Journal of Applied Analysis and Computation
Volume 17, Number 1, February 2027, 579–607

Website:http://www.jaac-online.com
DOI:10.11948/20260335

SOME BIFURCATION RESULTS IN NEAR-HAMILTONIAN SYSTEMS WITH FINITE-ORDER SMOOTHNESS*

Ci Kong[1], Maoan Han[1,†], Yun Tian[2] and Xianbo Sun[3]

Abstract This paper presents a systematic survey of bifurcation theory for near-Hamiltonian systems with finite-order smoothness. Unlike the classical C^∞ or C^ω framework, we collect some known results on limit cycle bifurcations for planar systems with finite-order smoothness, including Poincaré bifurcation, Hopf bifurcation, homoclinic bifurcation, double homoclinic bifurcation, heteroclinic bifurcation, and bifurcations involving nilpotent singularities. For each type, we summarize the relevant bifurcation theorems on properties of the Melnikov functions and the conditions for the existence and the number of limit cycles that can be produced.

Keywords Limit cycle, bifurcation, near-Hamiltonian system, finite-order smoothness, Melnikov function.

MSC(2010) 34C05, 34C07, 34C23.

1. Introduction

Various nonlinear wave equations and their traveling wave solutions have been widely investigated, see [2, 6, 8, 9, 22, 43, 47, 53, 56, 60, 61]. The important tools used in these studies are the geometric singular perturbation theory developed in Fenichel [9] (which is used to obtain a reduced system in the plane) and bifurcation theory of near-Hamiltonian systems (which is used to study the reduced system). However, some researchers neglected the smoothness of the reduced systems. In fact, the reduced systems have only finite-order smoothness, while bifurcation theory of near-Hamiltonian systems used requires C^∞ or C^ω smoothness sometimes. In this survey we collect some results on bifurcation theory for near-Hamiltonian systems with finite-order smoothness, presenting various bifurcation theorems which were established in the past few decades.

A fundamental result in the geometric singular perturbation theory is Fenichel's invariant manifold theorem. We now give this theorem.

Consider a system of the form

$$x' = f(x, y, \varepsilon), \quad y' = \varepsilon g(x, y, \varepsilon), \tag{1.1}$$

where $(x', y') = (\frac{dx}{dt}, \frac{dy}{dt})$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$ and ε is a real parameter. In (1.1) letting $\varepsilon \to 0$ we

[†]The corresponding author.
[1]School of Mathematical Sciences, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
[2]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[3]School of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 311121, China
*This work was supported by the National Natural Science Foundation of China (No. 12571187).
Email: cikong@zjnu.edu.cn(C. Kong), mahan@zjnu.edu.cn(M. Han),
ytian22@shnu.edu.cn(Y. Tian), xsun@hznu.edu.cn(X. Sun)

obtain the system

$$x' = f(x, y, 0), \quad y' = 0. \tag{1.2}$$

According to (1.2) the variable x will vary while y will remain constant. Thus x is called the fast variable.

In order to state Fenichel's invariant manifold theorem clearly we list the following three assumptions.

(H1) The functions f and g are both C^∞ on a set $U \times I$ where $U \subset \mathbb{R}^N$ is open, with $N = n + l$, and I is an open interval, containing 0.

(H2) There exists a compact l-dimensional manifold M_0, possibly with boundary and contained in the set $\{(x, y) \in U|\ f(x, y, 0) = 0\}$, such that the linearization of (1.2) at each point in M_0 has exactly l eigenvalues with zero real part. In other words, the compact manifold M_0 is normally hyperbolic.

(H3) The set M_0 is given as the graph of a C^∞ function $h_0(y)$ for $y \in K$ with K a compact, simply connected domain whose boundary is an $(l-1)$-dimensional C^∞ submanifold.

Then we have the following theorem whose proof can be found in [6, 9].

Theorem 1.1. (Fenichel's Invariant Manifold Theorem) *Let (H1) and (H2) hold. Then for all $|\varepsilon| > 0$ sufficiently small, there exists a manifold M_ε that lies within $O(\varepsilon)$ of M_0 and is diffeomorphic to M_0. Moreover it is locally invariant under the flow of* (1.1)*, and C^r, including in ε, for any $1 \le r < +\infty$. Further, if, in addition to (H1) and (H2), (H3) is also satisfied, then there is a C^r function $h_\varepsilon(y)$, defined on K, jointly in y and ε, such that the manifold M_ε can be represented as the graph of $h_\varepsilon(y)$*

$$M_\varepsilon = \{(x, y)|\ x = h_\varepsilon(y),\ y \in K\}.$$

We have several remarks below in order.

