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HOPF BIFURCATION FOR PIECEWISE SMOOTH LIÉNARD SYSTEMS*

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Abstract We consider a class of generalized Liénard systems and focus on the bifurcation of limit cycles near the origin as parameters vary. We derive the bifurcation function using a generalized polar coordinate transformation and analyze the cyclicity of the system at the origin, generalizing some known results. As an application, we discuss a special piecewise smooth system and study its cyclicity at the origin.

Keywords Liénard system, Hopf bifurcation, cyclicity, piecewise smooth.

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction and main results

The Liénard system is well known for its wide range of applications in various fields, including physics, engineering, and biology. The generation of limit cycles in planar dynamical systems is often driven by the Hopf bifurcation, a key mechanism that has been extensively studied. The authors [5, 6, 10] studied Hopf bifurcation of polynomial or smooth Liénard systems. In recent years, the bifurcation theory of limit cycles near a focus or a center was developed to non-smooth systems, see [9, 13, 14]. In [12] the author utilized the Picard–Fuchs equation to investigate and determine the upper bound on the number of small-amplitude limit cycles generated in general non-smooth Liénard systems, and verified the theoretical results through examples. Chen et al. [2] developed a higher-order Poincaré-Lyapunov method, successfully solving the problem of determining nilpotent center conditions in switching polynomial systems (particularly cubic switching Liénard systems), and proved that up to 5 limit cycles can bifurcate around the nilpotent center. B. Coll et al. [3] derived the general expressions of the Lyapunov constants for two classes of discontinuous Liénard differential equations.

In this paper, we consider the following piecewise smooth Liénard system

$$\begin{cases} \dot{x} = h\left(y\right) - F\left(x, a\right), \\ \dot{y} = -g\left(x, b\right), \end{cases} \tag{1.1}$$

where $(a, b) \in \mathbb{R}^{n_1}$ with n_1 being an positive integer, $h(y)$ is a C^{∞} function satisfying $h\left(y\right) = h_s y^{2s-1} + O\left(y^{2s}\right)$ with $s \in \mathbb{N}^{+}$, $h_s > 0$ and

$$F\left(x, a\right) = \begin{cases} F^{+}\left(x, a\right), \ x \geq 0, \\ F^{-}\left(x, a\right), \ x < 0, \end{cases} \qquad g\left(x, b\right) = \begin{cases} g^{+}\left(x, b\right), \ x \geq 0, \\ g^{-}\left(x, b\right), \ x < 0, \end{cases}$$

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where F^+, g^+ and F^-, g^- are C^∞ on $[0, x_0]$ and $[-x_0, 0]$, respectively, with $x_0 \in \mathbb{R}^+$. Suppose there exist integers $m \geq 1, n \geq 1, k \geq 0, l \geq 0$ such that

$$F^+(x,a) = \sum_{j \geq m} F_j^+(a) x^j, \; g^+(x,b) = \sum_{j \geq k} g_j^+(b) x^j, \; g_k^+(b) > 0 \tag{1.2}$$

for $0 \leq x \ll 1$ and

$$F^-(x,a) = \sum_{j \geq n} F_j^-(a) x^j, \; g^-(x,b) = g_l^-(b)(-x)^l + \sum_{j \geq l+1} g_j^-(b) x^j, \; g_l^-(b) < 0 \tag{1.3}$$

for $0 \leq -x \ll 1$. Without loss of generality, we set $h_s = 1$, which can be obtained by a suitable scaling transformation, so that

$$h(y) = y^{2s-1} + O\left(y^{2s}\right).$$

Further, in order to ensure that the origin of system (1.1) is either a center or a focus, by Theorem 1.2 and Remark 1.4 in [13], similar to (1.19) in [13], we suppose that system (1.1) satisfies the following assumptions:

(H_1) $m > \frac{2s-1}{2s}(k+1)$ or $m = \frac{2s-1}{2s}(k+1)$ and $|F_m^+(a)| < c\left(\left(g_k^+(b)\right)/(k+1)\right)^{\frac{2s-1}{2s}}$,

(H_2) $n > \frac{2s-1}{2s}(l+1)$ or $n = \frac{2s-1}{2s}(l+1)$ and $|F_n^-(a)| < c\left(\left(-g_l^-(b)\right)/(l+1)\right)^{\frac{2s-1}{2s}}$,

where $0 < c < 2s\left[2s/(2s-1)\right]^{(2s-1)/2s}$.

Let

$$H(y) = \int_0^y h(u)\,du = \frac{1}{2s} y^{2s} + O\left(y^{2s+1}\right),$$

and

$$G(x,b) = \int_0^x g(u,b)\,du = \begin{cases} G^+(x,b), & 0 \leq x \leq x_0, \\ G^-(x,b), & 0 < -x \leq x_0, \end{cases}$$

where

$$G^+(x,b) = \frac{g_k^+(b)}{k+1} x^{k+1} + \sum_{j \geq k+1} \frac{g_j^+(b)}{j+1} x^{j+1}, \tag{1.4}$$

for $0 \leq x \ll 1$ and

$$G^-(x,b) = \frac{-g_l^-(b)}{l+1}(-x)^{l+1} + \sum_{j \geq l+1} \frac{g_j^-(b)}{j+1} x^{j+1}, \tag{1.5}$$

for $0 \leq -x \ll 1$. Let $r = q/p$ where $q = k+1$ and $p = l+1$. Noting that $n \geq 1$ we can find an integer $\eta \in [0, p-1]$ such that $(n+\eta)r$ is an integer. Now, we introduce a set

$$S_{p,q,m,n} = \{(n+i)r + j | 0 \leq i \leq p-1, i \neq \eta, j \geq 0\} \cup \{j | j \geq \min\{m, (n+\eta)r\}.$$

Our main results are the following.

Theorem 1.1. *Let* (1.2), (1.3), (H_1) *and* (H_2) *hold. Then*

(*i*) $F^-(\alpha(x,b),a) - F^+(x,a)$ *has an expansion of the form for* $0 < x \ll 1$

$$F^-(\alpha(x,b),a) - F^+(x,a) = \sum_{j \geq 1} B_j(a,b) x^{k_j}, \tag{1.6}$$

where $\alpha(x,b)$ *satisfies* $G_l^-(\alpha(x,b),b) = G_k^+(x,b)$ *and* $x\alpha(x,b) < 0$ *for* $0 < x \ll 1$ *and* $\{k_j | j \geq 1\} \subset S_{p,q,m,n}$ *with* $k_1 < k_2 < \cdots$.

