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STABILITY OF 3D MHD EQUATIONS WITH ANISOTROPIC DISSIPATION NEAR A HORIZONTAL BACKGROUND MAGNETIC FIELD

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Abstract Both physical experiments and mathematical theories indicate that the background magnetic field can stabilize the conducting fluid. Existing results (e.g., [7, 20]) show that in the 3D case, either a background magnetic field in a single direction or a magnetic field that satisfies the Diophantine condition can provide a stabilizing mechanism for the fluid. In this paper, we examine a setting that lies between these two cases, specifically a horizontal background magnetic field $B_0 = (n_1, n_2, 0)$, $n_1, n_2 \neq 0$. We investigate the 3D incompressible magnetohydrodynamics (MHD) equations with anisotropic dissipation: Fractional viscosity in the velocity field in the x_1-direction and fractional magnetic diffusion in the x_2-direction. Since dissipation acts only in one direction and is fractional, establishing global stability in $H^3(\mathbb{T}^3)$ is nontrivial. We analyze the stabilizing effect of the horizontal magnetic field, combining both the velocity field and the weak dissipation mechanism of the magnetic field, and derive exponential decay in the horizontal directions, which is crucial for the stability proof.

Keywords 3D MHD equations, anisotropic dissipation, horizontal background magnetic field, stability.

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1. Introduction

It is well known that the incompressible Euler equations can exhibit rapid growth in time (see, e.g., [14, 32]). Even when dissipation acts in one direction, as in the system

$$\begin{cases} \partial_t V - \nu \partial_x^2 V + V \cdot \nabla V + \nabla P = 0, \\ \nabla \cdot V = 0, \end{cases}$$

the global well-posedness for small initial data remains highly nontrivial, primarily because dissipation acting in only one direction is too weak to control all the nonlinear terms. In fact, the system can be stable when dissipation is present in both directions or when it is fully Laplacian. In the fully Laplacian case, as demonstrated by Schonbek [21–23], the Navier–Stokes equations are asymptotically stable and exhibit explicit time decay rates. When dissipation is restricted to horizontal directions, due to its important physical relevance and special mathematical structure, the global well-posedness problem for small initial data has also attracted significant attention (see, e.g., [5, 6, 11, 12]). Moreover, the two-dimensional case is relatively easier than the three-dimensional one; for instance, Dong et al. [8] established stability in $H^2(\mathbb{T} \times \mathbb{R})$. It is worth

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noting that although periodic domains do not introduce additional dissipation, they enhance its efficiency through the discrete structure of the spectrum: On the torus the Laplacian has a spectral gap on the mean-zero subspace, in contrast to the whole space, where the spectrum is continuous down to zero. This spectral gap underlies the strong Poincaré-type inequalities and forces the nonzero modes to decay exponentially rather than algebraically, which will be discussed in detail later.

Both physical experiments and mathematical analysis show that a background magnetic field can stabilize conducting fluids (see, e.g., [1, 2]). This paper focuses on the three-dimensional MHD system with anisotropic fractional dissipation:

$$\begin{cases} \partial_t V + V\cdot\nabla V + \nu\partial_1^{2\alpha}V + \nabla P - B\cdot\nabla B = 0, \quad x\in\Omega=\mathbb{T}^3, \quad t>0,\\ \partial_t B + V\cdot\nabla B + \mu\partial_2^{2\beta}B - B\cdot\nabla V = 0,\\ \nabla\cdot V = \nabla\cdot B = 0, \end{cases}$$

where $\mathbb{T}$ denotes the 1D periodic box, $\alpha,\beta\in(0,1]$. Here $V=(V_1,V_2,V_3)$ denotes the velocity field, $B=(B_1,B_2,B_3)$ the magnetic field, P the total pressure and ν,μ denote the velocity viscosity and the magnetic diffusivity, respectively. The fractional derivative operator ∂_i^η is defined via the Fourier transform

$$\widehat{\partial_i^\eta f}(k_1,k_2,k_3) = |k_i|^\eta \widehat{f}(k_1,k_2,k_3), \quad k_i\in\mathbb{Z},\ i\in\{1,2,3\}.$$

The stability problem and related issues on the anisotropic MHD system near a background magnetic field have attracted considerable interest (see, e.g., [1–4, 9, 10, 15, 16, 19, 20, 28, 30, 31]). In this paper, we consider perturbations of the steady-state solution (U_0,B_0) near a horizontal background magnetic field, which is expected to have a stabilizing effect on $(\partial_2 u, \partial_1 b)$, i.e.,

$$U_0 = 0, \quad B_0 = (n_1,n_2,0) \stackrel{\text{def}}{=} (n_h, 0).$$

Without loss of generality, we assume $n_1, n_2 > 0$ in the rest of the paper. The perturbation (u,b) around the steady state (U_0,B_0) with $u=U-U_0$, $b=B-B_0$ is governed by

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu\partial_1^{2\alpha}u + \nabla p - b\cdot\nabla b = n_h\cdot\nabla_h b, \quad x\in\Omega, \quad t>0,\\ \partial_t b + u\cdot\nabla b + \mu\partial_2^{2\beta}b - b\cdot\nabla u = n_h\cdot\nabla_h u,\\ \nabla\cdot u = \nabla\cdot b = 0,\\ u(x,0)=u_0(x),\ b(x,0)=b_0(x). \end{cases} \tag{1.1}$$

This paper aims to establish a small data global well-posedness result for system (1.1) in the H^3 framework. Our main theorem is stated as follows:

Theorem 1.1. *Assume that $(u_0,b_0)\in H^3(\Omega)$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ satisfies the mean-zero condition on the vertical components*

$$\int_\Omega u_{03}\,\mathrm{d}x = \int_\Omega b_{03}\,\mathrm{d}x = 0. \tag{1.2}$$

Then there exists $\varepsilon_0>0$ such that for any initial data, if

$$\|(u_0,b_0)\|_{H^3} \le \varepsilon_0,$$

then (1.1) *admits a unique global solution* $(u,b) \in L^\infty(0,\infty;H^3(\Omega))$, *and there exists a positive constant* C_0 *such that, for any* $t \geq 0$,

$$\|(u,b)(t)\|_{H^3}^2 + \int_0^t (\nu\|\partial_1^\alpha u(\tau)\|_{H^3}^2 + \mu\|\partial_2^\beta b(\tau)\|_{H^3}^2)\,\mathrm{d}\tau \leq C_0\varepsilon_0^2. \tag{1.3}$$

Furthermore, $(\nabla_h u, \nabla_h b)$ *obeys the following exponential-in-time decay estimate*

$$\|(\nabla_h u, \nabla_h b)(t)\|_{H^2} \leq C_1 e^{-c_1 t}\|(\nabla_h u_0, \nabla_h b_0)\|_{H^2}, \tag{1.4}$$

where the constants $C_1, c_1 > 0$.

