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THE EXISTENCE NONTRIVIAL AND EXTREMAL SOLUTIONS FOR HADAMARD FRACTIONAL DIFFERNENTIAL EQUATION WITH INTEGRAL AND INFINITE-POINT BOUNDARY VALUE CONDITION ON AN INFINITE DOMAIN

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Abstract It is a qualitative leap to extend the boundary value problem from finite interval to infinite interval, this is because the fractional-order equation on infinite interval is more widely referenced, but the compactness of the interval on infinite interval is more difficult to prove, so there are not many studies on the upper value problem of infinite interval. In this paper, derivation of the expression of Green's function of fractional differential system on infinite interval, fractional derivative and its properties, then the problem of proving the compactness of operators on infinite intervals is overcome by establishing a suitable cone. Subsequently, the existence, uniqueness of nontrivial solution and positive extremal solutions are obtained by monotone iterative technique for Hadamard fractional differential system with integral and infinite-point boundary value condition on an infinite domain, moreover, the maximum and minimum solutions are obtained, and two iterative sequences which converge to the extremal solution pairs on an infinite interval. Finally, we give several examples to prove the validity of the theory.

Keywords Hadamard fractional derivative, extremal solutions, the existence and uniqueness of nontrivial solution, infinite domain.

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1. Introduction

Fractional calculus can more accurately reflect real physical phenomena, this is because fractional differential equations using fewer parameters can obtain good effect when describing complex physical and mechanical problems, and the physical meaning of fractional order model is clearer and more concise than that of integer order model. At present, fractional differential equations have been applied in various fields, such as dielectric dielectrics [1], porous media [19], nonlinear elasticity [21], diffusion and heat transfer [28], seismic wave propagation [34], signal processing [25], control systems [3], financial systems [2], nonlinear dynamics [4, 7, 16–18, 26, 29, 30, 35, 36], etc. It has been noticed that most of the work on the topic is based on Riemann-Liuville and

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Caputo derivatives, for more details, the reader is referred to [9,11,20,27,31] and the references therein.

The boundary value problems of differential equations about relevant literature are mostly studied on a finite interval, extending from a finite interval to an infinite interval is a qualitative flight. Compared with fractional-order differential model boundary value problem on a finite interval, fractional-order differential model boundary value problem on infinite interval originates from a wide range of applied mathematics and physics fields, so it has more important research value. However, the structure of fractional differential equations on infinite intervals develops slowly, which is caused by the complexity of the solution space and the difficulty of dealing with the compactness of the operator on an infinite intervals. Therefore, it is difficult and promising to study the structure of boundary-value problems of fractional differential models with nonlinear derivative terms on infinite intervals. Although the structure of the solution of fractional differential equation in infinite interval develops slowly, some mathematical scholars have still made some research achievements [22, 24, 32]. In [32], Zhang and Ntouyas got the minimal and maximal positive solutions of boundary value problem on the infinite interval. In the paper [23], the authors got nonnegative multiple solutions for nonlinear fractional differential equations by means of Leggett-Williams and Guo-Krasnoselskii's fixed point theorems. On infinite domain, Pei [22] studied the following fractional differential equation and got the positive minimal and maximal solutions. In [33], the authors get new existence, uniqueness, and multiplicity results of positive solutions by using Schauder's fixed point theorem, Banach's contraction mapping principle and so on. A body of our earlier work was dedicated to establishing existence and convergence results for several systems involving Caputo, Caputo-Hadamard, and Riemann-Liouville fractional derivatives [5,6,8,10,14,15].

Motivated by the excellent results above, on an infinite domain, we consider the following Hadamard fractional integro-differential equation

$$\varphi_p({}^{H}D^{\ell}_{1^+}u(\upsilon)) + \Im\left(\upsilon, u(\upsilon), {}^{H}D^{\ell-2}_{1^+}u(\upsilon), {}^{H}D^{\ell-1}_{1^+}u(\upsilon)\right) = 0, \tag{1.1}$$

with integral and infinite-point boundary value condition

$$\begin{aligned} &u(1) = u'(1) = \cdots = u^{(n-2)}(1) = 0, \\ &{}^{H}D^{\ell-1}_{1^+}u(+\infty) = \int_1^{\infty} h(\upsilon)u(\upsilon)dB(\upsilon) + b\sum_{i=1}^{\infty}\alpha_i u(\eta_i), \end{aligned} \tag{1.2}$$

where $1 < \upsilon < \infty$, $\ell \in \mathbb{R}^1_+ = [0,\infty)$, $n-1 < \ell \leq n$, $\alpha_i \geq 0 (i = 1, 2, \cdots, +\infty)$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s,\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, $1 < \eta_i < +\infty (i = 1, 2, \cdots, \infty)$, ${}^{H}D^{\ell}_{1^+}$ is the standard Hadamard fractional derivative, B is function of bounded variation, and $\int_1^{\infty} h(\upsilon)u(\upsilon)dB(\upsilon)$ denotes the Riemann-Stieltjes integral with respect to B, $h(\upsilon) \in L^1(1,\infty)$, $\int_1^{\infty} h(\upsilon)(\ln \upsilon)^{\ell-1}dB(\upsilon) + b\sum_{i=1}^{\infty}\alpha_i(\ln \eta_i)^{\ell-1} < \Gamma(\ell)$, $\Im(\upsilon, x, y, z) : [1,+\infty) \times \mathbb{R}^3 \to \mathbb{R}^1$. The existence, uniqueness of nontrivial solution, and the minimal and maximal positive extremal solutions for Hadamard fractional differential equation with integral and infinite-point boundary value conditions are obtained under some sufficient conditions on an infinite interval.

Compared with the relevant articles, the work introduces three key contributions:

(1) The equation we considered is Hadamard fractional differential equation with mixed integral and multi-point boundary conditions and the range of time variable we considered is on an finite domain.

(2) The result we obtained is more precise, we not only get the existence, uniqueness of nontrivial solution and get two explicit monotone iterative sequences which converge to the extremal solution on unbounded domains.

(3) The compactness of the boundary value problem on an infinite interval is difficult to guarantee. We overcame the compactness problem of the operator by establishing a special cone.

2. Preliminaries and lemmas

Now, some basic definitions and lemmas about Hadamard fractional integral, derivative and some of its operational properties can be found in [12], we omit them here. Now, we define the Banach space E by

$$E=\Bigg\{u(\upsilon),{}^H D_{1^+}^{\ell-2}u(\upsilon),{}^H D_{1^+}^{\ell-1}u(\upsilon)\in C([1,\infty)):\sup_{\upsilon\in[1,\infty)}\frac{|u(\upsilon)|}{1+(\ln\upsilon)^{\ell-1}}<\infty,$$
$$\sup_{\upsilon\in[1,\infty)}\frac{|{}^H D_{1^+}^{\ell-2}u(\upsilon)|}{1+\ln\upsilon}<\infty,\ \sup_{\upsilon\in[1,\infty)}|{}^H D_{1^+}^{\ell-1}u(\upsilon)|<\infty\Bigg\}. \tag{2.1}$$

Let

$$\|u\|_0=\sup_{\upsilon\in[1,\infty)}\frac{|u(\upsilon)|}{1+(\ln\upsilon)^{\ell-1}},\ \|{}^H D_{1^+}^{\ell-2}u\|_0=\sup_{\upsilon\in[1,\infty)}\frac{|{}^H D_{1^+}^{\ell-2}u(\upsilon)|}{1+\ln\upsilon},$$
$$\|{}^H D_{1^+}^{\ell-1}u\|_0=\sup_{\upsilon\in[1,\infty)}|{}^H D_{1^+}^{\ell-1}u(\upsilon)|,$$

and E is a Banach space with the norm

$$\|u\|=\max\left\{\|u\|_0,\|{}^H D_{1^+}^{\ell-2}u\|_0,\|{}^H D_{1^+}^{\ell-1}u\|_0|\right\},$$

moreover, define a cone

$$K=\{u\in E:u(\upsilon)\geq 0,\upsilon\in[1,\infty)\}.$$

For the convenience of the reader, let us first give out a conditon

(H_0) there exists a function $h(\upsilon)\in L[1,+\infty),\alpha_i\geq 0,\eta_i>1(i=1,2,\cdots,+\infty)$, such that

$$\int_1^{+\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)<+\infty,\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}<+\infty.$$

In the following, we present Green's function of the fractional differential equation.

Lemma 2.1. *Given $y\in C(1,+\infty)$, the linear problem*

$${}^H D_{1^+}^{\ell}u(\upsilon)+\varphi_q(y(\upsilon))=0,1<\upsilon<\infty, \tag{2.2}$$

with boundary value problem (1.2) *has a unique solution that can be expressed by*

$$u(\upsilon)=\int_1^{+\infty}\Phi(\upsilon,s)\frac{\varphi_q(y(s))}{s}ds, \tag{2.3}$$

where $\Phi(\upsilon,s)=\Phi_1(\upsilon,s)+\Phi_2(\upsilon,s)$, *and*

$$\Phi_1(\upsilon,s)=g(\upsilon,s)+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\upsilon)^{\ell-1}}{\Delta\Gamma(\ell)}g(\eta_i,s), \tag{2.4}$$

$$\Phi_2(\upsilon,s)=\frac{(\ln\upsilon)^{\ell-1}}{\Delta_1}\int_1^{+\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(s), \tag{2.5}$$

and

$$g(\upsilon,s)=\begin{cases}\dfrac{(\ln\upsilon)^{\ell-1}-(\ln\frac{\upsilon}{s})^{\ell-1}}{\Gamma(\ell)}, 1\le s\le\upsilon<+\infty,\\ \dfrac{(\ln\upsilon)^{\ell-1}}{\Gamma(\ell)}, 1\le\upsilon\le s<+\infty,\end{cases} \tag{2.6}$$

$$\Delta_1=\Delta-\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon), \Delta=\Gamma(\ell)-b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}.$$

Proof. By means of the Hadamard fractional integral, derivative of [12], we can reduce linear problem (2.2) to an equivalent integral equation

$$u(\upsilon)=-{}^HI_{1+}^{\ell}\varphi_q(y(\upsilon))+\zeta_1(\ln\upsilon)^{\ell-1}+\zeta_2(\ln\upsilon)^{\ell-2}+\cdots+\zeta_n(\ln\upsilon)^{\ell-n}, \tag{2.7}$$

for some $\zeta_1,\zeta_2,\cdots,\zeta_n\in\mathbb{R}^1$. Consequently, by (2.7), we get

$$u(\upsilon)=-\int_1^{\upsilon}\frac{(\ln\frac{\upsilon}{s})^{\ell-1}}{\Gamma(\ell)}\frac{\varphi_q(y(s))}{s}ds+\zeta_1(\ln\upsilon)^{\ell-1}+\zeta_2(\ln\upsilon)^{\ell-2}+\cdots+\zeta_n(\ln\upsilon)^{\ell-n}. \tag{2.8}$$

