

BÄCKLUND TRANSFORMATION AND WEIERSTRASS ELLIPTIC FUNCTION SOLUTIONS OF THE GENERALIZED (2+1)-DIMENSIONAL COUPLED DAVEY-STEWARTSON EQUATION IN SHALLOW WATER WAVES

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Abstract This paper investigates the generalized (2+1)-dimensional coupled Davey-Stewartson (DS) equation, which describes the motions of long and short waves in shallow waters and has extensive applications in plasma physics. First, we study the Weierstrass elliptic function solutions using the auxiliary function method. Then, the bilinear Bäcklund transformation (BT) and Bell polynomial Bäcklund transformation are constructed using Bell polynomial method. Moreover, explicit 1-soliton, explicit 2-soliton, and explicit 3-soliton solutions are obtained from the derived bilinear form. Finally, the physical behaviors of the solutions are illustrated through three-dimensional plots, density plots, and two-dimensional planar graphs.

Keywords Generalized (2+1)-dimensional coupled Davey-Stewartson equation, Bell polynomial method, Bäcklund transformation, Weierstrass elliptic function solutions.

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1. Introduction

Nonlinear evolution equations (NLEEs) are widely applied in the natural sciences. Solitons have been discovered in various fields such as fluid dynamics, plasma physics, nonlinear optics, astronomy and condensed matter physics. These fields can be addressed using mathematical models. In nonlinear optics, optical soliton communication has the advantages of large information capacity, reduced dispersion, long transmission distance, low loss, and strong confidentiality. The nonlinear Schrödinger equation (NLSE) [8, 41, 44, 48, 52]

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \quad (1.1)$$

is widely to describe the propagation of light pulses in nonlinear media. Here, $u = u(x, t)$ is a complex function. In the mid-20th century, A. Davey and K. Stewartson first proposed the DS equation [14]

$$iE_t + \delta E_{xx} + E_{yy} = a|E|^2 E + bEN_x, \quad (1.2a)$$

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$$N_{xx} + cN_{yy} = (|E|^2)_x. \quad (1.2b)$$

This equation is commonly adopted to describe quasi-monochromatic wave packets in water of finite depth. Here, δ, a, b, c are four undetermined parameters, $E = E(x, y, t)$ is a complex function, and $N = N(x, y, t)$ is a real function. The Eq. (1.2) describes the evolution of single-frequency wave packets, particularly in the context of finite-depth water waves. When $|\delta| = |a| = 1$, Eq. (1.2) reduces to a high-dimensional, fully integrable generalization of the NLSE. Subsequently, Fokas [16] extended the nonlocal NLSE to (2+1)-dimensional version and proposed a new (2+1)-dimensional DS equation

$$iE_t + E_{xx} - E_{yy} - 2(|E|^2 + N)E = 0, \quad (1.3a)$$

$$N_{xx} + N_{yy} + 2(|E|^2)_{xx} = 0, \quad (1.3b)$$

where $E = E(x, y, t)$ is a complex function, and $N = N(x, y, t)$ is a real function. In Refs. [38, 46], dark solitons and rogue wave solutions of Eq. (1.3), expressed in Gram determinant form, were constructed via the KP hierarchical approximation method and the Hirota bilinear method. For Eq. (1.3), solitary wave solutions were derived in Refs. [15, 37] using the G'/G method and the ansatz method. Meanwhile, exact solutions to the Fokas system and Eq. (1.3) were obtained in Refs. [3, 4] through the generalized auxiliary equation mapping method. The Hirota bilinear method was employed in Ref. [6] to construct line soliton and periodic soliton solutions of Eq. (1.3), while operator-like fixed-line solutions of Eq. (1.3) were established in Ref. [47] using the double exponential function method.

In this paper, we will use the Bell polynomial method to study the Bäcklund transformation and exact solutions for the generalized (2+1)-dimensional coupled DS equation

$$iE_t = \beta_1 E_{xx} + \beta_2 E_{yy} + (\delta|E|^2 - \beta_3 N)E, \quad (1.4a)$$

$$\alpha_1 N_{xx} - \alpha_2 N_{yy} = \delta(|E|^2)_{xx}, \delta = \pm 1, \quad (1.4b)$$

where $\beta_i, (i = 1, 2, 3)$ and $\alpha_j, (j = 1, 2)$ are arbitrary constants. Here, $E = E(x, y, t)$ is a complex function representing the amplitude of the water wave, while $N = N(x, y, t)$ is a real function corresponding to the speed of wave propagation. When $\alpha_1 = \beta_2 = 1$, the Eq. (1.4) can be reduced to the (2+1)-dimensional DSI equation, whereas for $\alpha_1 = \beta_2 = -1$, it can be reduced to the (2+1)-dimensional DSII equation. The (2+1)-dimensional DS equation [7, 9, 17, 23, 27–29, 31, 51] is widely used to describe the coupled dynamics of long and short waves in water of finite depth. It is also commonly applied in contexts where surface tension effects are significant. For these reasons, a systematic study of this equation is of considerable physical and mathematical interest.

The rest of this paper is organized as follows. In Section 2, Weierstrass elliptic function solutions of Eq. (1.4) are constructed via the traveling wave transformation method and the auxiliary equation method. In Section 3, the bilinear form of Eq. (1.4) is derived using the P -polynomial and $\mathcal{Y}$ -polynomial, followed by the presentation of its BT. In Section 4, soliton solutions are obtained based on the bilinear form derived in Section 3. In Section 5, the main results of this work are summarized and conclusions are provided.

2. Weierstrass elliptic function solutions

Introducing the traveling wave transformation

$$E = E(\xi) = u(\xi)e^{i\theta}, N = N(\xi) = \nu(\xi), \xi = x + \tau y + \lambda t, \theta = kx + ly + wt, \quad (2.1)$$

where $u(\xi)$ and $\nu(\xi)$ are real functions, and τ, λ, k, l, w are any constants. Substituting the traveling wave transformation (2.1) into Eq. (1.4), we can obtain

$$(\beta_1 + \beta_2\tau^2)u_{\xi\xi} + (2ik\beta_1 - i\lambda + 2i\tau l\beta_2)u_\xi + (w - k^2\beta_1 - l^2\beta_2)u + \delta u^3 - \beta_3 u\nu = 0, \quad (2.2a)$$

$$(\alpha_2\tau^2 - \alpha_1)\nu_{\xi\xi} + 2\delta u_\xi^2 + 2\delta u u_{\xi\xi} = 0. \quad (2.2b)$$

It can be obtained by integrating both sides of Eq. (2.2b) twice with respect to ξ . We have

$$(\alpha_1 - \alpha_2\tau^2)\nu = \delta u^2 + \overline{C}, \quad (2.3)$$

where $\overline{C}$ is the integration constant. Substituting Eq. (2.3) into Eq. (2.2a), we can obtain

$$\begin{aligned} & (\beta_1 + \beta_2\tau^2)u_{\xi\xi} + (w - k^2\beta_1 - l^2\beta_2)u + \delta u^3 \\ & - \beta_3 u \left(\frac{\delta}{\alpha_1 - \alpha_2\tau^2} u^2 + \frac{\overline{C}}{\alpha_1 - \alpha_2\tau^2} \right) + (2k\beta_1 - \lambda + 2\tau l\beta_2)iu_\xi = 0. \end{aligned} \quad (2.4)$$

We set the imaginary and real parts of the above expression to zero, respectively

$$(2k\beta_1 - \lambda + 2\tau l\beta_2)iu_\xi = 0, \quad (2.5a)$$

$$(\beta_1 + \beta_2\tau^2)u_{\xi\xi} + (w - k^2\beta_1 - l^2\beta_2)u + \delta u^3 - \beta_3 u \left(\frac{\delta}{\alpha_1 - \alpha_2\tau^2} u^2 + \frac{\overline{C}}{\alpha_1 - \alpha_2\tau^2} \right) = 0. \quad (2.5b)$$

From Eq. (2.5a), we have

$$k = \frac{\lambda - 2l\tau\beta_2}{2\beta_1}. \quad (2.6)$$

Substituting (2.6) into Eq. (2.5b), we have

$$(\beta_1 + \beta_2\tau^2)u_{\xi\xi} + \left[w - \frac{(\lambda - 2l\tau\beta_2)^2}{4\beta_1} - l^2\beta_2 \right] u + \delta u^3 - \beta_3 u \left(\frac{\delta}{\alpha_1 - \alpha_2\tau^2} u^2 + \frac{\overline{C}}{\alpha_1 - \alpha_2\tau^2} \right) = 0. \quad (2.7)$$

