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PERSISTENCE AND EXTINCTION OF A PEST-NATURAL ENEMY DIFFUSIVE MODEL WITH BIRTH AND HARVESTING IMPULSES*

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Abstract To understand how birth pulses and human interventions jointly influence the spatial dynamics of pests and their natural enemies, this paper develops an impulsive reaction-diffusion model for a pest-natural enemy system. We first investigate the principal eigenvalue of a corresponding single-species eigenvalue problem using the Krein-Rutman theorem and establish its monotonicity with respect to impulse intensities. Based on these eigenvalues, we analyze the threshold dynamical behavior of the full model. Sufficient conditions for co-extinction, single-species persistence, and two-species coexistence are derived. Compared with existing impulsive models that focus on either single species or single impulsive event, our work explicitly couples birth pulses and harvesting impulses in a two-species reaction-diffusion framework. The eigenvalue-based threshold criteria extend the classical logistic impulsive theory to predator-prey interactions with spatial heterogeneity. Finally, numerical simulations are presented to illustrate the theoretical results and to explore the impact of birth pulses and human interventions on population dynamics.

Keywords Diffusive model, pulse, threshold value, persistence, extinction.

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1. Introduction

Pest management is a globally recognized challenge, as effective control of pest outbreaks carries profound socio-economic implications [16]. In agriculturally intensive countries like China, optimizing pest management strategies are of particular importance. Achieving effective pest control is inherently complex, requiring an interdisciplinary integration of knowledge and methodologies. Among various approaches, chemical and biological control remain the most widely adopted strategies [3].

Biological control relies on natural enemies, entomopathogenic fungi, or introduced species to regulate pest populations. Its fundamental principle is to exploit naturally occurring ecological interactions such as predation, competition, and parasitism-to mitigate pest-induced damage. Compared to chemical control, this strategy is generally more environmentally benign and often provides longer-term suppression of pest outbreaks [3]. However, it typically requires higher labor inputs and exhibits a slower onset of observable effects.

Mathematically, the pest-natural enemy relationship is often described by the Lotka-Volterra

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predator-prey model:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(r - ax(t) - by(t)), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)(-d + cx(t)), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the population sizes of pest and natural enemy, respectively, r, a, b, c and d are positive constants, r represents the intrinsic growth rates of pest, d is death rate of natural enemy, a is intraspecific interaction coefficient, and b stands for the predation rate of natural enemies. The model assumes that the natural enemy survives by feeding on the pest at rate d.

In recent years, impulsive differential equations have been widely used to model abrupt interventions such as harvesting [6, 9, 20, 24], pesticide application and the release of natural enemies within integrated pest management (IPM) frameworks [17, 21]. These equations are well-suited for describing systems with long periods of smooth evolution punctuated by sudden changes, mirroring real-world phenomena like seasonal reproduction, periodic vaccination, or the stocking and harvesting of resources.

A pest-natural enemy model with periodic impulsive control was studied in [14]:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(r - ax(t) - by(t)), & t \neq nT, \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)(-d + cx(t)), & t \neq nT, \\ x(t^+) = (1 - p_1)x(t), & t = nT, \\ y(t^+) = (1 - p_2)y(t) + \mu, & t = nT, \\ x(0^+) = x_0,\ y(0^+) = y_0, & n = 1, 2, \cdots, \end{cases} \tag{1.2}$$

where p_1 and p_2 are mortality rates of pest and natural enemy caused by pesticide, μ is the release amount of natural enemy, and T is impulsive period. The model, which combines spraying pesticide and releasing natural enemies at discrete time points $t = nT$, demonstrates that the IPM strategy is more effective than biological or chemical control. A series of studies on impulsive ordinary differential equation models for pest control have also been conducted, see references in [2, 19, 23].

Motivated by the growing concern over the effects of spatial diffusion and impulsive intervention, Lewis and Li [11] proposed an impulsive reaction-diffusion equation model as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} + \alpha u - \gamma u^2, \quad -\infty < x < \infty, \\ u(x, 0) = g(N_n(x)), \\ N_{n+1}(x) = u(x, 1). \end{cases} \tag{1.3}$$

This model describes seasonal birth pulses combined with diffusion and nonlinear mortality. The wave speed in unbounded domains is $c^* = 2\sqrt{d \ln(g'(0)e^{\alpha})}$, and the minimal domain size for persistence in bounded domains is $I^* = \pi\sqrt{d/\ln(g'(0)e^{\alpha})}$. This work has been extended to higher dimensions [7], nonlocal dispersal [25], and evolving domains [15].

Recent research has shifted from single impulsive events to multiple impulsive strategies, reflecting periodic interventions in practice. Liang and Tang [12] presented a diffusive population model with birth pulses and pesticide application, showing that birth rates, pesticide efficacy, and application timing influence traveling waves, spreading speeds, and spatial pest distributions. Yuan et al. [28] studied a logistic diffusion model with birth and harvesting impulses, showing that population persistence is affected by growth, harvest, and the timing of interventions. Xu et al. [26] extended this to an age-structured fish model, demonstrating that the interplay between harvesting and birth perturbations determines extinction or persistence, with transitions achievable by adjusting impulse intensities and timing.

In recognition of the fact that real-world pest control involves multiple impulsive events, such as large-scale spring reproduction coupled with periodic pesticide applications, this paper characterizes the dynamics of the pest-natural enemy diffusive logistic model with impulses

$$\begin{cases} u_t - d_1\Delta u = u(-a_1(t,x) - b_{11}(t,x)u - b_{12}(t,x)v), & (t,x) \in \Omega_{nT,\tau}, \\ v_t - d_2\Delta v = v(-a_2(t,x) + b_{21}(t,x)u - b_{22}(t,x)v), & (t,x) \in \Omega_{nT,\tau}, \\ u(t,x) = v(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \overline{\Omega}, \\ u((nT)^+,x) = (1+\alpha)u(nT,x), & x \in \overline{\Omega}, \\ v((nT)^+,x) = (1+\gamma)v(nT,x), & x \in \overline{\Omega}, \\ u((nT+\tau)^+,x) = (1-\beta)u(nT+\tau,x), & x \in \overline{\Omega}, \\ v((nT+\tau)^+,x) = (1-\beta)v(nT+\tau,x), & x \in \overline{\Omega},\ n = 0,1,2,\cdots, \end{cases} \tag{1.4}$$

where $u(t,x)$ and $v(t,x)$ are the densities of pest population and natural enemy population at time t and location x, respectively. $a_1(t,x)$ and $a_2(t,x)$ are the net growth or mortality rates, $b_{11}(t,x)$ and $b_{22}(t,x)$ represent the intraspecific competition coefficients, and $b_{12}(t,x)$ and $b_{21}(t,x)$ imply the interspecific functional response coefficient between two species. Region

$$\Omega_{nT,\tau} := \{(t,x) : t \in (nT, nT+\tau] \bigcup (nT+\tau, (n+1)T],\ n = 0,1,2,\cdots,\ x \in \Omega\},$$

where constant τ satisfies $0 < \tau < T$ and Ω is a bounded and connected domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. When $t = nT(n = 0,1,2,\cdots)$, conditions $u((nT)^+,x) = (1+\alpha)u(nT,x)$ and $v((nT)^+,x) = (1+\gamma)v(nT,x)$ indicate that birth pulses occur in the pest population u and natural enemy population v with $\alpha,\gamma > 0$, respectively. When $t = nT + \tau(n = 0,1,2,\cdots)$, conditions $u((nT+\tau)^+,x) = (1-\beta)u(nT+\tau,x)$ and $v((nT+\tau)^+,x) = (1-\beta)v(nT+\tau,x)$ show that two species are harvested by the same tensity β with $0 < \beta < 1$. The functions $a_i(t,x), b_{ij}(t,x) (\in C^{\theta/2,\theta}([0,T] \times \overline{\Omega}))$ are periodic functions of time with period T, and $0 < b^m \leq b_{ij}(t,x) \leq b^M$ in $[0,T] \times \overline{\Omega}$ for $i,j = 1,2$. Initial functions $u_0(x), v_0(x)$ satisfy $u_0(x), v_0(x) \in C^2(\overline{\Omega})$, $u_0(x) \geq, \not\equiv 0$ and $v_0(x) \geq, \not\equiv 0$ for $x \in \overline{\Omega}$ and $u_0(x) = v_0(x) = 0$ for $x \in \partial\Omega$.