From $u(1)=0$, we have $\zeta_n=0$. By taking the derivative of formula (2.7), we have

$$u'(\upsilon)=-{}^HI_{1^+}^{\ell-1}\varphi_q(y(\upsilon))+\zeta_1(\ell-1)(\ln\upsilon)^{\ell-2}\frac{1}{\upsilon}+\zeta_2(\ell-2)(\ln\upsilon)^{\ell-3}\frac{1}{\upsilon}+\cdots, \tag{2.9}$$

by $u'(1)=0$, we get $\zeta_{n-1}=0$, the rest can be done in the same manner, we have $\zeta_2=\zeta_3=\cdots=\zeta_{n-2}=0$, then

$$u(\upsilon)=-\int_1^{t}\frac{(\ln\frac{\upsilon}{s})^{\ell-1}}{\Gamma(\ell)}\frac{\varphi_q(y(s))}{s}ds+\zeta_1(\ln\upsilon)^{\ell-1}, \tag{2.10}$$

$${}^HD_{1^+}^{\ell-1}u(\upsilon)=\zeta_1\Gamma(\ell)-\int_1^{\upsilon}\frac{\varphi_q(y(s))}{s}ds. \tag{2.11}$$

On the other hand, from ${}^HD_{1^+}^{\ell-1}u(\infty)=\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)+b\sum_{i=1}^{\infty}\alpha_iu(\eta_i)$, and

$${}^HD_{1^+}^{\ell-1}u(\infty)=-\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}+\Gamma(\ell)\zeta_1, \tag{2.12}$$

one has

$$\zeta_1\left(\Gamma(\ell)-b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}\right)=\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)+\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}$$

$$-\frac{b}{\Gamma(\ell)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i-\ln s)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}, \tag{2.13}$$

then

$$\zeta_1=\frac{1}{\Delta}\left(\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)+\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}-\frac{b}{\Gamma(\ell)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i-\ln s)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}\right). \tag{2.14}$$

By (2.11) and (2.14), we have

$$\begin{aligned}
u(\upsilon)=&\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}+\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)-\frac{1}{\Gamma(\ell)}\int_1^{\upsilon}(\ln\frac{\upsilon}{s})^{\ell-1}\varphi_q(y(s))\frac{ds}{s}\\
&-\frac{(\ln\upsilon)^{\alpha-1}b}{\Gamma(\ell)\Delta}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i-\ln s)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}\\
=&\frac{(\ln\upsilon)^{\ell-1}}{\Gamma(\alpha)}\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}+\frac{(\Gamma(\ell)-\Delta)(\ln t)^{\ell-1}}{\Delta\Gamma(\ell)}\int_1^{\infty}\varphi_q(y(s))\frac{ds}{s}\\
&+\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)\\
&-\frac{1}{\Gamma(\ell)}\int_1^{\upsilon}(\ln\frac{\upsilon}{s})^{\ell-1}\varphi_q(y(s))\frac{ds}{s}-\frac{(\ln\upsilon)^{\ell-1}b}{\Gamma(\alpha)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i-\ln s)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}\\
=&\int_1^{+\infty}g(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\upsilon)^{\ell-1}}{\Delta\Gamma(\alpha)}\int_1^{\infty}(\ln\eta_i)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}\\
&-\frac{(\ln\upsilon)^{\ell-1}b}{\Delta\Gamma(\ell)}\int_1^{\eta_i}(\ln\eta_i-\ln s)^{\ell-1}\varphi_q(y(s))\frac{ds}{s}+\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)\\
=&\int_1^{+\infty}g(\upsilon,s)y(s)\frac{ds}{s}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\upsilon)^{\ell-1}}{\Delta\Gamma(\ell)}\int_1^{+\infty}g(\eta_i,s)\varphi_q(y(s))\frac{ds}{s}\\
&+\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)\\
=&\int_1^{\infty}\Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}+\frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon).
\end{aligned} \tag{2.15}$$

Multiplying (2.15) by $h(\upsilon)$, and integrating from 1 to ∞, we have

$$\begin{aligned}
\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)=&\int_1^{\infty}h(\upsilon)\int_1^{\infty}\Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}dB(s)\\
&+\frac{\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)}{\Delta}\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon),
\end{aligned} \tag{2.16}$$

by (2.16), we have

$$\begin{aligned}
&\left(1-\frac{\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)}{\Delta}\right)\int_1^{\infty}h(\upsilon)u(\upsilon)dB(\upsilon)\\
=&\int_1^{\infty}h(\upsilon)\int_1^{\infty}\Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}dB(s).
\end{aligned} \tag{2.17}$$

Let $\Delta_1 = \Delta - \int_1^\infty h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)$, by (2.17), we have

$$\begin{aligned}\int_1^\infty h(\upsilon)u(\upsilon)dB(\upsilon) =&\frac{\Delta}{\Delta_1}\int_1^\infty h(\upsilon)\int_1^\infty \Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}dB(\upsilon)\\ =&\frac{\Delta}{\Delta_1}\int_1^\infty \varphi_q(y(s))\int_1^\infty \Phi_1(\upsilon,s)h(\upsilon)dB(\upsilon)\frac{ds}{s}.\end{aligned}\tag{2.18}$$

Hence, by (2.15) and (2.18), we get

$$\begin{aligned}u(\upsilon) =&\int_1^\infty \Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s} + \frac{(\ln\upsilon)^{\ell-1}}{\Delta}\int_1^\infty h(\upsilon)u(\upsilon)dB(\upsilon)\\ =&\int_1^\infty \Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s} + \frac{(\ln\upsilon)^{\ell-1}}{\Delta}\left(\frac{\Delta}{\Delta_1}\int_1^\infty \varphi_q(y(s))\int_1^\infty \Phi_1(\upsilon,s)h(\upsilon)dB(\upsilon)\frac{ds}{s}\right)\\ =&\int_1^\infty \Phi_1(\upsilon,s)\varphi_q(y(s))\frac{ds}{s} + \int_1^\infty \Phi_2(\upsilon,s)\varphi_q(y(s))\frac{ds}{s}\\ =&\int_1^\infty \Phi(\upsilon,s)\varphi_q(y(s))\frac{ds}{s},\end{aligned}\tag{2.19}$$

where $\Phi_1(\upsilon,s), \Phi_2(\upsilon,s), \Phi(\upsilon,s)$ are as (2.3)–(2.5). □

Lemma 2.2. *From* (2.3)*, by a simple computation, we get*

$${}^HD_{1^+}^{\ell-1}u(\upsilon) = \int_1^\infty \overline{\Phi}(\upsilon,s)\frac{\varphi_q(y(s))}{s}ds, {}^HD_{1^+}^{\ell-2}u(\upsilon) = \int_1^\infty \Phi_0(\upsilon,s)\frac{\varphi_q(y(s))}{s}ds,$$

where

$$\overline{\Phi}(\upsilon,s) = \vartheta(\upsilon,s) + \frac{b\sum_{i=1}^\infty \alpha_i g(\eta_i,s)}{\Delta} + \frac{\Gamma(\ell)}{\Delta_1}\int_1^\infty h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon),\tag{2.20}$$

$$\Phi_0(\upsilon,s) = {}^HD_{1^+}^{\ell-2}g(\upsilon,s) + \frac{b\sum_{i=1}^\infty \alpha_i g(\eta_i,s)\ln\upsilon}{\Delta} + \frac{\Gamma(\ell)\ln\upsilon}{\Delta_1}\int_1^\infty h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon),\tag{2.21}$$

and

$$\vartheta(\upsilon,s) = \begin{cases}0, 1\le s\le \upsilon<\infty,\\ 1, 1\le\upsilon\le s<\infty,\end{cases}\tag{2.22}$$

$${}^HD_{1^+}^{\ell-2}g(\upsilon,s) = \begin{cases}\ln\upsilon - \ln\dfrac{\upsilon}{s}, 1\le s\le\upsilon<+\infty,\\ \ln\upsilon, 1\le\upsilon\le s<+\infty.\end{cases}\tag{2.23}$$

□

Lemma 2.3. *The Green's function* $\Phi(\upsilon,s)$ *defiend by* (2.3) *has the following properties:*

(A_1): $\Phi(\upsilon,s)$ is continuous and $\Phi(\upsilon,s)\ge 0$ for $(\upsilon,s)\in[1,\infty)\times[1,\infty)$;

(A_2): $\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\le\Upsilon$, where

$$\Upsilon = \left(\frac{1}{\Gamma(\ell)} + \frac{b\sum_{i=1}^\infty \alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}\right)\left(1+\frac{1}{\Delta_1}\int_1^\infty h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)\right).$$

Proof. (A_1) is obviously hold, we only need to prove (A_2). By (2.3)–(2.5), we have

$$\begin{aligned}
&\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\\
=&\frac{\Phi_1(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}+\frac{\Phi_2(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\\
\leq&\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\upsilon)^{\ell-1}}{(1+(\ln\upsilon)^{\ell-1})\Delta\Gamma(\ell)}g(\eta_i,s)+\frac{(\ln\upsilon)^{\ell-1}}{\Delta_1(1+(\ln\upsilon)^{\ell-1}))}\int_1^{\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon)\\
\leq&\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}+\frac{1}{\Delta_1}\int_1^{\infty}h(\upsilon)(\frac{(\ln\upsilon)^{\ell-1}}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\upsilon)^{\ell-1}(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)})dB(\upsilon)\\
=&\left(\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)\right)\\
=&\Upsilon,
\end{aligned}\tag{2.24}$$

where

$$\Upsilon=\left(\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)\right),$$

hence, (A_2) holds. □

Remark 2.1. The Green's functions $\Phi(\upsilon,s)$,$\Phi_0(\upsilon,s)$ and $\overline{\Phi}(\upsilon,s)$ defined by (2.3) still have the following properties:

- (B_1): $\Phi(\upsilon,s)\leq\Upsilon(\ln\upsilon)^{\ell-1}$ for $(\upsilon,s)\in[1,\infty)\times[1,\infty)$;
- (B_2): $\Phi_0(\upsilon,s)\leq\Upsilon\Gamma(\ell)\ln\upsilon$ for $(\upsilon,s)\in[1,\infty)\times[1,\infty)$;
- (B_3): $\overline{\Phi}(\upsilon,s)\leq\Upsilon\Gamma(\ell)$, for $(\upsilon,s)\in[1,\infty)\times[1,\infty)$.

