x_1|,\\ &|g(t,x(\tau t)) - g(t,x_1(\tau t))| = (1/11)|x-x_1|,\end{aligned}$$

thus the hypothesis (H1) is satisfied. Additionally, we have $Q \approx 0.81310420$, $\gamma_1 \approx 0.55939441 + 0.4 = 0.95939441 < 1$, and $\gamma_2 \approx 0.55939441 < 1$. Since fractional q-DE (6.1) satisfies Theorem 3.1, we conclude that the problem has at least one solution. In addition, it can be proven that fractional q-DE (6.1) has a unique solution if $\gamma_1 - \gamma_2 = 0.4 < 1$ by applying the result in Lemma 2.4.

$$\begin{aligned}&\|\tilde{v}(t) - (\frac{(11/13)\Gamma_{3/4}(17/16) - t^{\gamma+\delta-1}}{0.81310420}[g(1,\tilde{v}(\tau)) - g(0,\tilde{v}(0)) + M] - \frac{1}{0.81310420}g(0,\tilde{v}(0))\\ &+ g(t,\tilde{v}(\tau t)) + \int_0^t f(s,\tilde{v}(s),\tilde{v}(\tau s))\Delta_q s + (\frac{11/13\lambda_1\Gamma_q(17/16) - t^{\gamma+\delta-1}}{0.81310420}[g(1,\tilde{v}(\tau))\\ &- g(1,x(\tau)) - g(0,\tilde{v}(0)) + g(0,x(0))] - \frac{1}{0.81310420}(g(0,\tilde{v}(0)) - g(0,x(0))) + g(t,\tilde{v}(\tau t))\\ &- g(t,x(\tau t)) + \int_0^t (f(s,\tilde{v}(s),\tilde{v}(\tau s)) - f(s,x(s),x(\tau s)))\Delta_q s)\|\\ \le&\varepsilon + [(\frac{2|11/13|\Gamma_{3/4}(17/16)+3}{0.81310420} + 1)0.1 + \frac{0.1}{\Gamma_{3/4}(2)}]\|\tilde{v} - x\|\\ \approx&\varepsilon + 0.95939441\|\tilde{v} - x\|.\end{aligned}$$

Since $\eta = \frac{1}{1-\gamma_1} \approx 24.6271509 > 0$. By Definition 3.1, we know that fractional q-DE (6.1) satisfies Hyers-Ulam stability.

Let $x_1 = 0.8$,

$$\begin{aligned}x(1) =& \frac{\frac{11}{13}\Gamma_{0.5}(\frac{17}{16})+1}{0.81310420}[1 + \frac{|x(\tau)|}{|x(\tau)|+11} - \frac{|x(0)|}{|x(0)|+11} + \hat{M} + \hat{N}] + 1 + \frac{|x(\tau)|}{|x(\tau)|+11}\\ &- \frac{1}{0.81310420}\frac{|x(0)|}{|x(0)|+11} + \int_0^1 H(1,s)BW^{-1}p(x)\Delta_q s\\ &+ \int_0^1 s^2 + \frac{|x(s)|}{|x(s)|+23} + \frac{|x(\tau s)|}{|x(\tau s)|+20}\Delta_q s\\ =&\frac{\frac{11}{13}\Gamma_{0.5}(\frac{17}{16})+1}{0.81310420}[1 + \frac{|x(\tau)|}{|x(\tau)|+11} - \frac{|x(0)|}{|x(0)|+11} + \hat{M} + \hat{N}] + 1 + \frac{|x(\tau)|}{|x(\tau)|+11}\\ &- \frac{1}{0.81310420}\frac{|x(0)|}{|x(0)|+11} + \int_0^1 s^2 + \frac{|x(s)|}{|x(s)|+23} + \frac{|x(\tau s)|}{|x(\tau s)|+20}\Delta_q s\\ &+ 1 + [0.8 - (\frac{\frac{11}{13}\Gamma_{0.5}(\frac{17}{16})+1}{0.81310420}[1 + \frac{|x(\tau)|}{|x(\tau)|+11} - \frac{|x(0)|}{|x(0)|+11} + \frac{|x(\tau)|}{|x(\tau)|+11}\\ &- \frac{1}{0.81310420}\frac{|x(0)|}{|x(0)|+11} + \int_0^1 s^2 + \frac{|x(s)|}{|x(s)|+23} + \frac{|x(\tau s)|}{|x(\tau s)|+20}\Delta_q s)]\\ =&0.8,\end{aligned}$$

so, $x(1) = x_1 = 0.8$, due to Lemma 3.2, we obtain that fractional q-DE (6.1) is controllable.

7. Conclusion

In this paper, we study the existence and uniqueness, Hyers-Ulam stability, and controllability of solutions for a class of Hilfer-type fractional q-dynamical equations with time delay on time scales. Firstly, we investigate the existence and uniqueness of solutions to Eq. (1.1) by using Krasnoselskii's fixed point theorem and Banach contraction mapping principle, and the obtained results are more general than those of the classical quantum difference equations. Secondly, we construct an appropriate perturbed equation to analyze the difference between the approximate solution and the exact solution of the original equation, and prove that this difference can be controlled by a constant linearly related to the perturbation amplitude, thus complete the proof of the Hyers-Ulam stability of a solution to Eq. (1.1). Differing from the controllability analysis of traditional fractional differential equations on time scales, we extend the controllability theory to fractional quantum dynamic equations on time scales (Eq. (1.2)). By means of Kuratowski's measure of noncompactness and Sadovskii's fixed point theorem, we discuss the controllability of the solution to Eq. (1.2) and establish an analytical method for quantum evolution problems. The results of this paper improve the theoretical framework of fractional q-dynamical equations on time scales, provide important theoretical support for the dynamic analysis and stability regulation of quantum systems, and can also be applied to practical problems such as signal delay compensation in quantum communication, evolution process control in quantum computing, and the modeling and optimization of multiscale hybrid systems. As the theory of quantum dynamical equations on time scales is still in a stage of continuous development, our future research will focus on exploring the solvability of fractional quantum dynamical equations defined on various infinite time scales. These include infinite continuous time scales such as $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = [0, +\infty)$, infinite discrete time scales $\mathbb{T} = h\mathbb{Z} = \{hk|k \in \mathbb{Z}\}$ and hybrid time scales $\mathbb{T} = \mathbb{R} \cup \{t_k\}_{k\in\mathbb{N}}$. The complexity of its mathematical treatment is significantly higher than that of finite time scales, providing a flexible modeling tool for multiscale hybrid systems and fields of theoretical physics.

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