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A FAMILY OF OPTIMAL EIGHTH-ORDER ITERATIVE METHODS FOR MULTIPLE ROOTS OF NONLINEAR EQUATIONS

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Abstract In this manuscript, we develop a family of optimal eighth-order iterative methods for approximating multiple roots of nonlinear equations with known multiplicity, derived by modifying an existing simple-root iterative method. The family is constructed by incorporating a multivariate weight function, where specific choices of this function yield distinct members while preserving eighth-order convergence. Convergence order is rigorously established through theoretical analysis. Comprehensive numerical experiments on real-life and academic problems demonstrate the accuracy and computational efficiency of the proposed methods relative to some well known existing methods. Dynamical analysis via basins of attraction in the complex plane further reveals the stability and convergence behavior of the proposed family.

Keywords Iterative method, multiple roots, nonlinear equations, basins of attraction.

MSC(2010) $65Z05, 65H05, 41A25$.

1. Introduction

Solving nonlinear equations is a fundamental problem in numerical analysis owing to its wider applications in science and engineering. Analytical solutions are often difficult or impossible to obtain, so iterative methods are commonly used to approximate solutions of nonlinear equations of the form

$$\mathfrak{f}(w) = 0 \tag{1.1}$$

where $\mathfrak{f} : \mathbb{D} \subseteq \mathbb{R} \to \mathbb{R}$ is a function defined on an open interval $\mathbb{D}$. The well known Newton-Raphson method for locating simple roots [15, 25] is given by

$$w_{n+1} = w_n - \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}. \tag{1.2}$$

The method (1.2) exhibits quadratic convergence when applied to simple roots but degrades to linear convergence for multiple roots. A number α is said to be multiple root with multiplicity m if $\mathfrak{f}^{(i)}(\alpha) = 0$ for $i = 0, 1 \cdots, m-1$ and $\mathfrak{f}^{(m)}(\alpha) \neq 0$. To overcome this reduced convergence

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rate for multiple roots, the modified Newton method, which achieves quadratic convergence for multiple roots, is given by [18, 19]:

$$w_{n+1} = w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}. \tag{1.3}$$

In recent literature, considerable attention has been devoted to the development of higher-order iterative methods for approximating multiple roots of nonlinear equations (see [6, 9, 14, 20, 23, 24]). Although these methods exhibit faster convergence than the modified Newton method for multiple roots, they do not always satisfy the optimality criteria given by the Kung–Traub conjecture [11]. According to the Kung–Traub conjecture, a multipoint iterative method without memory that uses l function evaluations per iteration can achieve at most an order of convergence 2^{l-1}.

Several optimal fourth-order methods for multiple roots have been proposed in the literature. Sharma et al. [21] introduced an optimal family of derivative-free fourth-order methods derived from the Traub–Steffensen approach. Chicharro et al. [5] proposed a family of fourth-order methods based on the arithmetic mean of Newton's and Chebyshev's methods. Notable contributions include those of Li et al. [12], Kansal et al. [8] and Behl et al. [3]. More recently, significant efforts have been directed toward the construction of optimal eighth-order methods for multiple roots. Zafar et al. [28] proposed eighth-order schemes for finding multiple roots. Geum et al. [7] developed modified Newton-type methods employing multivariate weight functions, whereas Sharma et al. [22] introduced efficient weighted Newton-type methods using bivariate weight functions. Further contributions include the eighth-order families proposed by Behl et al. [2], Kumar et al. [10], Akram et al. [1], Sharma et al. [22], and Bhavna and Bhatia [4], all of whom developed optimal schemes employing various combinations of univariate and multivariate weight functions.

In most cases, the conditions that guarantee optimal convergence for simple-root methods are not preserved when these methods are applied to multiple roots, often leading to a substantial reduction in the convergence order. In most cases, the conditions that guarantee optimal convergence for simple-root methods are not preserved when these methods are applied to multiple roots, often leading to a substantial reduction in the convergence order. Motivated by this observation, a new family of optimal eighth-order is proposed through a systematic extension of an existing simple-root method by introducing suitable multiplicity-dependent ratios together with an appropriate multivariate weight function. This construction generates a flexible class of methods that preserves eighth-order convergence while satisfying the Kung–Traub optimality criterion. Several particular members of the family are also derived, each requiring only three function evaluations and one derivative evaluation per iteration.

The effectiveness of the proposed methods is demonstrated through a collection of academic and real-world problems. Numerical experiments reveal that the proposed methods provide accurate approximations and exhibit superior computational performance when compared with several existing methods. Furthermore, basin-of-attraction analyses confirm the stability and robustness of the proposed methods from a dynamical perspective.

The remainder of the paper is organized as follows. Section 2 presents the construction of the proposed family and establishes its eighth-order convergence through a detailed theoretical analysis based on Taylor expansions. In Section 3, extensive numerical comparisons with existing methods are reported using several real life and academic problems. Section 4 investigates the corresponding basins of attraction to analyze the convergence dynamics and stability of the proposed methods. Finally, Section 5 summarizes the main findings and presents concluding remarks.

2. Method development and convergence analysis

In this section, we construct a family of optimal eighth-order methods for approximating multiple roots of nonlinear equations using an optimal eight-order iterative method for simple roots developed by Panday et al. [16], given by:

$$\begin{aligned}
y_n &= w_n - \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\\
z_n &= w_n - \frac{\mathfrak{f}(w_n)\left(\mathfrak{f}(w_n)^2 - 3\mathfrak{f}(w_n)\mathfrak{f}(y_n) + \mathfrak{f}(y_n)^2\right)}{\mathfrak{f}'(w_n)\left(\mathfrak{f}(w_n) - 3\mathfrak{f}(y_n)\right)\left(\mathfrak{f}(w_n) - \mathfrak{f}(y_n)\right)},\\
w_{n+1} &= z_n - \frac{\mathfrak{f}(z_n)}{\mathfrak{f}'(w_n)}\left[G(\kappa_n) + F(\mu_n,\nu_n)\right],\ n = 0,1,2,3,4\cdots,
\end{aligned} \tag{2.1}$$

where, $F(\mu_n,\nu_n)$ and $G(\kappa_n)$ are weight functions and $\mu_n = \frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}$, $\nu_n = \frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}$, $\kappa_n = \frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}$.