Proof. By (2.3)–(2.5) and (H_0), we easily get

$$\begin{aligned}
\Phi(\upsilon,s)\leq&\frac{(\ln\upsilon)^{\ell-1}}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_ig(\eta_i,s)}{\Delta\Gamma(\ell)}(\ln\upsilon)^{\ell-1}+\frac{(\ln\upsilon)^{\ell-1}}{\Delta_1}\int_1^{+\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon)\\
=&(\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_ig(\eta_i,s)}{\Delta\Gamma(\ell)}+\frac{1}{\Delta_1}\int_1^{+\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon))(\ln\upsilon)^{\ell-1}\\
\leq&\left(\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)\right)(\ln\upsilon)^{\ell-1}\\
\leq&\Upsilon(\ln\upsilon)^{\ell-1},
\end{aligned}\tag{2.25}$$

so (B_1) holds. Moreover,

$$\begin{aligned}
\Phi_0(\upsilon,s)\leq&\ln\upsilon+\frac{b\sum_{i=1}^{\infty}\alpha_ig(\eta_i,s)\ln\upsilon}{\Delta}+\frac{\Gamma(\ell)\ln\upsilon}{\Delta_1}\int_1^{\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon)\\
=&\left(1+\frac{b\sum_{i=1}^{\infty}\alpha_ig(\eta_i,s)}{\Delta}+\frac{\Gamma(\ell)}{\Delta_1}\int_1^{\infty}h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon)\right)\ln\upsilon\\
\leq&\Upsilon\Gamma(\ell).
\end{aligned}\tag{2.26}$$

Hence, (B_2) holds, similarly, (B_3) is easily checked, we omit the detail here. □

3. Results of existence and uniqueness of nontrivial solution

In this section, we assume that the following conditions hold:

(H_1): There exist nonnegative functions $p_1(\upsilon), p_2(\upsilon), p_3(\upsilon)$ and $q(\upsilon)$ defined on $[1,\infty)$ and constants $0 \le \sigma_i (i=1,2,3) \le 1$ such that

$$\varphi_q(\Im(\upsilon,x,y,z)) \le p_1(\upsilon)|x|^{\sigma_1} + p_2(\upsilon)|y|^{\sigma_2} + p_3(\upsilon)|z|^{\sigma_3} + q(\upsilon), \upsilon \in [1,\infty),$$

and

$$\begin{aligned}&\overline{p}_1 = \int_1^\infty p_1(\upsilon)[1+(\ln\upsilon)^{\ell-1}]^{\sigma_1}\frac{d\upsilon}{\upsilon} < \infty, \overline{p}_2 = \int_1^\infty p_2(\upsilon)(1+\ln\upsilon)^{\sigma_2}\frac{d\upsilon}{\upsilon} < \infty,\\ &\overline{p}_3 = \int_1^\infty p_3(\upsilon)\frac{d\upsilon}{\upsilon} < \infty, \overline{q} = \int_1^\infty q(\upsilon)\frac{d\upsilon}{\upsilon} < \infty.\end{aligned}$$

(H_2): $\Im$ is nondecreasing with respect to the second, third and last variables.

Lemma 3.1. *If (H_1) holds, then for any $u \in E$,*

$$\int_1^\infty \varphi_q\left(\Im(u(\upsilon), {}^H D_{1^+}^{\ell-2}u(\upsilon), {}^H D_{1^+}^{\ell-1}u(\upsilon))\right)\frac{d\upsilon}{\upsilon} \le \overline{p}_1\|u\|^{\sigma_1} + \overline{p}_2\|u\|^{\sigma_2} + \overline{p}_3\|u\|^{\sigma_3} + \overline{q}. \tag{3.1}$$

Proof. For any $u \in E$, by condition (H_1), we get

$$\begin{aligned}&\varphi_q\left(\Im(u(\upsilon), {}^H D_{1^+}^{\ell-2}u(\upsilon), {}^H D_{1^+}^{\ell-1}u(\upsilon))\right)\\ \le & p_1(\upsilon)|u(\upsilon)|^{\sigma_1} + p_2(\upsilon)|{}^H D_{1^+}^{\ell-2}u(\upsilon)|^{\sigma_2} + p_3(\upsilon)|{}^H D_{1^+}^{\ell-1}u(\upsilon)|^{\sigma_3} + q(\upsilon)\\ \le & p_1(\upsilon)(1+(\ln\upsilon)^{\ell-1})^{\sigma_1}\frac{|u(\upsilon)|^{\sigma_1}}{(1+(\ln\upsilon)^{\ell-1})^{\sigma_1}} + p_2(\upsilon)(1+\ln\upsilon)^{\sigma_2}\frac{|{}^H D_{1^+}^{\ell-2}u|^{\sigma_2}}{(1+\ln\upsilon)^{\sigma_2}}\\ &+ p_3(\upsilon)|{}^H D_{1^+}^{\ell-1}u(\upsilon)|^{\sigma_3} + q(\upsilon)\\ \le & p_1(\upsilon)[1+(\ln\upsilon)^{\ell-1}]^{\sigma_1}\|u\|^{\sigma_1} + p_2(\upsilon)(1+\ln\upsilon)^{\sigma_2}\|u\|^{\sigma_2} + p_3(\upsilon)\|u\|^{\sigma_3} + q(\upsilon).\end{aligned} \tag{3.2}$$

Hence, we have

$$\begin{aligned}&\int_1^\infty \varphi_q\left(\Im(u(\upsilon), {}^H D_{1^+}^{\ell-2}u(\upsilon), {}^H D_{1^+}^{\ell-1}u(\upsilon))\right)\frac{d\upsilon}{\upsilon}\\ \le & \int_1^\infty p_1(\upsilon)(1+(\ln\upsilon)^{\ell-1})^{\sigma_1}\|u\|^{\sigma_1}\frac{d\upsilon}{\upsilon}\\ &+ \int_1^\infty p_2(\upsilon)(1+\ln\upsilon)^{\sigma_2}\|u\|^{\sigma_2}\frac{d\upsilon}{\upsilon} + \int_1^\infty p_3(\upsilon)\|u\|^{\sigma_3}\frac{d\upsilon}{\upsilon} + \int_1^\infty q(\upsilon)\frac{d\upsilon}{\upsilon}\\ = & \overline{p}_1\|u\|^{\sigma_1} + \overline{p}_2\|u\|^{\sigma_2} + \overline{p}_3\|u\|^{\sigma_3} + \overline{q},\end{aligned} \tag{3.3}$$

the proof is completed. □

In view of Lemma 2.1, we define an operator A by

$$Au(\upsilon) = \int_1^\infty \Phi(\upsilon,s)\varphi_q\left(\Im(s,u(s), {}^H D_{1^+}^{\ell-2}u(s), {}^H D_{1^+}^{\ell-1}u(s))\right)\frac{ds}{s}, \tag{3.4}$$

obviously, the BVP(1.1) and (1.2) has a solution if and only if the operator equation $Au = u$ has a fixed point.

For convenience, we denote by

$$M = \max\{\Upsilon, \Gamma(\ell)\Upsilon\}. \tag{3.5}$$

Lemma 3.2. *Let $V \subset E$ be a bounded set. Then V is relatively compact in E if the following conditions hold:*

(i) For any $u(\upsilon) \in V$, $\frac{u(\upsilon)}{1+(\ln\upsilon)^{\ell-1}}$, $\frac{{}^H D_{1^+}^{\ell-2}u(\upsilon)}{1+\ln\upsilon}$ and ${}^H D_{1^+}^{\ell-1}u(\upsilon)$ are equicontinuous on any compact interval of $[1,\infty)$.

(ii) For any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 1$ such that

$$\left|\frac{u(\upsilon_1)}{1+(\ln\upsilon_1)^{\ell-1}} - \frac{u(\upsilon_2)}{1+(\ln\upsilon_2)^{\ell-1}}\right| < \varepsilon, \left|\frac{{}^H D_{1^+}^{\ell-2}u(\upsilon_1)}{1+\ln\upsilon_1} - \frac{{}^H D_{1^+}^{\ell-2}u(\upsilon_2)}{1+\ln\upsilon_2}\right| < \varepsilon,$$

and

$$|{}^H D_{1^+}^{\ell-1}u(\upsilon_1) - {}^H D_{1^+}^{\ell-1}u(\upsilon_2)| < \varepsilon,$$

for any $\upsilon_1, \upsilon_2 \geq T$ and $u \in V$.

Proof. The proof is similar with [22], we omit it here. □

Lemma 3.3. *Let (H_1) holds. Then the operator $A : E \to E$ is completely continuous.*

Proof. The whole process includes four steps.

Step 1. We show the operator $A : E \to E$ is uniformly bounded. Let Ω be any bounded subset of E, then there exists $\omega > 0$ such that $\|u\|_E \leq \omega$ for all $u \in \Omega$. From (A_2), it follows that

$$\begin{aligned}
\|Au\|_0 = &\sup_{t\in[1,\infty)} \int_1^\infty \frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}} \left|\varphi_q\left(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s))\right)\right| \frac{ds}{s} \\
\leq &\left(\frac{1}{\Gamma(\ell)} + \frac{b\sum_{i=1}^\infty \alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)}\right)\left(1 + \frac{1}{\Delta_1}\int_1^\infty h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon)\right) \\
&\times \int_1^\infty \left|\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\right| \frac{ds}{s} \\
\leq &\Upsilon \times [\overline{p}_1\|u\|^{\sigma_1} + \overline{p}_2\|u\|^{\sigma_2} + \overline{p}_3\|u\|^{\sigma_3} + \overline{q}] \\
\leq &M\,[\overline{p}_1\omega^{\sigma_1} + \overline{p}_2\omega^{\sigma_2} + \overline{p}_3\omega^{\sigma_3} + \overline{q}] \\
< &\infty,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\|{}^H D_{1^+}^{\ell-2}Au\|_0 = &\sup_{\upsilon\in[1,\infty)} \frac{|{}^H D_{1^+}^{\ell-2}Au(\upsilon)|}{1+\ln\upsilon} \\
= &\sup_{t\in[1,\infty)} \int_1^\infty \frac{\Phi_0(\upsilon,s)}{1+(\ln\upsilon)} \left|\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\right| \frac{ds}{s} \\
\leq &\Upsilon\Gamma(\ell) \times [\overline{p}_1\|u\|^{\sigma_1} + \overline{p}_2\|u\|^{\sigma_2} + \overline{p}_3\|u\|^{\sigma_3} + \overline{q}] \\
\leq &M\,[\overline{p}_1\omega^{\sigma_1} + \overline{p}_2\omega^{\sigma_2} + \overline{p}_3\omega^{\sigma_3} + \overline{q}] \\
< &\infty,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\|{}^H D_{1^+}^{\ell-1}Au\|_0 = &\sup_{\upsilon\in[1,\infty)} |{}^H D_{1^+}^{\ell-1}Au(\upsilon)| \\
= &\sup_{\upsilon\in[1,\infty)} \int_1^\infty \overline{\Phi}(\upsilon,s) \left|\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\right| \frac{ds}{s} \\
\leq &\Upsilon\Gamma(\ell) \times [\overline{p}_1\|u\|^{\sigma_1} + \overline{p}_2\|u\|^{\sigma_2} + \overline{p}_3\|u\|^{\sigma_3} + \overline{q}] \\
\leq &M\,[\overline{p}_1\omega^{\sigma_1} + \overline{p}_2\omega^{\sigma_2} + \overline{p}_3\omega^{\sigma_3} + \overline{q}] \\
< &\infty,
\end{aligned} \tag{3.8}$$

then, for $u \in E$, $A\Omega$ is uniformly bounded.