The equilibrium constant $m = 1$ can be obtained from Eq. (2.7). Therefore, the truncation series can be solved as

$$u(\xi) = a_0 + a_1 F(\xi). \quad (2.8)$$

We select the following auxiliary equation

$$(F'(\xi))^2 = c_0 + c_1 F(\xi) + c_2 F^2(\xi), \quad (2.9)$$

where c_0, c_1 and c_2 are undetermined constants. The auxiliary Eq. (2.9) has the following form of Weierstrass elliptic function solutions

$$F(\xi) = \begin{cases} -\frac{c_1}{2c_2} + \frac{\wp'(\xi, g_2, g_3)}{2c_2[\wp(\xi, g_2, g_3) - \frac{\Re}{12}]}, \\ -\frac{(2c_0 + c_1)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) - \wp'(\xi, g_2, g_3)}{(c_1 + 2c_2)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) + \wp'(\xi, g_2, g_3)}, \\ \frac{\frac{1}{2}c_0c_1[\wp(\xi, g_2, g_3) - \frac{\Re}{12}] - \frac{1}{2}c_0\wp'(\xi, g_2, g_3)}{[\wp(\xi, g_2, g_3) - \frac{c_0c_2}{6} - \frac{c_1^2}{12}]^2 - \frac{c_0^2c_2^2}{4}}, \end{cases} \quad (2.10)$$

with

$$g_2 = \frac{\Re^2}{12}, g_3 = -\frac{\Re^3}{216}, \Re = c_1^2 - 4c_0c_2. \quad (2.11)$$

Substituting formulas (2.8)-(2.11) into Eq. (2.7), and setting the coefficients with respect to $F^j(\xi)$ ($j = 0, 1, 2, 3, 4$) to be zero. We can obtain a set of over-determined equations as

$$\begin{cases} a_0w + a_0^3\delta + a_1c_0c_1\beta_1 - a_0l^2\beta_2 + a_1c_0c_1\tau^2\beta_2 \\ -\frac{a_0(\lambda - 2l\tau\beta_2)^2}{4\beta_1} + \frac{a_0\bar{C}\beta_3}{-\alpha_1 + l^2\alpha_2} + \frac{a_0^3\delta\beta_3}{-\alpha_1 + l^2\alpha_2} = 0, \\ a_1(c_1^2 + 2c_0c_2)\beta_1 - \frac{a_1(\lambda - 2l\tau\beta_2)^2}{4\beta_1} + a_1w + 3a_0^2a_1\delta \\ + [-l^2 + (c_1^2 + 2c_0c_2)\tau^2]a_1\beta_2 - \frac{(\bar{C} + 3a_0^2\delta)\beta_3a_1}{\alpha_1 - l^2\alpha_2} = 0, \\ 3a_1[c_1c_2\beta_1 + c_1c_2\tau^2\beta_2 + a_0a_1\delta(1 + \frac{\beta_3}{-\alpha_1 + l^2\alpha_2})] = 0, \\ 2a_1c_2^2\beta_1 + 2a_1c_2^2\tau^2\beta_2 + a_1^3\delta(1 + \frac{\beta_3}{-\alpha_1 + l^2\alpha_2}) = 0. \end{cases} \quad (2.12)$$

Solving the system of algebraic equations, the following two sets of solutions are obtained:

Case 1.

$$\begin{aligned} c_1 &= \frac{i\sqrt{2\delta(\alpha_1 - l^2\alpha_2 - \beta_3)}a_0}{(\alpha_1 - l^2\alpha_2)(\beta_1 + \tau^2\beta_2)}, \\ c_2 &= \frac{(\alpha_1 - l^2\alpha_2)[-4\beta_1(w + a_0^2\delta - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4(\bar{C} + a_0^2\delta)\beta_1\beta_3}{8c_0(\alpha_1 - l^2\alpha_2)\beta_1(\beta_1 + \tau^2\beta_2)}, \\ a_1 &= \frac{i\{(\alpha_1 - l^2\alpha_2)[-4\beta_1(w + a_0^2\delta - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4(\bar{C} + a_0^2\delta)\beta_1\beta_3\}}{4\sqrt{2}c_0\sqrt{\delta}\beta_1\sqrt{(\alpha_1 - l^2\alpha_2)(\beta_1 + \tau^2\beta_2)}\sqrt{\alpha_1 - l^2\alpha_2 - \beta_3}}. \end{aligned} \quad (2.13)$$

Case 2.

$$\begin{aligned} c_1 = 0, c_2 &= \frac{(\alpha_1 - l^2\alpha_2)[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\bar{C}\beta_1\beta_3}{8c_0(\alpha_1 - l^2\alpha_2)\beta_1(\beta_1 + \tau^2\beta_2)}, \\ a_0 = 0, a_1 &= \frac{i\{(\alpha_1 - l^2\alpha_2)[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\bar{C}\beta_1\beta_3\}}{4\sqrt{2}c_0^2\delta(\alpha_1 - l^2\alpha_2)\beta_1^2(\beta_1 + \tau^2\beta_2)(\alpha_1 - l^2\alpha_2 - \beta_3)}. \end{aligned} \quad (2.14)$$

From the above formula (2.13) and formula (2.11), we can obtain

$$\begin{aligned} g_2 &= \frac{\{\alpha_1[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] - l^2\alpha_2[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\bar{C}\beta_1\beta_3\}^2}{48(\alpha_1 - l^2\alpha_2)^2\beta_1^2(\beta_1 + \tau^2\beta_2)^2}, \\ g_3 &= \frac{\{\alpha_1[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] - l^2\alpha_2[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\bar{C}\beta_1\beta_3\}^3}{1728(\alpha_1 - l^2\alpha_2)^3\beta_1^3(\beta_1 + \tau^2\beta_2)^3}, \\ \Re &= \frac{-\alpha_1[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] + l^2\alpha_2[-4\beta_1(w - l^2\beta_2) + (\lambda - 2l\tau\beta_2)^2] - 4\bar{C}\beta_1\beta_3}{2(\alpha_1 - l^2\alpha_2)\beta_1(\beta_1 + \tau^2\beta_2)}. \end{aligned} \quad (2.15)$$

From formulas (2.8) to (2.13) and formula (2.15), the Weierstrass elliptic function solutions of Eq. (1.4) can be obtained as

$$E_1 = \left\{ -\frac{c_1}{2c_2} + \frac{\wp'(\xi, g_2, g_3)}{2c_2[\wp(\xi, g_2, g_3) - \frac{\Re}{12}]} \right\} e^{i\theta}. \quad (2.16)$$

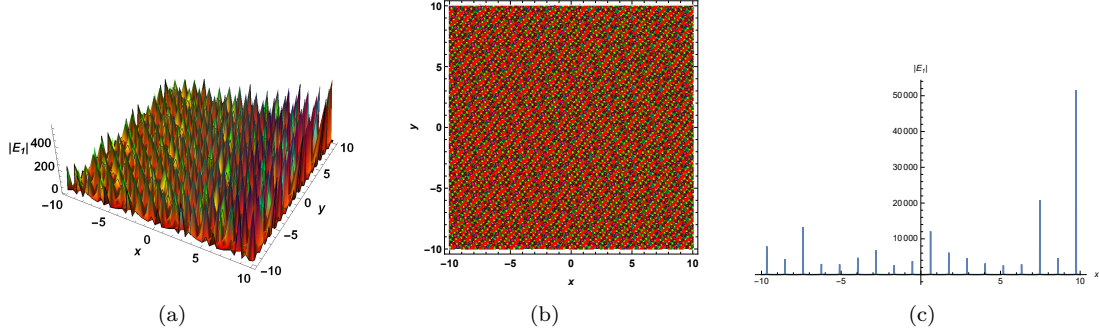


Figure 1. The propagation characteristics of expression E_1 when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \overline{C} = \lambda = 1, c_0 = 0.5, l = 4, \tau = 5, w = 3, \beta_3 = 2, \delta = -1$.

$$E_2 = \left\{ -\frac{(2c_0 + c_1)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) - \wp'(\xi, g_2, g_3)}{(c_1 + 2c_2)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) + \wp'(\xi, g_2, g_3)} \right\} e^{i\theta}. \quad (2.17)$$

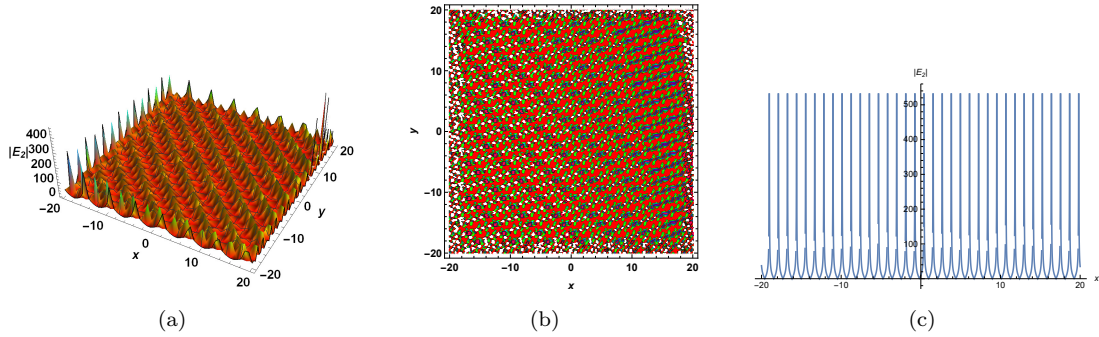


Figure 2. The propagation characteristics of expression E_2 when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \overline{C} = \lambda = 1, c_0 = 0.5, l = 4, \tau = 5, w = 3, \beta_3 = 2, \delta = -1$.