Now, we propose a family of optimal eighth-order iterative methods for roots with multiplicity $m > 1$, defined as:

$$\begin{aligned}
y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\\
z_n &= w_n - m\ \frac{1 - 3u_n + u_n^2}{(1-3u_n)(1-u_n)}\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\\
w_{n+1} &= z_n - m v_n\frac{1+u_n}{1-u_n}H(u_n,v_n,t_n)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\ n = 0,1,2,3,4\cdots,
\end{aligned} \tag{2.2}$$

where $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$ and $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$. And, H is a weight function in the neighborhood of $(0,0,0)$.

The necessary conditions for $H(u_n,v_n,t_n)$ under which the proposed method (2.2) has eighth-order convergence are shown in the following theorem.

Theorem 2.1. *Consider α as a multiple root of a sufficiently differentiable function $\mathfrak{f}:\mathbb{D}\subseteq\mathbb{R}\to\mathbb{R}$, with multiplicity $m>1$, where the domain $\mathbb{D}$ encloses the root α. The iterative method described in (2.2) achieve eighth-order convergence when it satisfies*

$$H_{000} = 1,\ H_{100} = 0,\ H_{001} = 1,\ H_{200} = 8,\ H_{010} = 0,\ H_{101} = 2,\ H_{300} = 60.$$

The error equation associated with the newly proposed method in (2.2) as follows:

$$\begin{aligned}
e_{n+1} = -\frac{1}{48m^7}\Bigg(& c_1\Big((m-1)c_1^2 - 2mc_2\Big)\Big(-24m^2c_1c_3 + 12m^2c_2^2\big(H_{002}-2\big)\\
& - 12mc_1^2c_2\Big(-12 + m(-4+H_{002}) - H_{002} + 2H_{110} + H_{201}\Big)\\
& + c_1^4\Big(-850 + 3H_{002} + m^2(3H_{002} - 14) - 12H_{110} - 6H_{201}
\end{aligned}$$

$$+\,6m(-12 - H_{002} + 2H_{110} + H_{201}) + H_{400}\Big)\Big)\Bigg)e_n^8 + \mathcal{O}(e_n^9), \tag{2.3}$$

where $e_n = w_n - \alpha$ *is the error at the* n^{th} *iteration and* $H_{jkl} = H^{(j,k,l)}(0,0,0), 0 \le j,k,l \le 3.$

Proof. Let us assume that error at the n^{th} iteration is $e_n = w_n - \alpha$. Using Taylor expansion, we obtain

$$\begin{aligned}\mathfrak{f}(w_n) = \frac{\mathfrak{f}^{(m)}(\alpha)}{m!}\, e_n^m & \left(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7\right.\\ & \left. + c_8 e_n^8\right) + \mathcal{O}(e_n^9),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathfrak{f}'(w_n) = \frac{\mathfrak{f}^{(m)}(\alpha)}{m!}\, e_n^{m-1} & \left(m + (1+m)c_1 e_n + (2+m)c_2 e_n^2 + (3+m)c_3 e_n^3\right.\\ & + (4+m)c_4 e_n^4 + (5+m)c_5 e_n^5 + (6+m)c_6 e_n^6 + (7+m)c_7 e_n^7\\ & \left. + (8+m)c_8 e_n^8\right) + \mathcal{O}(e_n^9),\end{aligned} \tag{2.5}$$

where $c_i = \frac{m!}{(m+i)!}\frac{\mathfrak{f}^{(m+i)}(\alpha)}{\mathfrak{f}^{(m)}(\alpha)}$, for $i = 1, 2, 3, \cdots$. Then, substituting the equations (2.4) and (2.5) in the first step of method given by equation (2.2), we have

$$y_n - \alpha = \frac{c_1}{m} e_n^2 + \frac{1}{m^2}\left(2mc_2 - (m+1)c_1^2\right) e_n^3 + \sum_{i=0}^{4} p_i e_n^{i+4} + \mathcal{O}(e_n^9), \tag{2.6}$$

where the first two coefficients are explicitly given by:

$$\begin{aligned} p_0 &= \frac{1}{m^3}\left(3c_3 m^2 + c_1^3(1+m)^2 - c_1 c_2 m(4+3m)\right),\\ p_1 &= \frac{1}{m^4}\left(-(1+m)^3 c_1^4 + 2m(1+m)(3+2m)\, c_1^2 c_2 - 2m^2(2+m)\, c_2^2\right.\\ &\quad \left. -2m^2(3+2m)\, c_1 c_3 + 4m^3 c_4\right), \end{aligned}$$

and p_2, p_3, p_4 are the functions of $m, c_1, c_2, \cdots, c_7$. Again, by using Taylor's series expansion, we get

$$\begin{aligned}\mathfrak{f}(y_n) = \frac{\mathfrak{f}^{(m)}(\alpha)}{m!}\, e_n^{2m} & \Bigg[\left(\frac{c_1}{m}\right)^m - \left(\frac{c_1}{m}\right)^m \frac{1}{c_1}\left((m+1)c_1^2 - 2mc_2\right) e_n\\ & + \left(\frac{c_1}{m}\right)^{m+1} \frac{1}{2c_1^3}\left((m^3 + 3m^2 + 3m + 3)c_1^4 - (3m + 2m^2 + 2)\, 2mc_1^2 c_2\right.\\ & \left. + 4(-1+m)m^2 c_2^2 + 6m^2 c_1 c_3\right) e_n^2 + \sum_{i=0}^{5} q_i e_n^{i+3}\Bigg] + \mathcal{O}(e_n^9),\end{aligned} \tag{2.7}$$

where q_0, q_1, q_2, q_3, q_4, q_5 are the functions of $m, c_1, c_2, \cdots, c_7$. The expansion of $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$ is given by

$$u_n = \frac{c_1}{m} e + \frac{-(2+m)c_1^2 + 2mc_2}{m^2} e^2 + \mathcal{O}(e^3). \tag{2.8}$$