Although impulsive differential equations and reaction-diffusion models have been widely used to study population dynamics under disturbances and spatial heterogeneity, most existing works focus either on temporal impulses alone [2, 19, 21] (e.g., birth pulses or pesticide application) or on spatial diffusion without considering periodic interventions. In particular, the joint effect of birth pulses and human interventions (such as natural enemy release or harvesting) on the spatial spread and persistence of pest-natural enemy systems remains largely unexplored. Moreover, while eigenvalue-based threshold criteria are well established for continuous-time reaction-diffusion systems, their extension to impulsive reaction-diffusion models with two interacting

species is nontrivial and has received limited attention. To fill this gap, the present paper develops an impulsive reaction-diffusion model that simultaneously incorporates periodic birth pulses and human interventions. Using the Krein-Rutman theorem, we analyze the principal eigenvalue of the associated single-species impulsive eigenvalue problem and establish its monotonicity with respect to impulse intensities. Based on these eigenvalue results, we obtain sharp threshold conditions for co-extinction, single-species persistence, and two-species coexistence-criteria that are not available in previous impulsive or non-impulsive models. Numerical simulations further reveal how the interplay between birth pulses, diffusion, and human interventions shapes spatial population dynamics, offering new insights into pest control strategies.

The paper is organized as follows. Section 2 studies a single-species impulsive diffusive model and its principal eigenvalue. Section 3 analyzes the threshold dynamics of the two-species model. Numerical simulations illustrate the effects of birth and harvesting impulses on species persistence or extinction.

2. The single-species impulsive model

Before studying the pest-natural enemy model with impulses (1.4), we first investigate a general single-species impulsive model.

$$\begin{cases} u_t = d\Delta u + a(t,x)u - b(t,x)u^p, & (t,x) \in \Omega_{nT,\tau}, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}, \\ u((nT)^+,x) = (1+\alpha)u(nT,x), & x \in \overline{\Omega}, \\ u((nT+\tau)^+,x) = (1-\beta)u(nT+\tau,x), & x \in \overline{\Omega},\ n = 0,1,2,\cdots, \end{cases} \tag{2.1}$$

where $a(t,x), b(t,x) (\in C^{\theta/2,\theta}([0,T]\times\overline{\Omega}))$ are periodic functions of time with period T, and $0 < b^m \le b(t,x) \le b^M$ in $[0,T]\times\overline{\Omega}$. Initial function $u_0(x)$ satisfies $u_0(x) \in C^2(\overline{\Omega})$, $u_0(x) \ge, \not\equiv 0$ for $x \in \overline{\Omega}$ and $u_0(x) = 0$ for $x \in \partial\Omega$, and p is a positive constant satisfying $p > 1$.

Problem (2.1) has been investigated in [28], we collect some results as follows.

Lemma 2.1. Problem (2.1) admits a unique solution $u(t,x)$ for all $t > 0$. Moreover,

$$u(t,x) \in PC^{1,2}((0,+\infty)\times\overline{\Omega}) := \bigcap_{n=0}^{\infty} [C^{1,2}((nT, nT+\tau]\times\overline{\Omega}) \bigcap C^{1,2}((nT+\tau,(n+1)T]\times\overline{\Omega})]$$

and

$$u(t,x) \le M_{[\frac{t}{T}]+1} := (1+\alpha)^{[\frac{t}{T}]+1} \max\{u_0^M,\ (a^M/b^m)^{1/(p-1)}\},\ (t,x) \in [0,nT]\times\overline{\Omega},$$

where $u_0^M = \max\limits_{x\in\overline{\Omega}} u_0(x)$, $a^M = \max\limits_{[0,T]\times\overline{\Omega}} a(t,x)$ and $b^m = \min\limits_{[0,T]\times\in\overline{\Omega}} b(t,x)$.

As in [4, 22], the longtime behavior of problem (2.1) is related to its corresponding periodic problem

$$\begin{cases} U_t = d\Delta U + a(t,x)U - b(t,x)U^p, & t \in (0^+,\tau]\cup(\tau^+,T],\ x \in \Omega, \\ U(t,x) = 0, & t \in [0,T],\ x \in \partial\Omega, \\ U(0,x) = U(T,x), & x \in \overline{\Omega}, \\ U(0^+,x) = (1+\alpha)U(0,x), & x \in \overline{\Omega}, \\ U(\tau^+,x) = (1-\beta)U(\tau,x), & x \in \overline{\Omega}, \end{cases} \tag{2.2}$$

and the existence of the positive solution to (2.2) depends on the principal eigenvalue $\mu_1(\alpha,\beta,d)$ of the following periodic eigenvalue problem

$$\begin{cases} \phi_t - d\Delta\phi - a(t,x)\phi = \mu_1\phi, & t\in(0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ \phi(t,x)=0, & t\in[0,T],\ \ x\in\partial\Omega,\\ \phi(0,x)=\phi(T,x), & x\in\overline{\Omega},\\ \phi(0^+,x)=(1+\alpha)\phi(0,x), & x\in\overline{\Omega},\\ \phi(\tau^+,x)=(1-\beta)\phi(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{2.3}$$

By applying the Krein-Rutman theorem [8,10,13] in the Banach space [27] containing pulses, the existence of μ_1 can be obtained.

Lemma 2.2. The following statements hold:

(*i*) $\mu_1(d,a(t,x),\alpha,\beta)$ is nonincreasing with respect to $a(t,x)$ for any given d, α and β;

(*ii*) $\mu_1(d,a(t,x),\alpha,\beta)$ is strictly monotonic decreasing with respect to α for any given d, $a(t,x)$ and β;

(*iii*) $\mu_1(d,a(t,x),\alpha,\beta)$ is strictly monotonic increasing with respect to β for any given d, $a(t,x)$ and α;

(*iv*) $\mu_1(d,a(t,x),\alpha,\beta)$ is nondecreasing with respect to d for any given $a(t,x)$, α and β.

Lemma 2.3. Assume that $\mu_1(d,a(t,x),\alpha,\beta)<0$. Then for each $u_0\in C^2(\overline{\Omega})$ such that $u_0(x)\geq,\not\equiv 0$ for $x\in\overline{\Omega}$ and $u_0(x)=0$ for $x\in\partial\Omega$, the following assertions hold:

(*i*) Periodic problem (2.2) admits a unique positive solution $U(t,x)$;

(*ii*) the solution $u(t,x)$ to (2.1) satisfies $\lim\limits_{m\to\infty}u(t+mT,x)=U(t,x)$ for any $t\geq 0$ and uniformly for $x\in\overline{\Omega}$, where $U(t,x)$ is the unique solution defined in (*i*).

The following result for $\mu_1(d,a(t,x),\alpha,\beta)>0$ has been given in [28] by constructing a upper solution which decays to zero in the long run, we here extend the corresponding results by using a approach technique.

Theorem 2.4. Assume that $\mu_1(d,a(t,x),\alpha,\beta)\geq 0$. Then for each $u_0\in C^2(\overline{\Omega})$ such that $u_0(x)=0$ for $x\in\partial\Omega$, the following assertions hold:

(*i*) Periodic problem (2.2) has no positive solution;

(*ii*) the solution $u(t,x)$ to (2.1) satisfies $\lim\limits_{t\to\infty}u(t,x)=0$ uniformly for $x\in\overline{\Omega}$.

Proof. (*i*) We first introduce the auxiliary problem of the eigenvalue problem (2.3)

$$\begin{cases} -\psi^*_t - d\Delta\psi^* - a(t,x)\psi^* = \lambda_1\psi^*, & t\in(0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ \psi^*(t,x)=0, & t\in[0,T],\ \ x\in\partial\Omega,\\ \psi^*(0^+,x)=\dfrac{1}{1+\alpha}\psi^*(0,x), & x\in\overline{\Omega},\\ \psi^*(\tau^+,x)=\dfrac{1}{1-\beta}\psi^*(\tau,x), & x\in\overline{\Omega},\\ \psi^*(0,x)=\psi^*(T,x), & x\in\overline{\Omega}. \end{cases} \tag{2.4}$$

It is easy to check that the principal eigenvalues λ_1 in (2.4) and μ_1 in (2.3) are the same, that is, $\lambda_1 = \mu_1$, see also Lemma 3.1 in [28].