$$E_3 = \left\{ \frac{\frac{1}{2}c_0c_1[\wp(\xi, g_2, g_3) - \frac{\Re}{12}] - \frac{1}{2}c_0\wp'(\xi, g_2, g_3)}{[\wp(\xi, g_2, g_3) - \frac{c_0c_2}{6} - \frac{c_1^2}{12}]^2 - \frac{c_0^2c_2^2}{4}} \right\} e^{i\theta}. \quad (2.18)$$

Based on the three-dimensional plots in Figures 1, 2, and 3, it is evident that the waves evolved by different Weierstrass elliptic function solutions for the same values are also significantly different. Based on the features of the contour plots in the figures the amplitude as well as the phase of the waves can be analyzed in depth for different values. In addition, it can be clearly found that Figures 1, 2, and 3 show periodic variations in their x -direction. Based on the density maps, the shape of the images can be analyzed as well as the amplitude for different values of y .

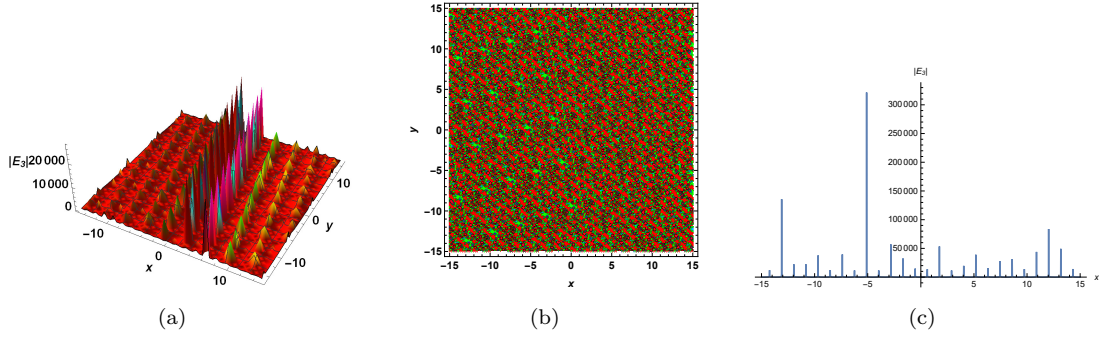


Figure 3. The propagation characteristics of expression $|E_3|$ when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \overline{C} = \lambda = 1, c_0 = 0.5, l = 4, \tau = 5, w = 3, \beta_3 = 2, \delta = -1$.

From formula (2.14) and formula (2.11), we have

$$\begin{aligned} g_2 &= \frac{\{(\alpha_1 - l^2 \alpha_2)[-4\beta_1(w - l^2 \beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\overline{C}\beta_1\beta_3\}^2}{48(\alpha_1 - l^2 \alpha_2)^2 \beta_1^2 (\beta_1 + \tau^2 \beta_2)^2}, \\ g_3 &= \frac{\{(\alpha_1 - l^2 \alpha_2)[-4\beta_1(w - l^2 \beta_2) + (\lambda - 2l\tau\beta_2)^2] + 4\overline{C}\beta_1\beta_3\}^2}{48(\alpha_1 - l^2 \alpha_2)^2 \beta_1^2 (\beta_1 + \tau^2 \beta_2)^2}, \\ \Re &= \frac{-4w + 4l^2 \beta_2 + \frac{(\lambda - 2l\tau\beta_2)^2}{\beta_1} + \frac{4C\beta_3}{\alpha_1 - l^2 \alpha_2}}{2(\beta_1 + \tau^2 \beta_2)}. \end{aligned} \quad (2.19)$$

From formulas (2.8) to (2.10) and (2.19), the Weierstrass elliptic function solutions of Eq. (1.4) can be obtained as

$$E_4 = \left\{ \frac{\wp'(\xi, g_2, g_3)}{2c_2[\wp(\xi, g_2, g_3) - \frac{\Re}{12}]} \right\} e^{i\theta}. \quad (2.20)$$

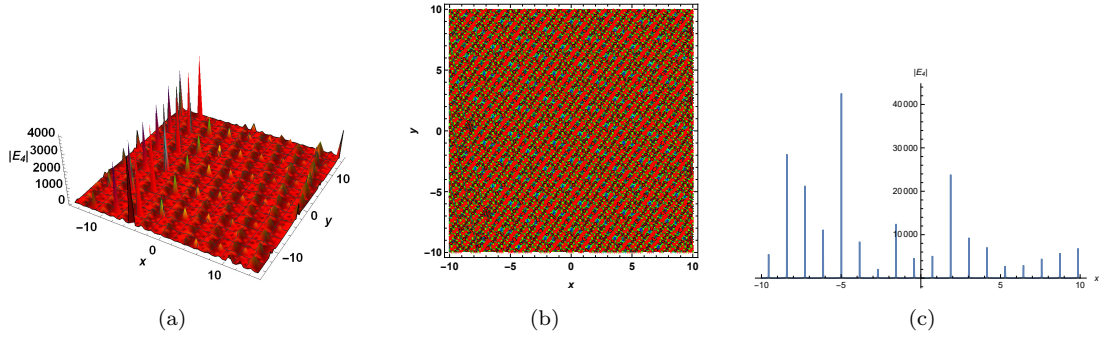


Figure 4. The propagation characteristics of expression E_4 when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \delta = \overline{C} = \lambda = c_0 = 1, l = 4, \tau = 5, w = 3, \beta_3 = 2$.

$$E_5 = \left\{ -\frac{(2c_0 + c_1)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) - \wp'(\xi, g_2, g_3)}{(c_1 + 2c_2)(\wp(\xi, g_2, g_3) - \frac{\Re}{12}) + \wp'(\xi, g_2, g_3)} \right\} e^{i\theta}, \quad (2.21)$$

$$E_6 = \left\{ \frac{\frac{1}{2}c_0c_1[\wp(\xi, g_2, g_3) - \frac{\Re}{12}] - \frac{1}{2}c_0\wp'(\xi, g_2, g_3)}{[\wp(\xi, g_2, g_3) - \frac{c_0c_2}{6} - \frac{c_1^2}{12}]^2 - \frac{c_0^2c_2^2}{4}} \right\} e^{i\theta}. \quad (2.22)$$

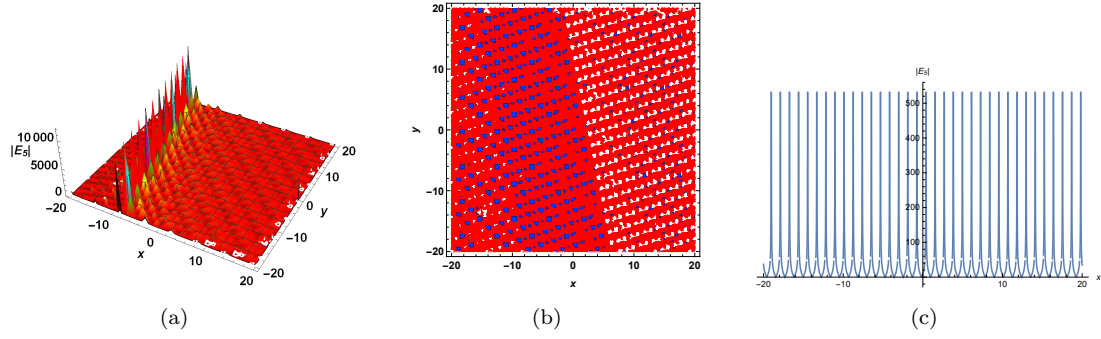


Figure 5. The propagation characteristics of expression E_5 when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \delta = \overline{C} = \lambda = c_0 = 1, l = 4, \tau = 5, w = 3, \beta_3 = 2$.