(ii) If there exist $(a_0, b_0) \in \mathbb{R}^{n_1}$ *and* $N \in \mathbb{N}^+$ *such that*

$$B_j(a_0, b_0) = 0,\ j = 1, 2, \cdots, N, \tag{1.7}$$

and $B_{N+1}(a_0, b_0) \neq 0$, *then system* (1.1) *has at most* N *limit cycles near the origin for all* (a, b) *near* (a_0, b_0). *If further*

$$\operatorname{rank} \frac{\partial(B_1, B_2, \cdots, B_N)}{\partial(a,b)}(a_0, b_0) = N, \tag{1.8}$$

then system (1.1) *has cyclicity* N *near the origin for all* (a, b) *near* (a_0, b_0).

Theorem 1.2. *Let* (1.2), (1.3), (H_1) *and* (H_2) *hold. If there exist* $(a_0, b_0) \in \mathbb{R}^{n_1}$ *and* $N \geq 1$ *such that* (1.7) *is satisfied and*

$$F^-\left(\alpha\left(x,b\right),a\right) \equiv F^+\left(x,a\right),\ 0 < x \ll 1 \tag{1.9}$$

as $B_1 = B_2 = \cdots = B_N = 0$, *then system* (1.1) *has at most* $N-1$ *limit cycles near the origin for all* (a, b) *near* (a_0, b_0). *If further* (1.8) *holds, then system* (1.1) *has cyclicity* $N-1$ *at the origin for all* (a, b) *near* (a_0, b_0).

The above theorems generalize main results of [13].

2. Proof of Theorems 1.1 and 1.2

In preparation for the proofs of Theorems 1.1 and 1.2, we define a bifurcation function d of system (1.1) and develop several preliminary lemmas, which will be used in the subsequent proofs. In [13], the author obtained the following result.

Lemma 2.1. ([13]) *Suppose* (1.4) *and* (1.5) *hold. Then the function* $\alpha(x,b)$ *defined in Theorem* 1.1 *has an expansion of the form*

$$\alpha\left(x,b\right) = \sum_{i=1}^{p} \sum_{j \geq 0} \alpha_{ij} x^{ir+j},$$

for $0 < x \ll 1$ *with* $\alpha_{10} = -\left[\left(\left(l+1\right)g_k^+\left(b\right)\right) / \left(-\left(k+1\right)g_l^-\left(b\right)\right)\right]^{1/(l+1)}$.

From Chapter 3 of [8], we have the following conclusion called the improved Taylor formula.

Lemma 2.2. ([8]) *Consider a function* $F : D \times G \to \mathbf{R}$, *where* $D = (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$, *and* $G \subset \mathbf{R}^n$ *is a domain. Let* k *and* m *be integers with* $1 \leq m \leq k+1$, $k \geq 0$, *such that* $\frac{\partial^m F}{\partial x^m} \in C^{k-m+1}(D \times G)$. *Then there exists a function* $\bar{R} \in C^{k-m+1}(D \times G)$ *such that*

$$F(x,y) = \sum_{j=0}^{m-1} \frac{1}{j!} \frac{\partial^j F}{\partial x^j}(0,y) x^j + x^m \bar{R}(x,y),$$

$$\bar{R}(0,y) = \frac{1}{m!} \frac{\partial^m F}{\partial x^m}(0,y).$$

If $F \in C^\infty(D \times G)$, *then* $\bar{R} \in C^\infty(D \times G)$.

Lemma 2.3. *Consider the function $y = x^{k+1}\varphi(x)$, where $k \in \mathbb{N}$, $\varphi(0) > 0$, and $\varphi \in C^\infty$ for $|x| \ll 1$. Then, for $|y| \ll 1$ as k is even, or $0 \le y \ll 1$ as k is odd, the function has a unique inverse of the form*

$$x = y^{\frac{1}{k+1}}\tilde{\varphi}\left(y^{\frac{1}{k+1}}\right),$$

where $\tilde{\varphi} \in C^\infty$ over its respective domain ($|\xi| \ll 1$ or $0 \le \xi \ll 1$), satisfying $\tilde{\varphi}(0) = (\varphi(0))^{-\frac{1}{k+1}} > 0$.

Proof. Let $f(x) = x^{k+1}\varphi(x)$. Observe that

$$f'(x) = x^k\left[(k+1)\varphi(x) + x\varphi'(x)\right].$$

Since $\varphi(0) > 0$, the bracketed term is positive for $|x| \ll 1$, meaning the sign of $f'(x)$ is determined by x^k. If k is odd, $f'(x) > 0$ for $x > 0$, yielding a unique local inverse restricted to $0 \le y \ll 1$. If k is even, $f'(x) > 0$ for all $x \ne 0$ small, yielding a unique local inverse defined for $|y| \ll 1$.

In both cases, within their respective valid domains, the equation $y = x^{k+1}\varphi(x)$ yields

$$y^{\frac{1}{k+1}} = x\left(\varphi(x)\right)^{\frac{1}{k+1}}.$$

Define $h(x) = (\varphi(x))^{\frac{1}{k+1}}$. Since $\varphi \in C^\infty$ and $\varphi > 0$ for $|x| \ll 1$, $h(x)$ is C^∞ near $x = 0$ with $h(0) > 0$.

Letting $z = y^{\frac{1}{k+1}}$, we have $z = xh(x)$. Noting $h(0) \ne 0$, the Inverse Function Theorem implies that the inverse $x = \psi(z)$ is C^∞ for $|z| \ll 1$.

Since $\psi(0) = 0$, according to Lemma 2.2, we can write $\psi(z) = z\tilde{\varphi}(z)$ for some function $\tilde{\varphi}$ satisfying $\tilde{\varphi}(0) = (\varphi(0))^{-\frac{1}{k+1}} > 0$. Substituting $z = y^{\frac{1}{k+1}}$ back, we obtain the desired form:

$$x = y^{\frac{1}{k+1}}\tilde{\varphi}\left(y^{\frac{1}{k+1}}\right).$$

This completes the proof. □

Using Lemma 2.2, we can obtain the following conclusion.