Now, using equations (2.4), (2.5), (2.7) and (2.8) in the second step of method given by equation (2.2), we obtain the expansion of z_n as

$$z_n - \alpha = \frac{(-1+m)c_1^3 - 2mc_1c_2}{2m^3}e^4 + \sum_{i=0}^{3} r_i e_n^{i+5} + \mathcal{O}(e^9), \tag{2.9}$$

where r_0, r_1, r_2, r_3 are the functions of $m, c_1, c_2, \cdots, c_7$. Now the expansion of $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$ and $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$ are given by:

$$\begin{aligned} v_n &= \frac{1}{2m^3}\left((-1+m)\,c_1^3 - 2m\,c_1c_2\right)e^3 + \mathcal{O}(e^4), \\ t_n &= \frac{1}{2m^2}\left((-1+m)\,c_1^2 - 2m\,c_2\right)e^2 + \mathcal{O}(e^3). \end{aligned} \tag{2.10}$$

Expanding the weight function $H(u_n, v_n, t_n)$ in the neighbourhood of $(0,0,0)$ using Taylor expansion, we get

$$\begin{aligned} H(u_n, v_n, t_n) = H_{000} + \big(H_{100}u_n + H_{010}v_n + H_{001}t_n\big) + \frac{1}{2!}\Big(H_{200}u_n^2 + 2H_{110}u_nv_n \\ + 2H_{101}u_nt_n + H_{020}v_n^2 + 2H_{011}v_nt_n + H_{002}t_n^2\Big) + \cdots. \end{aligned} \tag{2.11}$$

Now, by using equations (2.4), (2.5), (2.8), (2.9), (2.10) and (2.11) in the third step of method given by (2.2), we get

$$e_{n+1} = \lambda_1 e^4 + \lambda_2 e^5 + \lambda_3 e^6 + \lambda_4 e^7 + \lambda_5 e^8 + \mathcal{O}(e^9), \tag{2.12}$$

where

$$\begin{aligned} \lambda_1 &= -\frac{\big(H_{000}-1\big)\,c_1\Big((m-1)c_1^2 - 2mc_2\Big)}{2m^3}, \\ \lambda_2 &= \frac{1}{6m^4}\Bigg(\big(H_{000}-1\big)\Big((m-1)(1+7m)c_1^4 + 12(1-2m)mc_1^2c_2 + 12m^2c_2^2 \\ &\quad + 12m^2c_1c_3\Big) + 3c_1^2\Big(-(m-1)c_1^2 + 2mc_2\Big)H_{100}\Bigg), \end{aligned} \tag{2.13}$$

where the expressions $\lambda_3, \lambda_4, \lambda_5$ are the functions of $c_1, c_2, \cdots, c_5$ and $H^{(j,k,l)}(0,0,0)$. Now, substituting the values $H_{000} = 1, H_{100} = 0, H_{001} = 1, H_{200} = 8, H_{010} = 0,\ H_{101} = 2$ and $H_{300} = 60$ in equation (2.12), we get

$$\begin{aligned} e_{n+1} = -\frac{1}{48m^7}\Bigg(c_1\Big((m-1)c_1^2 - 2mc_2\Big)\Big(-24m^2c_1c_3 + 12m^2c_2^2\big(H_{002}-2\big) \\ - 12mc_1^2c_2\Big(-12 + m(-4+H_{002}) - H_{002} + 2H_{110} + H_{201}\Big) \\ + c_1^4\Big(-850 + 3H_{002} + m^2(3H_{002}-14) - 12H_{110} - 6H_{201} \\ + 6m(-12 - H_{002} + 2H_{110} + H_{201}) + H_{400}\Big)\Big)\Bigg)e_n^8 + \mathcal{O}(e_n^9). \end{aligned} \tag{2.14}$$

This completes the proof of the theorem. □

Particular cases of the weight functions

We observe from Theorem 1 that appropriate choices of $H(u_n, v_n, t_n)$ in our proposed method (2.2) give rise to several methods for finding multiple roots. Based on the conditions imposed on the weight function in Theorem 1, two cases of our proposed method are presented below.

Case 1. The weight function, which satisfy Theorem 2.1, are chosen as

$$H(u_n, v_n, t_n) = 1 + t_n + 4u_n^2 + 2u_n t_n + 10u_n^3.$$

For this choice, the newly proposed iterative method becomes

$$\begin{aligned}
y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \ n = 0, 1, 2, 3, 4 \cdots, \\
z_n &= w_n - m\ \frac{1 - 3u_n + u_n^2}{(1 - 3u_n)(1 - u_n)} \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\
w_{n+1} &= z_n - m v_n \frac{1 + u_n}{1 - u_n} \left(1 + t_n + 4u_n^2 + 2u_n t_n + 10u_n^3\right) \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}.
\end{aligned} \tag{2.15}$$

Case 2. The weight function, which satisfy Theorem 2.1, are chosen as

$$H(u_n, v_n, t_n) = \frac{1 + u_n + t_n + 4u_n^2 + 3u_n t_n + 14u_n^3}{1 + u_n}.$$

With this weight function, the iterative method becomes

$$\begin{aligned}
y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \ n = 0, 1, 2, 3, 4 \cdots, \\
z_n &= w_n - m\ \frac{1 - 3u_n + u_n^2}{(1 - 3u_n)(1 - u_n)} \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\
w_{n+1} &= z_n - m\frac{v_n}{1 - u_n} \left(1 + u_n + t_n + 4u_n^2 + 3u_n t_n + 14u_n^3\right) \frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}.
\end{aligned} \tag{2.16}$$

3. Numerical results

This section illustrates the effectiveness and computational efficiency of the proposed family of iterative method (2.2). To illustrate the applicability of the proposed iterative methods, we analyze two of its particular cases, referred to as NPMM1 (2.15) and NPMM2 (2.16), and apply them to several real-life and academic problems. For the purpose of numerical comparison, several eighth-order iterative methods are considered and summarized below.