Assume by contradiction, that problem (2.2) has a positive solution $U(t,x)$. Multiplying the first equation of problem (2.2) by ψ^* yields

$$\psi^* U_t - d\psi^* \Delta U - a(t,x)\psi^* U = b(t,x)U^p\psi^*,$$

while the first equation in problem (2.4) multiplied by $U(t,x)$ gives equation

$$-U\psi_t^* - dU\Delta\psi^* - a(t,x)U\psi^* = \lambda_1 U\psi^*.$$

Subtracting two equations implies

$$(U\psi^*)_t = d(\Delta U\psi^* - \Delta\psi^* U) - \lambda_1 U\psi^* - b(t,x)U^p\psi^*.$$

Integrating it in $x \in \Omega$ and t from 0^+ to τ, together with the other conditions in (2.2) and (2.4), we have

$$\begin{aligned}
&-\lambda_1 \int_{0^+}^{\tau}\int_{\Omega} U\psi^* dxdt - \int_{0^+}^{\tau}\int_{\Omega} b(t,x)U^p\psi^* dxdt \\
&= \int_{\Omega} [U(\tau,x)\psi^*(\tau,x) - U(0^+,x)\psi^*(0^+,x)]dx \\
&= \int_{\Omega} [U(0,x)\psi^*(0,x) - (1+\alpha)U(0,x)\frac{1}{(1+\alpha)}\psi^*(0,x)]dx \\
&= 0.
\end{aligned}$$

Since $b(t,x) > 0$ in $[0,\tau] \times \overline{\Omega}$, it can be obtained that $\lambda_1 < 0$. Therefore, $\mu_1 < 0$, which leads a contradiction.

(ii) We first consider the periodic problem

$$\begin{cases}
U_t = d\Delta U + a(t,x)U - b(t,x)U^p + \lambda U, & t \in (0^+,\tau] \cup (\tau^+,T],\ x \in \Omega, \\
U(t,x) = 0, & 0 \le t \le T,\ x \in \partial\Omega, \\
U(0,x) = U(T,x), & x \in \overline{\Omega}, \\
U(0^+,x) = (1+\alpha)U(0,x), & x \in \overline{\Omega}, \\
U(\tau^+,x) = (1-\beta)U(\tau,x), & x \in \overline{\Omega}.
\end{cases} \tag{2.5}$$

It is not difficult to verify that when $\lambda \le \mu_1(d, a(t,x), \alpha, \beta)$, periodic problem (2.5) has only the trivial solution. Similar to Lemma 2.3 (i), when $\lambda > \mu_1(d, a(t,x), \alpha, \beta)$, periodic problem (2.5) admits a unique positive solution $U_\lambda(t,x)$. Moreover, $U_\lambda(t,x)$ is monotonically increasing with respect to λ, and satisfies

$$\lim_{\lambda \to \mu_1(d,a(t,x),\alpha,\beta)} ||U_\lambda(t,x)||_{C([0,\tau]\times\overline{\Omega})} = 0. \tag{2.6}$$

Returning to the initial-boundary value problem (2.1), its solution $u(t,x)$ satisfies

$$u(t,x) \le u_\lambda(t,x), t \in [0,\infty), x \in \overline{\Omega}, \tag{2.7}$$

where $\lambda > \mu_1(d, a(t,x), \alpha, \beta)$, and $u_\lambda(t,x)$ is the solution to the following problem

$$\begin{cases} u_t = d\Delta u + a(t,x)u - b(t,x)u^p + \lambda u, & (t,x) \in \Omega_{nT,\tau}, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}, \\ u((nT)^+, x) = (1+\alpha)u(nT,x), & x \in \overline{\Omega}, \\ u((nT+\tau)^+, x) = (1-\beta)u(nT+\tau,x), & x \in \overline{\Omega}, n = 0,1,2,\cdots. \end{cases} \tag{2.8}$$

It follows from Lemma 2.3 (ii) that the

$$\lim_{m\to\infty} u_\lambda(t+m\tau,x) = U_\lambda(t,x), t \in [0,\infty), x \in \overline{\Omega},$$

which together with (2.7) yields that

$$\limsup_{m\to\infty} u(t+m\tau,x) \le U_\lambda(t,x), t \in [0,\infty), x \in \overline{\Omega}.$$

Thus, taking $\lambda \to \mu_1(d, a(t,x), \alpha, \beta) \ge 0$ and using (2.6), we obtain

$$\limsup_{t\to\infty} ||u(t,\cdot)||_{C(\overline{\Omega})} = 0,$$

i.e., $\lim\limits_{t\to\infty} ||u(t,\cdot)||_{C(\overline{\Omega})} = 0$. □

Next, we consider the impact factors of the longtime behavior of the solution which depends on the principal eigenvalue $\mu_1(d, a(t,x), \alpha, \beta)$ of problem (2.3).

Theorem 2.5. The following assertions hold:

(*i*) Assume that $a(x,t) := a(x)$. Then the principal eigenvalue of problem (2.3) can be precisely expressed as

$$\mu_1 = \frac{-\ln[(1+\alpha)(1-\beta)]}{T} + d\lambda_1, \tag{2.9}$$

where

$$\lambda_1 = \sup_{\substack{\psi\in H_0^1 \\ \psi\not\equiv 0}} \frac{\int_\Omega |\nabla\psi|^2 dx - \int_\Omega \frac{1}{d}a(x)\psi^2 dx}{\int_\Omega \psi^2 dx}. \tag{2.10}$$

(*ii*) Assume that $a(x,t) := a(t)$. Then the principal eigenvalue of problem (2.3) can be precisely expressed as

$$\mu_1 = \frac{-\ln[(1+\alpha)(1-\beta)]}{T} + d\lambda_1^* - \frac{\int_0^T a(t)dt}{T}, \tag{2.11}$$

where $\lambda_1^* (> 0)$ is the principal eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition.

Proof. We only prove (2.9), since (2.11) can be proved similarly.

Let $\phi(t,x) = f(t)\psi(x)$, then $\Delta\phi = f(t)\Delta\psi(x)$, which together with the first equation in (2.3) yields

$$f'(t)\psi(x) - f(t)[d\Delta\psi + a(x)\psi(x)] = \mu_1 f(t)\psi(x). \tag{2.12}$$

Let $(\lambda_1, \psi(x))$ be the eigen-pair to the eigenvalue problem

$$\begin{cases} -\Delta\psi = \dfrac{1}{d}a(x)\psi + \lambda_1\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.13}$$

then λ_1 satisfies (2.10) by variational method. Substituting (2.13) into equation (2.12) gives

$$\frac{f'(t)}{f(t)} + d\lambda_1 = \mu_1,\ t \in (0^+, \tau] \cup (\tau^+, T]$$

with conditions

$$\begin{cases} f(0) = f(T), \\ f(0^+) = (1+\alpha)f(0), \\ f(\tau^+) = (1-\beta)f(\tau). \end{cases} \tag{2.14}$$

Solving the equation, we have

$$f(t) = \begin{cases} c_1 e^{(\mu_1 - d\lambda_1)t}, & \mathrm{t} \in (0^+, \tau], \\ c_2 e^{(\mu_1 - d\lambda_1)t}, & \mathrm{t} \in (\tau^+, T], \end{cases}$$

which together with (2.14) gives

$$\begin{cases} f(0^+) = (1+\alpha)f(0) = (1+\alpha)f(T), \\ f(\tau^+) = (1-\beta)f(\tau), \end{cases}$$

that is,

$$\begin{cases} c_1 = (1+\alpha)c_2 e^{(\mu_1 - d\lambda_1)T}, \\ c_2 e^{(\mu_1 - d\lambda_1)\tau} = (1-\beta)c_1 e^{(\mu_1 - d\lambda_1)\tau}. \end{cases}$$

Hence,

$$\mu_1 = -\frac{\ln[(1+\alpha)(1-\beta)]}{T} + d\lambda_1.$$

□

Theorem 2.5, together with Lemma 2.3 and Theorem 2.4, shows that small diffusion rate (d), large birth rate (α) and small harvesting rate (β) are beneficial for the survival of species.