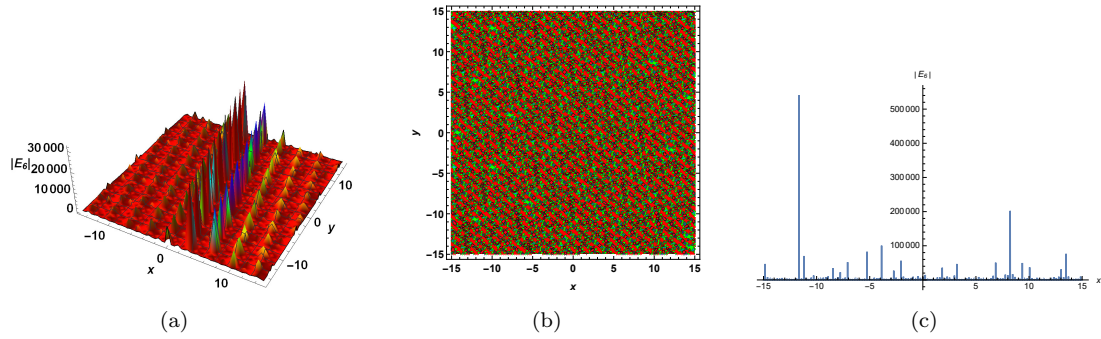


Figure 6. The propagation characteristics of expression E_6 when the parameters are $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \delta = \overline{C} = \lambda = c_0 = 1, l = 4, \tau = 5, w = 3, \beta_3 = 2$.

Figures 4, 5, and 6 show the dynamical profiles of the Weierstrass elliptic function solutions of Eq. (1.4). The solutions in Figures 5 and 6 are periodic steady waves propagating unidirectionally at a constant speed. The wave formed in Figure 6 is a regular periodic wave, whose amplitude and pulse width remain unchanged over time, and only undergo phase shifts with time evolution.

In the previous section, Weierstrass elliptic function solutions of Eq. (1.4) were directly constructed by using the homogeneous balance method and the auxiliary equation method. By selecting appropriate functional forms, these methods efficiently yield explicit exact solutions. However, in order to further investigate the integrability of the equation and derive a richer variety of solution structures, the Bell polynomial method is employed in Section 3. Unlike the approaches used in the previous section, the Bell polynomial method transforms the original equation into a bilinear form through an appropriate dependent variable transformation. This transformation not only facilitates the derivation of soliton solutions but also enables the construction of Bäcklund transformation, thereby revealing the integrability properties of the equation in a more systematic and rigorous manner.

3. Bilinear form and Bäcklund transformation

3.1. Bilinear form

We introduce the following transformations

$$N = N(x, y, t) = \sigma_2 q_{xx}, E = E(x, y, t) = \sigma_1 e^p, \quad (3.1)$$

where σ_1, σ_2 are arbitrary constants, and $q = q(x, y, t)$ is a real function. $p = p(x, y, t)$ is a complex function. Substituting the transformation (3.1) into Eq. (1.4), yields

$$i\sigma_1 p_t = \beta_1 \sigma_1 (p_{xx} + p_x^2) + \beta_2 \sigma_1 (p_{yy} + p_y^2) + (\delta |\sigma_1 e^p|^2 - \beta_3 \sigma_2 q_{xx}) \sigma_1, \quad (3.2a)$$

$$\alpha_1 \sigma_2 q_{xxxx} - \alpha_2 \sigma_2 q_{xxyy} = \delta (|\sigma_1 e^p|^2)_{xx}. \quad (3.2b)$$

Integrating Eq. (3.2b) simultaneously and taking the integration constant to zero, yields

$$\alpha_1 \sigma_2 q_{xx} - \alpha_2 \sigma_2 q_{yy} = \delta (|\sigma_1 e^p|^2). \quad (3.3)$$

Substituting Eq. (3.3) into Eq. (3.2a), we have

$$ip_t = \beta_1 (p_{xx} + p_x^2) + \beta_2 (p_{yy} + p_y^2) + \sigma_2 (\alpha_1 q_{xx} - \alpha_2 q_{yy} - \beta_3 q_{xx}). \quad (3.4)$$

Based on the Bell polynomial method and $q = \hbar - p$, the above formula is reduced to $\mathcal{Y}$ -polynomial form and $\mathcal{P}$ -polynomial form

$$\begin{aligned} i\mathcal{Y}(\hbar, p)_t &= \beta_1 \mathcal{Y}_{xx}(\hbar, p) + \beta_2 \mathcal{Y}_{yy}(\hbar, p), \\ \alpha_1 \sigma_2 \mathcal{P}_{xx}(q) - \alpha_2 \sigma_2 \mathcal{P}_{yy}(q) &= \delta (|\sigma_1 e^p|^2). \end{aligned} \quad (3.5)$$

The constraint conditions are

$$\beta_1 = \sigma_2 (\alpha_1 - \beta_3), \beta_2 = -\sigma_2 \alpha_2. \quad (3.6)$$

Introducing transformations as follows:

$$p = \ln\left(\frac{g}{f}\right), \hbar = \ln(gf), q = \hbar - p = 2 \ln(f). \quad (3.7)$$

Then Eq. (3.2) can be reduced to a bilinear form, which is

$$\begin{aligned} [\sigma_2 \alpha_2 D_y^2 + iD_t - \sigma_2 (\alpha_1 - \beta_3) D_x^2] g \cdot f &= 0, \\ (\alpha_1 \sigma_2 D_x^2 - \alpha_2 \sigma_2 D_y^2) f \cdot f &= \delta \sigma_1^2 g g^*, \end{aligned} \quad (3.8)$$

where $f = f(x, y, t)$ is a real function, and $g = g(x, y, t)$ is a complex function. Here, g^* denotes the complex conjugate of g , and D is a bilinear operator defined as follows

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x=x', t=t'}.$$

3.2. Bäcklund transformation

Based on Eq. (3.5), we can construct the BT of Eq. (1.4). Because $p, \hbar, q$ are solutions that satisfy Eq. (3.5), it may be helpful to introduce a new variable relationship by making $\tilde{p}, \tilde{\hbar}, \tilde{q}$ also solutions that satisfy Eq. (3.5) simultaneously

$$\tilde{p} = \ln\left(\frac{\tilde{g}}{\tilde{f}}\right), \tilde{\hbar}' = \ln(\tilde{g}\tilde{f}), \tilde{q} = \tilde{\hbar} - \tilde{p} = 2 \ln(\tilde{f}). \quad (3.9)$$

Consider the following mixed variable transformations

$$v_1 = \ln(\frac{\tilde{g}}{g}), v_2 = \ln(\frac{\tilde{f}}{f}), v_3 = \ln(\frac{\tilde{f}}{g}), v_4 = \ln(\frac{\tilde{g}}{f}), v_5 = \ln(\frac{\tilde{h}}{h}), v_6 = \ln(\frac{\tilde{f}}{h}), v_7 = \ln(\frac{\tilde{h}}{f}), \quad (3.10a)$$

$$\omega_1 = \ln(\tilde{g}g), \omega_2 = \ln(f\tilde{f}), \omega_3 = \ln(\tilde{f}g), \omega_4 = \ln(\tilde{g}f), \omega_5 = \ln(\tilde{h}h), \omega_6 = \ln(\tilde{f}h), \omega_7 = \ln(\tilde{h}f), \quad (3.10b)$$

$$\omega_5 = \ln(\tilde{h}h), \omega_6 = \ln(\tilde{f}h), \omega_7 = \ln(\tilde{h}f), \quad (3.10c)$$

$$\tilde{p} - p = v_1 - v_2 = \omega_4 - \omega_3, \tilde{p} + p = v_4 - v_3 = \omega_1 - \omega_2, \quad (3.10d)$$

$$\tilde{h} - h = v_1 + v_2 = v_4 + v_3, \tilde{h} + h = \omega_4 + \omega_3 = \omega_1 + \omega_2, \quad (3.10e)$$

$$\tilde{p} + \tilde{p}^* = v_1 - v_3 + v_7 - v_2, p + p^* = v_2 - v_3 + v_7 - v_5. \quad (3.10f)$$

Substituting the above transformations of variable into Eq.(3.5), there is

$$\begin{aligned} & \alpha_1 \sigma_2 (\tilde{h} - \tilde{p})_{xx} - \alpha_2 \sigma_2 (\tilde{h} - \tilde{p})_{yy} - \delta \sigma_1 e^{\tilde{p} + \tilde{p}^*} - [\alpha_1 \sigma_2 (\tilde{h} - p)_{xx} - \alpha_2 \sigma_2 (\tilde{h} - p)_{yy} - \delta \sigma_1 e^{p + p^*}] \\ &= \alpha_1 \sigma_2 (v_1 + v_2)_{xx} - \alpha_1 \sigma_2 (v_1 - v_2)_{xx} + \frac{3\delta \sigma_1}{2} (e^{v_2 - v_3 + v_7 - v_5} - e^{v_1 - v_3 + v_7 - v_2}) \\ & \quad - [\alpha_2 \sigma_2 (v_1 + v_2)_{yy} - \alpha_2 \sigma_2 (v_1 - v_2)_{yy} + \frac{\delta \sigma_1}{2} (e^{v_2 - v_3 + v_7 - v_5} - e^{v_1 - v_3 + v_7 - v_2})] \\ & \quad - (v_7 - v_3)_x + \frac{3\delta \sigma_1}{2} (e^{v_2 - v_5} - e^{v_1 - v_2}) - [-(v_7 - v_3)_y + \frac{\delta \sigma_1}{2} (e^{v_2 - v_5} - e^{v_1 - v_2})] \\ &= 0. \end{aligned} \quad (3.11)$$