Lemma 2.4. *Let (1.2) and (1.3) hold. Then system (1.1) is locally equivalent to*

$$\begin{cases} \dot{u} = v^{2s-1} - K(v)\,F^*(u,a,b), \\ \dot{v} = -u, \end{cases} \tag{2.1}$$

where $K(v) = 1/(2s)^{(2s-1)/2s} + O(v) \in C^\infty$, and

$$\begin{aligned} F^*(u,a,b) &= \begin{cases} F^+\left(X^+(u),a\right), & 0 \le u \ll 1, \\ F^-\left(X^-(u),a\right), & 0 < -u \ll 1, \end{cases} \\ &= \begin{cases} u^{2m/(k+1)}\bar{F}^+\left(u^{2/(k+1)},a,b\right), & 0 \le u \ll 1, \\ |u|^{2n/(l+1)}\,\bar{F}^-\left(|u|^{2/(l+1)},a,b\right), & 0 < -u \ll 1, \end{cases} \end{aligned}$$

where $\bar{F}^\pm(\xi,a,b)$ are C^∞ functions for $0 \le \xi \ll 1$, and $\bar{F}^+(0,a,b) = F_m^+(a)\left(\frac{s(k+1)}{g_k^+(b)}\right)^{m/(k+1)}$, $\tilde{F}^-(0,a,b) = F_n^-(a)\left(\frac{s(l+1)}{-g_l^-(b)}\right)^{n/(l+1)}$.

Proof. Similar to the formula (3) in [6], we make the transformation

$$\begin{cases} u = (\mathrm{sgn}x)\sqrt{G(x,b)/s} \triangleq \varphi(x), \\ v = (\mathrm{sgn}y)\sqrt[2s]{H(y)} \triangleq \psi(y), \end{cases} \tag{2.2}$$

together with the scaling of the time

$$d\tau = \left[g(X(u),b)\,h(Y(v))/(2suv^{2s-1})\right]dt$$

to system (1.1) so that (1.1) can be changed into

$$\begin{cases} \dot{u} = v^{2s-1} - K(v)F^*(u,a,b), \\ \dot{v} = -u, \end{cases}$$

where

$$K(v) = \frac{v^{2s-1}}{h(Y(v))}, \tag{2.3}$$

$F^*(u,a,b) = F(X(u),a)$ and $X(u), Y(v)$ denote the inverse of $\varphi(x), \psi(y)$, respectively.

First, we prove that F^* is a C^∞ function for $0 < \xi \ll 1$. According to (1.4), we have

$$\left.\frac{\partial G^+}{\partial x}\right|_{x=0} = \left.\frac{\partial^2 G^+}{\partial x^2}\right|_{x=0} = \cdots = \left.\frac{\partial^k G^+}{\partial x^k}\right|_{x=0} = 0.$$

By Lemma 2.2, we know that $G^+(x,b)$ can be written as

$$G^+(x,b) = x^{k+1}\hat{G}^+(x,b), \tag{2.4}$$

where $\hat{G}^+(0,b) = \frac{1}{(k+1)!}\frac{\partial^{k+1}G^+}{\partial x^{k+1}}(0,b) > 0$ and $\hat{G}^+$ is a C^∞ function for $0 \le x \ll 1$. Similarly, applying Lemma 2.2 to $F^+(x,a)$, we obtain

$$F^+(x,a) = x^m\hat{F}^+(x,a), \tag{2.5}$$

where $\hat{F}^+(0,a) = F_m^+(a)$ and $\hat{F}^+$ is a C^∞ function for $0 \le x \ll 1$.

According to (2.2) and (2.4), we have

$$su^2 = x^{k+1}\hat{G}^+(x,b), \tag{2.6}$$

for $0 \le x \ll 1$. Applying Lemma 2.3 to (2.6), we obtain

$$x = X^+(u) = u^{2/(k+1)}\tilde{\varphi}\left(u^{2/(k+1)}\right) \tag{2.7}$$

where $\tilde{\varphi}(\xi)$ is a C^∞ function for $0 \le \xi \ll 1$, satisfying $\tilde{\varphi}(0) = \left(\frac{s(k+1)}{g_k^+(b)}\right)^{1/(k+1)} > 0$.

Therefore, according to (2.5) and (2.7), we obtain

$$\begin{aligned} F^*(u,a,b) &= F^+\left(X^+(u),a\right) \\ &= \left[\tilde{\varphi}\left(u^{2/(k+1)}\right)\right]^m \hat{F}^+\left(\tilde{\varphi}\left(u^{2/(k+1)}\right),a,b\right) \\ &= u^{2m/(k+1)}\bar{F}^+\left(u^{2/(k+1)},a,b\right) \end{aligned} \tag{2.8}$$

for $0 \le u \ll 1$, where $\bar{F}^{+}(\xi,a,b)$ is C^{∞} for $0 \le \xi \ll 1$, and

$$\bar{F}^{+}(0,a,b)=F_{m}^{+}(a)\left(\frac{s(k+1)}{g_{k}^{+}(b)}\right)^{m/(k+1)}.$$

Using the same method, we can also obtain

$$F^{*}(u,a,b)=|u|^{2n/(l+1)}\,\bar{F}^{-}\left(|u|^{2/(l+1)},a,b\right) \tag{2.9}$$

for $0 \le -u \ll 1$, where $\bar{F}^{-}(\xi,a,b)$ is C^{∞} for $0 \le \xi \ll 1$, and

$$\tilde{F}^{-}(0,a,b)=F_{n}^{-}(a)\left(\frac{s(l+1)}{-g_{l}^{-}(b)}\right)^{n/(l+1)}.$$

Finally, we prove that $K(v) \in C^{\infty}$. Applying the Lemma 2.2 to $H(y)$, we obtain

$$H(y)=y^{2s}\hat{H}(y),$$

where $\hat{H}(0)=\frac{1}{2s}>0$ and $\hat{H} \in C^{\infty}$ for $|y| \ll 1$. Applying Lemma 2.3 to $v^{2s}=y^{2s}\hat{H}(y)$ yields

$$y=Y(v)=v\bar{H}(v), \tag{2.10}$$

where $\bar{H}$ is C^{∞} with respect to v for $|v| \ll 1$, and $\bar{H}(0)=(2s)^{1/2s}$. Substituting (2.10) into $h(y)$ and applying Lemma 2.2, we obtain

$$h(Y(v))=h\left(v\bar{H}(v)\right)=v^{2s-1}\tilde{H}(v) \tag{2.11}$$

where $\tilde{H}$ is C^{∞} with respect to v for $|v| \ll 1$, and $\tilde{H}(0)=(2s)^{(2s-1)/2s}$. Combining (2.3) and (2.11), we obtain

$$K(v)=\frac{1}{\tilde{H}(v)}.$$

Then K is C^{∞} with respect to v for $|v| \ll 1$, and $K(0)=\frac{1}{(2s)^{(2s-1)/2s}}$. □

Following the proof of Lemma 2.3 in [13], we can prove a similar result below.