Step 2. We will prove the operator $A : E \to E$ is equicontinuous on any compact interval of $[1, \infty)$. Let $\nu \subset [1, \infty)$ be any compact interval. We assume that $\forall \upsilon_1, \upsilon_2 \in \nu, \upsilon_2 > \upsilon_1$, then

$$\begin{aligned}
&\left|\frac{Au(\upsilon_2)}{1+(\ln \upsilon_2)^{\ell-1}} - \frac{Au(\upsilon_1)}{1+(\ln \upsilon_1)^{\ell-1}}\right| \\
=&\left|\int_1^\infty \left(\frac{\Phi(\upsilon_2, s)}{1+(\ln \upsilon_2)^{\ell-1}} - \frac{\Phi(\upsilon_1, s)}{1+(\ln \upsilon_1)^{\ell-1}}\right) \varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s))) \frac{ds}{s}\right| \\
\le&\int_1^\infty \left|\frac{\Phi(\upsilon_2, s)}{1+(\ln \upsilon_2)^{\ell-1}} - \frac{\Phi(\upsilon_1, s)}{1+(\ln \upsilon_1)^{\ell-1}}\right| \left|\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s)))\right| \frac{ds}{s}.
\end{aligned} \tag{3.9}$$

For any compact set $\nu \times \nu$, $\frac{\Phi(\upsilon,s)}{1+(\ln \upsilon)^{\ell-1}}$ is uniformly continuous as $\Phi(\upsilon, s) \in C([1, \infty) \times [1, \infty))$. Notice that this function is only related to υ when $s \ge \upsilon$, so it is uniformly continuous on $\nu \times \nu \in ([1, \infty) \times [1, \infty))$. Hence, for all $s \in [1, \infty)$, $\forall \varepsilon > 0, \exists \delta > 0$, $\upsilon_1, \upsilon_2 \in \nu$, $|\upsilon_1 - \upsilon_2| < \delta$, we have

$$\left|\frac{\Phi(\upsilon_2, s)}{1+(\ln \upsilon_2)^{\ell-1}} - \frac{\Phi(\upsilon_1, s)}{1+(\ln \upsilon_1)^{\ell-1}}\right| < \varepsilon. \tag{3.10}$$

Combined with Lemma 3.1, $\forall u \in \Omega$, we have

$$\int_1^\infty |\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s)))| \frac{ds}{s} < \infty, \forall u \in \Omega. \tag{3.11}$$

Together with (3.9), (3.10) and (3.11), the above inequality implies that $\frac{Au(\upsilon)}{1+(\ln \upsilon)^{\ell-1}}$ is equicontinuous on ν. Similarly, we can get that $\frac{{}^H D_{1^+}^{\ell-2} Au(\upsilon)}{1+\ln \upsilon}$ and ${}^H D_{1^+}^{\ell-1} Au(\upsilon)$ are equicontinuous on ν.

Step 3. we will show the operator A, ${}^H D_{1^+}^{\ell-1} A$ and ${}^H D_{1^+}^{\ell-2} A$ are equiconvergent at ∞, for any $u \in \Omega$, we get

$$\begin{aligned}
&\lim_{\upsilon\to\infty}\left|\frac{Au(\upsilon)}{1+(\ln \upsilon)^{\ell-1}}\right| \\
=&\int_1^\infty \lim_{\upsilon\to\infty} \frac{\Phi(\upsilon, s)}{1+(\ln \upsilon)^{\ell-1}} \left|\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s)))\right| \frac{ds}{s} \\
\le&\Upsilon \times [\overline{p}_1 \|u\|^{\sigma_1} + \overline{p}_2 \|u\|^{\sigma_2} + \overline{p}_3 \|u\|^{\sigma_3} + \overline{q}] \\
\le&M\,(\overline{p}_1 \omega^{\sigma_1} + \overline{p}_2 \omega^{\sigma_2} + \overline{p}_3 \omega^{\sigma_3} + \overline{q}) \\
<&\infty,
\end{aligned} \tag{3.12}$$

hence, $A\Omega$ is equiconvergent at ∞. From Remark 2.1 and Lemma 3.1, for any $u \in \Omega$,

$$\begin{aligned}
\lim_{\upsilon\to\infty}\left|\frac{{}^H D_{1^+}^{\ell-2} Au(\upsilon)}{1+\ln \upsilon}\right| \le& \lim_{\upsilon\to\infty} \int_1^\infty \left|\frac{{}^H D_{1^+}^{\ell-2} \Phi(\upsilon, s)}{1+\ln \upsilon}\right| \left|\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s)))\right| \frac{ds}{s} \\
\le& \lim_{\upsilon\to\infty} \frac{\Gamma(\ell)\Upsilon(\ln \upsilon)}{1+\ln \upsilon} \times [\overline{p}_1 \|u\|^{\sigma_1} + \overline{p}_2 \|u\|^{\sigma_2} + \overline{p}_3 \|u\|^{\sigma_3} + \overline{q}] \\
\le&M\,(\overline{p}_1 \omega^{\sigma_1} + \overline{p}_2 \omega^{\sigma_2} + \overline{p}_3 \omega^{\sigma_3} + \overline{q}) \\
\to&0,
\end{aligned} \tag{3.13}$$

this implies that ${}^HD^{\ell-2}_{1^+}Au(\upsilon)$ is equiconvergent at ∞. From Remark 2.1 and Lemma 3.1, for any $u \in \Omega$,

$$\begin{aligned}
\lim_{\upsilon\to\infty}\left|{}^HD^{\ell-1}_{1^+}Au(\upsilon)\right| \leq \lim_{\upsilon\to\infty}\int_1^\infty |{}^HD^{\ell-1}_{1^+}\Phi(\upsilon,s)|\left|\varphi_q(\Im(s,u(s),{}^HD^{\ell-2}_{1^+}u(s),{}^HD^{\ell-1}_{1^+}u(s)))\right|\frac{ds}{s}\\
\leq&\Gamma(\alpha)\Upsilon\times[\overline{p}_1\|u\|^{\sigma_1}+\overline{p}_2\|u\|^{\sigma_2}+\overline{p}_3\|u\|^{\sigma_3}+\overline{q}]\\
\leq&M\left(\overline{p}_1\omega^{\sigma_1}+\overline{p}_2\omega^{\sigma_2}+\overline{p}_3\omega^{\sigma_3}+\overline{q}\right)\\
<&+\infty,
\end{aligned}\tag{3.14}$$

this implies that ${}^HD^{\ell-1}_{1^+}Au(\upsilon)$ is equiconvergent at ∞.

Step 4. We will prove the operator $A : B \to B$ is continuous. Let $u_n, u \in B$, then by Lemma 3.1 and Remark 2.1, we have

$$\begin{aligned}
\left|\frac{Au_n(\upsilon)}{1+(\ln\upsilon)^{\ell-1}}\right| =& \int_1^\infty\left|\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\right|\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&\left(\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^\infty\alpha_i}{\Delta\Gamma(\ell)}g(\eta_i,s)+\frac{1}{\Delta_1}\int_1^\infty h(\upsilon)\Phi_1(\upsilon,s)dB(\upsilon)\right)\frac{(\ln\upsilon)^{\ell-1}}{1+(\ln\upsilon)^{\ell-1}}\\
&\times\int_1^\infty\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&\frac{\Upsilon(\ln\upsilon)^{\ell-1}}{1+(\ln\upsilon)^{\ell-1}}\times\int_1^\infty\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&M\left[\overline{p}_1\|u\|^{\sigma_1}+\overline{p}_2\|u\|^{\sigma_2}+\overline{p}_3\|u\|^{\sigma_3}+\overline{q}\right]\\
\leq&M\left(\overline{p}_1\omega^{\sigma_1}+\overline{p}_2\omega^{\sigma_2}+\overline{p}_3\omega^{\sigma_3}+\overline{q}\right)\\
<&+\infty,
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
\left|\frac{{}^HD^{\ell-2}_{1^+}Au_n(\upsilon)}{1+\ln\upsilon}\right| \leq& \int_1^\infty\left|\frac{\Phi_0(\upsilon,s)}{1+\ln\upsilon}\right|\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&\frac{\Upsilon\Gamma(\ell)\ln\upsilon}{1+\ln\upsilon}\times\int_1^\infty\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&M\left[\overline{p}_1\|u\|^{\sigma_1}+\overline{p}_2\|u\|^{\sigma_2}+\overline{p}_3\|u\|^{\sigma_3}+\overline{q}\right]\\
\leq&M\left(\overline{p}_1\omega^{\sigma_1}+\overline{p}_2\omega^{\sigma_2}+\overline{p}_3\omega^{\sigma_3}+\overline{q}\right)\\
<&+\infty,
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
\left|{}^HD^{\ell-1}_{1^+}Au_n(\upsilon)\right| \leq& \int_1^\infty|\overline{\Phi}(\upsilon,s)||\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))|\frac{ds}{s}\\
\leq&\Upsilon\Gamma(\ell)\times\int_1^\infty\left|\varphi_q(\Im(s,u_n(s),{}^HD^{\ell-2}_{1^+}u_n(s),{}^HD^{\ell-1}_{1^+}u_n(s)))\right|\frac{ds}{s}\\
\leq&M\left[\overline{p}_1\|u\|^{\sigma_1}+\overline{p}_2\|u\|^{\sigma_2}+\overline{p}_3\|u\|^{\sigma_3}+\overline{q}\right]\\
\leq&M\left(\overline{p}_1\omega^{\sigma_1}+\overline{p}_2\omega^{\sigma_2}+\overline{p}_3\omega^{\sigma_3}+\overline{q}\right)\\
<&+\infty.
\end{aligned}\tag{3.17}$$

By the Lebesgue dominated convergence theorem and continuity of $\Im$, for $u_n, u \in B, \upsilon, s \in (1, +\infty)$, we have

$$\begin{aligned} &\lim_{n\to\infty}\int_1^{\infty}\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\varphi_q\left(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s))\right)\frac{ds}{s}\\ =&\int_1^{\infty}\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\frac{ds}{s},\end{aligned}\tag{3.18}$$

$$\begin{aligned} &\lim_{n\to\infty}\int_1^{\infty}\frac{\Phi_0(\upsilon,s)}{1+\ln\upsilon}\varphi_q(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s)))\frac{ds}{s}\\ =&\int_1^{\infty}\frac{\Phi_0(t,s)}{1+\ln\upsilon}\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\frac{ds}{s},\end{aligned}\tag{3.19}$$

and

$$\begin{aligned} &\lim_{n\to\infty}\int_1^{\infty}\overline{\Phi}(\upsilon,s)\varphi_q(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s)))\frac{ds}{s}\\ =&\int_1^{\infty}\overline{\Phi}(\upsilon,s)\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))\frac{ds}{s}.\end{aligned}\tag{3.20}$$

So, for $u_n, u \in B, \upsilon, s \in (1, +\infty)$, we have

$$\begin{aligned} &\|Au_n - Au\|_0\\ =&\sup_{\upsilon\in[1,\infty)}\int_1^{\infty}\frac{\Phi(\upsilon,s)}{1+(\ln\upsilon)^{\ell-1}}|\varphi_q(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s)))\\ &-\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\ \to& 0(n\to\infty),\end{aligned}\tag{3.21}$$

$$\begin{aligned} &\|{}^H D_{1^+}^{\ell-2}Au_n - {}^H D_{1^+}^{\ell-2}Au\|_0\\ =&\sup_{\upsilon\in[1,\infty)}\int_1^{\infty}\frac{{}^H D_{1^+}^{\ell-2}\Phi(\upsilon,s)}{1+\ln\upsilon}|\varphi_q(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s)))\\ &-\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\ \to& 0(n\to\infty),\end{aligned}\tag{3.22}$$

and

$$\begin{aligned} &\|{}^H D_{1^+}^{\ell-1}Au_n - {}^H D_{1^+}^{\ell-1}Au\|_0\\ =&\sup_{\upsilon\in[1,\infty)}\int_1^{\infty}{}^H D_{1^+}^{\ell-1}\Phi(\upsilon,s)|\varphi_q(\Im(s,u_n(s),{}^H D_{1^+}^{\ell-2}u_n(s),{}^H D_{1^+}^{\ell-1}u_n(s)))\\ &-\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\ \to& 0(n\to\infty),\end{aligned}\tag{3.23}$$

in consequence, A is continuous. □

From the above, we get the operator $A : B \to B$ is completely continuous. The proof is completed.