3. The pest-natural enemy model with impulses

This section is devoted to dynamics of pest-natural enemy model (1.4). It is easy to see that, when $u = 0$, model (1.4) reduces to the following initial-boundary value problem

$$\begin{cases} v_t - d_2\Delta v = v(-a_2(t,x) - b_{22}(t,x)v), & (t,x) \in \Omega_{nT,\tau}, \\ v(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ v(0,x) = v_0(x), & x \in \overline{\Omega}, \\ v((nT)^+, x) = (1+\gamma)v(nT,x), & x \in \overline{\Omega}, \\ v((nT+\tau)^+, x) = (1-\beta)v(nT+\tau,x), & x \in \overline{\Omega},\ n = 0,1,2,\cdots, \end{cases} \tag{3.1}$$

and its corresponding periodic problem is

$$\begin{cases} V_t - d_2\Delta V = V(-a_2(t,x) - b_{22}(t,x)V), & t \in (0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ V(t,x) = 0, & t\in[0,T],\ x\in\partial\Omega,\\ V(0,x) = V(T,x), & x\in\overline{\Omega},\\ V(0^+,x) = (1+\gamma)V(0,x), & x\in\overline{\Omega},\\ V(\tau^+,x) = (1-\beta)V(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{3.2}$$

Using the same method as in Section 2, we have the following results.

Corollary 3.1. Let $v(t,x)$ be the solution to problem (3.1), and $\mu_v := \mu_1(d_2, -a_2(t,x), \gamma, \beta)$.

(*i*) If $\mu_v \geq 0$, then $\lim\limits_{t\to\infty} v(t,x) = 0$ uniformly for $x\in\overline{\Omega}$.

(*ii*) If $\mu_v < 0$, then $\lim\limits_{m\to\infty} v(t+mT,x) = V^*(t,x)$ for any $t\geq 0$ and uniformly for $x\in\overline{\Omega}$, where $V^*(t,x)$ is the unique solution of the periodic problem (3.2).

On the other hand, for the case when $v = 0$, model (1.4) reduces to the following initial value problem

$$\begin{cases} u_t - d_1\Delta u = u(-a_1(t,x) - b_{11}(t,x)u), & (t,x)\in\Omega_{nT,\tau},\\ u(t,x) = 0, & t>0,\ x\in\partial\Omega,\\ u(0,x) = u_0(x), & x\in\overline{\Omega},\\ u((nT)^+,x) = (1+\alpha)u(nT,x), & x\in\overline{\Omega},\\ u((nT+\tau)^+,x) = (1-\beta)u(nT+\tau,x), & x\in\overline{\Omega},\ n=0,1,2,\cdots, \end{cases} \tag{3.3}$$

and the corresponding periodic problem is

$$\begin{cases} U_t - d_1\Delta U = U(-a_1(t,x) - b_{11}(t,x)U), & t \in (0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ U(t,x) = 0, & t\in[0,T],\ x\in\partial\Omega,\\ U(0,x) = U(T,x), & x\in\overline{\Omega},\\ U(0^+,x) = (1+\alpha)U(0,x), & x\in\overline{\Omega},\\ U(\tau^+,x) = (1-\beta)U(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{3.4}$$

Similarly as above, the following conclusion can be drawn.

Corollary 3.2. Let $u(t,x)$ be the solution to problem (3.3), and $\mu_u := \mu_1(d_1, -a_1(t,x), \alpha, \beta)$.

(*i*) If $\mu_u \geq 0$, then $\lim\limits_{t\to\infty} u(t,x) = 0$ uniformly for $x\in\overline{\Omega}$.

(*ii*) If $\mu_u < 0$, then $\lim\limits_{m\to\infty} u(t+mT,x) = U^*(t,x)$ for any $t\geq 0$ and uniformly for $x\in\overline{\Omega}$, where $U^*(t,x)$ is the unique solution of the periodic problem (3.4).

Based on the about results for single species, we present our main result for problem (1.4).

Theorem 3.3. The dynamics of the pest and natural enemy populations (u,v) to problem (1.4) are shown as follows.

(*i*) If $\mu_u \geq 0$ and $\mu_v > 0$, then $\lim\limits_{t\to\infty} u(t,x) = 0$ and $\lim\limits_{t\to\infty} v(t,x) = 0$ uniformly for $x\in\overline{\Omega}$. That is, both pest and natural enemy populations tend to extinction;

(*ii*) if $\mu_u \geq 0$ and $\mu_v < 0$, then $\lim\limits_{t\to\infty} u(t,x) = 0$ uniformly for $x \in \overline{\Omega}$, and $\lim\limits_{m\to\infty} v(t+mT,x) = V^*(t,x)$ for any $t \geq 0$ and uniformly for $x \in \overline{\Omega}$, where $V^*(t,x)$ is the unique solution of the periodic problem (3.2). That is, the pest population tends to extinction while the natural enemy population tends to a stable periodic state;

(*iii*) if $\mu_u < 0$ and $\mu_v^* > 0$, then $\lim\limits_{m\to\infty} u(t+mT,x) = U^*(t,x)$ for any $t \geq 0$ and uniformly for $x \in \overline{\Omega}$, and $\lim\limits_{t\to\infty} v(t,x) = 0$ uniformly for $x \in \overline{\Omega}$, where μ_v^* is defined in (3.11) and $U^*(t,x)$ is the unique solution to the periodic problem (3.4). That is, the pest population tends to a stable periodic state while the predator population tends to extinction;

(*iv*) if $\mu_u^* < 0$ and $\mu_v^{**} < 0$, where μ_u^* and μ_v^{**} are defined in (3.20) and (3.27), respectively, then the solution $(u(t,x), v(t,x))$ to problem (1.4) satisfies

$$U_*(t,x) \leq \liminf_{m\to+\infty} u(t+mT,x) \leq \limsup_{m\to+\infty} u(t+mT,x) \leq U^*(t,x),\ t>0,\ x\in\overline{\Omega},$$

and

$$V_*(t,x) \leq \liminf_{m\to+\infty} v(t+mT,x) \leq \limsup_{m\to+\infty} v(t+mT,x) \leq V^*(t,x),\ t>0,\ x\in\overline{\Omega},$$

where $U^*(t,x)$ and $V^*(t,x)$ are defined as above in (3.4) and (3.2), and $U_*(t,x)$ and $V_*(t,x)$ are the unique positive solutions to periodic problems (3.22) and (3.29), respectively. The result implies that, the pest and the natural enemy populations persist in a long run.

Proof. (i) and (ii) are directly from Corollaries 3.1 and 3.2, we omit the details since it is standard.

(iii) It is obvious that the solution $(u(t,x), v(t,x))$ to problem (1.4) satisfies

$$u(t,x) \leq \overline{u}(t,x), t \geq 0, x \in \overline{\Omega}, \tag{3.5}$$

where $\overline{u}(t,x)$ is the solution to problem (3.3). Since $\mu_u < 0$, it follows from Corollary 3.2 (ii) that

$$\lim_{m\to\infty} \overline{u}(t+mT,x) = U^*(t,x), t \geq 0, x \in \overline{\Omega}, \tag{3.6}$$

where $U^*(t,x)$ is the unique positive solution to periodic problem (3.4). Then, for any $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$u(t,x) \leq \overline{u}(t,x) \leq U^*(t,x) + \varepsilon, \forall t \geq T_1, x \in \overline{\Omega},$$

which gives that

$$v_t - d_2\Delta v \leq v[-a_2(t,x) + b_{21}(t,x)(U^*(t,x)+\varepsilon) - b_{22}(t,x)v] \tag{3.7}$$

holds for $t \geq T_1$ and $x \in \Omega$. It follows from comparison principle that

$$v(t,x) \leq \overline{v}_\varepsilon(t,x), t \geq T_1, x \in \overline{\Omega}, \tag{3.8}$$