Remark 1.1. The set M_0 is called the critical manifold of (1.2), and the manifold M_ε is called the slow manifold of (1.1). The dynamics of (1.1) on M_ε is determined by the reduced C^r differential system

$$\dot{y} = g(h_\varepsilon(y), y, \varepsilon), \quad y \in K, \tag{1.3}$$

where $\dot{y} = \frac{dy}{d(\varepsilon t)}$.

Remark 1.2. By the local invariance of M_ε, we have the equality below

$$f(h_\varepsilon(y), y, \varepsilon) = \varepsilon \frac{\partial h_\varepsilon}{\partial y}(y) g(h_\varepsilon(y), y, \varepsilon).$$

It follows that if

$$h_\varepsilon(y) = h_0(y) + \varepsilon h_1(y) + O(\varepsilon^2),$$

then the function h_1 satisfies

$$\frac{\partial f}{\partial x}(h_0(y), y, 0) h_1(y) + \frac{\partial f}{\partial \varepsilon}(h_0(y), y, 0) = \frac{\partial h_0}{\partial y}(y) g(h_0(y), y, 0).$$

Thus, the function h_1 can be solved from the above equation uniquely under (H1)-(H3). Obviously, $h_1 \in C^\infty(K)$.

Remark 1.3. System (1.3) can be rewritten as

$$\dot{y} = g_0(y) + \varepsilon g_1(y) + o(\varepsilon), \quad y \in K,$$

where g_0 and g_1 are C^∞ functions for $y \in K$ with $g_0(y) = g(h_0(y), y, 0)$. If $l = 2$ and $\mathrm{div} g_0 \equiv 0$, then (1.3) will be a near-Hamiltonian system on the plane.

Remark 1.4. According to [6, 9], in the same way as in [25] it is possible to prove, under (H1)-(H3), that if (1.1) satisfies

$$f(-x, -y, \varepsilon) = -f(x, y, \varepsilon),\ g(-x, -y, \varepsilon) = -g(x, y, \varepsilon),$$

or

$$f(x, -y, \varepsilon) = f(x, y, \varepsilon),\ g(x, -y, \varepsilon) = -g(x, y, \varepsilon),$$

then M_ε can be chosen to satisfy

$$h_\varepsilon(-y) = -h_\varepsilon(y)$$

or

$$h_\varepsilon(-y) = h_\varepsilon(y)$$

respectively. This implies that (1.3) is centrally symmetric with respect to the origin.

Remark 1.5. According to [6, 9], the integer r in the above theorem can not be taken as $+\infty$. If the functions f and g in (1.1) are C^k for some integer $k \geq 2$, then the integer r will be required to satisfy $1 \leq r \leq k$.

The near-Hamiltonian system we consider in this paper has the form

$$\begin{aligned} \dot{x} &= H_y(x, y) + \varepsilon p(x, y, \varepsilon, \delta), \\ \dot{y} &= -H_x(x, y) + \varepsilon q(x, y, \varepsilon, \delta), \end{aligned} \tag{1.4}$$

where the functions H, p and q are smooth functions in $(x, y) \in G \subset \mathbb{R}^2$ with G open, $\varepsilon \geq 0$ is a small parameter and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. The order of smoothness of these functions will be clarified in the next sections.

A powerful tool for finding limit cycles in system (1.4) is the Melnikov function

$$M(h, \delta) = \oint_{L_h} q(x, y, 0, \delta)\mathrm{d}x - p(x, y, 0, \delta)\mathrm{d}y, \tag{1.5}$$

where L_h denotes a periodic orbit of the unperturbed system $(1.4)|_{\varepsilon=0}$ defined by $H(x, y) = h$ for $h \in J$ with J an open interval.

In the following sections we will collect some known general results on Hopf, Poincaré, homoclinic and heteroclinic bifurcations respectively. Most of these results are given based on the properties of the function M given by (1.5).