Lemma 2.5. *Suppose* (1.2) *and* (1.3) *hold. Then we have*
(i) $F^{-}(\alpha(x,b),a)-F^{+}(x,a)$ *has an expansion of the form* (1.6) *for* $0 \le x \ll 1$.
(ii) Let $F_0(u,a,b)=F^{*}(-u,a,b)-F^{*}(u,a,b)$. *Then* F_0 *has the expansion*

$$F_0(u,a,b)=\sum_{j\ge1}A_j(a,b)\,u^{2k_j/(k+1)},\ 0\le u\ll 1, \tag{2.12}$$

where

$$A_1(a,b)=B_1(a,b)\,N_1\left(x_1^{+}\right),$$
$$A_j(a,b)=B_j(a,b)\,N_j\left(x_1^{+}\right)+O\left(|B_1,B_2,\cdots,B_{j-1}|\right),\ j\ge2,$$

$N_j (j\ge1)$ *are positive* C^{∞} *functions in* x_1^{+}, *and* $k_j \in S_{p,q,m,n}$.

Before introducing the bifurcation function, we introduce a transformation used in [1]. In [1], Gasull introduced the generalized trigonometrical functions, $x(t) = Sn(t)$, $y(t) = Cs(t)$. These functions are defined as the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} = y^{2s-1}, \\ \dot{y} = -x, \end{cases} \tag{2.13}$$

with initial conditions $x(0) = 0$, $y(0) = 1$. Some properties of $Cs(t)$ and $Sn(t)$ were given in [4] and [11] which are listed as follows.

Proposition 2.1. ([4,11]) *The solution $(Sn(t), Cs(t))$ of (2.13) with initial value $(0, 1)$ at $t = 0$ satisfies*
(1) $Cs^{2s}(t) + sSn^2(t) = 1.$
(2) *$Cs(t)$ and $Sn(t)$ are $T-$periodic functions with*

$$T = 2\sqrt{\frac{\pi}{s}}\frac{\Gamma(1/2s)}{\Gamma((s+1)/2s)}.$$

(3) $\int_0^T Sn^p(t)\,Cs^q(t)\,dt = 0$ *if p or q is odd.*
(4) $\int_0^T Sn^p(t)\,Cs^q(t)\,dt = \frac{2}{\sqrt{s^{p+1}}}\frac{\Gamma((p+1)/2)\Gamma((q+1)/2s)}{\Gamma((p+1)/2+(q+1)/2s)}$ *if both p and q are even.*

In addition, we will utilize Young's inequality to prove a new property of $Cs(t)$ and $Sn(t)$. Young's inequality states that for any non-negative real numbers $a, b \geq 0$ and conjugate exponents $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, it holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

By this inequality we can prove

Proposition 2.2. *The periodic functions $Cs(t)$ and $Sn(t)$ satisfy*

$$\left(\frac{s}{2s-1}\right)^{\frac{2s-1}{2s}} Cs(t)\,[Sn(t)]^{\frac{2s-1}{s}} \leq \frac{1}{2s}, \quad t \in \mathbb{R}.$$

Proof. Let $p = 2s$ and $q = 2s/(2s-1)$. Then, by Young's inequality, we have

$$\begin{aligned}\left(\frac{s}{2s-1}\right)^{\frac{2s-1}{2s}} Cs(t)\,[Sn(t)]^{\frac{2s-1}{s}} &\leq \frac{1}{p}[Cs(t)]^p + \frac{1}{q}\left\{[Sn(t)]^{\frac{2s-1}{s}}\left(\frac{s}{2s-1}\right)^{\frac{2s-1}{2s}}\right\}^q \\ &= \frac{1}{2s}[Cs(t)]^{2s} + \frac{1}{2}[Sn(t)]^2 \\ &= \frac{1}{2s}\left\{[Cs(t)]^{2s} + s\,[Sn(t)]^2\right\}.\end{aligned}$$

The conclusion can be obtained directly from (1) in Proposition 2.1. □

Next, we establish the bifurcation function d of system (2.1). By applying the following transformation to the system (2.1)

$$\begin{cases} u = r^s Sn(\theta), \\ v = rCs(\theta), \end{cases}$$

with $0<r\ll 1$ and $|\theta|\le T/2$, we obtain

$$\begin{cases}\dot r=\dfrac{v^{2s-1}\dot v+u\dot u}{r^{2s-1}},\\ \dot\theta=\dfrac{v\dot u-su\dot v}{r^{s+1}}.\end{cases}$$

Therefore, we have

$$\frac{dr}{d\theta}=\frac{-Sn\left(\theta\right)K\left(rCs\left(\theta\right)\right)F^{*}\left(r^{s}Sn\left(\theta\right),a,b\right)}{r^{2s-2}-Cs\left(\theta\right)K\left(rCs\left(\theta\right)\right)F^{*}\left(r^{s}Sn\left(\theta\right),a,b\right)/r}. \tag{2.14}$$

Now, we define

$$R^{*}\left(\theta,r,a,b\right)=\begin{cases}\dfrac{-Sn\left(\theta\right)K\left(rCs\left(\theta\right)\right)F^{*}\left(r^{s}Sn\left(\theta\right),a,b\right)}{r^{2s-2}-Cs\left(\theta\right)K\left(rCs\left(\theta\right)\right)F^{*}\left(r^{s}Sn\left(\theta\right),a,b\right)/r}, & 0<r\ll 1,\\ 0, & r=0.\end{cases} \tag{2.15}$$