Now we give out several theorems of solution for fractional differential equations (1.1) and (1.2).

Theorem 3.1. *Suppose* $(H_0), (H_1)$ *and* $\Im(\upsilon, 0, 0, 0) \not\equiv 0$ *for any* $\upsilon \in [1, +\infty)$. *When* $\sigma_1 = \sigma_2 = \sigma_3 = 1$, *that is, there exist nonnegative functions* $p_1, p_2, p_3, q \in L^1[1, +\infty)$ *such that*

$$\begin{aligned} &|\varphi_q(\Im(\upsilon, u_1, u_2, u_3))| \leq \sum_{k=1}^{3} p_k(\upsilon)|u_k| + q(\upsilon), \\ &a.e.(\upsilon, u_1, u_2, u_3) \in [1, +\infty) \times \mathbb{R}^3, \end{aligned} \tag{3.24}$$

and

$$\Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \frac{ds}{s} < 1, \tag{3.25}$$

where Υ *is defined by Lemma 2.3. Then BVP*(1.1) *and* (1.2) *has at least one nontrivial solution.*

Proof. Since $\Im(\upsilon, 0, 0, 0) \not\equiv 0$, there exists $[\sigma, \tau] \in [0, +\infty)$ such that

$$\min_{\upsilon \in [\sigma, \tau]} |\varphi_q(\Im(\upsilon, 0, 0, 0))| > 0.$$

From (3.24), we have $|\varphi_q(\Im(\upsilon, 0, 0, 0))| \leq q(\upsilon), a.e. \upsilon \in [1, +\infty)$, then

$$\int_1^{+\infty} q(s) \frac{ds}{s} > 0.$$

Let

$$r = \frac{\Upsilon \int_1^{+\infty} q(s) \frac{ds}{s}}{1 - \Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \frac{ds}{s}},$$

from (3.25), we have $r > 0$.

Now let $\Omega_r = \{u \in C[0, 1] : \|u\| < r\}$, suppose $u \in \partial\Omega_r, \lambda > 1$ such that $Au = \lambda u$, then

$$\begin{aligned} \lambda r &= \lambda\|u\| \\ &= \|Au\| \\ &= \max\left\{\|Au\|_0, \|{}^H D_{1^+}^{\ell-2} Au\|_0, {}^H D_{1^+}^{\ell-1} Au\|_0\right\} \\ &= \max\left\{\max_{\upsilon \in [1, +\infty)} \frac{|Au(\upsilon)|}{1 + (\ln \upsilon)^{\ell-1}}, \max_{\upsilon \in [1, +\infty)} \frac{|D_{1^+}^{\ell-2} Au(\upsilon)|}{1 + \ln \upsilon}, \max_{\upsilon \in [1, +\infty)} |D_{1^+}^{\ell-1} Au(\upsilon)|, \right\} \\ &\leq \Upsilon \int_1^{+\infty} \varphi_q(\Im(s, u(s), D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s)) \frac{ds}{s}. \end{aligned} \tag{3.26}$$

By (3.24), for $p_1, p_2, p_3, q \in L^1[1, +\infty)$, we have

$$\begin{aligned} &|\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s))| \\ \leq &p_1(s)|u(s)| + p_2(s)|{}^H D_{1^+}^{\ell-2} u(s)| + p_3(s)|{}^H D_{1^+}^{\ell-1} u(s)| + q(s) \\ = &p_1(s)(1 + (\ln s)^{\ell-1}) \frac{|u(s)|}{1 + (\ln s)^{\ell-1}} + p_2(s)(1 + \ln s) \frac{|{}^H D_{1^+}^{\ell-2} u(s)|}{1 + \ln s} + p_3(s)^H |D_{1^+}^{\ell-1} u(s)| \\ \leq &(p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s))\|u\| + q(s). \end{aligned} \tag{3.27}$$

Hence, by (3.26) and (3.27), for $p_1, p_2, p_3, q \in L^1[1, +\infty)$, we get

$$\begin{aligned}\lambda r &\le \Upsilon \int_1^{+\infty} |\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s))| \frac{ds}{s} \\ &\le \Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \|u\| \frac{ds}{s} + \Upsilon \int_1^{+\infty} q(s) \frac{ds}{s} \\ &\le r\Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \frac{ds}{s} + \Upsilon \int_1^{+\infty} q(s) \frac{ds}{s}. \end{aligned} \tag{3.28}$$

Therefore, for $p_1, p_2, p_3, q \in L^1[1, +\infty)$, by (3.28), we have

$$\begin{aligned}\lambda &\le \Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \frac{ds}{s} + \frac{\Upsilon \int_1^{+\infty} q(s) \frac{ds}{s}}{r} \\ &= 1. \end{aligned} \tag{3.29}$$

This contradicts $\lambda > 1$. By Schauder fixed point theorem, A has a fixed point $u^\star \in \overline{\Omega}$. By $\Im(\upsilon, 0, 0, 0) \not\equiv 0$ we get that BVP(1.1) and (1.2) has a nontrivial solution $u^\star$. This completes the proof. □

Theorem 3.2. *Suppose* $(H_0), (H_1)$ *and* $\Im(\upsilon, 0, 0, 0) \not\equiv 0$ *for any* $\upsilon \in [1, +\infty)$. *Moreover, there exist nonnegative functions* $p_1, p_2, p_3, q \in L^1[1, +\infty)$ *such that*

$$\begin{aligned}&|\varphi_q(\Im(\upsilon, u_1, u_2, u_3))| \le \sum_{k=1}^{3} p_k(\upsilon)|u_k|^{\sigma_k} + q(\upsilon), \\ &a.e.(\upsilon, u_1, u_2, u_3) \in [0, +\infty) \times \mathbb{R}^3, \end{aligned} \tag{3.30}$$

where $0 < \sigma_1, \sigma_2, \sigma_3 < 1$ *(or* $\sigma_1, \sigma_2, \sigma_3 > 1$*) are nonnegative constants. Then BVP*(1.1) *and* (1.2) *has at least one nontrivial solution.*

Proof. Here we only proof the case for $0 < \sigma_1, \sigma_2, \sigma_3 < 1$, the case of $\sigma_1, \sigma_2, \sigma_3 > 1$ is similar with $0 < \sigma_1, \sigma_2, \sigma_3 < 1$, we omit it here.

By Lemma 3.3, we know $A : E \to E$ is a completely continuous operator.

Let

$$a = \Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s)) \frac{ds}{s}, \tag{3.31}$$

$$\overline{a} = \Upsilon \int_1^{+\infty} q(s) \frac{ds}{s}. \tag{3.32}$$

Choose $R > \{4\overline{a}, (4a)^{\frac{1}{1-\sigma_1}}, (4a)^{\frac{1}{1-\sigma_2}}, (4a)^{\frac{1}{1-\sigma_3}}\}$ and define a ball $B = \{u \in C[1, +\infty) : \|u\| \le R, \upsilon \in [1, +\infty)\}$. For any $u \in B$, one has

$$\begin{aligned}\|Au\| &= \max\left\{\|Au\|_0, \|{}^H D_{1^+}^{\ell-2} Au\|_0, {}^H D_{1^+}^{\ell-1} Au\|_0\right\} \\ &= \max\left\{\max_{\upsilon \in [1,+\infty)} \frac{|Au(\upsilon)|}{1 + (\ln \upsilon)^{\alpha-1}}, \max_{\upsilon \in [1,+\infty)} \frac{|D_{1^+}^{\ell-2} Au(\upsilon)|}{1 + \ln \upsilon}, \max_{\upsilon \in [1,+\infty)} |D_{1^+}^{\ell-1} Au(\upsilon)|\right\} \\ &\le \Upsilon \int_1^{+\infty} |\varphi_q(\Im(s, u(s), {}^H D_{1^+}^{\ell-2} u(s), {}^H D_{1^+}^{\ell-1} u(s))| \frac{ds}{s}. \end{aligned} \tag{3.33}$$

On the other hand, from (3.30), for any $u \in B$, $p_1, p_2, p_3, q \in L^1[1, +\infty)$, one has

$$\begin{aligned}
&|\varphi_q(\Im(s, u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s))| \\
\leq & p_1(s)|u(s)|^{\sigma_1} + p_2(s)|{}^H D_{1^+}^{\ell-2}u(s)|^{\sigma_2} + p_3(s)|{}^H D_{1^+}^{\ell-1}u(s))|^{\sigma_3} + q(s) \\
\leq & p_1(s)\|u\|^{\sigma_1} + p_2(s)\|u\|^{\sigma_2} + \cdots + p_{i+1}(s)\|u\|^{\sigma_{i+1}} + q(s) \\
= & p_1(s)(1 + (\ln \upsilon)^{\ell-1})^{\sigma_1} \cdot \left|\frac{u(s)}{(1 + (\ln \upsilon))}\right|^{\sigma_1} + p_2(s)(1 + (\ln \upsilon))^{\sigma_2} \cdot \left|\frac{{}^H D_{1^+}^{\ell-2}u(s)}{(1 + \ln \upsilon)}\right|^{\sigma_2} \\
& + p_3(s)|{}^H D_{1^+}^{\ell-1}u(s))|^{\sigma_3} + q(s) \\
\leq & \sum_{k=1}^{3} \|u\|^{\sigma_k}(p_1(s)(1 + (\ln \upsilon)^{\ell-1})^{\sigma_1} + p_2(s)(1 + (\ln \upsilon))^{\sigma_2} + p_3(s)) + q(s).
\end{aligned} \tag{3.34}$$