The selection constraints are

$$\begin{aligned} v_{2,x} &= -\frac{1}{2\alpha_1 \sigma_2} e^{v_7 - v_3}, v_{2,y} = -\frac{1}{2\alpha_2 \sigma_2} e^{v_7 - v_3}, \\ v_{7,x} &= \frac{3\delta \sigma_1}{2} e^{v_2 - v_5}, v_{3,x} = \frac{3\delta \sigma_1}{2} e^{v_1 - v_2}, \\ v_{7,y} &= \frac{\delta \sigma_1}{2} e^{v_2 - v_5}, v_{3,y} = \frac{\delta \sigma_1}{2} e^{v_1 - v_2}. \end{aligned} \quad (3.12)$$

Subject to the constraints, we further have

$$\begin{aligned} & [\sigma_2 \alpha_2 \mathcal{Y}_{yy}(p', \tilde{h}') + i \mathcal{Y}_t(p', \tilde{h}') - \sigma_2 (\alpha_1 - \beta_3) \mathcal{Y}_{xx}(p', \tilde{h}')] - [\sigma_2 \alpha_2 \mathcal{Y}_{yy}(p, \tilde{h}) + i \mathcal{Y}_t(p, \tilde{h}) \\ & \quad - \sigma_2 (\alpha_1 - \beta_3) \mathcal{Y}_{xx}(p, \tilde{h})] \\ &= \sigma_2 \alpha_2 [(\omega_1 - \omega_2)_{yy} + v_{1,y}^2 - v_{2,y}^2] + i(v_{1,t} - v_{2,t}) - \sigma_2 (\alpha_1 - \beta_3) [(\omega_1 - \omega_2)_{xx} + v_{1,x}^2 - v_{2,x}^2] \\ & \quad + 2\sigma_2 \alpha_2 [v_{3,yy} + v_{3,y}(v_2 - v_1)_y] - \sigma_2 (\alpha_1 - \beta_3) [v_{3,xx} + v_{3,x}(v_2 - v_1)_x] \\ &= -(v_7 - v_3)_x + \frac{3\delta \sigma_1}{2} (e^{v_2 - v_5} - e^{v_1 - v_2}) - [-(v_7 - v_3)_y + \frac{\delta \sigma_1}{2} (e^{v_2 - v_5} - e^{v_1 - v_2})] \\ &= 0. \end{aligned} \quad (3.13)$$

The Bell polynomial BT of Eq. (1.4) is

$$\begin{aligned} & \sigma_2 \alpha_2 \mathcal{Y}_{yy}(v_1, \omega_1) + i \mathcal{Y}_t(v_1, \omega_1) - \sigma_2 (\alpha_1 - \beta_3) \mathcal{Y}_{xx}(v_1, \omega_1) = 0, \\ & \sigma_2 \alpha_2 \mathcal{Y}_{yy}(v_2, \omega_2) + i \mathcal{Y}_t(v_2, \omega_2) - \sigma_2 (\alpha_1 - \beta_3) \mathcal{Y}_{xx}(v_2, \omega_2) = 0, \\ & \mathcal{Y}(v_{2,x}) = -\frac{1}{2\alpha_1 \sigma_2} e^{v_7 - v_3}, \mathcal{Y}(v_{2,y}) = -\frac{1}{2\alpha_2 \sigma_2} e^{v_7 - v_3}, \end{aligned}$$

$$\begin{aligned}\mathcal{Y}(v_{7,x}) &= \frac{3\delta\sigma_1}{2}e^{v_2-v_5}, \mathcal{Y}(v_{3,x}) = \frac{3\delta\sigma_1}{2}e^{v_1-v_2}, \\ \mathcal{Y}(v_{7,y}) &= \frac{\delta\sigma_1}{2}e^{v_2-v_5}, \mathcal{Y}(v_{3,y}) = \frac{\delta\sigma_1}{2}e^{v_1-v_2}.\end{aligned}\quad (3.14)$$

According to the relationship between the $\mathcal{Y}$ -polynomial and the Hirota bilinear operator, the bilinear BT can be obtained as

$$\begin{aligned}[\sigma_2\alpha_2D_y^2 + iD_t - \sigma_2(\alpha_1 - \beta_3)D_x^2]\tilde{f} \cdot f &= 0, D_x\tilde{f} \cdot g = \frac{3\sigma_1\delta}{2}\tilde{g}f, \\ [\sigma_2\alpha_2D_y^2 + iD_t - \sigma_2(\alpha_1 - \beta_3)D_x^2]\tilde{g} \cdot g &= 0, D_y\tilde{f} \cdot g = \frac{\delta\sigma_1}{2}f\tilde{g}, \\ D_x\tilde{f} \cdot f &= -\frac{1}{2\alpha_1\sigma_2}\tilde{h}g, D_y\tilde{f} \cdot f = -\frac{1}{2\alpha_2\sigma_2}\tilde{h}g, D_x\tilde{h} \cdot f = \frac{3\delta\sigma_1}{2}f\tilde{h}, D_y\tilde{h} \cdot f = \frac{\delta\sigma_1}{2}f\tilde{h}.\end{aligned}\quad (3.15)$$

4. Soliton solutions

We expand f, g to the following form:

$$f = f(x, y, t) = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \cdots, \quad (4.1)$$

$$g = g(x, y, t) = \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \cdots, \quad (4.2)$$

where ε is a small parameter. Substituting Eq. (4.1) and Eq. (4.2) into Eq. (3.8) and collecting the coefficients of the same power of each order of ε^j , ($j = 0, 1, 2, 3, \dots$) can be obtained

$$\begin{aligned}\varepsilon^0 : (\alpha_1\sigma_2D_x^2 - \alpha_2\sigma_2D_y^2)1 \cdot 1 &= 0, \\ \varepsilon^1 : [\sigma_2\alpha_2D_y^2 + iD_t - \sigma_2(\alpha_1 - \beta_3)D_x^2]g_1 \cdot 1 &= 0, \\ \varepsilon^2 : (\alpha_1\sigma_2D_x^2 - \alpha_2\sigma_2D_y^2)(1 \cdot f_2 + f_2 \cdot 1) - \delta\sigma_1^2g_1 \cdot g_1^* &= 0, \\ \varepsilon^3 : [\sigma_2\alpha_2D_y^2 + iD_t - \sigma_2(\alpha_1 - \beta_3)D_x^2](g_1 \cdot f_2 + g_3 \cdot 1 + 1 \cdot g_3 + f_2 \cdot g_1) &= 0, \\ \cdots.\end{aligned}\quad (4.3)$$

4.1. Explicit 1-soliton solution

In order to obtain the explicit 1-soliton solution of Eq. (1.4), the formulas (4.1) and (4.2) are truncated as follows:

$$f = 1 + \varepsilon^2 f_2, g = \varepsilon g_1. \quad (4.4)$$

Suppose the functions f_2 and g_1 take the following exponential forms

$$f_2 = \chi e^{\Upsilon_1 + \Upsilon_1^*}, g_1 = \mu e^{\Upsilon_1}, \Upsilon_1 = s_1 x + l_1 y + d_1 t + \xi_1, \quad (4.5)$$

where μ, χ are arbitrary constants, and s_1, l_1, d_1, ξ_1 are complex constants. Without loss of generality, let $\varepsilon = 1$. Substituting expressions (4.4) and (4.5) into Eq. (4.3), the dispersion relation is given by

$$d_1 = i(-s_1^2\alpha_1\sigma_2 + l_1^2\alpha_2\sigma_2 + s_1^2\beta_3\sigma_2), \chi = \frac{\delta\mu^2\sigma_1^2}{2\sigma_2[-\alpha_2(l_1 + l_1^*)^2 + \alpha_1(s_1 + s_1^*)^2]}. \quad (4.6)$$

Simultaneous substitution of (4.4)-(4.6) into (3.1) yields the explicit single soliton solution of Eq. (35) as

$$E_1 = \frac{\sigma_1 \mu e^{\Upsilon_1}}{1 + \chi e^{\Upsilon_1 + \Upsilon_1^*}}, N_1 = 2\sigma_2 \left\{ \ln \left[1 + \frac{\delta \mu^2 \sigma_1^2}{-2\alpha_2 \sigma_2 (l_1 + l_1^*)^2 + 2\alpha_1 \sigma_2 (s_1 + s_1^*)^2} e^{(\Upsilon_1 + \Upsilon_1^*)} \right] \right\}_{xx}, \quad (4.7)$$

where $\mu, \chi, \xi_1, s_1, l_1, d_1$ satisfies expression (4.6).