From the three-step eighth-order family of iterative methods given by Akram et al. [1] (AKM), the following form is considered:

$$\begin{aligned}
y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\
z_n &= y_n - m(2u_n^2 + u_n)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\
w_{n+1} &= z_n - m\left(t_n(1 + t_n + 2t_n u_n + u_n^2) - 6v_n u_n^2\right)\left(2u_n^2 + u_n\right)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},
\end{aligned} \tag{3.1}$$

where $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$, $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$.

Sharma et al. [22] (SHM) proposed a family of eighth-order iterative method, we consider the following expression from this family:

$$\begin{aligned} y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ z_n &= y_n - mu_n(1+2u_n)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ w_{n+1} &= z_n - mu_nt_n\left(1+2u_n+t_n+u_n^2+4u_nt_n-4u_n^3\right)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \end{aligned} \tag{3.2}$$

where $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$ and $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$.

Kumar et al. [10] (DM) introduced a family of optimal eighth-order iterative method in 2020. The method takes the following form:

$$\begin{aligned} y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ z_n &= y_n - m(1-u_n^2+2u_n)u_n\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ w_{n+1} &= z_n - m\frac{1+u_n}{1-4v_n}v_n\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)} - m\frac{u_n+t_n}{1+6u_n+6u_n^2}(1+6u_n)v_n\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \end{aligned} \tag{3.3}$$

where $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$.

Zafar et al. [28] (ZM) proposed a three-step optimal eighth-order iterative method in 2018. We consider a particular case from their family, given as:

$$\begin{aligned} y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ z_n &= y_n - m\left(1+6u_n^3-u_n^2+2u_n\right)u_n\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ w_{n+1} &= z_n - m(2v_n+1)(t_n+1)(2u_n+1)u_nt_n\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \end{aligned} \tag{3.4}$$

where $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$ and $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$.

Bhavna and Bhatia [4] (BBM) introduced a family of optimal eighth-order iterative method, we consider the following expression from this family :

$$\begin{aligned} y_n &= w_n - m\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ z_n &= y_n - mu_n\frac{1+\lambda u_n}{1+(\lambda-2)u_n}\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)}, \\ w_{n+1} &= z_n - m\frac{z_n-y_n}{t_n}\frac{\left(\frac{5-13u_n}{5-13u_n-5u_n^2+(3+20\lambda)u_n^2}+2v_n\right)}{t_n-1}, \end{aligned} \tag{3.5}$$

where $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $v_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$, $t_n = \left(\frac{\mathfrak{f}(z_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$ and $\lambda = 0$.

Behl et al. [2] (BAMS) introduced a family of eighth-order iterative methods, we consider the following expression from this family:

$$\begin{aligned}
y_n &= w_n - m\,\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\\
z_n &= y_n - u_n\left(\frac{m(\theta-\beta+2p_n-2)}{\theta-\beta}\right)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},\\
w_{n+1} &= w_n - k_n u_n\bigg(m\big(1+2u_n+(1-2\beta)u_n^2+2(\beta^2-2\beta-2)u_n^3\big)\\
&\qquad +\frac{mq_n}{1-4u_n}\bigg)\frac{\mathfrak{f}(w_n)}{\mathfrak{f}'(w_n)},
\end{aligned} \tag{3.6}$$

where $\theta = \frac{1}{2}$, $\beta = \frac{-3}{2}$, $p_n = \frac{1+\theta u_n}{1+\beta u_n}$, $q_n = \left(\frac{\mathfrak{f}(w_n)}{\mathfrak{f}(y_n)}\right)^{1/m}$ and $u_n = \left(\frac{\mathfrak{f}(y_n)}{\mathfrak{f}(w_n)}\right)^{1/m}$. All numerical computations were carried out using Mathematica 13.3 with 4000 significant digits to ensure high numerical accuracy. Tables 1–5 summarize the comparative performance of all the examined methods, including SHM, AKM, ZM, DM, BBM, BAMS, NPMM1 and NPMM2. The numerical results presented in these tables are: The estimated roots (w_n), absolute residual error ($|\mathfrak{f}(w_n)|$), absolute difference of consecutive iterations ($|w_n - w_{n-1}|$), number of iterations (n) and the computational order of convergence (COC). The COC [17] is computed as

$$COC \approx \frac{\log|\mathfrak{f}(w_n)/\mathfrak{f}(w_{n-1})|}{\log|\mathfrak{f}(w_{n-1})/\mathfrak{f}(w_{n-2})|}.$$

Now we discussed the applicability of our newly proposed methods on several real life and academic problems.