Remark 1.1. The condition (1.2) is preserved by the flow, i.e.,

$$\int_\Omega u_3(t)\,\mathrm{d}x = \int_\Omega u_{03}\,\mathrm{d}x = 0, \quad \int_\Omega b_3(t)\,\mathrm{d}x = \int_\Omega b_{03}\,\mathrm{d}x = 0, \quad \forall\, t > 0.$$

In particular, Lemma 2.3 applies to $(u,b)(t)$ throughout the proof of Theorem 1.1.

The key to the proof consists of two parts, which are the stabilizing effect of the background magnetic field and the bootstrap. We introduce them separately.

1.1. Stabilizing effect of the background magnetic field

In this subsection, we compare the weak dissipation mechanisms introduced by different background magnetic fields. Let us first consider the 3D incompressible MHD system near a constant background magnetic field $\omega \in \mathbb{R}^3$. The perturbation system reads:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mathcal{L}_1 u + \nabla p - b\cdot\nabla b = \omega\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b - \mathcal{L}_2 b - b\cdot\nabla u = \omega\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases}$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ represent the partial dissipation operators or the full Laplacian operator. We now introduce dissipation via $\omega\cdot\nabla u$. A natural approach is to take the inner product of the magnetic equation with $\omega\cdot\nabla u$:

$$\begin{aligned} \|\omega\cdot\nabla u\|_{\dot{H}^s}^2 \leq{}& \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} \nabla^s u\cdot\nabla^s(\omega\cdot\nabla b)\,\mathrm{d}x \\ &+ C\|\sqrt{-\mathcal{L}_1}u\|_{\dot{H}^s}^2 + C\|\sqrt{-\mathcal{L}_2}b\|_{\dot{H}^s}^2 + \text{nonlinear terms.} \end{aligned} \tag{1.5}$$

Consider the simplest case: $\omega = (0,1,0)$. In this setting, let $\mathcal{L}_1 = \partial_1^2$ and $\mathcal{L}_2 = \Delta_h$. Then the inequality (1.5) reduces to:

$$\frac{1}{2}\|\partial_2 u\|_{\dot{H}^s}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^s\partial_2 u\cdot\nabla^s b\,\mathrm{d}x + C\|(\nabla_h b, \partial_1 u)\|_{H^{s+1}}^2 + \text{nonlinear terms.}$$

Using this dissipative mechanism, Lin et al. [19] achieved stability in $H^4(\mathbb{R}^3)$. In later work [20], they optimized their stability result under the assumption that the initial data (u_0,b_0) satisfy $\|(u_0,b_0)\|_{H^3} + \sum_{k=0}^2 \|\partial_3^k(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2x_3}} \leq \varepsilon$. More recently, Lai et al. [16] further improved this result in $H^3(\mathbb{R}^3)$ without requiring the additional smallness condition $\sum_{k=0}^2 \|\partial_3^k(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2x_3}}$.

Furthermore, there are also abundant results in the 2D case, please refer to [10, 15, 17, 18, 28, 30, 31].

Next, consider the case where each component of ω is nonzero. Reviewing the dissipation term, it is clear that the quantity $\|\omega \cdot \nabla u\|_{\dot{H}^s}^2$ on the left-hand side is insufficient to serve as a conventional dissipative term. This is because we cannot directly control the nonlinear term in the form $\|\omega \cdot \nabla u\|_{\dot{H}^s}^2$. To make this term effective, additional conditions must be imposed. In particular, when the domain is $\mathbb{T}^3$ and the background magnetic field ω satisfies a Diophantine condition, one can establish the following Poincaré-type inequality:

$$\|f\|_{H^s} \leq C\|\omega \cdot \nabla f\|_{H^{s+r}}, \tag{1.6}$$

for functions f satisfying the mean-zero condition:

$$\int_{\mathbb{T}^3} f \, \mathrm{d}x = 0.$$

Here, f stands for either the velocity field u or the magnetic field b. It is worth noting that the spatial integrals of both the velocity and magnetic fields are conserved in time. That is,

$$\int_{\mathbb{T}^3} u(x,t) \, \mathrm{d}x = \int_{\mathbb{T}^3} u(x,0) \, \mathrm{d}x, \quad \int_{\mathbb{T}^3} b(x,t) \, \mathrm{d}x = \int_{\mathbb{T}^3} b(x,0) \, \mathrm{d}x. \tag{1.7}$$

Using (1.6), global well-posedness for small initial data was proved by Chen et al. [7] in the case $\mathcal{L}_1 = 0,\ \mathcal{L}_2 = \Delta$ or $\mathcal{L}_1 = \Delta,\ \mathcal{L}_2 = 0$. The key lies in a weak dissipation mechanism. For example, in the case $\mathcal{L}_1 = 0,\ \mathcal{L}_2 = \Delta$, the following inequality holds:

$$\begin{aligned} \frac{1}{2}\|\omega \cdot \nabla u\|_{H^{r+3}}^2 \leq & \sum_{0 \leq s \leq r+3} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} \nabla^s b \cdot \nabla^s (\omega \cdot \nabla u) \, \mathrm{d}x \\ & + C\|b\|_{H^{r+5}}^2 + \text{nonlinear terms}. \end{aligned} \tag{1.8}$$

There has also been a large body of work on global well-posedness results for background magnetic fields satisfying the Diophantine condition. These include studies in two, three, and even n-dimensional cases, both compressible and incompressible settings, and have also been extended to other models such as the magneto-micropolar fluid equations. We refer the reader to [13, 24–27, 29] for more details. Notably, in the incompressible MHD system on $\mathbb{T}^n$, Xie et al. [25] established the optimal decay rate, consistent with the decay rate in the linearized case.