2. Poincaré and Hopf bifurcations

Consider the following $C^k (1 \leq k \leq \infty)$ near-Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y, \varepsilon, \delta), \\ \dot{y} &= -H_x + \varepsilon g(x, y, \varepsilon, \delta), \end{aligned} \tag{2.1}$$

where $H \in C^{k+1}$, $f, g \in C^k$, $\varepsilon \geq 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. Suppose that the unperturbed system $(2.1)|_{\varepsilon=0}$ has a family of periodic orbits $L_h : H(x,y) = h$, where $h \in (\alpha, \beta) := J$. Without loss of generality we assume that L_h is oriented clockwise. Define a period annular Γ by

$$\Gamma = \bigcup_{h \in J} L_h.$$

Obviously, Γ does not contain any singular point of $(2.1)|_{\varepsilon=0}$. For any given $h_0 \in J$, take a point A_0 on L_{h_0}, see Figure 1. Let l be a cross section passing through A_0. Let $\vec{n}$ be the unit vector parallel to l satisfying $H'(A_0)\cdot\vec{n} > 0$, where $H'(A_0) = (H_x, H_y)|_{A_0}$. Since the cross section l can be taken as a C^k curve and smoothly extended such that the intersection $A(h) = L_h \bigcap l$ exists for all $h \in J$. Then we can write

$$A(h) = A_0 + a(h)\vec{n}, \quad a \in C^k.$$

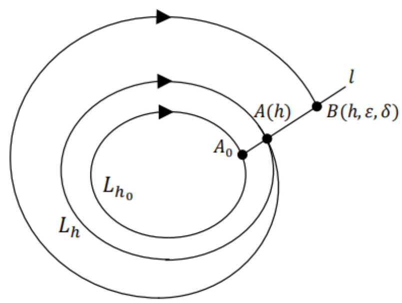


Figure 1. Poincaré map.

Denote by $B(h, \varepsilon, \delta)$ the first intersection point of the positive orbit of system (2.1) starting from $A(h)$ with l (see Figure 1). Then

$$\begin{aligned} H(B) - H(A) &= \int_{\widehat{AB}} H_x \mathrm{d}x + H_y \mathrm{d}y, \\ &= \varepsilon \int_{\widehat{AB}} (H_x f + H_y g) \mathrm{d}t \\ &\equiv \varepsilon F(h, \varepsilon, \delta). \end{aligned} \tag{2.2}$$

Clearly,

$$F(h, 0, \delta) = \oint_{L_h} (H_x f + H_y g)|_{\varepsilon=0} \mathrm{d}t = \oint_{L_h} (g\mathrm{d}x - f\mathrm{d}y)|_{\varepsilon=0} \equiv M(h, \delta), \quad h \in J.$$

It is well known that $F(h, \varepsilon, \delta)$ and $M(h, \delta)$ are called bifurcation function and the first order Melnikov function of system (2.1), respectively.

By [28], we have that $M \in C^k$ in $h \in J$ for $\delta \in D$ and that for any closed interval $I \subset J$, $F \in C^k$ in $h \in I$ for $|\varepsilon|$ small and $\delta \in D$. Thus, for any $h \in J$ and sufficiently small $|\varepsilon|$, the bifurcation function $F(h, \varepsilon, \delta)$ in (2.2) has an expansion of the form [29]

$$F(h, \varepsilon, \delta) = \sum_{j=1}^{k} M_j(h, \delta)\varepsilon^{j-1} + o\left(\varepsilon^{k-1}\right), \tag{2.3}$$

where $M_j(h,\delta)$ is called the jth order Melnikov function of system (2.1) with

$$M_j(h,\delta) = \frac{1}{(j-1)!}\frac{\partial^{j-1}F}{\partial\varepsilon^{j-1}}(h,0,\delta) \in C^{k-j+1}.$$

The following theorem was proved in [29].