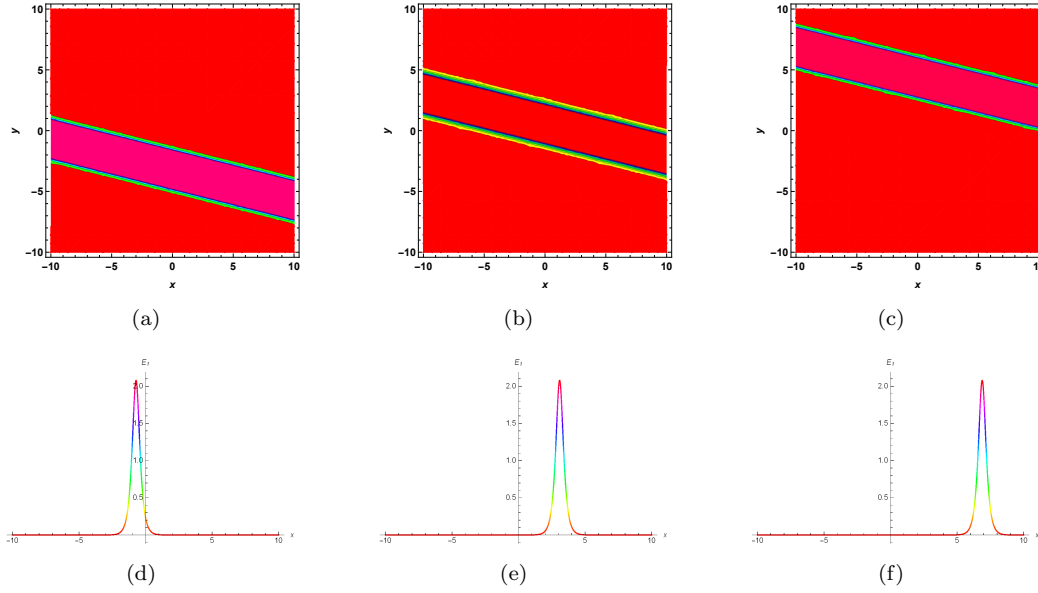


Figure 7. Profiles of the explicit 1-soliton solution given by expression (4.7), with parameters: $\sigma_1 = 0.1, \sigma_2 = 0.2, \mu = 0.2, \alpha_1 = 0.4, \alpha_2 = 1, \delta = 1, s_1 = 4 - i, l_1 = 1 + i, \beta_1 = 2 - i, \beta_2 = 3 + 5i, \beta_3 = -0.3, \xi_1 = 3 + 2i$. (a, d) $t = -10$; (b, e) $t = 0$; (c, f) $t = 10$.

4.2. Explicit 2-soliton solution

Similarly, for the explicit double soliton solution of the construction Eq. (1.4), the the solutions (4.1) and (4.2) are truncated as follows:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, g = \varepsilon g_1 + \varepsilon^3 g_3. \quad (4.8)$$

Suppose

$$\begin{aligned} g_1 &= \mu_1 e^{\Upsilon_1} + \mu_2 e^{\Upsilon_1}, g_3 = \mu_{121} e^{\Upsilon_1 + \Upsilon_2 + \Upsilon_1^*} + \mu_{122} e^{\Upsilon_1 + \Upsilon_2 + \Upsilon_2^*}, \\ f_2 &= \chi_{11} e^{\Upsilon_1 + \Upsilon_1^*} + \chi_{12} e^{\Upsilon_1 + \Upsilon_2^*} + \chi_{21} e^{\Upsilon_2 + \Upsilon_1^*} + \chi_{22} e^{\Upsilon_2 + \Upsilon_2^*}, \\ f_4 &= \chi_{1212} e^{\Upsilon_1 + \Upsilon_2 + \Upsilon_1^* + \Upsilon_2^*}, \Upsilon_j = s_j x + l_j y + d_j t + \xi_j, (j = 1, 2), \end{aligned} \quad (4.9)$$

where $\mu_1, \mu_2, \mu_{121}, \mu_{122}, \chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}, \chi_{1212}$ are arbitrary constants. By substituting expression (4.8) and expression (4.9) into Eq. (4.3) at the same time gives the dispersion relation:

$$\begin{aligned} d_1 &= i(-s_1^2 \alpha_1 \sigma_2 + l_1^2 \alpha_2 \sigma_2 + s_1^2 \beta_3 \sigma_2), d_2 = i(-s_2^2 \alpha_1 \sigma_2 + l_2^2 \alpha_2 \sigma_2 + s_2^2 \beta_3 \sigma_2), \\ \chi_{11} &= \frac{\delta \mu_1^2 \sigma_1^2}{-2\alpha_2 \sigma_2 (l_1 + l_1^*)^2 + 2\alpha_1 \sigma_2 (s_1 + s_1^*)^2}, \chi_{12} = \frac{\delta \mu_1 \mu_2 \sigma_1^2}{2\sigma_2 [-\alpha_2 (l_1 + l_2^*)^2 + \alpha_1 (s_1 + s_2^*)^2]}, \end{aligned}$$

$$\begin{aligned}
\chi_{21} &= \frac{\delta\mu_1\mu_2\sigma_1^2}{2\sigma_2[-\alpha_2(l_2+l_1^*)^2+\alpha_1(s_2+s_1^*)^2]}, \chi_{22} = \frac{\delta\mu_1\mu_2\sigma_1^2}{2\sigma_2[-\alpha_2(l_2+l_2^*)^2+\alpha_1(s_2+s_2^*)^2]}, \\
\mu_{121} &= \frac{(d_1-d_2)(\mu_2\chi_{11}-\mu_1\chi_{21})+(\mu_2\chi_{11}+\mu_1\chi_{21})d_1^*+\mathrm{i}\sigma_2\alpha_2\Lambda_1-\mathrm{i}\sigma_2(\alpha_1-\beta_3)\Lambda_2}{2[d_1+d_2+d_1^*-\mathrm{i}\sigma_2\alpha_2(l_1+l_2+l_1^*)-\mathrm{i}\sigma_2\alpha_2(s_1+s_2+s_1^*)^2]}, \\
\mu_{122} &= \frac{(d_1-d_2)(\mu_2\chi_{12}-\mu_1\chi_{22})+(\mu_2\chi_{12}+\mu_1\chi_{22})d_1^*+\mathrm{i}\sigma_2\alpha_2\Lambda_3-\mathrm{i}\sigma_2(\alpha_1-\beta_3)\Lambda_4}{2[d_1+d_2+d_1^*-\mathrm{i}\sigma_2\alpha_2(l_1+l_2+l_1^*)^2-\mathrm{i}\sigma_2\alpha_2(s_1+s_2+s_1^*)^2]}, \\
\chi_{1212} &= \frac{\delta(\mu_1\mu_{122}+\mu_{121}\mu_2)\sigma_1^2+\sigma_2(\alpha_2\Lambda_5-\alpha_1\Lambda_6)}{\sigma_2[-\alpha_2(l_1+l_2+l_1^*+l_2^*)^2+\mathrm{i}\alpha_1(s_1+s_2+s_1^*+s_2^*)^2]}, \tag{4.10}
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= (l_1-l_2)^2(\mu_2\chi_{11}+\mu_1\chi_{21})+2(l_1-l_2)(\mu_2\chi_{11}-\mu_1\chi_{21})l_1^*+(\mu_2\chi_{11}+\mu_1\chi_{21})l_1^{*2}, \\
\Lambda_2 &= (s_1-s_2)^2(\mu_2\chi_{11}+\mu_1\chi_{21})+2(s_1-s_2)(\mu_2\chi_{11}-\mu_1\chi_{21})s_1^*+(\mu_2\chi_{11}+\mu_1\chi_{21})s_1^{*2}, \\
\Lambda_3 &= (l_1-l_2)^2(\mu_2\chi_{12}+\mu_1\chi_{22})+2(l_1-l_2)(\mu_2\chi_{12}-\mu_1\chi_{22})l_1^*+(\mu_2\chi_{12}+\mu_1\chi_{22})l_2^{*2}, \\
\Lambda_4 &= (s_1-s_2)^2(\mu_2\chi_{12}+\mu_1\chi_{22})+2(s_1-s_2)(\mu_2\chi_{12}-\mu_1\chi_{22})s_1^*+(\mu_2\chi_{12}+\mu_1\chi_{22})s_2^{*2}, \\
\Lambda_5 &= (l_1-l_2)^2(\chi_{12}\chi_{21}+\chi_{11}\chi_{22})+[-2(l_1-l_2)(\chi_{12}\chi_{21}-\chi_{11}\chi_{22}) \\
&\quad +(\chi_{12}\chi_{21}+\mu_{11}\chi_{22})(l_1^*-l_2^*)](l_1^*-l_2^*), \\
\Lambda_6 &= (s_1-s_2)^2(\chi_{12}\chi_{21}+\mu_{11}\chi_{22})+[-2(s_1-s_2)(\chi_{21}\chi_{12}-\chi_{11}\chi_{22}) \\
&\quad +(\chi_{12}\chi_{21}+\chi_{11}\chi_{22})(s_1^*-s_2^*)](s_1^*-s_2^*).
\end{aligned}$$