A natural next question is: What type of dissipation mechanism does a horizontal background magnetic field provide? Our answer is that a horizontal background magnetic field behaves similarly to the single-directional case. Specifically, when $\mathcal{L}_1 = \partial_1^{2\alpha},\ \mathcal{L}_2 = \partial_2^{2\beta}$, it provides dissipation mechanisms for both the velocity and magnetic fields. Precisely, from Lemmas 2.6 and 2.7 we obtain the following inequality:

$$\begin{aligned} & \frac{n_1}{4}\|\partial_1 b\|_{\dot{H}^2}^2 + \frac{n_2}{4}\|\partial_2 u\|_{\dot{H}^2}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 \partial_2 u \cdot \nabla^2 b \, \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 \partial_1 b \cdot \nabla^2 u \, \mathrm{d}x \\ & \leq C\|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2 + C\|(\partial_2^{1+\beta} b, \partial_1^{1+\alpha} u)\|_{\dot{H}^2}^2. \end{aligned} \tag{1.9}$$

This inequality is effective for any positive horizontal background magnetic field (i.e., $n_1, n_2 > 0$). It is easy to see that the background magnetic field compensates for the lack of dissipation in the horizontal directions for both the velocity and magnetic fields. Moreover, the terms on the right-hand side can be controlled by the existing dissipative terms. For more details, please refer to Section 2. Finally, we present an interesting problem. When $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = \Delta_h$, can the horizontal background magnetic field also provide a stabilizing effect similar to (1.8)? This seems to be difficult because (1.7) is no longer a conserved quantity over time in the horizontal direction.

1.2. Periodic domain and bootstrap argument

Numerous results have shown that periodic domains exhibit elegant properties. Although periodic domains do not directly introduce dissipation, dissipation in the periodic directions is more effective compared to the whole space. For example, by decomposing u into zero mode and nonzero modes, and combining it with the strong form Poincaré-type inequalities, some nonlinear terms that are difficult to control can be managed. Additionally, nonzero modes can exhibit exponential decay. For more details, we refer to [8, 11, 18, 30, 31].

Compared to the recent works [16, 20], establishing small-data stability in $H^3(\mathbb{T}^3)$ is significantly more challenging due to the presence of fractional dissipation acting in only one direction. This dissipation is particularly weak, not only is it anisotropic, but it also has low regularity, as the fractional orders satisfy $0 < \alpha, \beta \leq 1$. A natural strategy is to attempt to prove the following a priori estimate:

$$E(t) \lesssim E(0) + E(t)^{1+c_0}, \quad c_0 > 0, \tag{1.10}$$

where the energy functional is defined as $E(t) = E_1(t) + E_2(t)$, with

$$E_1(t) = \sup_{0\leq\tau\leq t} \|(u,b)(\tau)\|_{H^3}^2 + \int_0^t \left(2\nu\|\partial_1^\alpha u\|_{H^3}^2 + 2\mu\|\partial_2^\beta b\|_{H^3}^2\right) \mathrm{d}\tau,$$

and

$$E_2(t) = \int_0^t \|(\partial_2 u, \partial_1 b)\|_{H^2}^2 \, \mathrm{d}\tau.$$

However, it turns out to be very difficult to obtain the estimate (1.10) by relying solely on anisotropic inequalities, especially when the dissipation is of fractional order. Fortunately, when the domain is periodic, inspired by [30, 31], we adopt a different approach that replaces (1.10) and allows us to derive the necessary a priori bounds more directly and efficiently, without the use of anisotropic inequalities. Specifically, we observe that the following inequality holds:

$$E_1(t) \leq E_1(0) + C \sup_{0\leq\tau\leq t} \|(u,b)(\tau)\|_{H^3}^2 \int_0^t \|(\nabla_h u, \nabla_h b)(\tau)\|_{H^2} \, \mathrm{d}\tau.$$

Under a smallness assumption on the initial data, we can show that $(\nabla_h u, \nabla_h b)$ decays exponentially in time, i.e.

$$\|(\nabla_h u, \nabla_h b)(t)\|_{H^2} \leq e^{-\frac{1}{2}\delta_0 t}\|(\nabla_h u, \nabla_h b)(0)\|_{H^2}.$$

This allows us to control $E_1(t)$, thereby closing the bootstrap argument.

To establish the exponential decay in the horizontal directions, or equivalently in the course of proving (1.9), the most difficult nonlinear term is

$$\int_\Omega \partial_3^2\partial_2\mathbf{f} \cdot g_3\, \partial_3^3\mathbf{h} \, \mathrm{d}x, \qquad \mathbf{f}, \mathbf{g}, \mathbf{h} \in \{u, b\},$$

where g_3 denotes the vertical component of $\mathbf{g}$; see the estimate of M_3 in the proof of Lemma 2.6. Since no dissipation acts in the x_3-direction, this term carries the third-order vertical derivative $\partial_3^3\mathbf{h}$ together with the factor g_3, and cannot be closed by the available dissipation. Thanks to the conservation of $\int_\Omega u_3\,\mathrm{d}x$ and $\int_\Omega b_3\,\mathrm{d}x$, which follows from (1.2), we may apply Lemma 2.3,

$$\|g_3\|_{H^2} \le \|\nabla_h \mathbf{g}\|_{H^2},$$

so that the term is closed. Compared with related works on periodic domains that impose a symmetry assumption on the initial data, the condition (1.2) is weaker and is preserved in time merely by this conservation.

Throughout this paper, we use C to denote various positive constants, which may differ from line to line. For simplicity, we write $f \lesssim g$ to indicate that there exists a constant $C > 0$ such that $f \le Cg$. For a vector field $\mathbf{f} = (f_1, f_2, f_3)$, we denote its horizontal component by $\mathbf{f}_h = (f_1, f_2)$.

2. Preliminaries

In this section, we present Poincaré-type inequalities on the periodic domain and several prior estimates.

2.1. Poincaré-type inequalities

The following lemmas show that the oscillation part satisfies Poincaré-type inequalities, where Lemma 2.1 and Lemma 2.2 can be found in [10]. And Lemma 2.3 shows that, for a divergence-free field with vanishing mean, the vertical component is controlled by the horizontal gradient.