By (3.33), (3.34), for any $u \in B$, $p_1, p_2, p_3, q \in L^1[1, +\infty)$, one arrives

$$\begin{aligned}
\|Au\| \leq & \Upsilon \int_1^{+\infty} (p_1(s)(1 + (\ln s)^{\ell-1}) + p_2(s)(1 + \ln s) + p_3(s))\frac{ds}{s} + \Upsilon \int_1^{+\infty} q(s)\frac{ds}{s} \\
= & a\sum_{k=1}^{3} \|u\|^{\sigma_k} + \overline{a} \\
\leq & \frac{3R}{4} + \frac{R}{4} \\
= & R.
\end{aligned}$$

Hence, $\|Au\| \leq R$. Thus we get $A : B \to B$. By the Schauder fixed point theorem we get that BVP(1.1) and (1.2) has a solution. Since $\Im(\upsilon, 0, 0, 0) \not\equiv 0$, then BVP(1.1) and (1.2) has a nontrivial solution $u^\star$. The proof is completed. □

Theorem 3.3. *Suppose* $(H_0), (H_1)$ *and* $\Im(\upsilon, 0, 0, 0) \not\equiv 0$ *for any* $\upsilon \in [1, +\infty)$. *Moreover, there exist nonnegative functions* $p_1, p_2, p_3 \in L^1[1, +\infty)$ *such that*

$$|\varphi_q(\Im(\upsilon, u_1, u_2, u_3)) - \varphi_q(\Im(\upsilon, v_1, v_2, v_3))| \leq \sum_{k=1}^{3} p_k(\upsilon)|u_k - v_k| \tag{3.35}$$

holds for a.e.$(\upsilon, u_1, u_2, u_3), (\upsilon, v_1, v_2, v_3) \in [1, e] \times \mathbb{R}^3$ *and* (3.24) *holds. Then BVP*(1.1) *and* (1.2) *has a unique nontrivial solution.*

Proof. For (3.35), if $v_1 = v_2 = v_3 \equiv 0$, then we have

$$\varphi_q(\Im(\upsilon, u_1, u_2, u_3)) \leq \sum_{k=1}^{3} p_k(\upsilon)|u_k| + |\varphi_q(\Im(\upsilon, 0, 0, 0))|,$$

then by Theorem 3.1, we get BVP(1.1) and (1.2) has a nontrivial solution.

In the following, we prove the uniqueness of a nontrivial solution for BVP(1.1) and (1.2). Let A be given in (3.4), we shall show that A is a contraction. In fact, by (3.35), a similar method to Theorem 3.1, for *a.e.*$(s, u_1, u_2, u_3), (s, v_1, v_2, v_3) \in [1, e] \times \mathbb{R}^3$, $p_1, p_2, p_3 \in L^1[1, +\infty)$, we get

$$|\varphi_q(\Im(s, u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)) - \varphi_q(\Im(s, v(s),{}^H D_{1^+}^{\ell-2}v(s),{}^H D_{1^+}^{\ell-1}v(s))|$$

$$
\begin{aligned}
&\leq p_1(s)|u(s)-v(s)|+p_2(s)|^H D_{1^+}^{\ell-2}u(s)-^H D_{1^+}^{\ell-2}v(s)|+p_3(s)|^H D_{1^+}^{\ell-1}u(s)-^H D_{1^+}^{\ell-1}v(s)| \\
&= p_1(s)(1+(\ln \upsilon)^{\ell-1})\frac{|u(s)-v(s)|}{1+(\ln \upsilon)^{\ell-1}}+p_2(s)(1+(\ln \upsilon))\frac{|^H D_{1^+}^{\ell-2}u(s)-^H D_{1^+}^{\ell-2}v(s)|}{1+(\ln \upsilon)} \\
&\quad + p_3(s)|^H D_{1^+}^{\ell-1}u(s)-^H D_{1^+}^{\ell-1}v(s)|.
\end{aligned}
$$

So, for $a.e.(\upsilon, u_1, u_2, u_3), (\upsilon, v_1, v_2, v_3) \in [1, e] \times \mathbb{R}^3$, $p_1, p_2, p_3 \in L^1[1, +\infty)$, we have

$$
\|Au-Av\| \leq \Upsilon \int_1^{+\infty} (p_1(s)(1+(\ln \upsilon)^{\ell-1})+p_2(s)(1+\ln t)+p_3(s))\frac{ds}{s}\|u-v\|. \tag{3.36}
$$

Then (3.25) implies that A is indeed a contraction. Finally, we get a unique nontrivial solution to BVP(1.1) and (1.2) by the Banach fixed point theorem. □

4. Result of iterative solutions

In this section, we discuss the iterative solution of BVP(1.1) and (1.2).

Theorem 4.1. *Under the conditions (H_1) and (H_2), the positive minimal and maximal solution $\vartheta^\star, u^\star$ of the problems (1.1) and (1.2) can be obtained by the following two explicit monotone iterative sequences:*

$$
\begin{cases}
\vartheta_{n+1} = \displaystyle\int_1^\infty \Phi(\upsilon, s)\varphi_q(\Im(s, \vartheta_n(s), ^H D_{1^+}^{\ell-2}\vartheta_n(s), ^H D_{1^+}^{\ell-1}\vartheta_n(s)))\frac{ds}{s}, \\
u_{n+1} = \displaystyle\int_1^\infty \Phi(\upsilon, s)\varphi_q(\Im(s, u_n(s), ^H D_{1^+}^{\ell-2}u_n(s), ^H D_{1^+}^{\ell-1}u_n(s)))\frac{ds}{s},
\end{cases} \tag{4.1}
$$

moreover, for initial values $u_0(\upsilon) = \chi(\ln \upsilon)^{\ell-1}$, $\vartheta_0(\upsilon) = 0$, $\upsilon \in [1, +\infty)$, define the iterative sequences $\{u_n\}$ and $\{\vartheta_n\}$ by

$$
u_n = Au_{n-1} = A^n u_0, \ \vartheta_n = A\vartheta_{n-1} = A^n \vartheta_0,
$$

then

$$
\lim_{n\to\infty} u_n = \lim_{n\to\infty} A^n u_0 = u^\star, \ \lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^\star,
$$

and we have the relation

$$
\vartheta_0 \leq \vartheta_1 \leq \cdots \leq \vartheta_n \leq \cdots \vartheta^\star \leq \cdots \leq u^\star \leq u_n \leq \cdots \leq u_1 \leq u_0.
$$

Proof. We can easily know the fact that $Au(\upsilon) \geq 0$ for any $u \in K, \upsilon \in [1, \infty)$. Hence, we get $A(K) \subset K$.

For $0 \leq \sigma_1, \sigma_2, \sigma_3 < 1$, choose

$$
\chi \geq \max\left\{4M\bar{a}, (4M\bar{b})^{\frac{1}{1-\sigma_1}}, (4M\bar{c})^{\frac{1}{1-\sigma_2}}, (4M\bar{d})^{\frac{1}{1-\sigma_3}}\right\},
$$

and define $K_\chi = \{u \in E, \|u\|_E \leq \chi\}$. Next we show $A(K_\chi) \subset K_\chi$.

For any $u \in K_\chi$, we have

$$
\|Au\|_0 = \sup_{\upsilon\in[1,\infty)} \int_1^\infty \frac{\Phi(\upsilon, s)}{1+(\ln \upsilon)^{\ell-1}}|\varphi_q(\Im(s, u(s), ^H D_{1^+}^{\ell-2}u(s), ^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}
$$

$$
\begin{aligned}
&\leq \Upsilon \int_1^{\infty} |\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\
&\leq \Upsilon[\overline{q}+\overline{p}_1\|u\|_E^{\sigma_1}+\overline{p}_2\|u\|_E^{\sigma_2}+\overline{p}_3\|u\|_E^{\sigma_3}]\\
&\leq \Upsilon[\overline{q}+\overline{p}_1\chi^{p}+\overline{p}_2\chi^{q}+\overline{p}_3\chi^{e}]\\
&\leq M[\overline{q}+\overline{p}_1\chi^{\sigma_1}+\overline{p}_2\chi^{\sigma_2}+\overline{p}_3\chi^{\sigma_3}]\\
&\leq \chi,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\|{}^H D_{1^+}^{\ell-2}Au\|_0 = &\sup_{\upsilon\in[1,\infty)}\int_1^{\infty}\frac{{}^H D_{1^+}^{\ell-2}\Phi(\upsilon,s)}{1+\ln\upsilon}|\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\
&\leq \Upsilon\Gamma(\ell)\int_1^{\infty} |\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\
&\leq \Upsilon\Gamma(\ell)[\overline{q}+\overline{p}_1\|u\|_E^{p}+\overline{p}_2\|u\|_E^{q}+\overline{p}_3\|u\|_E^{e}]\\
&\leq \Upsilon\Gamma(\ell)[\overline{q}+\overline{p}_1\chi^{\sigma_1}+\overline{p}_2\chi^{\sigma_2}+\overline{p}_3\chi^{\sigma_3}]\\
&\leq M[\overline{q}+\overline{p}_1\chi^{\sigma_1}+\overline{p}_2\chi^{\sigma_2}+\overline{p}_2\chi^{\sigma_3}]\\
&\leq \chi,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
\|{}^H D_{1^+}^{\ell-1}Au\|_0 = &\sup_{\upsilon\in[1,\infty)}\int_1^{\infty}{}^H D_{1^+}^{\ell-1}\Phi(\upsilon,s)|\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\
&\leq \Upsilon\Gamma(\ell)\int_1^{\infty} |\varphi_q(\Im(s,u(s),{}^H D_{1^+}^{\ell-2}u(s),{}^H D_{1^+}^{\ell-1}u(s)))|\frac{ds}{s}\\
&\leq \Upsilon\Gamma(\ell)[\overline{q}+\overline{p}_1\|u\|_E^{\sigma_1}+\overline{p}_2\|u\|_E^{\sigma_2}+\overline{p}_3\|u\|_E^{\sigma_3}]\\
&\leq \Upsilon\Gamma(\ell)[\overline{q}+\overline{p}_1\chi^{\sigma_1}+\overline{p}_2\chi^{\sigma_2}+\overline{p}_3\chi^{\sigma_3}]\\
&\leq M[\overline{q}+\overline{p}_1\chi^{\sigma_1}+\overline{p}_2\chi^{\sigma_2}+\overline{p}_2\chi^{\sigma_3}]\\
&\leq \chi,
\end{aligned}
\tag{4.4}
$$

this implies that $\|Au\| \leq \chi$, $\forall u \in K_\chi$. Thus, $A(K_\chi) \subset K_\chi$.

The definition of the operator A and condition (H_2) lead A to be a nondecreasing operator.