Without loss of generality, let $\varepsilon = 1$. Substituting expressions (4.8)-(4.10) into Eq. (4.1), the explicit 2-soliton solution of Eq. (4.1) can be obtained as

$$\begin{aligned}
E_2 &= \frac{\sigma_1(\mu_1\mathrm{e}^{\Upsilon_1}+\mu_2\mathrm{e}^{\Upsilon_2}+\mu_{121}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_1^*}+\mu_{122}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_2^*})}{1+\chi_{11}\mathrm{e}^{\Upsilon_1+\Upsilon_1^*}+\chi_{12}\mathrm{e}^{\Upsilon_1+\Upsilon_2^*}+\chi_{21}\mathrm{e}^{\Upsilon_2+\Upsilon_1^*}+\chi_{22}\mathrm{e}^{\Upsilon_2+\Upsilon_2^*}+\chi_{1212}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_1^*+\Upsilon_2^*}}, \\
N_2 &= 2\sigma_2\{\ln(1+\chi_{11}\mathrm{e}^{\Upsilon_1+\Upsilon_1^*}+\chi_{12}\mathrm{e}^{\Upsilon_1+\Upsilon_2^*}+\chi_{21}\mathrm{e}^{\Upsilon_2+\Upsilon_1^*}+\chi_{22}\mathrm{e}^{\Upsilon_2+\Upsilon_2^*}+\chi_{1212}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_1^*+\Upsilon_2^*})\}_{xx}, \tag{4.11}
\end{aligned}$$

where $\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}, \chi_{1212}, \Upsilon_j, (j = 1, 2)$ satisfies expressions (4.10).

4.3. Explicit 3-soliton solution

By the same method, we truncate Eq. (4.2) to $g_5^{(j)}$. To proceed, we take

$$g = \mu_1\mathrm{e}^{\Upsilon_1} + \mu_2\mathrm{e}^{\Upsilon_2} + \mu_3\mathrm{e}^{\Upsilon_3}, \Upsilon_j = b_j t + a_j x, (j = 1, 2, 3), \tag{4.12}$$

where $\Upsilon_j, s_j, l_j, d_j, \xi_j$ are arbitrary constants. Substituting expression (4.12) into the bilinear form (4.3) gives corresponding the numerator and denominator to the Explicit 3-soliton solution, respectively

$$\begin{aligned}
g &= \mu_1\mathrm{e}^{\Upsilon_1} + \mu_2\mathrm{e}^{\Upsilon_2} + \mu_3\mathrm{e}^{\Upsilon_3} + \mu_{121}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_1^*} + \mu_{122}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_2^*} \\
&\quad + \mu_{123}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_3^*} + \mu_{131}\mathrm{e}^{\Upsilon_1+\Upsilon_1+\Upsilon_1^*} \\
&\quad + \mu_{132}\mathrm{e}^{\Upsilon_1+\Upsilon_3+\Upsilon_2^*} + \mu_{133}\mathrm{e}^{\Upsilon_1+\Upsilon_3+\Upsilon_3^*} + \mu_{231}\mathrm{e}^{\Upsilon_3+\Upsilon_2+\Upsilon_1^*} + \mu_{232}\mathrm{e}^{\Upsilon_3+\Upsilon_2+\Upsilon_2^*} + \mu_{233}\mathrm{e}^{\Upsilon_3+\Upsilon_2+\Upsilon_3^*} \\
&\quad + \mu_{12312}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_2^*+\Upsilon_1^*} + \mu_{12313}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_3^*+\Upsilon_1^*} + \mu_{12323}\mathrm{e}^{\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_2^*+\Upsilon_3^*},
\end{aligned}$$

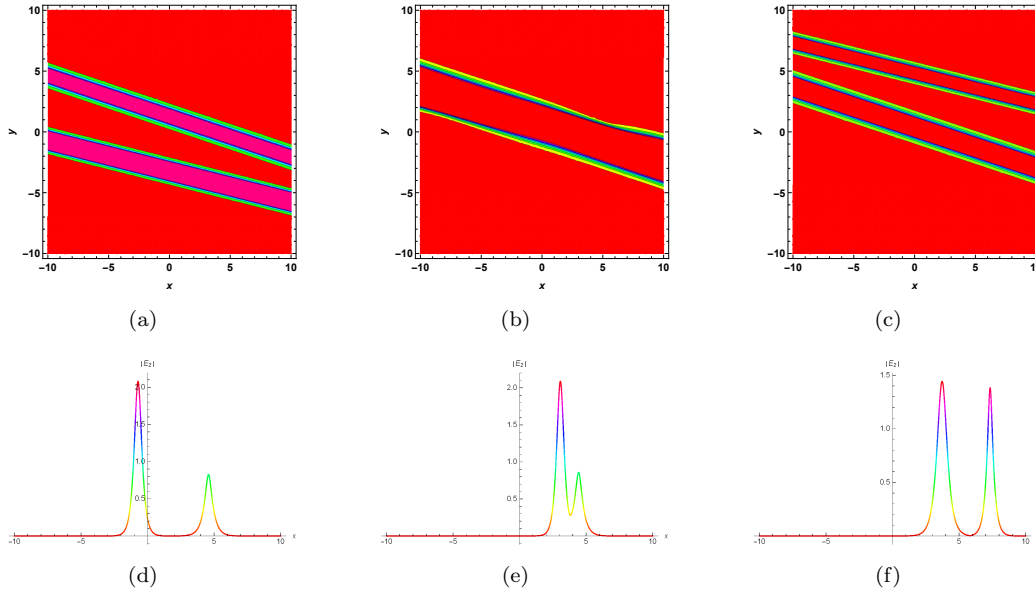


Figure 8. Profiles of the explicit 2-soliton solution given by expression (4.11), with parameters: $\sigma_1 = 0.1, \sigma_2 = 0.2, \mu = 0.2, \alpha_1 = 0.4, \alpha_2 = 1, \delta = 1, s_1 = 4 - i, l_1 = 1 + i, \beta_1 = 2 - i, \beta_2 = 3 + 5i, \beta_3 = -0.3, \xi_1 = 3 + 2i, \xi_2 = 3 - 2i$. (a, d) $t = -10$; (b, e) $t = 0$; (c, f) $t = 10$.

$$\begin{aligned}
 f = & 1 + \chi_{11}e^{\Upsilon_1 + \Upsilon_1^*} + \chi_{22}e^{\Upsilon_2 + \Upsilon_2^*} + \chi_{12}e^{\Upsilon_1 + \Upsilon_2^*} + \chi_{21}e^{\Upsilon_2 + \Upsilon_1^*} + \chi_{13}e^{\Upsilon_1 + \Upsilon_3^*} + \chi_{23}e^{\Upsilon_2 + \Upsilon_3^*} + \chi_{31}e^{\Upsilon_3 + \Upsilon_1^*} \\
 & + \chi_{32}e^{\Upsilon_3 + \Upsilon_2^*} + \vartheta_{33}e^{\theta_3 + \theta_3^*} \chi_{1212}e^{\Upsilon_1 + \Upsilon_1^* + \Upsilon_2 + \Upsilon_2^*} + \chi_{1213}e^{\theta_1 + \theta_1^* + \theta_2 + \theta_2^*} + \chi_{1223}e^{\Upsilon_1 + \Upsilon_2^* + \Upsilon_2 + \Upsilon_3^*} \\
 & + \chi_{1312}e^{\Upsilon_1 + \Upsilon_1^* + \Upsilon_3 + \Upsilon_2^*} + \chi_{1313}e^{\Upsilon_1 + \Upsilon_1^* + \Upsilon_3 + \Upsilon_3^*} \\
 & + \chi_{1323}e^{\theta_1 + \theta_3^* + \theta_3 + \theta_2^*} \chi_{123123}e^{\Upsilon_1 + \Upsilon_1^* + \Upsilon_2 + \Upsilon_3 + \Upsilon_2^* + \Upsilon_3^*}.
 \end{aligned} \tag{4.13}$$

Because there are too many parameters in Eq. (4.13), we only list some of them here.