Lemma 2.1. *Let $f : \Omega \to \mathbb{R}$ satisfy*

$$\int_{\mathbb{T}} f(x_1, x_2, x_3)\,\mathrm{d}x_i = 0, \ \ i = 1, 2, 3.$$

Then for any $\eta > 0$, it holds that

$$\|f\|_{L^2} \le \|\partial_i^\eta f\|_{L^2}.$$

Lemma 2.2. *Let $f, g : \Omega \to \mathbb{R}$, then it holds that*

$$\left|\int_\Omega \partial_{x_i} f g\,\mathrm{d}x\right| \le \|\partial_{x_i} f\|_{L^2} \|\partial_{x_i} g\|_{L^2}.$$

Lemma 2.3. *Let $\mathbf{g} = (g_1, g_2, g_3) : \Omega \to \mathbb{R}^3$ satisfy $\nabla \cdot \mathbf{g} = 0$ and*

$$\int_\Omega g_3\,\mathrm{d}x = 0.$$

Then for any $s \ge 0$,

$$\|g_3\|_{H^s} \le \|\nabla_h g_3\|_{H^s}.$$

Proof. Expand $\mathbf{g}$ in Fourier series with frequencies $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$. The divergence-free condition $\nabla \cdot \mathbf{g} = 0$ reads

$$k_1\widehat{g}_1(k) + k_2\widehat{g}_2(k) + k_3\widehat{g}_3(k) = 0, \qquad k \in \mathbb{Z}^3.$$

Choosing $k=(0,0,k_3)$ with $k_3\neq 0$ gives $k_3\widehat{g_3}(0,0,k_3)=0$, and hence $\widehat{g_3}(0,0,k_3)=0$. Together with the mean-zero condition

$$\widehat{g_3}(0,0,0)=\int_\Omega g_3\,\mathrm{d}x=0,$$

we conclude that $\widehat{g_3}(k)=0$ whenever $k_h\overset{\text{def}}{=}(k_1,k_2)=0$. Consequently, for any $s\geq 0$, we have

$$\|g_3\|_{H^s}^2=\sum_{|k_h|\geq 1}\langle k\rangle^{2s}\,|\widehat{g_3}(k)|^2\leq\sum_{|k_h|\geq 1}|k_h|^2\,\langle k\rangle^{2s}\,|\widehat{g_3}(k)|^2=\|\nabla_h g_3\|_{H^s}^2\,,$$

where $\langle k\rangle=(1+|k|^2)^{1/2}$. This completes the proof. □

2.2. Energy estimates

In this subsection, Lemmas 2.4 and 2.5 provide direct energy estimates, Lemma 2.6 and Lemma 2.7 reveal the hidden dissipation of $(\partial_2 u,\partial_1 b)$ induced by the background magnetic field.

Lemma 2.4. *Let $(u,b)\in L^\infty(0,\infty;H^3)$ be a solution to the system* (1.1). *Then the following inequality holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b)(t)\|_{H^3}^2+\nu\|\partial_1^\alpha u\|_{H^3}^2+\mu\|\partial_2^\beta b\|_{H^3}^2\lesssim\|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2.\tag{2.1}$$

Proof. It is easy to derive the following standard L^2 energy estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b)\|_{L^2}^2+\nu\|\partial_1^\alpha u\|_{L^2}^2+\mu\|\partial_2^\beta b\|_{L^2}^2=0.\tag{2.2}$$

Next, applying the operator ∂_i^3 for $i=1,2,3$ to equation (1.1) and dotting with $(\partial_i^3 u,\partial_i^3 b)$ in L^2, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^3\|(\partial_i^3 u,\partial_i^3 b)\|_{L^2}^2+\nu\sum_{i=1}^3\|\partial_i^3\partial_1^\alpha u\|_{L^2}^2+\mu\sum_{i=1}^3\|\partial_i^3\partial_2^\beta b\|_{L^2}^2=\sum_{j=1}^4 H_j,$$

due to the elimination

$$\int_\Omega(u\cdot\nabla\partial_i^3 u)\cdot\partial_i^3 u\,\mathrm{d}x=\int_\Omega(u\cdot\nabla\partial_i^3 b)\cdot\partial_i^3 b\,\mathrm{d}x=0,$$
$$\int_\Omega(b\cdot\nabla\partial_i^3 b)\cdot\partial_i^3 u\,\mathrm{d}x+\int_\Omega(b\cdot\nabla\partial_i^3 u)\cdot\partial_i^3 b\,\mathrm{d}x=0.$$

Hence we can set

$$H_1\overset{\text{def}}{=}-\sum_{i=1}^3\int_\Omega\left[\partial_i^3(u\cdot\nabla u)-u\cdot\nabla\partial_i^3 u\right]\cdot\partial_i^3 u\,\mathrm{d}x,$$

$$H_2\overset{\text{def}}{=}\sum_{i=1}^3\int_\Omega\left[\partial_i^3(b\cdot\nabla b)-b\cdot\nabla\partial_i^3 b\right]\cdot\partial_i^3 u\,\mathrm{d}x,$$

$$H_3\overset{\text{def}}{=}-\sum_{i=1}^3\int_\Omega\left[\partial_i^3(u\cdot\nabla b)-u\cdot\nabla\partial_i^3 b\right]\cdot\partial_i^3 b\,\mathrm{d}x,$$

$$H_4\overset{\text{def}}{=}\sum_{i=1}^3\int_\Omega\left[\partial_i^3(b\cdot\nabla u)-b\cdot\nabla\partial_i^3 u\right]\cdot\partial_i^3 b\,\mathrm{d}x.$$

Note that H_1, H_2, H_3, and H_4 share a similar structure. Thus, it suffices to estimate the general form

$$\begin{aligned}I &= \sum_{i=1}^{2}\int_\Omega \partial_i^3\mathbf{f}\cdot\nabla\mathbf{g}\cdot\partial_i^3\mathbf{h}\,\mathrm{d}x + 3\sum_{i=1}^{2}\int_\Omega \partial_i^2\mathbf{f}\cdot\nabla\partial_i\mathbf{g}\cdot\partial_i^3\mathbf{h}\,\mathrm{d}x\\ &\quad + 3\sum_{i=1}^{2}\int_\Omega \partial_i\mathbf{f}\cdot\nabla\partial_i^2\mathbf{g}\cdot\partial_i^3\mathbf{h}\,\mathrm{d}x + \int_\Omega \partial_3^3\mathbf{f}\cdot\nabla\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x\\ &\quad + 3\int_\Omega \partial_3^2\mathbf{f}\cdot\nabla\partial_3\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x + 3\int_\Omega \partial_3\mathbf{f}\cdot\nabla\partial_3^2\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x\\ &\overset{\text{def}}{=} \sum_{j=1}^{6} I_j,\end{aligned}$$

where $\mathbf{f},\mathbf{g},\mathbf{h}\in\{u,b\}$, and each vector field is given by $\mathbf{f}=(f_1,f_2,f_3)$, etc.