where $\overline{v}_\varepsilon(t,x)$ satisfies

$$\begin{cases} \overline{v}_t - d_2\Delta\overline{v} = \overline{v}(-a_2 + b_{21}(U^*+\varepsilon) - b_{22}\overline{v}), & (t,x) \in \Omega_{nT,\tau}, t \geq T_1, \\ \overline{v}(t,x) = 0, & t \geq T_1,\ x \in \partial\Omega, \\ \overline{v}(T_1,x) = v(T_1,x), & x \in \overline{\Omega}, \\ \overline{v}((nT)^+,x) = (1+\gamma)\overline{v}(nT,x), & x \in \overline{\Omega}, \\ \overline{v}((nT+\tau)^+,x) = (1-\beta)\overline{v}(nT+\tau,x), & x \in \overline{\Omega},\ n = [\frac{T_1}{T}], [\frac{T_1}{T}]+1, \cdots, \end{cases} \tag{3.9}$$

the corresponding periodic problem is

$$\begin{cases} \overline{V}_t - d_2\Delta\overline{V} = \overline{V}(-a_2 + b_{21}(U^* + \varepsilon) - b_{22}\overline{V}), & t \in (0^+, \tau] \cup (\tau^+, T],\ x \in \Omega, \\ \overline{V}(t, x) = 0, & t \in [0, T],\ x \in \partial\Omega, \\ \overline{V}(0, x) = V(T, x), & x \in \overline{\Omega}, \\ \overline{V}(0^+, x) = (1 + \gamma)\overline{V}(0, x), & x \in \overline{\Omega}, \\ \overline{V}(\tau^+, x) = (1 - \beta)\overline{V}(\tau, x), & x \in \overline{\Omega}. \end{cases} \tag{3.10}$$

Define the eigenvalue μ_v^* for the following eigenvalue problem

$$\begin{cases} \phi_t - d_2\Delta\phi = \phi(-a_2(t, x) + b_{21}(t, x)U^*) + \mu_v^*\phi, & t \in (0^+, \tau] \cup (\tau^+, T],\ x \in \Omega, \\ \phi(t, x) = 0, & t \in [0, T],\ x \in \partial\Omega, \\ \phi(0, x) = \phi(T, x), & x \in \overline{\Omega}, \\ \phi(0^+, x) = (1 + \gamma)\phi(0, x), & x \in \overline{\Omega}, \\ \phi(\tau^+, x) = (1 - \beta)\phi(\tau, x), & x \in \overline{\Omega}. \end{cases} \tag{3.11}$$

From the condition $\mu_v^* > 0$ and Corollary 3.1(i), periodic problem (3.10) has only the trivial solution provided that ε is sufficiently small such that $b_{21}^M \cdot \varepsilon \le \mu_v^*$, which gives

$$\lim_{t\to+\infty} \overline{v}_\varepsilon(t, x) = 0,\ x \in \overline{\Omega}.$$

Then, for the above $\varepsilon > 0$, there exists a $T_2 \ge T_1$ such that

$$\overline{v}_\varepsilon(t, x) \le \varepsilon, t \ge T_2, x \in \overline{\Omega},$$

which combined with (3.8) yields

$$v(t, x) \le \overline{v}_\varepsilon(t, x) \le \varepsilon, t \ge T_2, x \in \overline{\Omega}.$$

Thus,

$$\lim_{t\to 0} v(t, x) = 0, x \in \overline{\Omega}. \tag{3.12}$$

Recalling the first equation for pest u in problem (1.4), we have

$$\begin{aligned} u_t - d_1\Delta u &= u(-a_1(t, x) - b_{11}(t, x)u - b_{12}(t, x)v) \\ &\ge u(-a_1(t, x) - b_{11}(t, x)u - b_{12}(t, x)\varepsilon) \text{ for } t \ge T_2, x \in \overline{\Omega}, \end{aligned}$$

which by comparison principle implies that

$$u(t, x) \ge \underline{u}_\varepsilon(t, x), t \ge T_2, x \in \overline{\Omega}, \tag{3.13}$$

where $\underline{u}_\varepsilon(t, x)$ satisfies

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = \underline{u}(-a_1(t, x) - b_{11}(t, x)\underline{u} - b_{12}(t, x)\varepsilon), & (t, x) \in \Omega_{nT,\tau}, t \ge T_2, \\ \underline{u}(t, x) = 0, & t \ge T_2,\ x \in \partial\Omega, \\ \underline{u}(T_2, x) = u(T_2, x), & x \in \overline{\Omega}, \\ \underline{u}((nT)^+, x) = (1 + \alpha)\underline{u}(nT, x), & x \in \overline{\Omega}, \\ \underline{u}((nT + \tau)^+, x) = (1 - \beta)\underline{u}(nT + \tau, x), & x \in \overline{\Omega},\ n = [\frac{T_2}{T}], [\frac{T_2}{T}] + 1, \cdots, \end{cases}$$

the periodic problem corresponding to the above initial value problem is

$$\begin{cases} \underline{U}_t - d_1\Delta\underline{U} = \underline{U}(-a_1 - b_{11}\underline{U} - b_{12}\varepsilon), & t \in (0^+, \tau] \cup (\tau^+, T],\ x \in \Omega, \\ \underline{U}(t,x) = 0, & t \in [0,T],\ x \in \partial\Omega, \\ \underline{U}(0,x) = \underline{U}(T,x), & x \in \overline{\Omega}, \\ \underline{U}(0^+,x) = (1+\alpha)\underline{U}(0,x), & x \in \overline{\Omega}, \\ \underline{U}(\tau^+,x) = (1-\beta)\underline{U}(\tau,x), & x \in \overline{\Omega}. \end{cases}$$

Denote the periodic solution to the above periodic problem as $\underline{U}_\varepsilon(t,x)$. Under the condition $\mu_u < 0$, we have, for sufficiently small ε,

$$\lim_{m\to\infty} \underline{u}_\varepsilon(t+mT,x) = \underline{U}_\varepsilon(t,x), t \geq 0, x \in \overline{\Omega},$$

which together with (3.13) gives

$$\liminf_{m\to\infty} u(t+mT,x) \geq \underline{U}_\varepsilon(t,x), t \geq 0, x \in \overline{\Omega}.$$

As $\varepsilon \to 0$, we have $\underline{U}_\varepsilon(t,x) \to U^*(t,x)$, then

$$\liminf_{m\to\infty} u(t+mT,x) \geq U^*(t,x), t \geq 0, x \in \overline{\Omega}.$$

Recalling (3.5) and (3.6), we have $\limsup_{m\to\infty} u(t+mT,x) \leq U^*(t,x), t \geq 0, x \in \overline{\Omega}$. Then, it can be obtained that

$$\lim_{m\to\infty} u(t+mT,x) = U^*(t,x), t \geq 0, x \in \overline{\Omega},$$

which combined with (3.12) deduces that

$$(u(t+mT,x), v(t,x)) \to (U^*(t,x), 0) \text{ for } m \to \infty, t \geq 0, x \in \overline{\Omega}.$$

(iv) It is easy to check that $\mu_u \leq \mu_u^*$ and $\mu_v^* \leq \mu_v^{**}$, where μ_u^* is defined in (3.20) and μ_v^{**} is defined in (3.27). Similarly as in (iii), by assumption that $\mu_u^* < 0$, we then can prove that there exist $T_1 > 0$ and $\varepsilon > 0$ such that

$$u(t,x) \leq U^*(t,x) + \varepsilon,\ v(t,x) \leq \overline{v}_\varepsilon(t,x) \text{ for } t \geq T_1, x \in \overline{\Omega}. \tag{3.14}$$

Moreover, due to $\mu_v^* < 0$ and Corollary 3.1(ii), as long as ε is sufficiently small such that $b_{21}^M \cdot \varepsilon \leq -\mu_v^*$, periodic problem (3.10) has a positive periodic solution $\overline{V}_\varepsilon(t,x)$, and the solution $\overline{v}_\varepsilon(t,x)$ to problem (3.9) satisfies

$$\lim_{m\to\infty} \overline{v}_\varepsilon(t+mT,x) \to \overline{V}_\varepsilon(t,x), t \geq 0, x \in \overline{\Omega},$$

which together with $v(t,x) \leq \overline{v}_\varepsilon(t,x)$ for $t \geq T_1$ and $x \in \overline{\Omega}$ deduces that

$$\lim_{m\to\infty} \sup v(t+mT,x) \leq \overline{V}_\varepsilon(t,x), t \geq 0, x \in \overline{\Omega}.$$