Theorem 2.1. (Poincaré Bifurcation Theorem) *Consider C^k system (2.1). Suppose that there exist integers $r \geq 1$, $l \geq 1$ with $r+l \leq k+1$ such that $M_1 = \cdots = M_{r-1} = 0$, $M_r \neq 0$ in (2.3). If M_r has at most l zeros in $h \in J$ for all $\delta \in D \subset \mathbb{R}^m$, multiplicity taken into account, then for any compact set $V \subset \Gamma$ there is $\varepsilon_0 = \varepsilon_0(V) > 0$ such that (2.1) has at most l limit cycles in V for $\delta \in D$ and $0 < |\varepsilon| < \varepsilon_0$, multiplicity taken into account. Moreover, if M_r has l simple zeros in $h \in J$ for some $\delta = \delta_0 \in D$, then (2.1) has l limit cycles in Γ for $|\delta - \delta_0| + |\varepsilon|$ small with $\varepsilon \neq 0$. In this case, we say that the period annulus Γ has cyclicity l for (2.1).*

The author [14] (see also [18]) proved the following formula

$$M_h(h,\delta) = \oint_{L_h} (f_x + g_y)|_{\varepsilon=0} dt.$$

Consider a special case of M

$$M(h,\delta) = I(h)(P(h) - \lambda), \quad h \in J,$$

where $I(h) \neq 0$ for $h \in J$. In this case, if $P'(h) \neq 0$ for all $h \in J$, then the period annulus Γ generates at most one limit cycle by Theorem 2.1. Further, if the limits

$$\lambda_1 = \lim_{h\to\alpha} P(h), \quad \lambda_2 = \lim_{h\to\beta} P(h)$$

both exist, then the period annulus Γ generates a unique limit cycle if and only if λ lies between λ_1 and λ_2.

If M has a zero with multiplicity 2, then saddle-node bifurcation may occur as stated in the following theorem (see [12]).

Theorem 2.2. (Saddle-Node Bifurcation of Limit Cycles) *Consider system (2.1) with $f, g \in C^k$ and $H \in C^{k+1}$. Let $k \geq 2$ and $m = 1$. Suppose that there exist $h_0 \in J$ and $\delta_0 \in D$ such that*

$$M(h_0,\delta_0) = M_h(h_0,\delta_0) = 0, \quad \mu \equiv M_\delta(h_0,\delta_0)M_{hh}(h_0,\delta_0) \neq 0.$$

Then there exist $\varepsilon_0 > 0$, a differentiable function $\delta(\varepsilon) = \delta_0 + O(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_0$, and a neighborhood U of L_{h_0} such that system (2.1) has precisely two limit cycles (a unique limit cycle of multiplicity 2, no limit cycle, repectively) in U for $0 < |\varepsilon| < \varepsilon_0$, $|\delta - \delta_0| \leq \varepsilon_0$ if $\mu(\delta - \delta(\varepsilon)) < 0 (= 0, > 0$, respectively).

The above theorem can be proved by applying the implicit function theorem to the equations

$$F(h,\varepsilon,\delta) = 0, \quad F_h(h,\varepsilon,\delta) = 0$$

near the point $(h,\varepsilon,\delta) = (h_0,0,\delta_0)$.

There are several different approaches to study the number of zeros of M on the interval J, see [3, 7, 11, 41].

Now we assume that the period annulus Γ has an elementary center of $(2.1)|_{\varepsilon=0}$ as its inner boundary. Without loss of generality, we assume that (2.1) has a singular point at the origin for all small $|\varepsilon|$ and that the function H has the following form in a small neighborhood of the origin

$$H(x,y) = K(x^2+y^2) + o(x^2+y^2), \quad K>0. \tag{2.4}$$

In this case, we can take $A(h) = (a(h), 0)$ for small $h>0$. From the definition of M and F, one finds that

$$\lim_{h\to 0} M(h,\delta) = \lim_{h\to 0} F(h,\varepsilon,\delta) = 0.$$

Then we can define $M(0,\delta) = F(0,\varepsilon,\delta) = 0$ so that they are continuous at $h=0$. The point $B(h,\varepsilon,\delta)$ can be written as $B(h,\varepsilon,\delta) = (b(h,\varepsilon,\delta), 0)$. From Theorem 3 in [28], we know that there exists a C^k function $P(x,\varepsilon,\delta) = O(x)$, which is called a Poincaré map of (2.1) near the origin, such that

$$b(h,\varepsilon,\delta) = P(a(h),\varepsilon,\delta), \quad P(a(h),0,\delta) = a(h).$$

On the Hopf bifurcation theory of system (2.1), from [28] (Theorem 16) and Remark 2.1 in [39], we have the following Hopf bifurcation theorem for (2.1).