Problem 3.1. (Newton's beam problem) *"The problem of the positioning of the beam is considered as described in [26]. A beam whose length is r meters rests against the edge of a cubic box whose sides each measure 2 unit. One end of the beam is in contact with the wall, and the other lies on the floor, as depicted in figure below Let $BC = y$ and $AB = w$. The required*

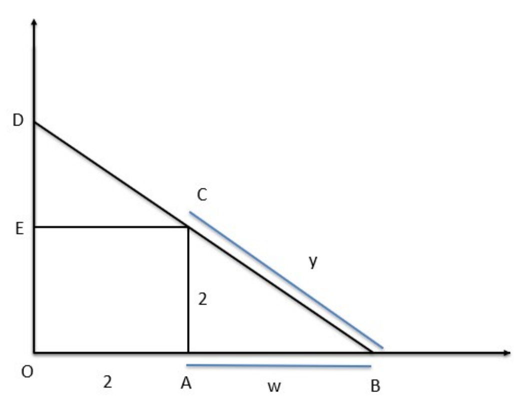


Figure 1. Positioning of the beam against the cubic box.

distance is therefore $w + 2$ meters (see Figure 1). By the similarity of triangles $\triangle CAB$ and

$\triangle DEC$, we obtain

$$\begin{aligned} \frac{w}{y} &= \frac{2}{r-y}, \\ y &= \frac{rw}{w+2}. \end{aligned} \tag{3.7}$$

Applying the Pythagoras theorem in the right triangle $\triangle CAB$, we obtain

$$w^2 + 2^2 = y^2. \tag{3.8}$$

Upon substituting y from (3.7) into (3.8), we obtain

$$\begin{aligned} & w^2 + 4 = \frac{r^2 w^2}{(w+2)^2}, \\ & w^4 + 4w^3 + 8w^2 + 16w + 16 = r^2 w^2, \\ & w^4 + 4w^3 + (8 - r^2)w^2 + 16w + 16 = 0. \end{aligned} \tag{3.9}$$

Putting $r^2 = 32$ in (3.9), we obtain

$$\mathfrak{f}_1(w) = w^4 + 4w^3 - 24w^2 + 16w + 16.$$

This equation has roots $\alpha \approx 2$ with multiplicity $m = 2$". In the numerical illustration, we take the initial approximation $w_0 = 3$. The numerical results are summarized in Table 1.

As evident from Table 1, the proposed methods NPMM1 and NPMM2 produce absolute lower residual errors and absolute smaller successive iterate differences than SHM, AKM, BAMS, BBM, ZM and DM, while maintaining the expected convergence order.

Table 1. Comparison of methods for $\mathfrak{f}_1(w)$.

Method	$\lvert\mathfrak{f}(w_n)\rvert$	$\lvert w_n - w_{n-1}\rvert$	n	COC
SHM	1.3231×10^{-465}	1.2188×10^{-29}	3	8.0000
AKM	2.7251×10^{-473}	4.1329×10^{-30}	3	8.0000
BAMS	1.8815×10^{-140}	4.2621×10^{-9}	3	15.412
BBM	1.3289×10^{-667}	5.3181×10^{-42}	3	8.0000
ZM	4.2627×10^{-476}	4.8281×10^{-30}	3	8.0000
DM	1.0049×10^{-479}	1.6338×10^{-30}	3	8.0000
NPMM1	7.1656×10^{-645}	2.4376×10^{-40}	3	8.0000
NPMM2	7.8085×10^{-637}	3.6965×10^{-40}	3	8.0000

Problem 3.2. (Van Der Waal's equation) *"The Van der Waals equation [13] for real gases is given by*

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT, \tag{3.10}$$

where a and b are van der Waals parameters that vary depending on type of gas, with ‘a’ representing the intermolecular attractive forces and ‘b’ corresponding to the molecular size. These parameters allow the calculation of pressure P, absolute temperature T and number of moles n. Expanding the left-hand side and rearranging terms, we obtain the equivalent Van der Waals–type cubic equation

$$PV^3 - (nbP + nRT)V^2 + an^2V = abn^3. \tag{3.11}$$

We substitute the given numerical values: $n = 0.1807$, $P = 1$ *atm*, $T = 313$ *K*, $a = 278.3$ *atm* $\cdot L^2/mol^2$, $b = 3.2104$ *L/mol*, $R = 0.08206$ *atm* $\cdot L/(mol \cdot K)$. *Hence,* (3.11) *can now be expressed as a nonlinear function* $\mathfrak{f}_1(w)$ *that can establish the volume V of the gas as a function of w*

$$\mathfrak{f}_2(w) = w^3 - \frac{261}{50}w^2 + \frac{3633}{400}w - \frac{2107}{400}.$$

This equation has the multiple roots $\alpha \approx 1.75$ with multiplicity $m = 2$”, using initial approximation w_0 =2. The numerical results are summarized in Table 2.

The results in Table 2 demonstrate that NPMM1 and NPMM2 are not only convergent but also demonstrate competitive numerical performance compared with the existing methods SHM, AKM, BAMS, BBM, ZM and DM.

Table 2. Comparison of methods for $\mathfrak{f}_2(w)$.

Method	$\lvert\mathfrak{f}(w_n)\rvert$	$\lvert w_n - w_{n-1}\rvert$	n	COC
SHM	2.6894×10^{-74}	1.3110×10^{-6}	3	7.4908
AKM	2.7860×10^{-76}	1.0102×10^{-6}	3	7.5215
BAMS	4.5116×10^{-39}	5.3930×10^{-11}	3	1.0194
BBM	1.6963×10^{-122}	2.1310×10^{-9}	3	7.8740
ZM	2.9464×10^{-78}	7.3386×10^{-7}	3	7.5208
DM	1.2456×10^{-78}	7.1259×10^{-7}	3	7.5480
NPMM1	7.3585×10^{-111}	9.4459×10^{-9}	3	7.8087
NPMM2	1.2284×10^{-108}	1.2647×10^{-8}	3	7.7955

Problem 3.3. (Higher Multiplicity) *Consider the nonlinear equation defined by*

$$\mathfrak{f}_3(w) = \left(-\frac{2}{3}w^3 + 16w^2 - 96w\right)^{30}. \tag{3.12}$$

This equation has two roots located at $\alpha \approx 0$ and $\alpha \approx 12$, where the latter is a multiple root of multiplicity $m = 60$. This high-multiplicity root problem is considered to evaluate the robustness of the methods for roots of high multiplicity. For the numerical experiments, the initial approximation is chosen as $w_0 = 10.5$. Using this initial guess, the corresponding numerical results are summarized in Table 3. From Table 3, it is observed that the proposed methods NPMM1 and NPMM2 demonstrate competitive convergence behavior relative to SHM, AKM, BBM, ZM and DM. Moreover, the obtained absolute residual errors and observed COC values indicate their effectiveness for the considered problem.