It follows from the continuity of the solution to the coefficient functions that $\overline{V}_\varepsilon(t,x) \to V^*(t,x)$ as $\varepsilon \to 0$. Then,

$$\limsup_{m\to\infty} v(t+mT,x) \leq V^*(t,x), t \geq 0, x \in \overline{\Omega}, \tag{3.15}$$

where $V^*(t,x)$ is the unique solution of the periodic problem (3.2). We then have, there exists a time $T_3 \geq T_1$ such that

$$v(t,x) \leq V^*(t,x) + \varepsilon, \forall t \geq T_3, x \in \overline{\Omega}. \tag{3.16}$$

Substituting (3.16) into the first equation for pest u in problem (1.4) yields

$$u_t - d_1\Delta u \geq u(-a_1(t,x) - b_{11}(t,x)u - b_{12}(t,x)(V^*(t,x)+\varepsilon)), \tag{3.17}$$

which by comparison principle deduces that

$$u(t,x) \geq \underline{u}_\varepsilon(t,x), t \geq T_3, x \in \overline{\Omega}, \tag{3.18}$$

where $\underline{u}_\varepsilon(t,x)$ satisfies

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = \underline{u}(-a_1 - b_{11}\underline{u} - b_{12}(V^*+\varepsilon)), & (t,x) \in \Omega_{nT,\tau}, t \geq T_3, \\ \underline{u}(t,x) = 0, & t \geq T_3,\ x \in \partial\Omega, \\ \underline{u}(T_3,x) = u(T_3,x), & x \in \overline{\Omega}, \\ \underline{u}((nT)^+,x) = (1+\alpha)\underline{u}(nT,x), & x \in \overline{\Omega}, \\ \underline{u}((nT+\tau)^+,x) = (1-\beta)\underline{u}(nT+\tau,x), & x \in \overline{\Omega},\ n = [\dfrac{T_3}{T}], [\dfrac{T_3}{T}]+1, \cdots, \end{cases}$$

and its corresponding periodic problem is

$$\begin{cases} \underline{U}_t - d_1\Delta\underline{U} = \underline{U}(-a_1 - b_{11}\underline{U} - b_{12}(V^*+\varepsilon)), & t \in (0^+,\tau] \cup (\tau^+,T],\ x \in \Omega, \\ \underline{U}(t,x) = 0, & t \in [0,T],\ x \in \partial\Omega, \\ \underline{U}(0,x) = \underline{U}(T,x), & x \in \overline{\Omega}, \\ \underline{U}(0^+,x) = (1+\alpha)\underline{U}(0,x), & x \in \overline{\Omega}, \\ \underline{U}(\tau^+,x) = (1-\beta)\underline{U}(\tau,x), & x \in \overline{\Omega}. \end{cases} \tag{3.19}$$

Define the eigenvalue μ_u^* for the following eigenvalue problem,

$$\begin{cases} \phi_t - d_1\Delta\phi = \phi(-a_1 - b_{12}V^*(t,x)) + \mu_u^*\phi, & t \in (0^+,\tau] \cup (\tau^+,T],\ x \in \Omega, \\ \phi(t,x) = 0, & t \in [0,T],\ x \in \partial\Omega, \\ \phi(0,x) = \phi(T,x), & x \in \overline{\Omega}, \\ \phi(0^+,x) = (1+\alpha)\phi(0,x), & x \in \overline{\Omega}, \\ \phi(\tau^+,x) = (1-\beta)\phi(\tau,x), & x \in \overline{\Omega}. \end{cases} \tag{3.20}$$

From the condition $\mu_u^* < 0$ and Corollary 3.2(ii), as long as ε is sufficiently small such that $b_{12}^M \cdot \varepsilon < -\mu_u^*$, the periodic problem (3.19) admits a positive periodic solution $U_{*,\varepsilon}(t,x)$ and for $t > 0, x \in \overline{\Omega}$,

$$\underline{u}_\varepsilon(t+mT,x) \to U_{*,\varepsilon}(t,x), m \to \infty.$$

Recalling (3.18), we have

$$\liminf_{m\to\infty} u(t+mT,x) \geq U_{*,\varepsilon}(t,x), t \geq 0, x \in \overline{\Omega},$$

and letting $\varepsilon \to 0$ yields

$$\liminf_{m\to\infty} u(t+mT,x) \geq U_*(t,x), t \geq 0, x \in \overline{\Omega}, \tag{3.21}$$

where $U_*(t,x)$ satisfies

$$\begin{cases} \underline{U}_t - d_1\Delta\underline{U} = \underline{U}(-a_1 - b_{11}\underline{U} - b_{12}V^*), & t \in (0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ \underline{U}(t,x) = 0, & t\in[0,T],\ x\in\partial\Omega,\\ \underline{U}(0,x) = \underline{U}(T,x), & x\in\overline{\Omega},\\ \underline{U}(0^+,x) = (1+\alpha)\underline{U}(0,x), & x\in\overline{\Omega},\\ \underline{U}(\tau^+,x) = (1-\beta)\underline{U}(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{3.22}$$

Combing (3.21) and (3.15) obtain that

$$U_*(t,x) \le \liminf_{m\to+\infty} u(t+mT,x) \le \limsup_{m\to+\infty} u(t+mT,x) \le U^*(t,x),\ t>0,\ x\in\overline{\Omega}.$$

Moreover, using (3.21), we have there exists a $T_4 \ge T_3$ such that

$$u(t,x) \ge U_*(t,x) - \varepsilon, \forall t \ge T_4, x\in\overline{\Omega}. \tag{3.23}$$

Recalling the second equation for v in problem (1.4), we have

$$v_t - d_2\Delta v \ge v[-a_2(t,x) + b_{21}(t,x)(U_*(t,x)-\varepsilon) - b_{22}(t,x)v]. \tag{3.24}$$

It is easy to see from comparison principle that

$$v(t,x) \ge \underline{v}_\varepsilon(t,x), t\ge T_4, x\in\overline{\Omega}, \tag{3.25}$$

where $\underline{v}_\varepsilon(t,x)$ satisfies

$$\begin{cases} \underline{v}_t - d_2\Delta\underline{v} = \underline{v}(-a_2 + b_{21}(U_*(t,x)-\varepsilon) - b_{22}\underline{v}), & (t,x)\in\Omega_\tau, t\ge T_4,\\ \underline{v}(t,x) = 0, & t\ge T_4,\ x\in\partial\Omega,\\ \underline{v}(T_4,x) = v(T_4,x), & x\in\overline{\Omega},\\ \underline{v}((nT)^+,x) = (1+\gamma)\underline{v}(nT,x), & x\in\overline{\Omega},\\ \underline{v}((nT+\tau)^+,x) = (1-\beta)\underline{v}(nT+\tau,x), & x\in\overline{\Omega},\ n=[\dfrac{T_4}{T}],[\dfrac{T_4}{T}]+1,\cdots, \end{cases}$$

and the periodic problem corresponding to the above initial value problem is

$$\begin{cases} \underline{V}_t - d_2\Delta\underline{V} = \underline{V}(-a_2 + b_{21}(U_*(t,x)-\varepsilon) - b_{22}\underline{V}), & t\in(0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ \underline{V}(t,x) = 0, & t\in[0,T],\ x\in\partial\Omega,\\ \underline{V}(0,x) = \underline{V}(T,x), & x\in\overline{\Omega},\\ \underline{V}(0^+,x) = (1+\gamma)\underline{V}(0,x), & x\in\overline{\Omega},\\ \underline{V}(\tau^+,x) = (1-\beta)\underline{V}(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{3.26}$$