In order to ensure that (2.14) and (2.15) is well-defined, we need to verify that

$$1-\frac{Cs\left(\theta\right)K\left(rCs\left(\theta\right)\right)F^{*}\left(r^{s}Sn\left(\theta\right),a,b\right)}{r^{2s-1}}>0 \tag{2.16}$$

for $r>0$ small under (H1) and (H2). In fact, for $-T/2\le\theta\le 0$, the left side of inequality (2.16) becomes

$$1-Cs\left(\theta\right)K\left(0\right)F_{m}^{+}\left(a\right)\left[Sn\left(\theta\right)\right]^{\frac{2s-1}{s}}\left[\frac{s\left(k+1\right)}{g_{k}^{+}\left(b\right)}\right]^{\frac{2s-1}{2s}}\left(1+O\left(r\right)\right). \tag{2.17}$$

From (2.17) and Proposition 2.2, we find that

$$\begin{aligned}&1-Cs\left(\theta\right)K\left(0\right)F_{m}^{+}\left(a\right)\left[Sn\left(\theta\right)\right]^{\frac{2s-1}{s}}\left[\frac{s\left(k+1\right)}{g_{k}^{+}\left(b\right)}\right]^{\frac{2s-1}{2s}}\\ &=1-\frac{1}{\left(2s\right)^{\frac{2s-1}{2s}}}Cs\left(\theta\right)\left[Sn\left(\theta\right)\right]^{\frac{2s-1}{s}}F_{m}^{+}\left(a\right)\left[\frac{s\left(k+1\right)}{g_{k}^{+}\left(b\right)}\right]^{\frac{2s-1}{2s}}\\ &>1-\frac{1}{\left(2s\right)^{\frac{2s-1}{2s}}}Cs\left(\theta\right)\left[Sn\left(\theta\right)\right]^{\frac{2s-1}{s}}s^{\frac{2s-1}{2s}}c\\ &>1-2s\left(\frac{s}{2s-1}\right)^{\frac{2s-1}{2s}}Cs\left(\theta\right)\left[Sn\left(\theta\right)\right]^{\frac{2s-1}{s}}\\ &\ge 0.\end{aligned}$$

Hence, for $-T/2\le\theta\le 0$, (2.16) holds. In a similar way, for $0<\theta<T/2$, (2.16) also holds. Through the above verification, it is shown that (2.14) and (2.15) is well-defined.

Now, let $\rho=r^{1/\beta}$ with $\beta=(k+1)(l+1)$. Then we obtain

$$R\left(\theta,\rho,a,b\right)=\begin{cases}\dfrac{\rho}{\beta}\times\dfrac{-Sn\left(\theta\right)K\left(\rho^{\beta}Cs\left(\theta\right)\right)F_{1}\left(\theta,\rho,a,b\right)}{1-Cs\left(\theta\right)K\left(\rho^{\beta}Cs\left(\theta\right)\right)F_{1}\left(\theta,\rho,a,b\right)}, & 0<\rho\ll 1,\\ 0, & \rho=0,\end{cases} \tag{2.18}$$

and $F_1(\theta,\rho,a,b) = F^*\left(\rho^{s\beta} Sn(\theta), a, b\right)/\rho^{\beta(2s-1)}$. By Lemma2.4, we have

$$F_1(\theta,\rho,a,b) = \begin{cases} \sum_{j\geq m} f_j^+ (Sn(\theta))^{\frac{2j}{k+1}} \rho^{2(l+1)sj-\beta(2s-1)}, & -T/2 \leq \theta \leq 0, \\ \sum_{j\geq n} f_j^- |Sn(\theta)|^{\frac{2j}{l+1}} \rho^{2(k+1)sj-\beta(2s-1)}, & 0 < \theta < T/2, \end{cases}$$

for $0 < \rho \ll 1$. Clearly, $R(\theta,\rho,a,b)$ is a T-periodic function with respect to θ. Moreover, for any $\theta \in [-T/2, T/2)$, R is C^∞ with respect to ρ for $0 \leq \rho \ll 1$.

Consider the system below:

$$\frac{d\rho}{d\theta} = R(\theta,\rho,a,b). \tag{2.19}$$

Let $\rho(\theta,\rho_0,a,b)$ denote the solution of (2.19) satisfying $\rho(0,\rho_0,a,b) = \rho_0$. For convenience, let

$$\rho\left(-\frac{T}{2},\rho_0,a,b\right) = \rho_1,\ \rho\left(\frac{T}{2},\rho_0,a,b\right) = \rho_2.$$

Define a function as follows:

$$d(\rho_0,a,b) = \rho\left(-\frac{T}{2},\rho_0,a,b\right) - \rho\left(\frac{T}{2},\rho_0,a,b\right) = \rho_1 - \rho_2. \tag{2.20}$$

Then system (1.1), (2.1) or (2.19) has a periodic orbit near the origin if and only if $d(\rho_0,a,b)$ in (2.20) has a zero in ρ_0 for $\rho_0 > 0$ sufficiently small.

Hence, the function d in (2.20) is called a displacement function or bifurcation function of system (1.1), (2.1) or (2.19). About the bifurcation function d, we have the following lemma.

Lemma 2.6. *Let* (1.2), (1.3), (H$_1$) *and* (H$_2$) *hold. Then the bifurcation function d in* (2.20) *has the form*

$$d(\rho_0,a,b) = \sum_{j\geq 1} C_j(a,b)\,\rho_0^{r_j}(1 + d_j(\rho_0)), \tag{2.21}$$

where $r_j = 2(l+1)sk_j + 1 - (2s-1)\beta$ are positive integers satisfying $1 \leq r_1 < r_2 < \cdots$,

$$\begin{aligned} C_1(a,b) &= B_1(a,b)\,N_1^*(f_n^-, f_m^+), \\ C_j(a,b) &= B_j(a,b)\,N_j^*(f_n^-, f_m^+) + O(|B_1, B_2, \cdots, B_{j-1}|), j \geq 2 \end{aligned} \tag{2.22}$$

with $N_j^* \in C^\infty, N_j(0,0) > 0$ for $j \geq 1$ and $d_j (j \geq 1)$ are C^∞ functions in ρ_0 satisfying $d_i(0) = 0$.