Theorem 2.3. (Hopf Bifurcation Theorem) ([28]) *Let system* (2.1) *satisfy* (2.4) *with* $f, g \in C^k$ *and* $H \in C^{k+2}$ $(k \geq 1)$. *Then the following statements hold.*

(a) There exists a C^k *function* $\Psi(x,\varepsilon,\delta)$ *satisfying* $\Psi(x,\varepsilon,\delta) = O(x^2)$ *such that for* $h>0$ *small*

$$F(h,\varepsilon,\delta) = \Psi(\sqrt{h},\varepsilon,\delta).$$

(b) There exists a function $\psi(x,\delta) = O(x^2) \in C^k$ *even in* x *for* $|x|$ *small such that* $\psi_x^{(k+1)}(0,\delta)$ *exists and* $M(h,\delta) = F(h,0,\delta) = \psi(\sqrt{h},\delta)$. *Hence, the first order Melnikov function* $M(h,\delta)$ *is* $C^{[\frac{k+1}{2}]}$ *at* $h=0$ *and has the expansion*

$$M(h,\delta) = b_1(\delta)h + b_2(\delta)h^2 + \cdots + b_l(\delta)h^l + o(h^l), \quad 2l \leq k+1.$$

(c) If there exist $\delta_0 \in D$ *and integer* $l \geq 1$ *such that*

$$b_l(\delta_0) \neq 0, \quad b_i(\delta_0) = 0, \quad i = 1, \cdots, l-1, \quad 2 \leq 2l \leq k+1,$$

then system (2.1) *has at most* $l-1$ *limit cycles near the origin for* $|\varepsilon| + |\delta - \delta_0|$ *small.*

(d) If $\delta = (\delta_1, \delta_2, \cdots, \delta_m) \in \mathbb{R}^m$ *with* $m \geq l-1$, *and*

$$\operatorname{rank} \frac{\partial(b_1, b_2, \cdots, b_{l-1})}{\partial(\delta_1, \cdots, \delta_m)}(\delta_0) = l-1,$$

then system (2.1) *does have* $l-1$ *limit cycles for some* (ε, δ) *near* $(0, \delta_0)$.

Similar to Theorem 2.2 in [39], by the results in Theorem 2.3, we can obtain

Theorem 2.4. ([39]) *Let system* (2.1) *satisfy* (2.4) *with* $f, g \in C^k$ *and* $H \in C^{k+2}$. *Suppose that there exists an integer* $l \geq 1$ *satisfying* $2l \leq k+1$ *such that for each* $\delta \in D$, $M(h,\delta)$ *has the zero* $h=0$ *with multiplicity not exceeding* l. *Then there exist* $\varepsilon_0 > 0$ *and a neighborhood* U *of the origin such that system* (2.1) *has at most* $l-1$ *limit cycles in* U *for* $0 < |\varepsilon| < \varepsilon_0$, $\delta \in D$, *multiplicity taken into account.*

There are two ways to obtain conditions under which (2.1) has limit cycles near the origin. One is to use the coefficients $\{b_j\}$ in the expansion of M to produce zeros of M and F in h, respectively. To see this clearly, we assume that there exists $\delta_0 \in D$ such that $b_1(\delta_0) = 0$, $b_2(\delta_0) \neq 0$. Then noting $F(h, 0, \delta) = M(h, \delta)$ (2.1) will have a unique limit cycle near the origin if

$$0 < |\varepsilon| \ll |b_1| \ll |b_2(\delta_0)|, \quad b_1 b_2 < 0.$$

Another way to find limit cycles is to use the Poincaré map $P(x, \varepsilon, \delta)$ to change the stability of the origin. For (2.1), if $f, g \in C^k$ and $H \in C^{k+2}$, then we have

$$P(x, \varepsilon, \delta) = \varepsilon d(x, \varepsilon, \delta),$$

where $d \in C^k$ from [28] (Theorem 6). For $k \geq 3$ we have

$$d(x, \varepsilon, \delta) = v_1(\varepsilon, \delta)x + v_2(\varepsilon, \delta)x^2 + v_3(\varepsilon, \delta)x^3 + o(x^3)$$

for $|x|$ small. From [30], $v_2(\varepsilon, \delta) = O(v_1(\varepsilon, \delta))$, and there are positive constants k_1 and k_2 such that

$$v_1(0, \delta) = k_1 b_1(\delta), \quad v_3(0, \delta) = k_2 b_2(\delta) + O(b_1).$$

Then it is direct that if $b_1(\delta_0) = 0$, $b_2(\delta_0) \neq 0$, then for $|\delta - \delta_0| + |\varepsilon|$ small (2.1) has a unique limit cycle near the origin if and only if $v_1(\varepsilon, \delta) b_2(\delta_0) < 0$.