Now we estimate each I_j. For I_1 and I_2, by the Sobolev embeddings $H^2\hookrightarrow L^\infty$ and $H^1\hookrightarrow L^4$, we obtain

$$\begin{aligned}I_1 &\lesssim \|\nabla\mathbf{g}\|_{L^\infty}\left\|\nabla_h^3\mathbf{f}\right\|_{L^2}\left\|\nabla_h^3\mathbf{h}\right\|_{L^2}\\ &\lesssim \|\nabla\mathbf{g}\|_{H^2}\left\|\nabla_h^3\mathbf{f}\right\|_{L^2}\left\|\nabla_h^3\mathbf{h}\right\|_{L^2}\\ &\lesssim \|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2\end{aligned}$$

and

$$\begin{aligned}I_2 &\lesssim \left\|\nabla_h^3\mathbf{h}\right\|_{L^2}\left\|\nabla_h^2\mathbf{f}\right\|_{L^4}\|\nabla\nabla_h\mathbf{g}\|_{L^4}\\ &\lesssim \left\|\nabla_h^3\mathbf{h}\right\|_{L^2}\left\|\nabla_h^2\mathbf{f}\right\|_{H^1}\|\nabla\nabla_h\mathbf{g}\|_{H^1}\\ &\lesssim \|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2.\end{aligned}$$

To estimate I_4, I_5, we use divergence-free conditions $\nabla\cdot\mathbf{f}=\nabla\cdot\mathbf{g}=\nabla\cdot\mathbf{h}=0$. For I_4, we compute

$$\begin{aligned}I_4 &= \int_\Omega \partial_3^3 f_3\partial_3\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x + \int_\Omega \partial_3^3\mathbf{f}_h\cdot\nabla_h\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x\\ &\lesssim \|\partial_3\mathbf{g}\|_{L^\infty}\left\|\partial_3^2\nabla_h\mathbf{f}\right\|_{L^2}\left\|\partial_3^3\mathbf{h}\right\|_{L^2} + \|\nabla_h\mathbf{g}\|_{L^\infty}\left\|\partial_3^3\mathbf{f}_h\right\|_{L^2}\left\|\partial_3^3\mathbf{h}\right\|_{L^2}\\ &\lesssim \|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2.\end{aligned}$$

Similarly, for I_5

$$\begin{aligned}I_5 &= \int_\Omega \partial_3^2 f_3\partial_3^2\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x + \int_\Omega \partial_3^2\mathbf{f}_h\cdot\nabla_h\partial_3\mathbf{g}\cdot\partial_3^3\mathbf{h}\,\mathrm{d}x\\ &\lesssim \left\|\partial_3^3\mathbf{h}\right\|_{L^2}\|\partial_3\nabla_h\mathbf{f}\|_{L^4}\left\|\partial_3^2\mathbf{g}\right\|_{L^4} + \left\|\partial_3^3\mathbf{h}\right\|_{L^2}\|\nabla_h\partial_3\mathbf{g}\|_{L^4}\left\|\partial_3^2\mathbf{f}_h\right\|_{L^4}\\ &\lesssim \|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2.\end{aligned}$$

The estimate for I_3 is similar to I_1, and that for I_6 is similar to I_4. Collecting all the estimates, we conclude that

$$H_j \lesssim \|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2,\quad j\in\{1,2,3,4\}. \tag{2.3}$$

The proof then follows directly from estimates (2.2) and (2.3). □

Lemma 2.5. *Let $(u,b)\in L^\infty(0,\infty;H^3)$ be a solution to the system (1.1). Then the following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla_h u,\nabla_h b)(t)\|_{\dot H^2}^2 + \nu\|\nabla_h\partial_1^\alpha u\|_{\dot H^2}^2 + \mu\left\|\nabla_h\partial_2^\beta b\right\|_{\dot H^2}^2\\ &\lesssim \|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2.\end{aligned} \tag{2.4}$$

Proof. Applying ∇_h to (1.1) and dotting with $(\nabla_h u, \nabla_h b)$ in $\dot{H}^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla_h u, \nabla_h b)\|_{\dot{H}^2}^2 + \nu\|\nabla_h\partial_1^\alpha u\|_{\dot{H}^2}^2 + \mu\|\nabla_h\partial_2^\beta b\|_{\dot{H}^2}^2 = \sum_{j=1}^4 J_j,$$

where

$$\begin{aligned}
J_1 &\overset{\text{def}}{=} -\int \nabla^2\nabla_h(u\cdot\nabla u)\cdot\nabla^2\nabla_h u\,\mathrm{d}x,\\
J_2 &\overset{\text{def}}{=} \int \left[\nabla^2\nabla_h(b\cdot\nabla b) - b\cdot\nabla\nabla^2\nabla_h b\right]\cdot\nabla^2\nabla_h u\,\mathrm{d}x,\\
J_3 &\overset{\text{def}}{=} -\int \nabla^2\nabla_h(u\cdot\nabla b)\cdot\nabla^2\nabla_h b\,\mathrm{d}x,\\
J_4 &\overset{\text{def}}{=} \int \left[\nabla^2\nabla_h(b\cdot\nabla u) - b\cdot\nabla\nabla^2\nabla_h u\right]\cdot\nabla^2\nabla_h b\,\mathrm{d}x.
\end{aligned}$$

Similar to the proof of Lemma 2.4, we only need to estimate

$$\begin{aligned}
K = &\int \nabla^2\nabla_h\mathbf{f}\cdot\nabla\mathbf{g}\cdot\nabla^2\nabla_h\mathbf{h}\,\mathrm{d}x + 2\int \nabla\nabla_h\mathbf{f}\cdot\nabla\nabla\mathbf{g}\cdot\nabla^2\nabla_h\mathbf{h}\,\mathrm{d}x\\
&+\int \nabla_h\mathbf{f}\cdot\nabla\nabla^2\mathbf{g}\cdot\nabla^2\nabla_h\mathbf{h}\,\mathrm{d}x + \int \nabla^2\mathbf{f}\cdot\nabla\nabla_h\mathbf{g}\cdot\nabla^2\nabla_h\mathbf{h}\,\mathrm{d}x\\
&+2\int \nabla\mathbf{f}\cdot\nabla\nabla\nabla_h\mathbf{g}\cdot\nabla^2\nabla_h\mathbf{h}\,\mathrm{d}x\\
\overset{\text{def}}{=}&\sum_{j=1}^5 K_j.
\end{aligned}$$