Proof. Let $\bar\rho(\theta,\rho_0,a,b) = \rho(-\theta,\rho_0,a,b)$. Then, we can find that

$$\bar\rho\left(-\frac{T}{2},\rho_0,a,b\right) = \rho\left(\frac{T}{2},\rho_0,a,b\right) \tag{2.23}$$

and

$$\frac{d\bar\rho}{dt} = -R(-\theta,\rho,a,b).$$

Let $\bar R(\theta,\rho,a,b) = -R(-\theta,\rho,a,b)$. Then, $\bar\rho(\theta,\rho_0,a,b)$ is a solution of the equation

$$\frac{d\rho}{dt} = \bar R(\theta,\rho,a,b) \tag{2.24}$$

satisfying $\bar{\rho}(0,\rho_0,a,b)=\rho_0$. Let $\theta\in[-T/2,0]$. Then $-\theta\in[0,T/2]$ and

$$\begin{aligned} R(\theta,\rho,a,b) &= R_1(\theta,a,b)\,\rho^{2(l+1)sm-(2s-1)\beta+1}\left(1+O(\rho)\right), \\ \bar{R}(\theta,\rho,a,b) &= \bar{R}_1(\theta,a,b)\,\rho^{2(k+1)sn-(2s-1)\beta+1}\left(1+O(\rho)\right), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} R_1(\theta,a,b) &= \begin{cases} -\dfrac{K(0)\,f_m^+}{\beta}\left(Sn(\theta)\right)^{2m/(k+1)+1}, & m>\dfrac{2s-1}{2s}(k+1), \\ -\dfrac{\left(Sn(\theta)\right)^{s+1}K(0)\,f_m^+}{\beta\left[1-Cs(\theta)\,K(0)\,f_m^+\left(Sn(\theta)\right)^{2m/(k+1)}\right]}, & m=\dfrac{2s-1}{2s}(k+1), \end{cases} \\ \bar{R}_1(\theta,a,b) &= \begin{cases} -\dfrac{K(0)\,f_n^-}{\beta}\left(Sn(\theta)\right)^{2n/(l+1)+1}, & n>\dfrac{2s-1}{2s}(l+1), \\ -\dfrac{\left(Sn(\theta)\right)^{s+1}K(0)\,f_n^-}{\beta\left[1-Cs(\theta)\,K(0)\,f_n^-\left(Sn(\theta)\right)^{2n/(l+1)}\right]}, & n=\dfrac{2s-1}{2s}(l+1). \end{cases} \end{aligned} \tag{2.26}$$

Thus, by (2.25), (2.26) and the mean value theorem, we have

$$\rho(\theta,\rho_0,a,b)=\begin{cases} \rho_0+O\left(\rho_0^{2(l+1)sm-(2s-1)\beta+1}\right), & m>\dfrac{2s-1}{2s}(k+1), \\ \rho_0\exp\left(\displaystyle\int_0^\theta R_1(u,a,b)\,du\right)+O\left(\rho_0^2\right), & m=\dfrac{2s-1}{2s}(k+1), \end{cases} \tag{2.27}$$

$$\bar{R}(\theta,\rho(\theta,\rho_0,a,b),a,b)-\bar{R}(\theta,\bar{\rho}(\theta,\rho_0,a,b),a,b)=\left[\rho(\theta,\rho_0,a,b)-\bar{\rho}(\theta,\rho_0,a,b)\right]\Delta(\theta,\rho_0) \tag{2.28}$$

where

$$\Delta(\theta,\rho_0)=O\left(\rho_0^{2(k+1)sn-(2s-1)\beta}\right).$$

Further by (2.18), (2.24) and Lemma (2.5), we can obtain

$$R(\theta,\rho,a,b)-\bar{R}(\theta,\rho,a,b)=\begin{cases} \dfrac{Sn(\theta)\,K\left(\rho^\beta Cs(\theta)\right)F_0\left(\rho^{s\beta}Sn(\theta),a,b\right)}{\beta M_1(\theta,\rho,a,b)\,M_2(\theta,\rho,a,b)\,\rho^{\beta(2s-1)-1}}, & 0<\rho\ll 1, \\ 0, & \rho=0, \end{cases}$$

with

$$\begin{aligned} M_1(\theta,\rho,a,b) &= 1-Cs(\theta)\,K\left(\rho^\beta Cs(\theta)\right)F^*\left(\rho^{s\beta}Sn(\theta),a,b\right)/\rho^{\beta(2s-1)}, \\ M_2(\theta,\rho,a,b) &= 1-Cs(\theta)\,K\left(\rho^\beta Cs(\theta)\right)F^*\left(-\rho^{s\beta}Sn(\theta),a,b\right)/\rho^{\beta(2s-1)}. \end{aligned}$$

Therefore, we have

$$R(\theta,\rho,a,b)-\bar{R}(\theta,\rho,a,b)=Sn(\theta)\,\rho^{1-(2s-1)\beta}F_0\left(\rho^{s\beta}Sn(\theta),a,b\right)S(\theta,\rho), \tag{2.29}$$

where $S(\theta,\rho)$ is a C^∞ function in ρ with

$$S(\theta,0)=\begin{cases}\dfrac{K(0)}{\beta}, & m>\dfrac{(2s-1)(k+1)}{2s}, n>\dfrac{(2s-1)(l+1)}{2s},\\ \dfrac{K(0)}{\beta A^{+}(\theta)}, & m=\dfrac{(2s-1)(k+1)}{2s}, n>\dfrac{(2s-1)(l+1)}{2s},\\ \dfrac{K(0)}{\beta A^{-}(\theta)}, & m>\dfrac{(2s-1)(k+1)}{2s}, n=\dfrac{(2s-1)(l+1)}{2s},\\ \dfrac{K(0)}{\beta A^{+}(\theta)A^{-}(\theta)}, & m=\dfrac{(2s-1)(k+1)}{2s}, n=\dfrac{(2s-1)(l+1)}{2s},\end{cases}$$

$$A^{+}(\theta)=1-Cs(\theta)K(0)f_m^{+}(Sn(\theta))^{\frac{2m}{k+1}},\ A^{-}(\theta)=1-Cs(\theta)K(0)f_n^{-}(Sn(\theta))^{\frac{2m}{l+1}},$$

which implies $S(\theta,0)>0$ by the above discussion. According to (2.12), (2.27) and (2.29), we get