If $b_1(\delta_0) = b_2(\delta_0) = 0$, $b_3(\delta_0) \neq 0$, then saddle-node bifurcation of limit cycles may occur in Hopf bifurcation.

For the computing formulas of b_1, b_2 and b_3, one can see [18]. Also, it was proved in [16, 18] that M is C^∞ (C^ω, resp.) at $h = 0$ if the functions H and $(f, g)|_{\varepsilon=0}$ are C^∞ (C^ω, resp.) near the origin. A new proof for this conclusion as well as a generalization of it was given in [42].

3. Homoclinic bifurcation

Consider the following C^k system

$$\begin{aligned} \dot{x} &= f_0(x, y) + \varepsilon f(x, y, \varepsilon, \delta), \\ \dot{y} &= g_0(x, y) + \varepsilon g(x, y, \varepsilon, \delta), \end{aligned} \tag{3.1}$$

where $f_0, g_0, f, g \in C^k$ with $2 \leq k \leq \infty$, $\varepsilon \geq 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. Suppose that unperturbed system $(3.1)|_{\varepsilon=0}$ has a hyperbolic saddle point S_0 and a homoclinic orbit

$$L_0 = \{(x(t), y(t)) \mid t \in \mathbb{R}\},$$

which tends to S_0 as $t \to +\infty$ and $t \to -\infty$, i.e., $\omega(L_0) = \alpha(L_0) = S_0$. We call $L_0 \cup S_0$ a homoclinic loop. For definiteness, we assume that L_0 is oriented clockwise and the Poincaré map is well defined on the interior side of L_0. See Figure 2. When $|\varepsilon|$ is sufficiently small, system (3.1) has a hyperbolic saddle point $S(\varepsilon, \delta)$ near S_0. Moreover, in a neighborhood of L_0, system (3.1) admits separatrices $L^s(\varepsilon, \delta)$ and $L^u(\varepsilon, \delta)$, which lie on the stable and unstable manifolds of $S(\varepsilon, \delta)$, respectively. Take an arbitrary fixed point $A_0 \in L_0$, and let l be a transversal of system (3.1) passing through A_0, with the direction vector chosen as

$$n_0 = \frac{1}{|f_0(A_0), g_0(A_0)|}\big(-g_0(A_0), f_0(A_0)\big).$$

Then the separatrices L^s and L^u intersect l at points denoted by A^s and A^u, respectively. The author [45] proved the following lemma (whose proof can be also found in [5, 12, 17, 18]).

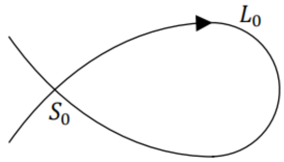


Figure 2. Clockwise-oriented homoclinic orbit.

Lemma 3.1. ([45]) *There exist C^k functions $a^{s,u}(\varepsilon,\delta)=O(\varepsilon)$ such that*

$$A^{s,u}=A_0+a^{s,u}(\varepsilon,\delta)n_0.$$

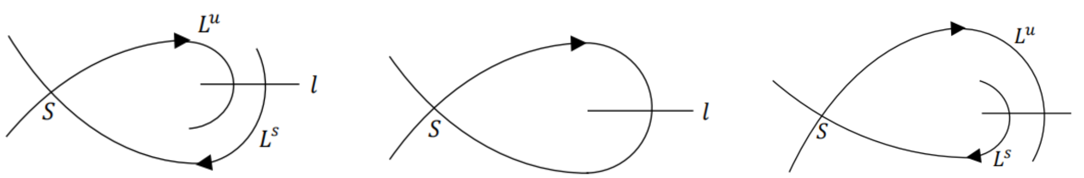
Let $d(\varepsilon,\delta,A_0)=a^u(\varepsilon,\delta)-a^s(\varepsilon,\delta)$. Then

$$d(\varepsilon,\delta,A_0)=\frac{\varepsilon M(\delta)}{|f_0(A_0),g_0(A_0)|}+O(\varepsilon^2),$$

where

$$M(\delta)=\int_{-\infty}^{+\infty}(f_0g-g_0f)\exp\left(-\int_0^t(f_{0x}+g_{0y})\,\mathrm{d}\tau\right)\bigg|_{\varepsilon=0,\,L_0}\mathrm{d}t\in\mathbb{R}. \tag{3.2}$$

The sign of the quantity d determines the relative position of the separatrices L^s and L^u, as illustrated in Figure 3.