Define the eigenvalue μ_v^{**} for the following eigenvalue problem,

$$\begin{cases} \phi_t - d_2\Delta\phi = \phi(-a_2(t,x) + b_{21}(t,x)U_*(t,x)) + \mu_v^{**}\phi, & t\in(0^+,\tau]\cup(\tau^+,T],\ x\in\Omega,\\ \phi(t,x) = 0, & t\in[0,T],\ x\in\partial\Omega,\\ \phi(0,x) = \phi(T,x), & x\in\overline{\Omega},\\ \phi(0^+,x) = (1+\gamma)\phi(0,x), & x\in\overline{\Omega},\\ \phi(\tau^+,x) = (1-\beta)\phi(\tau,x), & x\in\overline{\Omega}. \end{cases} \tag{3.27}$$

Recalling $\mu_v^{**} < 0$ and Corollary 3.1(ii), as long as ε is sufficiently small such that $b_{21}^M \cdot \varepsilon \le -\mu_v^{**}$, periodic problem (3.26) has a positive periodic solution $V_{*,\varepsilon}(t,x)$ and

$$\underline{v}_\varepsilon(t+mT,x) \to V_{*,\varepsilon}(t,x), m \to \infty, t \ge 0, x \in \overline{\Omega},$$

which by (3.25) deduces that

$$\liminf_{m\to\infty} v(t+mT,x) \ge V_{*,\varepsilon}(t,x), t \ge 0, x \in \overline{\Omega}.$$

It follows from the arbitrary of ε that

$$\liminf_{m\to\infty} v(t+mT,x) \ge V_*(t,x), t \ge 0, x \in \overline{\Omega}, \tag{3.28}$$

where $V_*(t,x)$ is a unique positive periodic solution to

$$\begin{cases} \underline{V}_t - d_2\Delta\underline{V} = \underline{V}(-a_2 + b_{21}U_*(t,x) - b_{22}\underline{V}), & t \in (0^+,\tau] \cup (\tau^+,T],\ x \in \Omega, \\ \underline{V}(t,x) = 0, & t \in [0,T],\ x \in \partial\Omega, \\ \underline{V}(0,x) = \underline{V}(T,x), & x \in \overline{\Omega}, \\ \underline{V}(0^+,x) = (1+\gamma)\underline{V}(0,x), & x \in \overline{\Omega}, \\ \underline{V}(\tau^+,x) = (1-\beta)\underline{V}(\tau,x), & x \in \overline{\Omega}. \end{cases} \tag{3.29}$$

Thus, combing (3.15) and (3.28) implies that

$$V_*(t,x) \le \liminf_{m\to+\infty} v(t+mT,x) \le \limsup_{m\to+\infty} v(t+mT,x) \le V^*(t,x),\ t > 0,\ x \in \overline{\Omega}.$$

□

4. Numerical simulations and discussions

Numerical simulations are presented to verify our theoretical results and to gain a more direct sense of the specific effects of different parameters on the impulse differential equation.

First, the spatial domain is taken as $\Omega = (0,\pi)$ and the period as $T = 1$. It is easy to see that the eigenvalue problem

$$\begin{cases} -\Delta\psi = \lambda_0\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

admits the principal eigenvalue $\lambda_0 = 1$.

Recalling (2.13), we know that $(\lambda_u, \psi(x))$ is the eigen-pair to the eigenvalue problem

$$\begin{cases} -\Delta\psi = \dfrac{1}{d}a\psi + \lambda_u\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

and direct calculation shows that

$$\lambda_u = 1 + \frac{a}{d}.$$

If $a_i(t,x) \equiv a_i$ for $i = 1,2$, it follows from (2.11) that

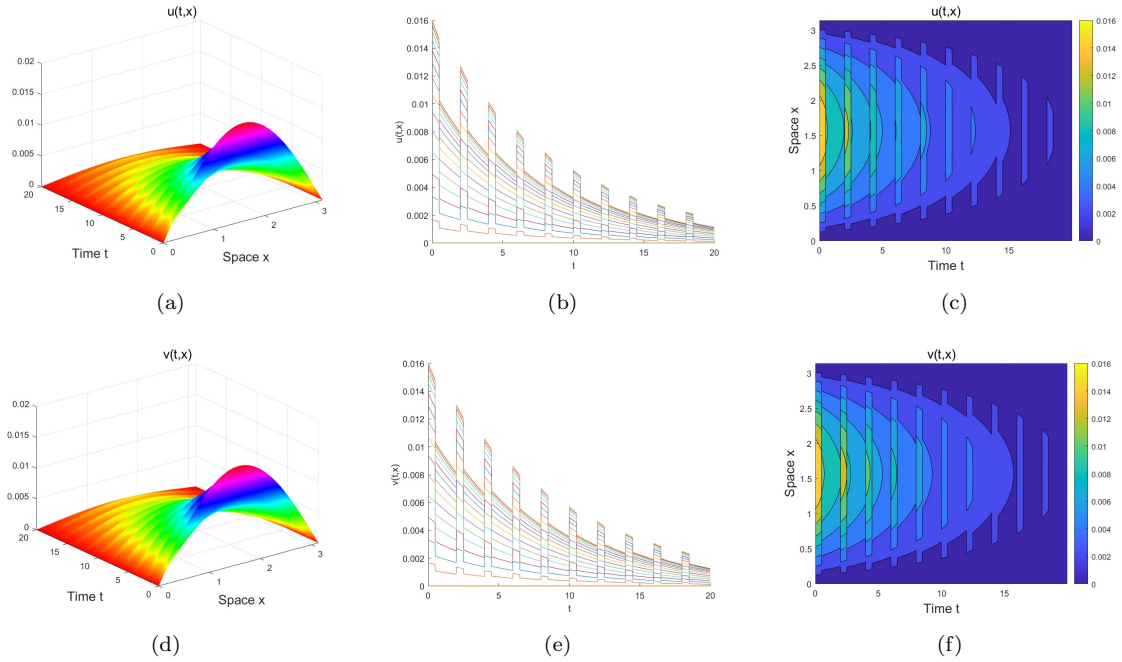
$$\mu_u = \mu_1(d_1, -a_1, \alpha, \beta) = -\ln[(1+\alpha)(1-\beta)] + d_1 + a_1, \tag{4.3}$$

and similarly,

$$\mu_v = \mu_1(d_2, -a_2, \alpha, \beta) = -\ln[(1+\gamma)(1-\beta)] + d_2 + a_2. \tag{4.4}$$

Next, we let $p = 2$, fix the mortality rates $a_1 = 0.15, a_2 = 0.15$, the intrinsic reproduction rate $\alpha = 0.6$, the diffusion rates $d_1 = 0.01, d_2 = 0.01$, and the intraspecific and interspecific functional response coefficient between two species $b_{11} = 0.6, b_{12} = 0.6, b_{21} = 0.6, b_{22} = 0.6$. We will consider the imparts of the birth pulse of the natural enemy population γ and the harvesting pulses of both populations β. The birth pulse of the natural enemy γ is adjusted by periodically releasing natural enemies into the ecosystem, and the harvesting pulses of both populations β are controlled by periodically manually preying on pests and natural enemies.

Now, take $\beta = 0.3$ and $\gamma = 0.6$. In this case, $\mu_u \approx 0.047 > 0$ and $\mu_v \approx 0.047 > 0$. Using Matlab programming, we can see from Figure 1 that the populations of both species gradually decrease to 0, and both populations tend to extinction, which verifies case (i) of Theorem 3.3.