It is easy to see that each K_j contains at least two terms containing ∇_h, so it can be proved by Hölder's and Young's inequalities and embedding theory directly

$$\begin{aligned}
K_1 &\le \|\nabla\mathbf{g}\|_{L^\infty}\left\|\nabla^2\nabla_h\mathbf{f}\right\|_{L^2}\left\|\nabla^2\nabla_h\mathbf{h}\right\|_{L^2} \lesssim \|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2,\\
K_2 &\le 2\left\|\nabla^2\nabla_h\mathbf{h}\right\|_{L^2}\|\nabla\nabla_h\mathbf{f}\|_{L^4}\left\|\nabla^2\mathbf{g}\right\|_{L^4} \lesssim \|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2,\\
K_3 &\le \|\nabla_h\mathbf{f}\|_{L^\infty}\left\|\nabla^3\mathbf{g}\right\|_{L^2}\left\|\nabla^2\nabla_h\mathbf{h}\right\|_{L^2} \lesssim \|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2,\\
K_4 &\le \left\|\nabla^2\nabla_h\mathbf{h}\right\|_{L^2}\left\|\nabla^2\mathbf{f}\right\|_{L^4}\|\nabla\nabla_h\mathbf{g}\|_{L^4} \lesssim \|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2,\\
K_5 &\le 2\|\nabla\mathbf{f}\|_{L^\infty}\left\|\nabla^2\nabla_h\mathbf{g}\right\|_{L^2}\left\|\nabla^2\nabla_h\mathbf{h}\right\|_{L^2} \lesssim \|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2.
\end{aligned}$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $(u,b)\in L^\infty(0,\infty;H^3)$ be a solution to the system* (1.1). *Then the following inequality holds*

$$\begin{aligned}
&\frac{n_2}{2}\|\partial_2 u\|_{\dot{H}^2}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2\partial_2 u\cdot\nabla^2 b\,\mathrm{d}x\\
\le{}& C\|(u,b)\|_{H^3}\|(\nabla_h u, \nabla_h b)\|_{H^2}^2 + \frac{n_1}{4}\|\partial_1 b\|_{\dot{H}^2}^2 + C\|(\partial_2^{1+\beta}b, \partial_1^{1+\alpha}u)\|_{\dot{H}^2}^2.
\end{aligned} \tag{2.5}$$

Proof. Applying ∇^2 to the second equation of (1.1) and taking the inner product with $\nabla^2\partial_2 u$, we obtain

$$n_2\|\partial_2 u\|_{\dot{H}^2}^2 = \int_\Omega \nabla^2\partial_t b\cdot\nabla^2\partial_2 u\,\mathrm{d}x + \mu\int_\Omega \nabla^2\partial_2^{2\beta}b\cdot\nabla^2\partial_2 u\,\mathrm{d}x$$

$$
\begin{aligned}
&- n_1 \int_\Omega \nabla^2 \partial_1 u \cdot \nabla^2 \partial_2 u \, \mathrm{d}x + \int_\Omega \nabla^2 (u \cdot \nabla b) \cdot \nabla^2 \partial_2 u \, \mathrm{d}x \\
&- \int_\Omega \nabla^2 (b \cdot \nabla u) \cdot \nabla^2 \partial_2 u \, \mathrm{d}x \\
&\overset{\text{def}}{=} \sum_{j=1}^{5} L_j.
\end{aligned}
$$

For the linear terms L_2 and L_3, by applying Hölder's and Young's inequalities directly, we have

$$
L_2 \le \frac{n_2}{4} \left\| \partial_2 u \right\|_{\dot{H}^2}^2 + C \left\| \partial_2^{2\beta} b \right\|_{\dot{H}^2}^2, \quad L_3 \le \frac{n_2}{4} \|\partial_2 u\|_{\dot{H}^2}^2 + C \|\partial_1 u\|_{\dot{H}^2}^2.
$$

For the term L_1, by a direct computation, we get

$$
\begin{aligned}
L_1 &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 u \, \mathrm{d}x - \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 \partial_t u \, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 u \, \mathrm{d}x \\
&\quad - \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 \Big(- \nu \partial_1^{2\alpha} u - \nabla p - u \cdot \nabla u + b \cdot \nabla b + n_h \cdot \nabla_h b \Big) \, \mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 u \, \mathrm{d}x + C \left\| \partial_1^{2\alpha} u \right\|_{\dot{H}^2}^2 + C \left\| \partial_2 b \right\|_{\dot{H}^2}^2 + \frac{n_1}{4} \left\| \partial_1 b \right\|_{\dot{H}^2}^2 \\
&\quad - \int_\Omega \nabla^2 \partial_2 b \cdot \nabla^2 \left(u \cdot \nabla u - b \cdot \nabla b \right) \, \mathrm{d}x \\
&\overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \nabla^2 b \cdot \nabla^2 \partial_2 u \, \mathrm{d}x + C \left\| \partial_1^{2\alpha} u \right\|_{\dot{H}^2}^2 + C \left\| \partial_2 b \right\|_{\dot{H}^2}^2 + \frac{n_1}{4} \left\| \partial_1 b \right\|_{\dot{H}^2}^2 + L_{11} + L_{12}.
\end{aligned}
$$

Note that the terms L_{11}, L_{12}, L_4, and L_5 share the same structure. Similar to the proof of Lemma 2.4, we only need to estimate

$$
\begin{aligned}
M &= \int_\Omega \nabla^2 \partial_2 \mathbf{f} \cdot \nabla^2 (\mathbf{g} \cdot \nabla \mathbf{h}) \, \mathrm{d}x \\
&= \int_\Omega \nabla_h^2 \partial_2 \mathbf{f} \cdot \nabla_h^2 (\mathbf{g} \cdot \nabla \mathbf{h}) \, \mathrm{d}x + \int_\Omega \partial_3^2 \partial_2 \mathbf{f} \cdot \partial_3^2 (\mathbf{g}_h \cdot \nabla_h \mathbf{h}) \, \mathrm{d}x \\
&\quad + \int_\Omega \partial_3^2 \partial_2 \mathbf{f} \cdot \partial_3^2 (g_3 \partial_3 \mathbf{h}) \, \mathrm{d}x \\
&\overset{\text{def}}{=} M_1 + M_2 + M_3.
\end{aligned}
$$