$$\begin{aligned}&R(\theta,\rho(\theta,\rho_0,a,b),a,b)-\bar{R}(\theta,\rho(\theta,\rho_0,a,b),a,b)\\ &=\sum_{j\geq 1}A_j(a,b)(Sn(\theta))^{2k_j/(k+1)+1}\rho_0^{2s(l+1)k_j-(2s-1)\beta+1}q_j(\theta,\rho_0)\end{aligned}$$

where $q_j(\theta,\rho_0)$ are C^∞ in ρ_0 with

$$q_j(\theta,0)=\begin{cases}S(\theta,0), & m>\dfrac{2s-1}{2s}(k+1),\\ S(\theta,0)\exp\left(\displaystyle\int_0^\theta R_1(u,a,b)\,du\right)^{\left(\frac{2s\beta k_j}{k+1}-(2s-1)\beta+1\right)}, & m=\dfrac{2s-1}{2s}(k+1),\end{cases}$$

which shows that $q_j(\theta,0)>0$ for all $j\geq 1$. From (2.19), (2.24) and (2.28), we get

$$\frac{d}{d\theta}(\rho-\bar{\rho})=(\rho-\bar{\rho})\mathit{\Delta}(\theta,\rho_0)+R(\theta,\rho,a,b)-\bar{R}(\theta,\rho,a,b), \tag{2.30}$$

where $\bar{\rho}=\bar{\rho}(\theta,\rho_0,a,b)$ and $\rho=\rho(\theta,\rho_0,a,b)$. In fact, we find that

$$\rho(0,\rho_0,a,b)=\bar{\rho}(0,\rho_0,a,b)=\rho_0. \tag{2.31}$$

By (2.30), (2.31) and the formula of variation of constants, we obtain

$$\begin{aligned}&\rho(\theta,\rho_0,a,b)-\bar{\rho}(\theta,\rho_0,a,b)\\ &=\int_0^\theta\left(R(\delta,\rho(\delta,\rho_0,a,b),a,b)-\bar{R}(\delta,\rho(\delta,\rho_0,a,b),a,b)\right)\times\exp\left(\int_\delta^\theta\mathit{\Delta}(\tau,\rho_0)\,d\tau\right)d\delta.\end{aligned}$$

Taking $\theta=-T/2$, by (2.20) and (2.23), we can get

$$\begin{aligned}d(\rho_0,a,b)=&\sum_{j\geq 1}A_j(a,b)\rho_0^{2s(l+1)k_j-(2s-1)\beta+1}\int_0^{-T/2}(Sn(\theta))^{2k_j/(k+1)+1}q_j(\theta,0)\\ &\times\exp\left(\int_\theta^{-T/2}\mathit{\Delta}(\tau,0)\,d\tau\right)(1+O(\rho_0))\,d\theta,\end{aligned}$$

which implies (2.21). This ends the proof. $\square$

In order to prove Theorems 1.1 and 1.2, we present a result from [7].

Lemma 2.7. ([7]) *Consider a function of the form*

$$F(r,\delta) = \sum_{j=1}^{k+1} b_j(\delta) r^{l_j} P_j(r,\delta),$$

where $0 \le r \ll 1$, $\delta \in V \subset \mathbb{R}^m$ *is a vector parameter with* V *compact,* $P_j(r,\delta) = 1 + O(r) \in C^\infty$, $0 \le l_1 < l_2 < \cdots < l_{k+1}$ *and* $k \ge 0$. *Let*

$$b_j(\delta_0) = 0,\ j = 1, 2, \cdots, k$$

for some $\delta_0 \in V$.

(i) If $b_{k+1}(\delta_0) \neq 0$, *then there exist* $\varepsilon > 0$ *and* $r_0 > 0$ *such that the function* F *has at most* k *zeros in* $(0, r_0)$ *for* $|\delta - \delta_0| < \varepsilon$ *and* $\delta \in V$, *multiplicity taken into account. If further*

$$\operatorname{rank} \frac{\partial(b_1, b_2, \cdots, b_k)}{\partial(\delta_1, \delta_2, \cdots, \delta_m)}(0, \delta_0) = k, \tag{2.32}$$

then for any $\varepsilon_0 > 0$ *there exist* δ *with* $|\delta - \delta_0| < \varepsilon_0$ *and* $\delta \in V$ *such that the function* F *has* k *simple zeros in* $r \in (0, \varepsilon_0)$.

(ii) If $b_{k+1}(\delta) \equiv 0$, *then there exist* $\varepsilon_0 > 0$ *and* $r_0 > 0$ *such that the function* F *has at most* $k-1$ *isolated zeros in* $(0, r_0)$ *for* $|\delta - \delta_0| < \varepsilon_0$ *and* $\delta \in V$, *multiplicity taken into account. Moreover, this number can be reached for some* δ *near* δ_0 *under* (2.32).

In the following part, we verify Theorem 1.1 and 1.2 using the above lemmas.

First, it follows from Lemmas 2.1 and 2.5 that conclusion (i) of Theorem 1.1 holds. Second, we prove conclusion (ii) of Theorem 1.1. From (2.22), we find that $C_j(a_0, b_0) = 0$ for $1 \le j \le N$ and

$$C_{N+1}(a_0, b_0) = B_{N+1}(a_0, b_0) N^*_{N+1} \neq 0. \tag{2.33}$$

Hence, we know that $C_{N+1}(a, b) \neq 0$ for all (a, b) near (a_0, b_0).

According to (2.21) and (2.33), we have

$$\begin{aligned} d(\rho_0, a, b) &= \sum_{j=1}^{N+1} C_j(a, b) \rho_0^{r_j} (1 + d_j(\rho_0)) + \rho_0^{r_{N+2}} Q(\rho_0, a, b) \\ &= \sum_{j=1}^{N+1} C_j(a, b) \rho_0^{r_j} \left(1 + \bar{d}_j(\rho_0)\right), \end{aligned} \tag{2.34}$$

where

$$Q(\rho_0, a, b) = \sum_{j \ge N+2} C_j(a, b) \rho_0^{r_j - r_{N+2}}$$

and

$$\bar{d}_j = d_j, j = 1, 2, \cdots, N,\ \bar{d}_{N+1} = d_{N+1} + \frac{Q(\rho_0, a, b)}{C_{N+1}(a, b)} \rho_0^{r_{N+2} - r_{N+1}},$$

which implies $\bar{d}_j$ in (2.34) are C^∞ in ρ_0 satisfying $\bar{d}_j(0) = 0$ for all (a, b) near (a_0, b_0).