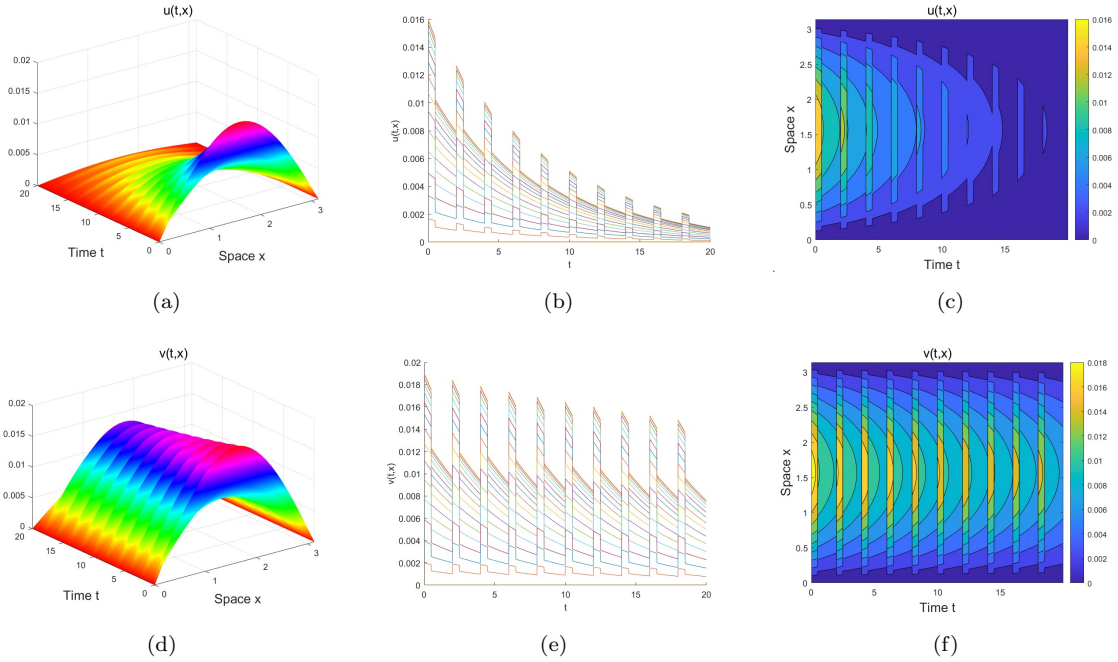
(a) (b) (c)

(d) (e) (f)

Figure 1. Simulation for $\beta = 0.3$, $\gamma = 0.6$ ($\mu_u \approx 0.047 > 0$, $\mu_v \approx 0.047 > 0$). Both populations tend to extinction. (a,d): u and v surfaces; (b,e): Cross-sections at $x = \pi/2$; (c,f): Top views.

We next verify the impact of the birth pulse of the natural enemy population γ. Keep $\beta = 0.3$ and other parameter values unchanged, take $\gamma = 0.9$. Direct calculations show that $\mu_u \approx 0.047 > 0$ and $\mu_v \approx -0.125 < 0$. It can be seen from Figure 2 that the pest population tends to extinction, while the natural enemy population tends to stability, which corresponds to case (ii) of Theorem 3.3.

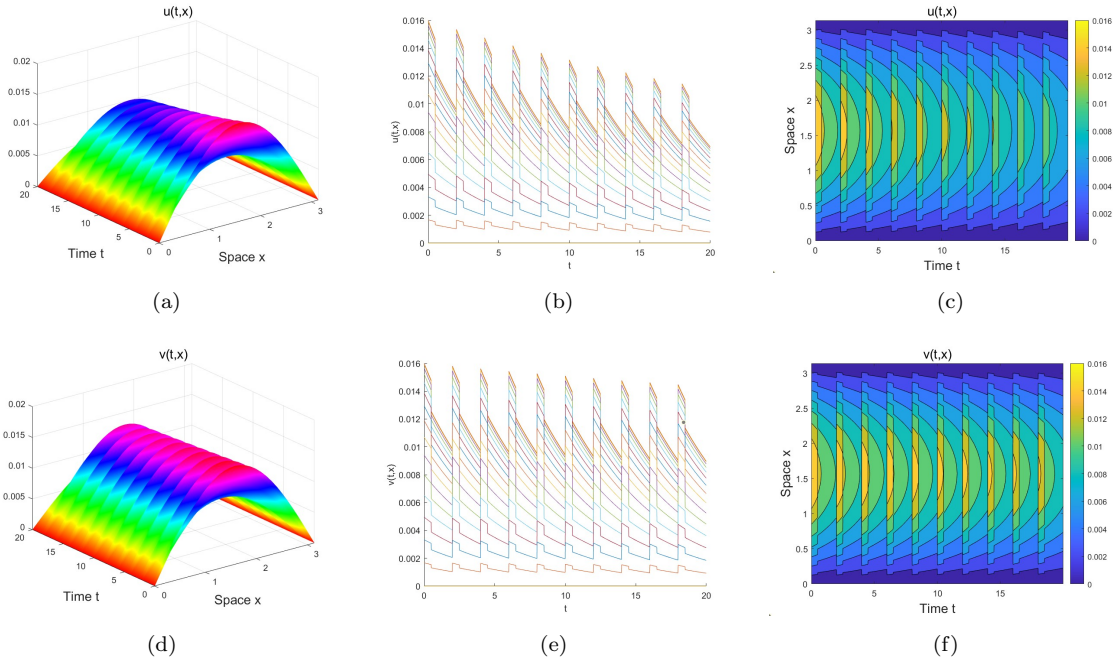
To verify the impact of the harvesting pulses of the pest and natural enemy populations, we keep $\gamma = 0.6$ and other parameter values unchanged, take $\beta = 0.15$. In this case, $\mu_u \approx -0.147 < 0$ and $\mu_v \approx -0.147 < 0$. According to monotonicity and the definitions of $\mu_u^*, \mu_v^*, \mu_v^{**}$ in (3.11), (3.20) and (3.27), we can obtain $\mu_u^* < 0, \mu_v^* < 0$ and $\mu_v^{**} < 0$. By (iv) of Theorem 3.3, both pest and natural enemy populations persist. See also Figure 3, the solution tends to a positive periodic state.



(a) (b) (c)

(d) (e) (f)

Figure 2. Simulations for $\beta = 0.3$ and $\gamma = 0.9$ ($\mu_u \approx 0.047 > 0, \mu_v \approx -0.125 < 0$). Graphs $(a) - (f)$ show the pest population tends to extinction and the natural enemy population tends to a positive periodic steady state.



(a) (b) (c)

(d) (e) (f)

Figure 3. Simulations for $\beta = 0.15$ and $\gamma = 0.6$. In this case, $\mu_u^* < 0, \mu_v^* < 0$ and $\mu_v^{**} < 0$. Graphs $(a) - (f)$ show both populations persist.

Recently, diffusive models [1] and fractional-order epidemic models [5, 18] have attracted much attention for describing the spread of infections. Our impulsive reaction-diffusion framework captures the combined effects of birth and harvesting pulses in a predator-prey context, which is directly relevant to integrated pest management.

Unlike the single-species impulsive logistic model in [28] and the travelling wave analysis in [12], our model incorporates interspecific interactions and yields sharp threshold conditions for coexistence or extinction of both species. In contrast to the ordinary differential equation impulsive model in [14], we include spatial diffusion and heterogeneous coefficients, allowing us to study how domain size and diffusion rates affect persistence via eigenvalue expressions (e.g., Eqs. (4.3)-(4.4)). The monotonicity results in Lemma 2.2 and the eigenvalue formulae in Theorem 2.5 generalize the work of Lewis and Li [11] to two species with distinct impulse timings (birth at nT, harvest at $nT + \tau$).

The analytical toolkit includes using the Krein-Rutman theorem for impulsive eigenvalue problems, constructing upper and lower solutions via periodic comparison, and deriving threshold criteria from sign conditions of principal eigenvalues, and can be readily adapted to other systems with periodic abrupt changes. For example: Epidemic models with impulsive vaccination or treatment: One can study a reaction-diffusion SIR model where vaccination pulses are applied at fixed times, and the basic reproduction number becomes an impulsive eigenvalue problem. Competition models with pulsed disturbances: Two competing species subject to periodic harvesting or resource pulses could be analysed using the same comparison and eigenvalue techniques to determine competitive exclusion or coexistence. Age-structured or stage-structured populations with pulsed reproduction and harvesting: The approach in [26] can be combined with our spatial eigenvalue analysis to handle both age structure and spatial diffusion under impulsive interventions.

Our numerical simulations (Figures 1-3) demonstrate that by adjusting the natural enemy birth pulse γ (e.g., via periodic releases) and the harvesting rate β (e.g., manual removal), one can achieve different ecological outcomes: Co-extinction (Figure 1), pest extinction with natural enemy persistence (Figure 2), or two-species coexistence (Figure 3). These findings highlight the potential for designing optimal impulse strategies to maintain ecological balance and suggest that careful tuning of impulse intensities can shift the system from undesirable extinction to stable coexistence.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability. No data was used for the research described in the article.

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