For M_1, by a direct application of Hölder's inequality, Young's inequality, and Sobolev embedding, we obtain

$$
\begin{aligned}
M_1 &= \int_\Omega \nabla_h^2 \partial_2 \mathbf{f} \cdot \nabla_h^2 \mathbf{g} \cdot \nabla \mathbf{h} \, \mathrm{d}x + 2 \int_\Omega \nabla_h^2 \partial_2 \mathbf{f} \cdot \nabla_h \mathbf{g} \cdot \nabla \nabla_h \mathbf{h} \, \mathrm{d}x \\
&\quad + \int_\Omega \nabla_h^2 \partial_2 \mathbf{f} \cdot \mathbf{g} \cdot \nabla \nabla_h^2 \mathbf{h} \, \mathrm{d}x \\
&\le \|\nabla \mathbf{h}\|_{L^\infty} \left\| \nabla_h^2 \partial_2 \mathbf{f} \right\|_{L^2} \left\| \nabla_h^2 \mathbf{g} \right\|_{L^2} + 2 \left\| \nabla_h^2 \partial_2 \mathbf{f} \right\|_{L^2} \left\| \nabla_h \mathbf{g} \right\|_{L^4} \left\| \nabla \nabla_h \mathbf{h} \right\|_{L^4} \\
&\quad + \|\mathbf{g}\|_{L^\infty} \left\| \nabla_h^2 \partial_2 \mathbf{f} \right\|_{L^2} \left\| \nabla \nabla_h^2 \mathbf{h} \right\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|\mathbf{h}\|_{H^3}\|\nabla_h\mathbf{f}\|_{H^2}\|\nabla_h\mathbf{g}\|_{H^2}+\|\mathbf{f}\|_{H^3}\|\nabla_h\mathbf{g}\|_{H^2}\|\nabla_h\mathbf{h}\|_{H^2}\\
&\quad+\|\mathbf{g}\|_{H^3}\|\nabla_h\mathbf{f}\|_{H^2}\|\nabla_h\mathbf{h}\|_{H^2}\\
&\lesssim \|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2.
\end{aligned}
$$

The estimate for M_2 is similar to that for M_1

$$
\begin{aligned}
M_2 &= \int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot(\partial_3^2\mathbf{g}_h\cdot\nabla_h)\mathbf{h}\,\mathrm{d}x+2\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot(\partial_3\mathbf{g}_h\cdot\nabla_h)\partial_3\mathbf{h}\,\mathrm{d}x\\
&\quad+\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot(\mathbf{g}_h\cdot\nabla_h)\partial_3^2\mathbf{h}\,\mathrm{d}x\\
&\le \|\nabla_h\mathbf{h}\|_{L^\infty}\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\left\|\partial_3^2\mathbf{g}\right\|_{L^2}+2\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|\partial_3\mathbf{g}\|_{L^4}\|\partial_3\nabla_h\mathbf{h}\|_{L^4}\\
&\quad+\|\mathbf{g}\|_{L^\infty}\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\left\|\nabla_h\partial_3^2\mathbf{h}\right\|_{L^2}\\
&\lesssim \|\mathbf{g}\|_{H^3}\|\nabla_h\mathbf{f}\|_{H^2}\|\nabla_h\mathbf{h}\|_{H^2}\\
&\lesssim \|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2.
\end{aligned}
$$

For M_3, a more refined estimate is required

$$
\begin{aligned}
M_3 &= \int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot\partial_3^2 g_3\partial_3\mathbf{h}\,\mathrm{d}x+2\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot\partial_3 g_3\partial_3^2\mathbf{h}\,\mathrm{d}x\\
&\quad+\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot g_3\partial_3^3\mathbf{h}\,\mathrm{d}x\\
&= -\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot\partial_3(\nabla_h\cdot\mathbf{g}_h)\partial_3\mathbf{h}\,\mathrm{d}x-2\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot(\nabla_h\cdot\mathbf{g}_h)\partial_3^2\mathbf{h}\,\mathrm{d}x\\
&\quad+\int_\Omega \partial_3^2\partial_2\mathbf{f}\cdot g_3\partial_3^3\mathbf{h}\,\mathrm{d}x\\
&\le \|\partial_3\mathbf{h}\|_{L^\infty}\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|\partial_3\nabla_h\mathbf{g}\|_{L^2}+2\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|\nabla_h\mathbf{g}\|_{L^4}\left\|\partial_3^2\mathbf{h}\right\|_{L^4}\\
&\quad+\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|g_3\|_{L^\infty}\left\|\partial_3^3\mathbf{h}\right\|_{L^2}\\
&\lesssim \|\partial_3\mathbf{h}\|_{H^2}\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|\partial_3\nabla_h\mathbf{g}\|_{L^2}+\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|\nabla_h\mathbf{g}\|_{H^1}\left\|\partial_3^2\mathbf{h}\right\|_{H^1}\\
&\quad+\left\|\partial_3^2\partial_2\mathbf{f}\right\|_{L^2}\|g_3\|_{H^2}\left\|\partial_3^3\mathbf{h}\right\|_{L^2}\\
&\lesssim \|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2,
\end{aligned}
$$

where we have used Lemma 2.3 to bound $\|g_3\|_{H^2}\le\|\nabla_h\mathbf{g}\|_{H^2}$. Therefore, combining the conditions $0<\alpha,\beta\le 1$ with the estimates for the term M completes the proof. □

Lemma 2.7. *Let $(u,b)\in L^\infty(0,\infty;H^3)$ be a solution to the system (1.1). Then the following inequality holds*

$$
\begin{aligned}
&\frac{n_1}{2}\|\partial_1 b\|_{\dot H^2}^2-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2\partial_1 b\cdot\nabla^2 u\,\mathrm{d}x\\
&\le C\|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2+C\|(\partial_2^{1+\beta}b,\partial_1^{1+\alpha}u)\|_{\dot H^2}^2.
\end{aligned}\tag{2.6}
$$

Proof. Applying ∇^2 to the first equation of (1.1) and taking the inner product with $\nabla^2\partial_1 b$, we obtain

$$
n_1\|\partial_1 b\|_{\dot H^2}^2=\int_\Omega \nabla^2\partial_t u\cdot\nabla^2\partial_1 b\,\mathrm{d}x+\nu\int_\Omega \nabla^2\partial_1^{2\alpha}u\cdot\nabla^2\partial_1 b\,\mathrm{d}x
$$

$$
\begin{aligned}
&- n_2\int_\Omega \nabla^2\partial_2 b\cdot\nabla^2\partial_1 b\,\mathrm{d}x+\int_\Omega \nabla^2(u\cdot\nabla u)\cdot\nabla^2\partial_1 b\,\mathrm{d}x\\
&-\int_\Omega \nabla^2(b\cdot\nabla b)\cdot\nabla^2\partial_1 b\,\mathrm{d}x\\
&\stackrel{\text{def}}{=}\sum_{j=1}^5 N_j.
\end{aligned}
$$