According to (2.22), we can get

$$\begin{aligned} \frac{\partial(C_1, C_2, \cdots, C_N)}{\partial(a, b)} &= \frac{\partial(C_1, C_2, \cdots, C_N)}{\partial(B_1, B_2, \cdots, B_N)} \frac{\partial(B_1, B_2, \cdots, B_N)}{\partial(a, b)} \\ &= M_{N \times N} \cdot \frac{\partial(B_1, B_2, \cdots, B_N)}{\partial(a, b)} \end{aligned} \tag{2.35}$$

where

$$M_{N\times N}=\begin{pmatrix} N_1^* & & & \\ * & N_2^* & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & N_N^* \end{pmatrix}$$

is a lower triangular matrix and $\det M\neq 0$. If (1.8) holds, then by (2.35) we get

$$\operatorname{rank}\left(\frac{\partial(C_1,C_2,\cdots,C_N)}{\partial(a,b)}(a_0,b_0)\right)=N.$$

Therefore, using Lemma 2.7, we can obtain the conclusion (ii) of Theorem 1.1.

Next, we prove Theorem 1.2. By (1.7), (1.9) and (2.22), we find that

$$C_j\left(a,b\right)=0, j=N+1,N+2,\cdots$$

as long as $C_1=C_2=\cdots=C_N=0$. Therefore, using Lemma (2.7), we can obtain Theorem 1.2.

From the above proof of Theorems 1.1 and 1.2, we have the following corollaries.

Corollary 2.1. *Let* (1.2), (1.3), (H_1) *and* (H_2) *hold. If there exist a point* (a_0,b_0) *and an integer* $N\geq 1$ *such that*

$$B_j\left(a_0,b_0\right)=0,\ j=1,2,\cdots,N,\ B_{N+1}(a_0,b_0)<0\,(resp.,>0)\,,$$

then the origin is a stable(resp.,unstable) focus of system (1.1) *for* $(a,b)=(a_0,b_0)$ *and there has at most* N *limit cycles at the origin for all* (a,b) *near* (a_0,b_0).

Corollary 2.2. *Let* (1.2), (1.3), (H_1) *and* (H_2) *hold. If there exist* $N\geq 1$ *such that*

$$B_j=O\left(|B_1,B_2,\cdots,B_N|\right)$$

for $j\geq N+1$*, then for any given* $G>0$ *there exist a neighborhood* U *of the origin such that for all* $|B_j|<G, j=1,2,\cdots,N$ *system* (1.1) *has at most* $N-1$ *limit cycles in* U.

3. Application

In this section, we give an application of our main results.

Consider a non-smooth Rayleigh-Liénard equation of the form

$$\begin{cases}\dot{x}=y,\\ \dot{y}=-g\left(x\right)-yf\left(y\right),\end{cases}\tag{3.1}$$

where $g(x)=g_0x^{2s-1}+O(x^{2s}), s\geq 1$ and

$$f\left(y\right)=\begin{cases}\displaystyle\sum_{j=0}^{m}a_j^+y^j,\ \ y>0,\\ \displaystyle\sum_{j=0}^{n}a_j^-y^j,\ \ y\leq 0,\end{cases}$$

with $g_0 > 0$ and $a = (a_0^+, a_1^+, \cdots, a_m^+, a_0^-, a_1^-, \cdots, a_n^-) \in \mathbb{R}^{m+n+2}$.

We claim that the cyclicity of system (3.1) at the origin is $\max\{m, n\}$.

Applying the following coordinate transformation to (3.1)

$$x = v, \quad y = -g_0 u,$$

we get

$$\begin{cases} \dot{u} = h(v) - F(u, a), \\ \dot{v} = -\tilde{g}(u), \end{cases}$$

where $h(v) = g(v)/h_0 = v^{2s-1} + O(v^{2s})$, $\tilde{g}(u) = g_0 u$ and

$$F(u, a) = \begin{cases} F^+(u, a) = \sum_{j=0}^{n} a_j^- (-g_0 u)^{j+1}, \ u \geq 0, \\ F^-(u, a) = \sum_{j=0}^{m} a_j^+ (-g_0 u)^{j+1}, \ u < 0. \end{cases}$$

By Lemma 2.1, we find that $\alpha(u) = -u$. Hence, we obtain

$$F^-(\alpha(u), a) - F^+(u, a) = \sum_{j=0}^{m} a_j^+ (g_0 u)^{j+1} - \sum_{j=0}^{n} a_j^- (-g_0 u)^{j+1}. \tag{3.2}$$

From (3.2) it is not difficult to find that $F^-(\alpha(u), a) \equiv F^+(u, a)$ if and only if $B_j = 0$ for $j = 1, 2, \cdots, \max\{m, n\} + 1$. Using Theorem 1.2, we know that the cyclicity of system (3.1) less than $\max\{m, n\}$. For $m > n$, we can obtain

$$B_j = g_0^j \left(a_{j-1}^+ - (-1)^j a_{j-1}^- \right), \ j = 1, \cdots, n, \ B_j = g_0^j a_{j-1}^+, \ j = n+1, \cdots, m+1. \tag{3.3}$$

Take $a = a_0$ such that $a_{j0}^+ = 0$, $j = 0, 1 \cdots, m-1$, $a_{j0}^- = 0$, $j = 0, 1, \cdots, n$ and $a_{m,0}^+ \neq 0$. Then by (3.3), we have

$$B_j(a_0) = 0, \ j = 1, 2, \cdots, m, \ \operatorname{rank} \left(\frac{\partial (B_1, B_2, \cdots, B_m)}{\partial \left(a_0^+, a_1^+, \cdots, a_{m-1}^+ \right)} \right) = m.$$

Using Theorem 1.1, we know that m limit cycles can appear near the origin for a near a_0. Hence, the cyclicity of system (3.1) is m. For $n \geq m$, we can discuss similarly. Therefore, the cyclicity of system (3.1) at the origin is $\max\{m, n\}$.

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