(a) $d<0$ (b) $d=0$ (c) $d>0$

Figure 3. The relative positions of L^s and L^u.

Any point A near the left side of A^s on the transversal l can be expressed as

$$A=A_0+an_0,\quad a\le a^s(\varepsilon,\delta).$$

Let B denote the subsequent intersection point of the positive semi-orbit of system (3.1) starting from A with the transversal l. Then B can be written as

$$B=A_0+P(a,\varepsilon,\delta)n_0,\quad a<a^s(\varepsilon,\delta).$$

The mapping $P:l\to l$ is called the Poincaré map of system (3.1) near L_0. Define

$$F(a,\varepsilon,\delta)=P(a,\varepsilon,\delta)-a$$

which is called the bifurcation function of system (3.1) near L_0. From the properties of saddle points, it is easy to see that

$$\lim_{a\to a^s-}P(a,\varepsilon,\delta)=a^u(\varepsilon,\delta).$$

We can define

$$F(a^s,\varepsilon,\delta)=a^u(\varepsilon,\delta)-a^s(\varepsilon,\delta)=d(\varepsilon,\delta,A_0).$$

From Lemma 3.4 in [17], there exists a C^k function $\mu(F,a,\varepsilon,\delta)=O(F)$ such that

$$\frac{\partial F}{\partial a}=-1+\frac{1}{1+\mu(F,a,\varepsilon,\delta)}\cdot\exp\int_{AB}\big[f_{0x}+g_{0y}+\varepsilon(f_x+g_y)\big]\mathrm{d}t.$$

By this formula and using Lemma 3.1 one can obtain the following theorem.

Theorem 3.1. ([45]) *Let $\sigma_0=(f_{0x}+g_{0y})(S_0)$. If $\sigma_0\neq 0$, then*

(i) When $\sigma_0<0$ (>0), the homoclinic orbit L_0 is inward stable (unstable).

(ii) There exist $\varepsilon_0>0$ and a neighbourhood V of L_0 such that for $0<|\varepsilon|<\varepsilon_0$, $\delta\in D$, system (3.1) has at most one limit cycle in V. Moreover, a limit cycle exists if and only if $\sigma_0 d(\varepsilon,\delta,A_0)>0$, and its stability is determined by the sign of σ_0.

The theorem above was first obtained in [45]. Its proof can also be found in [5,17] and in [18] for the case $k\geq 3$.

When $\sigma_0=(f_{0x}+g_{0y})(S_0)=0$, one can prove that $\sigma_1=\oint_{L_0}(f_{0x}+g_{0y})\mathrm{d}t\in\mathbb{R}$, and that L_0 is stable (unstable) as $\sigma_1<0$ (>0). In this case, it was proved in [17] that the $C^k(k\geq 2)$ system (3.1) has at most two limit cycles near L_0 for $|\varepsilon|>0$ small.

If $f_{0x}+g_{0y}\equiv 0$, then (3.2) becomes

$$M(\delta)=\oint_{L_0}\big(f_0g-g_0f\big)\big|_{\varepsilon=0}\mathrm{d}t=\oint_{L_0}\big(g\mathrm{d}x-f\mathrm{d}y\big)\big|_{\varepsilon=0}. \tag{3.3}$$

Let

$$\sigma(\delta)=(f_x+g_y)(S_0,0,\delta).$$

The authors [44] obtained the following two theorems.

Theorem 3.2. ([44]) *Consider C^k system (3.1) with $k\geq 2$. Suppose there exists a C^{k+1} function $H(x,y)$ satisfying*

$$H_y=f_0(x,y),\quad H_x=-g_0(x,y). \tag{3.4}$$

Then

(i) For sufficiently small $|\varepsilon|$ and $\delta\in D$, a necessary condition for system (3.1) to have a limit cycle near L_0 is $M(\delta_0)=0$ for some $\delta_0\in D$.