For the linear terms N_2 and N_3, by applying Hölder's and Young's inequalities directly, we have

$$
N_2\le \frac{n_1}{4}\,\|\partial_1 b\|_{\dot H^2}^2+C\left\|\partial_1^{2\alpha}u\right\|_{\dot H^2}^2,\quad N_3\le\frac{n_1}{4}\|\partial_1 b\|_{\dot H^2}^2+C\|\partial_2 b\|_{\dot H^2}^2.
$$

For the term N_1, by a direct computation, we get

$$
\begin{aligned}
N_1&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 u\cdot\nabla^2\partial_1 b\,\mathrm{d}x-\int_\Omega\nabla^2 u\cdot\nabla^2\partial_1\partial_t b\,\mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 u\cdot\nabla^2\partial_1 b\,\mathrm{d}x\\
&\quad-\int_\Omega \nabla^2 u\cdot\nabla^2\partial_1(-\mu\partial_2^{2\beta}b-(u\cdot\nabla b)+(b\cdot\nabla u)+n_h\cdot\nabla_h u)\,\mathrm{d}x\\
&\le\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 u\cdot\nabla^2\partial_1 b\,\mathrm{d}x+C\left\|\partial_2^{2\beta}b\right\|_{\dot H^2}^2+C\,\|\partial_1 u\|_{\dot H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2\\
&\quad-\int_\Omega \nabla^2\partial_1 u\cdot\nabla^2(u\cdot\nabla b-b\cdot\nabla u)\,\mathrm{d}x\\
&\stackrel{\text{def}}{=}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2 u\cdot\nabla^2\partial_1 b\,\mathrm{d}x+C\left\|\partial_2^{2\beta}b\right\|_{\dot H^2}^2+C\,\|\partial_1 u\|_{\dot H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2+N_{11}+N_{12}.
\end{aligned}
$$

The remaining nonlinear part of the proof is similar to Lemma 2.6 and we omit it. □

3. Proof of Theorem 1.1

This section concerns the global well-posedness result stated in Theorem 1.1. Since the energy estimates have been established, it remains to complete the bootstrap argument.

First, from Lemma 2.6 and Lemma 2.7, we have

$$
\begin{aligned}
&\frac{n_1}{4}\|\partial_1 b\|_{\dot H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nabla^2\partial_2 u\cdot\nabla^2 b\,\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\nabla^2\partial_1 b\cdot\nabla^2 u\,\mathrm{d}x\\
&\le C\|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2+C\|(\partial_2^{1+\beta}b,\partial_1^{1+\alpha}u)\|_{\dot H^2}^2.
\end{aligned}\tag{3.1}
$$

Now, multiplying (2.4) by a positive constant A and adding it to (3.1), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{2}\|(\nabla_h u,\nabla_h b)(t)\|_{\dot H^2}^2-\int_\Omega \nabla^2\partial_2 u\cdot\nabla^2 b\,\mathrm{d}x-\int_\Omega \nabla^2\partial_1 b\cdot\nabla^2 u\,\mathrm{d}x\right)\\
&\quad+A\nu\,\|\nabla_h\partial_1^\alpha u\|_{\dot H^2}^2+A\mu\left\|\nabla_h\partial_2^\beta b\right\|_{\dot H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2+\frac{n_1}{4}\|\partial_1 b\|_{\dot H^2}^2\\
&\le(A+1)C\|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2+C\|(\partial_1^{1+\alpha}u,\partial_2^{1+\beta}b)\|_{\dot H^2}^2.
\end{aligned}
$$

Define

$$
F(t)=\frac{A}{2}\|(\nabla_h u,\nabla_h b)(t)\|_{\dot H^2}^2-\int_\Omega \nabla^2\partial_2 u\cdot\nabla^2 b\,\mathrm{d}x-\int_\Omega\nabla^2\partial_1 b\cdot\nabla^2 u\,\mathrm{d}x
$$

and

$$D(t)=\frac{A\nu}{2}\|\nabla_h\partial_1^{\alpha}u\|_{\dot H^2}^2+\frac{A\mu}{2}\|\nabla_h\partial_2^{\beta}b\|_{\dot H^2}^2+\frac{n_2}{4}\|\partial_2 u\|_{\dot H^2}^2+\frac{n_1}{4}\|\partial_1 b\|_{\dot H^2}^2.$$

By choosing A sufficiently large and $\delta_0>0$ sufficiently small, we can ensure that the following inequalities hold

$$\begin{aligned}&\|(\nabla_h u,\nabla_h b)(t)\|_{H^2}^2\le F(t)\le A\|(\nabla_h u,\nabla_h b)(t)\|_{H^2}^2,\\&\delta_0 F(t)\le D(t).\end{aligned}$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t)+\delta_0F(t)\le C_0\|(u,b)\|_{H^3}\|(\nabla_h u,\nabla_h b)\|_{H^2}^2.$$

To start the bootstrap argument process, we make the ansatz

$$E_1(t)\le 2\varepsilon_0^2.$$

By choosing $\varepsilon_0>0$ small, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t)+\frac{\delta_0}{2}F(t)\le 0.$$

Solving this differential inequality yields

$$F(t)\le e^{-\frac{1}{2}\delta_0 t}F(0),$$

which implies

$$\int_0^t\|(\nabla_h u,\nabla_h b)(\tau)\|_{H^2}\,\mathrm{d}\tau\le C\|(\nabla_h u_0,\nabla_h b_0)\|_{H^2}.$$

From Lemma 2.4, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b)(t)\|_{H^3}^2+\nu\|\partial_1^{\alpha}u\|_{H^3}^2+\mu\|\partial_2^{\beta}b\|_{H^3}^2\le C\|(\nabla_h u,\nabla_h b)\|_{H^2}\|(u,b)\|_{H^3}^2.$$

Integrating in t, we obtain

$$E_1(t)\le E_1(0)+CE_1(t)\|(\nabla_h u_0,\nabla_h b_0)\|_{H^2}.$$

By choosing $\varepsilon_0>0$ sufficiently small, we can deduce a stronger bound for $E_1(t)$. The bootstrap argument then concludes that the stronger bound holds for all $t>0$. This completes the proof of Theorem 1.1.

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