Define $\vartheta_0(\upsilon) = 0, \vartheta_1 = A0 = A\vartheta_0, \vartheta_2 = A^2 0 = A\vartheta_1, \forall \upsilon \in [1,\infty)$. In view of $\vartheta_0(\upsilon) = 0 \in K$ and $A : K \to K$, then we can get $\vartheta_1, \vartheta_2 \in A(K) \subset K$. Thus, we get

$$\vartheta_1(\upsilon) = (A0)(\upsilon) \geq 0 = \vartheta_0(\upsilon), \forall \upsilon \in [1,\infty).$$

By the nondecreasing property of the operator A, we have

$$\vartheta_2(\upsilon) = (A\vartheta_1)(\upsilon) \geq (A\vartheta_0)(\upsilon) = \vartheta_1(\upsilon), \forall \upsilon \in [1,\infty).$$

Through induction, define a sequence $\vartheta_{n+1} = A\vartheta_n, n = 0,1,2,\cdots$. Obviously, the sequence $\{\vartheta_n\}_{n=1}^{\infty} \subset A(K) \subset K$ and satisfies

$$\vartheta_{n+1} \geq \vartheta_n(\upsilon), \forall \upsilon \in [1,\infty), n = 0,1,2,\cdots. \tag{4.5}$$

Since the complete continuity of the operator A, there exists a $\vartheta^\star \in K$ such that $\vartheta_{n_k} \to \vartheta^\star$ as $k \to \infty$, this is because $\{\vartheta_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{\vartheta_{n_k}\}_{k=1}^{\infty}$. This implies that $\lim_{n\to\infty}\vartheta_n = \vartheta^\star$.

Moreover, A is continuous and $\vartheta_{n+1} = A\vartheta_n$, it implies $A\vartheta^\star = \vartheta^\star$, i.e. $\vartheta^\star$ is a fixed point of the operator A.

Denote $u_0(\upsilon) = \chi(\ln\upsilon)^{\ell-1}$, $u_1 = Au_0, u_2 = A^2u_0 = Au_1$. For $u_0(\upsilon) \in K$ and $A : K \to K$, we have $u_1, u_2 \in A(K) \subset K$. By Lemma 2.3 and (3.4), we have

$$\begin{aligned} u_1(\upsilon) =& \int_1^\infty \Phi(\upsilon,s)\varphi_q(\Im(s,u_0(s),{}^H D_{1^+}^{\ell-2}u_0(s),{}^H D_{1^+}^{\ell-1}u_0(s)))\frac{ds}{s} \\ \le& \int_1^\infty \Upsilon(\ln\upsilon)^{\ell-1}\varphi_q(\Im(s,u_0(s),{}^H D_{1^+}^{\ell-2}u_0(s),{}^H D_{1^+}^{\ell-1}u_0(s)))\frac{ds}{s} \\ \le& \Upsilon(\ln\upsilon)^{\ell 1}\left[\overline{q} + \overline{p}_1\|u\|_E^{\sigma_1} + \overline{p}_2\|u\|_E^{\sigma_2} + \overline{p}_3\|u\|_E^{\sigma_3}\right] \\ \le& \chi(\ln\upsilon)^{\alpha-1} \\ =& u_0(\upsilon). \end{aligned}$$

Moreover, A is nondecreasing, we have

$$u_2(\upsilon) = Au_1(\upsilon) \le (Au_0)(\upsilon) = u_1(\upsilon),$$

then, we define $u_{n+1} = Au_n, (n = 0, 1, 2, \cdots .)$. Then the sequence $\{u_n\}_{n=1}^\infty \subset A(K) \subset K$ and satisfies the following relation

$$u_{n+1} \le u_n(\upsilon), n = 0, 1, 2, \cdots . \tag{4.6}$$

Similarly, we can know that there exists a $u^\star \in K$ such that $\lim_{n\to\infty} u_n = u^\star$.

Since A is continuous and $u_{n+1} = Au_n$, it implies $Au^\star = u^\star$, i.e. $u^\star$ is a fixed point of the operator A.

Then, we show that $u^\star$ and $\vartheta^\star$ are the positive maximal and minimal solutions of (1.1) and (1.2) respectively in $(0, \chi(\ln\upsilon)^{\ell-1}]$.

Firstly, we assume that $\overline{u} \in (0, \chi(\ln\upsilon)^{\ell-1}]$ is any solution of (1.1) and (1.2), then $\vartheta_0(\upsilon) = 0 \le \overline{u}(\upsilon) \le Au_0(\upsilon) = u_1(\upsilon)$.

Iterating the whole process several times, we have

$$\vartheta_n(\upsilon) \le \overline{u}(\upsilon) \le u_n(\upsilon), n = 0, 1, 2, \cdots . \tag{4.7}$$

Since $u^\star = \lim_{n\to\infty} u_n$ and $\vartheta^\star = \lim_{n\to\infty} \vartheta_n$, from (4.6), (4.7), we have

$$\vartheta_0 \le \vartheta_1 \le \cdots \le \vartheta_n \le \cdots \le \vartheta^\star \le \overline{u} \le u^\star \le \cdots \le u_n \le \cdots \le u_1 \le u_0. \tag{4.8}$$

$\Im(\upsilon, 0, 0, 0) \not\equiv 0, \forall \upsilon \in [1, \infty)$, then 0 is not a solution of problems (1.1) and (1.2). Then the Hadamard type fractional integro-differential equations (1.1) and (1.2) has the positive maximal and minimal solutions $u^\star, v^\star \in (0, \chi(\ln\upsilon)^{\ell-1}]$, which can be obtained by two monotone iterative sequences in (3.36). □

5. Examples

Example 5.1. Consider the following infinite-point $p-$Laplacian fractional differential equations

$$\varphi_3({}^H D_{1^+}^{\frac{5}{2}} u(\upsilon)) = \left(\frac{1}{4}\upsilon e^{-\upsilon} + \frac{\frac{1}{4}\upsilon e^{-\upsilon}x}{1+(\ln\upsilon)^{\frac{3}{2}}} + \left(\frac{\frac{1}{4}\upsilon e^{-\upsilon}y}{1+\upsilon}\right)e^{-\upsilon} + \frac{\upsilon}{(3+\upsilon)^2}e^{-\nu}z\right)^{\frac{1}{2}} \tag{5.1}$$

with integral and infinite-point boundary value problem on an infinite interval

$$u(1)=u'(1)=0,^{H}D_{1^+}^{\frac{3}{2}}u(+\infty)=\int_0^\infty h(\upsilon)u(\upsilon)dB(\upsilon)+1.2\sum_{i=1}^{m-2}\frac{1}{2j^7}u(\eta_i),0<\upsilon<\infty, \tag{5.2}$$

where $\ell=\frac{5}{2}, h(\upsilon)=\frac{1}{2}e^{-\upsilon}(\ln\upsilon)^{-\frac{3}{2}}$, $\alpha_i=\frac{1}{2j^7},\eta_i=e^{j^2},b=1.2,$

$$B(\upsilon)=\begin{cases}0, & \upsilon\in[0,1),\\ 1, & \upsilon\in[1,e),\\ 0.8, & \upsilon\in[e,\infty),\end{cases} \tag{5.3}$$

hence,

$$\begin{aligned}\int_1^\infty h(\upsilon)(\ln\upsilon)^{\alpha-1}dB(\upsilon)=&\int_1^\infty\frac{1}{2}e^{-\upsilon}(\ln\upsilon)^{-\frac{3}{2}}(\ln\upsilon)^{\frac{3}{2}}dB(\upsilon)\\ =&\int_1^\infty\frac{1}{2}e^{-\upsilon}dB(t)\\ =&1\times\frac{1}{2}e^{-1}-0.2e^{-e}\\ \approx&0.1707\\ <&\infty,\end{aligned}$$

$$\begin{aligned}\Delta=&\Gamma(\ell)-b\sum_{i=1}^\infty\alpha_i(\ln\eta_i)^{\ell-1}=\Gamma(\frac{3}{2})-1.2\sum_{i=1}^\infty\frac{1}{2j^7}\times(\ln e^{j^2})^{\frac{3}{2}}\approx0.2368>0,\\ \Delta_1=&\Delta-\int_1^\infty h(\upsilon)(\ln\upsilon)^{\alpha-1}dB(\upsilon)\approx0.2368-0.1707\approx0.0661,\end{aligned}$$

hence,

$$\begin{aligned}&\frac{1}{\Gamma(\ell)}+\frac{b\sum_{i=1}^\infty\alpha_i(\ln\eta_i)^{\frac{3}{2}}}{\Delta\Gamma^2(\alpha)}=\frac{1}{\Gamma(\ell)}+\frac{b\frac{1}{2}\sum_{i=1}^\infty\frac{1}{j^4}}{\Delta\Gamma^2(\ell)}\\ =&\frac{1}{\Gamma(\frac{5}{2})}+\frac{1.2\times0.5412}{\Delta\Gamma^2(\frac{5}{2})}\\ \approx&1.5520\\ <&\infty,\end{aligned}$$

and

$$\varphi_q(\Im(\upsilon,u,\psi,w))=\Im^2(\upsilon,u,\psi,w)=\frac{\frac{1}{4}\upsilon e^{-\upsilon}u}{1+(\ln\upsilon)^{\frac{3}{2}}}+\frac{\frac{1}{4}\upsilon e^{-\upsilon}\psi}{1+\ln\upsilon}+\frac{\upsilon}{(3+\upsilon)^2}e^{-\upsilon}w+\frac{1}{4}\upsilon e^{-\upsilon},$$

$$\Im(\upsilon,x,y,z)=\left(\frac{1}{4}\upsilon e^{-\upsilon}+\frac{\frac{1}{4}\upsilon e^{-\upsilon}x}{1+(\ln\upsilon)^{\frac{3}{2}}}+\left(\frac{\frac{1}{4}\upsilon e^{-\upsilon}y}{1+\upsilon}\right)e^{-\upsilon}+\frac{\upsilon}{(3+\upsilon)^2}e^{-\upsilon}z\right)^{\frac{1}{2}}.$$

Taking

$$p_1(\upsilon)=\frac{\frac{1}{4}\upsilon e^{-\upsilon}}{1+(\ln\upsilon)^{\frac{3}{2}}},p_2(t)=\frac{\frac{1}{4}\upsilon e^{-\upsilon}}{1+\ln\upsilon},p_3(\upsilon)=\frac{\upsilon}{(3+\upsilon)^2}e^{-\upsilon},q(\upsilon)=\frac{1}{4}\upsilon e^{-\upsilon}$$

we show that

$$\varphi_q(\Im(\upsilon,u,\psi,w)) \leq p_1(\upsilon)|u| + p_2(\upsilon)|\psi| + p_3(\upsilon)|w| + q(\upsilon).$$