Table 3. Comparison of methods for $\mathfrak{f}_3(w)$.

Method	$\lvert\mathfrak{f}(w_n)\rvert$	$\lvert w_n - w_{n-1}\rvert$	n	COC
SHM	6.4916×10^{-3319}	8.4725×10^{-7}	3	7.8708
AKM	5.5144×10^{-3341}	7.8140×10^{-7}	3	7.8726
BAMS	6.2937×10^{-964}	9.6853×10^{-9}	3	1.0232
BBM	1.2418×10^{-933}	0.000019310	3	3.0075
ZM	5.5750×10^{-3066}	2.8216×10^{-6}	3	7.8212
DM	1.9802×10^{-2741}	0.000013717	3	7.7696
NPMM1	4.4183×10^{-3776}	1.4900×10^{-7}	3	7.8991
NPMM2	5.8940×10^{-3744}	1.4900×10^{-7}	3	7.9969

Problem 3.4. (Eigenvalue problem) *A fundamental challenge in linear algebra is determining the eigenvalues corresponding to a square matrix. Let us consider a* 6×6 *integer matrix given by*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 11 & -5 & 4 & 1 & 0 & 0 \\ -28 & 12 & -3 & 3 & 0 & 0 \\ 46 & -20 & 5 & -3 & 3 & 0 \\ -55 & 24 & -6 & 5 & -1 & 4 \end{pmatrix}.$$

The characteristic polynomial of A *is*

$$\mathfrak{f}_4(w) = w^6 - 15w^5 + 92w^4 - 296w^3 + 528w^2 - 496w + 192.$$

This equation has a multiple root $\alpha \approx 2$ *with multiplicity* $m = 4$*, and using the initial approximation* $w_0 = 2.1$*, the numerical results are summarized in Table 4.*

The results reported in Table 4 indicate that NPMM1 and NPMM2 attain observed COC values close to the theoretical order together with smaller absolute residual errors, demonstrating competitive numerical performance compared with the existing methods.

Table 4. Comparison of methods for $\mathfrak{f}_4(w)$.

Method	$\lvert\mathfrak{f}(w_n)\rvert$	$\lvert w_n - w_{n-1}\rvert$	n	COC
SHM	4.5531×10^{-364}	4.9545×10^{-12}	3	7.9768
AKM	1.2257×10^{-366}	4.2219×10^{-12}	3	7.9773
BAMS	3.3283×10^{-424}	1.0439×10^{-13}	3	7.9681
BBM	4.0407×10^{-98}	6.9460×10^{-9}	3	3.0008
ZM	2.6195×10^{-353}	1.0643×10^{-11}	3	7.9711
DM	5.7077×10^{-360}	6.7327×10^{-12}	3	7.9755
NPMM1	1.2645×10^{-399}	5.2206×10^{-13}	3	7.9818
NPMM2	1.4211×10^{-396}	6.3240×10^{-13}	3	7.9811

Problem 3.5. (Predatory prey Model) *"We examine a predator–prey system where l adybugs act as predators that feed on aphids. Let w be quantity of aphids consumed by ladybugs in a region of unit size over a unit time period; this is termed the predatory rate. The predation rate per unit area and unit time is defined as*

$$J(w) = K\,\frac{w^n}{w^n + a^n}, \quad a, K > 0,$$

where K is the predation constant. As given in [27], the equilibrium between growth and predation leads to

$$-sw^{n+1} + Kw^n - sa^n w = 0.$$

For the specific case, $n = 2$, $s = \frac{1}{2}$ per hour, $K = 20$, and $a = 20$, this reduces to

$$\mathfrak{f}_5(w) = -\frac{1}{2}w^3 + 20w^2 - 200w.$$

This equation has the multiple roots $\alpha \approx 20$ with multiplicity $m = 2$". Taking the initial approximation $w_0 = 17.5$, then the corresponding numerical results are presented in Table 5.

As observed from Table 5, NPMM1 and NPMM2 attain the theoretical convergence order while producing smaller absolute residual errors and absolute successive iterate differences than SHM, AKM, BAMS, BBM, ZM and DM, demonstrating the effectiveness of the proposed methods for the considered problem.

Table 5. Comparison of methods for $\mathfrak{f}_5(w)$.

Method	$\lvert\mathfrak{f}(w_n)\rvert$	$\lvert w_n - w_{n-1}\rvert$	n	COC
SHM	1.0265×10^{-905}	2.9321×10^{-56}	4	8.0000
AKM	8.9616×10^{-912}	1.2570×10^{-56}	4	8.0000
BAMS	4.3068×10^{-281}	5.2114×10^{-17}	4	14.651
BBM	5.3601×10^{-99}	1.6668×10^{-16}	4	3.0000
ZM	3.4019×10^{-838}	4.8166×10^{-52}	4	8.0000
DM	8.1087×10^{-752}	1.2357×10^{-46}	4	8.0000
NPMM1	6.5261×10^{-1030}	7.0425×10^{-64}	4	8.0000
NPMM2	3.7685×10^{-1021}	2.4163×10^{-63}	4	8.0000

The results presented in Tables 1–5 are consistent with the theoretical convergence analysis and demonstrate the effectiveness of the proposed method for solving nonlinear equations with multiple roots of varying multiplicities. Although the performance of iterative methods is inherently influenced by the choice of initial approximations, leading to noticeable variations in the computed COC values of some existing methods across the test problems, the proposed method consistently exhibits stable convergence behaviour and, in several cases, outperforms the competing methods. These results indicate that the proposed family of iterative methods is computationally efficient, achieves competitive convergence speed and high accuracy and is practically effective for a broad range of academic and real-world applications.