$$\begin{aligned}
 d_1 &= i(-s_1^2 \alpha_1 \sigma_2 + l_1^2 \alpha_2 \sigma_2 + s_1^2 \beta_3 \sigma_2), \\
 d_2 &= i(-s_2^2 \alpha_1 \sigma_2 + l_2^2 \alpha_2 \sigma_2 + s_2^2 \beta_3 \sigma_2), \\
 d_3 &= i(-s_3^2 \alpha_1 \sigma_2 + l_3^2 \alpha_2 \sigma_2 + s_3^2 \beta_3 \sigma_2), \\
 \chi_{11} &= \frac{\delta \mu_1^2 \sigma_1^2}{2\sigma_2[s_1^2 \alpha_1 - l_1^2 \alpha_2 - 2l_1 \alpha_2 l_1^* - \alpha_2(l_1^*)^2 + 2s_1 \alpha_1 s_1^* + \alpha_1(s_1^*)^2]}, \\
 \chi_{12} &= \frac{\delta \mu_1 \mu_2 \sigma_1^2}{2\sigma_2[s_1^2 \alpha_1 - l_1^2 \alpha_2 - 2l_1 \alpha_2 l_2^* - \alpha_2(l_2^*)^2 + 2s_1 \alpha_1 s_2^* + \alpha_1(s_2^*)^2]}, \\
 \chi_{13} &= \frac{\delta \mu_1 \mu_3 \sigma_1^2}{2\sigma_2[s_1^2 \alpha_1 - l_1^2 \alpha_2 - 2l_1 \alpha_2 l_3^* - \alpha_2(l_3^*)^2 + 2s_1 \alpha_1 s_3^* + \alpha_1(s_3^*)^2]}, \\
 \chi_{21} &= \frac{\delta \mu_1 \mu_2 \sigma_1^2}{2\sigma_2[s_2^2 \alpha_1 - l_2^2 \alpha_2 - 2l_2 \alpha_2 l_1^* - \alpha_2(l_1^*)^2 + 2s_2 \alpha_1 s_1^* + \alpha_1(s_1^*)^2]}, \\
 \chi_{22} &= \frac{\delta \mu_1 \mu_2 \sigma_1^2}{2\sigma_2[s_2^2 \alpha_1 - l_2^2 \alpha_2 - 2l_2 \alpha_2 l_2^* - \alpha_2(l_2^*)^2 + 2s_2 \alpha_1 s_2^* + \alpha_1(s_2^*)^2]}, \\
 \chi_{31} &= \frac{\delta \mu_1 \mu_3 \sigma_1^2}{2\sigma_2[s_3^2 \alpha_1 - l_3^2 \alpha_2 - 2l_3 \alpha_2 l_1^* - \alpha_2(l_1^*)^2 + 2s_3 \alpha_1 s_1^* + \alpha_1(s_1^*)^2]}.
 \end{aligned} \tag{4.14}$$

Therefore, the explicit 3-soliton solution of the equation is given by

$$E_3 = \sigma_1 \frac{g}{f}, N_3 = 2\sigma_2 (\ln f)_{xx}. \quad (4.15)$$

Then the graph of the solution is shown in the following Figure 9.

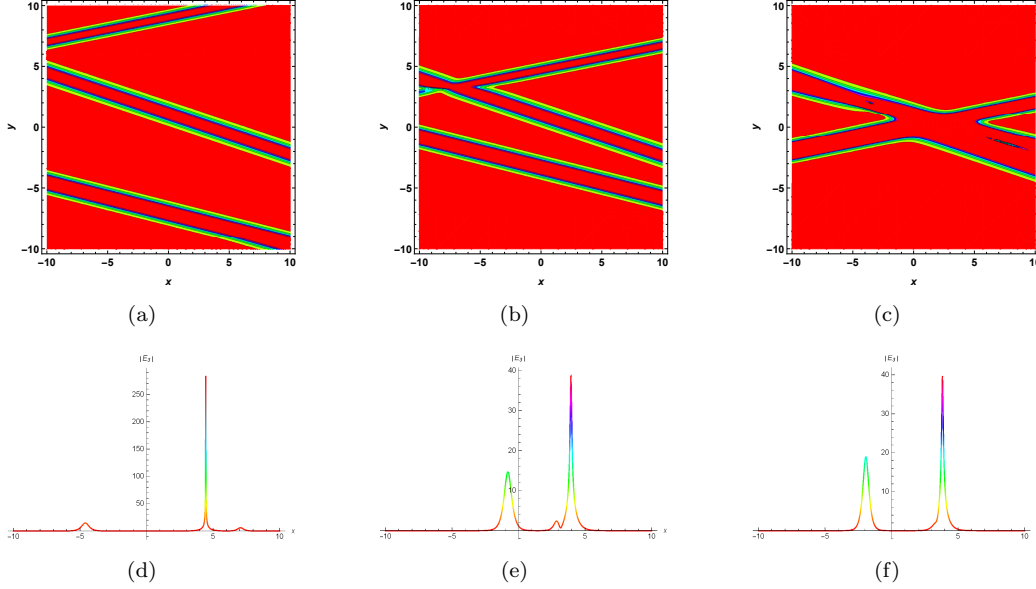


Figure 9. Profiles of the explicit 3-soliton solution given by expression (4.15), with parameters: $\sigma_1 = 0.1, \sigma_2 = 0.2, \mu = 0.2, \alpha_1 = 0.4, \alpha_2 = 1, \delta = 1, s_1 = 4 - i, l_1 = 1 + i, \beta_1 = 2 - i, \beta_2 = 3 + 5i, \beta_3 = -0.3, \xi_1 = 3 + 2i, \xi_2 = 3 - 2i, \xi_3 = 2$. (a, d) $t = -20$; (b, e) $t = -10$; (c, f) $t = 0.5$.

This subsection constructs explicit soliton solutions of Eq. (1.4) using the small parameter perturbation method. The analysis of Figure 7 shows that σ_2 affects the degree of undulation of the wave, with a gradual increase in amplitude when $\sigma_2 \rightarrow +\infty$. β_2 affects the motion of the wave. When β_2 increases and the wave moves in a positive direction, the fluctuation degree of the wave at this time is almost unchanged. In fact, it is clear from the expression that there exists a characteristic line of the soliton wave solution $s_1x + l_1y + i[-s_1^2\sigma_2(\alpha_1 - \beta_3) + l_1^2\sigma_2\alpha_2]t + \xi_1 = 0$. The propagation speed of the isolated wave on the x -axis is given by $v_x = \frac{i[s_1^2\sigma_2(\alpha_1 - \beta_3) - l_1^2\sigma_2\alpha_2]}{s_1}$ and the shape of the image varies with time. The contour and density surfaces of the explicit 1-soliton wave solution (4.7) are shown in Figure 7. It's clear that the soliton wave also moves in the direction $x \rightarrow +\infty$ as time t goes on. Figure 8 shows the trajectory diagram of the explicit 2-soliton solution to Eq. (1.4). The Figure 8 illustrates the dynamic evolution of the solitary waves on the (x, y) plane at times $t = -10, t = 0$, and $t = 10$. At time $t = -10$, the two solitary waves do not intersect. At time $t = 0$, the two solitary waves intersect and almost merge into a single wave. At time $t = 10$, the two solitary waves separate again. Figure 9 shows the motion diagram for the explicit 3-soliton solution of Eq. (1.4), illustrating the dynamic evolution of the solitary waves on the plane at $t = -20, t = -10$, and $t = 0.5$. As shown in the figure, as time increases, the three solitary waves undergo a transition from an initial separate state, to a state where two waves overlap, forming an "X" shape while the third remains separate, and finally to a state where all three overlap.

5. Conclusion

In this paper, the integrability of the generalized (2+1)-dimensional coupled DS equation is investigated via the Bell polynomial method. Utilizing the intrinsic connection between the Bell polynomial method and the Hirota bilinear method, the corresponding bilinear form of the equation is derived. Explicit 1-soliton, explicit 2-soliton solution and explicit 3-soliton solution are constructed using the Mathematica symbolic computation system, and contour plots of the solutions are generated by selecting appropriate parameters.

Furthermore, the Weierstrass elliptic function solutions of Eq. (1.4) are explored through the homogeneous balance method and auxiliary function method. These solutions differ from those reported in previous studies: although they are also Weierstrass elliptic function solutions, their mathematical expressions and propagation behavior are distinct. Notably, the derived Weierstrass elliptic function solutions can degenerate into trigonometric and hyperbolic function solutions, demonstrating that the Weierstrass elliptic function method offers a versatile approach for finding exact solutions of nonlinear evolution equations (NLEEs). Specifically, the solutions presented herein include the special cases of solutions (125) and (166) in Refs. [4, 37].

In addition, the Bell polynomial method is applied to derive the bilinear form of Eq. (1.4) via suitable variable transformations and constraints. Based on this bilinear form and newly introduced variable relations, the BT of Eq. (1.4) is constructed. Leveraging the commutation relation and fundamental properties of Bell polynomials, a bilinear Bell polynomial BT consisting of 4 equations with 5 arbitrary parameters is further obtained.

In summary, the Weierstrass elliptic function method, commutation relation, and Bell polynomial method are powerful tools for investigating the integrability and exact solutions of the generalized (2+1)-dimensional coupled DS equation. Notably, the Bell polynomial method not only enables the derivation of exact solutions for the equation but also facilitates the construction of its BT, making it a robust and favorable tool for the study of integrable NLEEs.

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