By derivation, we have

$$\begin{aligned}
\overline{q} &= \int_1^{\infty} q(\upsilon)\frac{d\upsilon}{\upsilon} \leq \int_1^{\infty} \frac{\upsilon}{4}e^{-\upsilon}\frac{d\upsilon}{\upsilon} = \frac{1}{16e} < \infty,\\
\overline{p}_1 &= \int_1^{\infty} p_1(\upsilon)[1+(\ln\upsilon)^{\frac{3}{2}}]\frac{d\upsilon}{\upsilon} = \int_1^{\infty} \frac{\frac{1}{4}\upsilon e^{-\upsilon}}{[1+(\ln\upsilon)^{\frac{3}{2}}]}[1+(\ln\upsilon)^{\frac{3}{2}}]\frac{d\upsilon}{\upsilon} \leq \frac{1}{16e} < \infty,\\
\overline{p}_2 &= \int_1^{\infty} p_2(\upsilon)(1+(\ln\upsilon))^q\frac{d\upsilon}{\upsilon} = \int_1^{\infty} \frac{\frac{1}{4}\upsilon e^{-\upsilon}}{[1+(\ln\upsilon)]^q}[1+(\ln\upsilon)]^q\frac{d\upsilon}{\upsilon} \leq \frac{1}{16e} < +\infty,\\
\overline{p}_3 &= \int_1^{\infty} p_3(\upsilon)\frac{d\upsilon}{\upsilon} = \int_1^{\infty} \frac{\upsilon}{(3+\upsilon)^2}e^{-\upsilon}\frac{d\upsilon}{\upsilon} = 0.0161,
\end{aligned}$$

which imply that (H_1) holds. By a simple computation, we get

$$\begin{aligned}
\Upsilon =&(\frac{1}{\Gamma(\ell)} + \frac{b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}}{\Delta\Gamma^2(\ell)})(1+\frac{1}{\Delta_1}\int_1^{\infty} h(\upsilon)(\ln\upsilon)^{\ell-1}dB(t))\\
=&(\frac{1}{\Gamma(\frac{3}{2})} + \frac{1.2\times 0.5412}{0.2368\Gamma^2(\frac{5}{2})})(1+\frac{1}{0.0661}\times 0.1707)\\
\approx&9.6016,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&\Gamma(\ell) - \int_1^{\infty} h(\upsilon)(\ln\upsilon)^{\ell-1}dB(\upsilon) - b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\ell-1}\\
=&\Gamma(\frac{5}{2}) - \int_1^{\infty}\frac{1}{2}e^{-\upsilon}(\ln\upsilon)^{-\frac{3}{2}}(\ln\upsilon)^{\frac{3}{2}}dB(\upsilon) - \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{2j^2}(\frac{1}{j^{\frac{4}{3}}})^{\frac{3}{2}}\\
=&\frac{3}{4}\sqrt{\pi} - 0.1707 - \frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{j^4}\\
\approx&1.3293 - 0.1707 - 0.2706\\
=&0.888.
\end{aligned} \tag{5.5}$$

By calculation, we get

$$\begin{aligned}
&\Upsilon\int_1^{+\infty}(p_1(s)(1+(\ln s)^{\ell-1}) + p_2(s)(1+\ln s) + p_3(s) + q(s))\frac{ds}{s}\\
=&\Upsilon(\frac{3}{16e} + \int_1^{\infty}\frac{\frac{1}{4}s}{(3+s)^2}e^{-s}\frac{ds}{s})\\
=&9.6016(\frac{3}{16e} + 0.0161)\\
=&0.8169\\
<&1.
\end{aligned} \tag{5.6}$$

Thus, the condition (3.25) in Theorem 3.1 and Theorem 3.3 are satisfied, and from Theorem 3.1 and Theorem 3.3, BVP(5.1) and (5.2) has at least a nontrivial solution and unique solution under the condition of $\Im$ is continuous.

Example 5.2. We consider the following Hadamard fractional integro-differential equation

$$\begin{aligned}\varphi_3({}^H D_{1^+}^{\frac{5}{2}} u(\upsilon)) = \Bigg(& \frac{\upsilon e^{-\upsilon}|u(\upsilon)|^{\sigma_1}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^{\sigma_1}} + (\frac{|{}^H D^{\frac{1}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{1}{2}}})^{\sigma_2} \upsilon e^{-\upsilon} \\ & + (\frac{|{}^H D^{\frac{3}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{3}{2}}})^{\sigma_3} \upsilon e^{-\upsilon} + e^{-\upsilon}\Bigg)^{\frac{1}{2}}\end{aligned} \tag{5.7}$$

with integral and infinite-point boundary value problem (5.4) on an infinite interval, where $\ell, h(\upsilon)$, $\sigma_i (i = 1, 2, 3)$, $\alpha_i, B(\upsilon)$ are as Example 5.1, and

$$\begin{aligned}&\varphi_q(\Im(\upsilon, u(\upsilon), {}^H D_{1^+}^{\ell-2} u(\upsilon), {}^H D_{1^+}^{\ell-1} u(\upsilon))) \\ =&\Im^2(\Im(\upsilon, u(\upsilon), {}^H D_{1^+}^{\ell-2} u(\upsilon), {}^H D_{1^+}^{\ell-1} u(\upsilon)) \\ =&\frac{\upsilon e^{-\upsilon}|u(\upsilon)|^{\sigma_1}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^{\sigma_1}} + (\frac{|{}^H D^{\frac{1}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{1}{2}}})^{\sigma_2} \upsilon e^{-\upsilon} + (\frac{|{}^H D^{\frac{3}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{3}{2}}})^{\sigma_3} \upsilon e^{-\upsilon} + \upsilon e^{-\upsilon}.\end{aligned}$$

Taking

$$p_1(\upsilon) = \frac{\upsilon e^{-\upsilon}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^{\sigma_1}}, p_2(\upsilon) = \frac{\upsilon e^{-\upsilon}}{(1+\ln\upsilon)^{\sigma_2}}, p_3(\upsilon) = \frac{\upsilon}{(3+\upsilon)^2}\upsilon e^{-\upsilon}, q(\upsilon) = \upsilon e^{-\upsilon},$$

then

$$\begin{aligned}\varphi_q(\Im(\upsilon, u, \psi, w)) =& \frac{\upsilon e^{-\upsilon}|u|^{\sigma_1}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^{\sigma_1}} + (\frac{|\psi|}{1+(\ln\upsilon)^{\frac{1}{2}}})^{\sigma_2} \upsilon e^{-\upsilon} + (\frac{|w|}{1+(\ln\upsilon)^{\frac{3}{2}}})^{\sigma_3} \upsilon e^{-\upsilon} + \upsilon e^{-\upsilon} \\ =& \frac{\upsilon e^{-\upsilon}|u(\upsilon)|^{\sigma_1}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^{\sigma_1}} + (\frac{|{}^H D^{\frac{1}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{1}{2}}})^{\sigma_2} \upsilon e^{-\upsilon} + (\frac{|{}^H D^{\frac{3}{2}} u(\upsilon)|}{1+(\ln\upsilon)^{\frac{3}{2}}})^{\sigma_3} \upsilon e^{-\upsilon} + \upsilon e^{-\upsilon} \\ \le& p_1(\upsilon)|u|^{\sigma_1} + p_2(\upsilon)|\psi|^{\sigma_2} + p_3(\upsilon)|w|^{\sigma_3} + q(\upsilon), \\ \overline{q}(\upsilon) =& \int_1^\infty q(\upsilon)\frac{d\upsilon}{\upsilon} \le \int_1^\infty e^{-\upsilon} d\upsilon = \frac{1}{e} < \infty, \\ \overline{p}_1 =& \int_1^\infty p_1(\upsilon)[1+(\ln\upsilon)^{\alpha-1}]^p \frac{d\upsilon}{\upsilon} = \int_1^\infty \frac{e^{-\upsilon}}{[1+(\ln\upsilon)^{\frac{3}{2}}]^p}[1+(\ln\upsilon)^{\frac{3}{2}}]^p \frac{d\upsilon}{\upsilon} \le \frac{1}{e} < \infty, \\ \overline{p}_2 =& \int_1^\infty p_2(\upsilon)(1+(\ln\upsilon))^q \frac{d\upsilon}{\upsilon} = \int_1^\infty \frac{e^{-\upsilon}}{[1+(\ln\upsilon)]^q}[1+(\ln\upsilon)]^q \frac{d\upsilon}{\upsilon} \le \frac{1}{e} < +\infty, \\ \overline{p}_3 =& \int_1^\infty p_3(\upsilon)\frac{d\upsilon}{\upsilon} = \int_1^\infty \frac{\upsilon}{(3+\upsilon)^2} e^{-\upsilon}\frac{d\upsilon}{\upsilon} = 0.0161 < \infty,\end{aligned}$$

which imply that (H_1) holds. Taking $R = 8e$, by simple calculation, we get

$$\begin{aligned}\Upsilon\overline{p}_1 = \Upsilon\overline{p}_2 = \Upsilon\overline{q} = \int_1^\infty p_1(\upsilon)[1+(\ln\upsilon)^{\ell-1}]^p \frac{d\upsilon}{\upsilon} \approx 9.6016 \times \frac{1}{e} \approx 3.5322 < \frac{R}{4}, \\ \Upsilon\overline{p}_3 = \Upsilon \int_1^\infty \frac{\upsilon}{(3+\upsilon)^2} e^{-\upsilon}\frac{d\upsilon}{\upsilon} \approx 9.6016 \times 0.0161 \approx 0.1546 < \frac{R}{4},\end{aligned}$$

so the condition of Theorem 3.2 is satisfied, the boundary value problem (5.7) with boundary condition (5.2) has one nontrivial solution. The proof of Theorem 3.2 is completed.

Example 5.3. The Hadamard fractional differential equation (5.7) with boundary value condition (5.2) is considered once again, where $\ell, h(\upsilon)$, $\sigma_i (i = 1, 2, 3)$, $\alpha_i, B(\upsilon)$ are as Example 5.1, and $\varphi_q(\Im(\upsilon, u, v, w)), p_i(\upsilon)(i = 1, 2, 3), q(\upsilon)$ is as Example 5.2. From the expression of function $\Im$, we know that $\Im$ is nondecreasing with respect to the second, third and last variables. Hence, by Theorem 4.1, we can get that Hadamard fractional integral-differential boundary value problem (5.7) with boundary value (5.2) has the positive minimal and maximal solutions $\vartheta^\star, u^\star$ respectively in $(0, \chi(\log \upsilon)^{\frac{3}{2}}]$, which can be approximated by the following iterative sequence:

$$\vartheta_{n+1} = \int_1^\infty \Phi(\upsilon, s)\varphi_q(\Im(s, \vartheta_n(s), {}^H D^{\frac{1}{2}}_{1^+}\vartheta_n(s), {}^H D^{\frac{3}{2}}_{1^+}\vartheta_n(s)))\frac{ds}{s},$$

with initial value $\vartheta_0(t) = 0$,

$$u_{n+1} = \int_1^\infty \Phi(\upsilon, s)\varphi_q(\Im(s, u_n(s), {}^H D^{\frac{1}{2}}_{1^+}u_n(s), {}^H D^{\frac{3}{2}}_{1^+}u_n(s)))\frac{ds}{s},$$

with initial value $u_0(\upsilon) = \chi(\ln \upsilon)^{\frac{3}{2}}$.

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