4. Basins of attraction

This section analyzes dynamical behavior of the proposed eighth-order iterative methods (NPMM1 (2.15) and NPMM2 (2.16)) as well as SHM (3.2), AKM (3.1), ZM (3.4), DM (3.3), BBM (3.5) and BAMS (3.6) designed for locating multiple roots of nonlinear equations. We focus on the analysis of the complex dynamics of these methods by visually examining the basins of attraction corresponding to the zeros of a polynomial $p(z)$ in the complex plane, which illustrate the influence of initial guesses on the convergence behavior and stability of the associated iterative methods.

To examine the convergence behavior of the proposed iterative method, we consider the following two nonlinear functions in the complex plane for evaluation:

$$p_1(z) = (z^2 - z)^2, \qquad p_2(z) = (z^3 - 1)^3, \qquad z \in \mathbb{D},$$

where $p_1(z)$ has two roots 0 and 1 each of multiplicity two. And $p_2(z)$ has three roots $-0.5 - 0.866025\,i$, $-0.5 + 0.866025\,i$ and 1 each of multiplicity three. The computational region is defined as $\mathbb{D} = [-3, 3] \times [-3, 3] \subset \mathbb{C}$, discretized into a 601×601 mesh.

For each starting point $z_0 \in \mathbb{D}$, z_0 is considered divergent if the convergence condition $|\mathfrak{f}(z_n)| \leq 10^{-3}$ is not satisfied during the first 100 iterations of the iterative method. The roots are indicated by small white circles in the complex plane. An initial point z_0 is rendered in black when the iterative method fails to converge to any root and such points are classified as divergent points. In each basin of attraction, brighter regions correspond to faster convergence, while darker regions represent slower convergence. The resulting basins of attraction are shown in Figures 1 and 2 for the test polynomials $p_1(z)$ and $p_2(z)$, respectively.

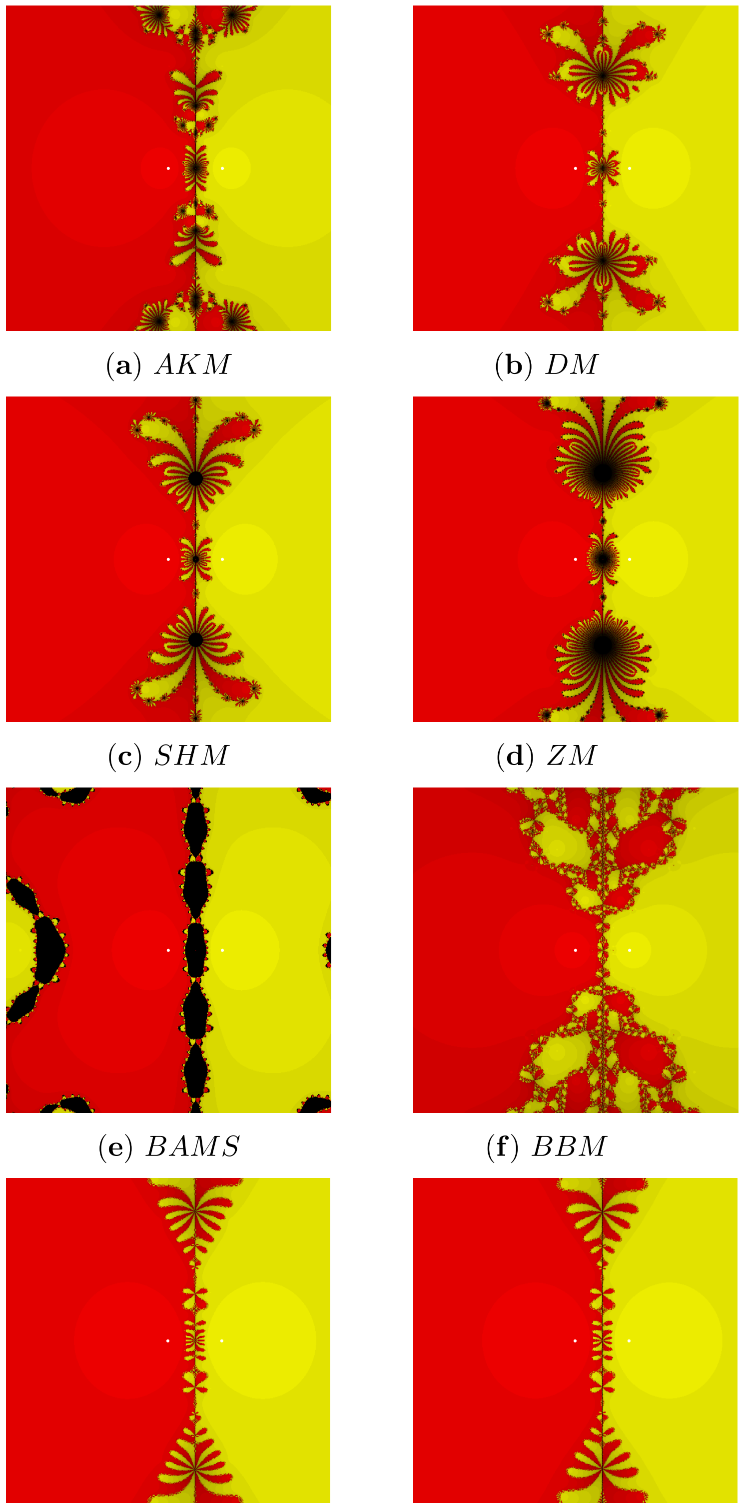
The proposed methods NPMM1 and NPMM2 produce wider basins of attraction and exhibit fewer dark regions compared with AKM, DM, SHM, ZM, BBM and BAMS. These visual observations are further supported by the number of divergent points reported in Table 6, where NPMM1 and NPMM2 exhibit fewer divergent points among the methods under consideration. From these basins of attraction, it is observed that NPMM1 and NPMM2 exhibit stable convergence behavior with fewer black points.

Table 6. Number of divergent points corresponding to eighth-order methods.

$p_n(z)$	SHM	AKM	BAMS	BBM	ZM	DM	NPMM1	NPMM2
$p_1(z)$	1971	3086	34867	15	3110	104	2	2
$p_2(z)$	2103	1579	1119	1	1719	7605	1	1

5. Conclusion

In this work, we proposed a family of optimal eighth-order iterative methods for solving nonlinear equations having multiple roots. The proposed methods are constructed by incorporating appropriate ratios and a multivariate weight function. A convergence analysis was carried out to verify the optimal order of convergence of this family. Two particular members of the proposed family are compared with well-known methods available in the literature. The numerical results indicate that the newly proposed methods compete efficiently with existing methods in terms of the absolute residual errors and absolute differences between consecutive iterates. Furthermore, basins of attraction illustrate the stable convergence behavior of the proposed methods



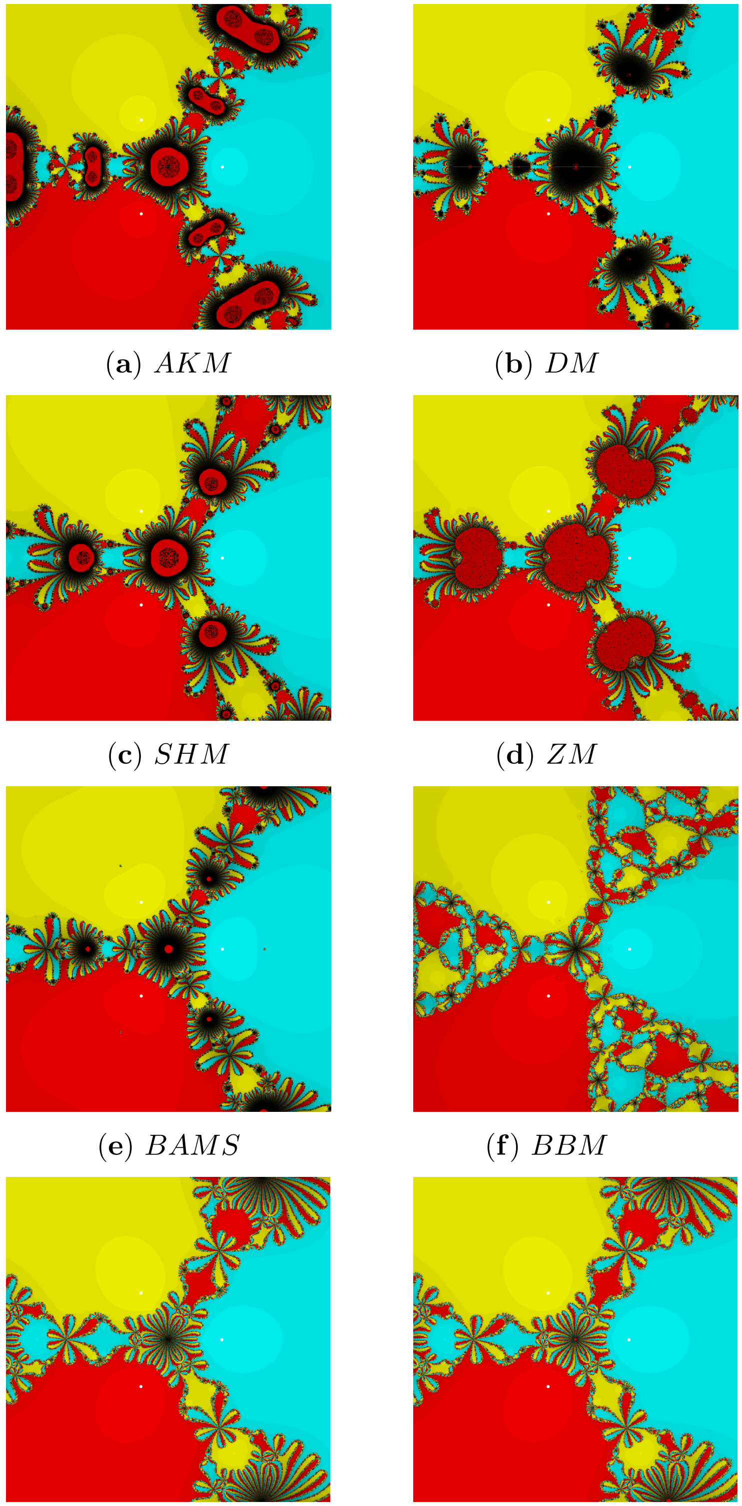
(**a**) AKM (**b**) DM

(**c**) SHM (**d**) ZM

(**e**) $BAMS$ (**f**) BBM

(**g**) $NPMM1$ (**h**) $NPMM2$

Figure 2. Basins of attraction associated with the methods for the polynomial $p_1(z) = (z^2 - z)^2$.



(**a**) AKM (**b**) DM

(**c**) SHM (**d**) ZM

(**e**) $BAMS$ (**f**) BBM

(**g**) $NPMM1$ (**h**) $NPMM2$

Figure 3. Basins of attraction associated with the methods for the polynomial $p_2(z) = (z^3 - 1)^3$.

with fewer divergent regions. Overall, the dynamical analysis and numerical results confirm that the proposed family constitutes an efficient and competitive approach for solving nonlinear equations with multiple roots.

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