(ii) Assume $M(\delta_0)=0$, and further suppose $\sigma(\delta_0)\neq 0$. Then there exist $\varepsilon_0>0$ and a neighbourhood V of L_0 such that for $0<|\varepsilon|<\varepsilon_0$ and $|\delta-\delta_0|<\varepsilon_0$, system (3.1) has at most one limit cycle in V. Moreover, a limit cycle exists if and only if $\sigma(\delta_0)d^*(\varepsilon,\delta)>0$, and its stability is determined by the sign of $\varepsilon\sigma(\delta_0)$, where

$$d^*(\varepsilon,\delta)=\frac{1}{\varepsilon}d(\varepsilon,\delta,A_0)=\frac{M(\delta)}{\big|f_0(A_0),\,g_0(A_0)\big|}+O(\varepsilon).$$

Theorem 3.3. ([44]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$ *and* $H \in C^{k+1}$ *with* $k \geq 2$. *Suppose* $D \subset \mathbb{R}$. *If there exists* $\delta_0 \in D$ *such that*

$$M(\delta_0) = 0, \quad \mu = M'(\delta_0)\sigma(\delta_0) \neq 0,$$

then there exist $\varepsilon_0 > 0$, *a differentiable function* $\delta^*(\varepsilon) = \delta_0 + O(\varepsilon)$ *defined for* $|\varepsilon| < \varepsilon_0$, *and a neighbourhood* V *of* L_0 *such that system* (3.1) *has a unique limit cycle in* V *if and only if* $\mu(\delta - \delta^*(\varepsilon)) > 0$ *for* $0 < |\varepsilon| < \varepsilon_0$ *and* $|\delta - \delta_0| < \varepsilon_0$, *and that system* (3.1) *has a homoclinic loop near* L_0 *if and only if* $\delta = \delta^*(\varepsilon)$.

One can see [17, 18, 36] for a proof of Theorem 3.2, and [18, 36] for a proof of Theorem 3.3. The authors [36] proved the following theorem which was also proved in [17].

Theorem 3.4. ([36]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$, $H \in C^{k+1}$ *and* $k \geq 2$. *Suppose there exists* $\delta_0 \in D$ *such that*

$$M(\delta_0) = 0, \quad \sigma(\delta_0) = (f_x + g_y)(S_0, 0, \delta_0) = 0,$$

and

$$\sigma_1(\delta_0) = \oint_{L_0} (f_x + g_y)(x, y, 0, \delta_0)\mathrm{d}t \neq 0.$$

Then there exist $\varepsilon_0 > 0$ *and a neighbourhood* U *of* L_0 *such that*

(i) system (3.1) *has at most two limit cycles in* U *for all* $0 < |\varepsilon| < \varepsilon_0$ *and* $|\delta - \delta_0| < \varepsilon_0$;

(ii) if $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$ *and*

$$\det \frac{\partial(M, \sigma)}{\partial(\delta_1, \delta_2)}(\delta_0) \neq 0,$$

then for any neighbourhood $U_0 \subset U$ *there exists some* (ε, δ) *near* $(0, \delta_0)$ *such that system* (3.1) *has two limit cycles in* U_0.

In [36] the authors further proved that there exists a number, denoted by $\sigma_2(\delta)$, such that for the C^8 system (3.1) with (3.4) satisfied, L_0 generates at most three limit cycles for $|\varepsilon| + |\delta - \delta_0|$ small if $\sigma(\delta_0) = \sigma_1(\delta_0) = 0$, $\sigma_2(\delta_0) \neq 0$ for some $\delta_0 \in D$. For a formula of $\sigma_2(\delta)$, see [17, 18, 36].

Under (3.4) we can suppose that there is an interval $J = (\alpha, 0)$ such that $(3.1)|_{\varepsilon=0}$ possesses a family of closed orbits L_h given by $H(x, y) = h$ near L_0 for $h \in J$. In this case, L_h tends to the homoclinic orbit L_0 as $h \to 0$. The first order Melnikov function of (3.1) has the form

$$M_1(h, \delta) = \oint_{L_h} (g\mathrm{d}x - f\mathrm{d}y)\big|_{\varepsilon=0}.$$

The authors [33] proved the following theorem. See also [17] for its proof.

Theorem 3.5. ([33]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$ *and* $H \in C^{k+1}$ *with* $k \geq 2$. *Then* $M_1(h, \delta)$ *is twice continuously differentiable with respect to* $h \in J$, *and has the asymptotic expansion for* $0 < -h \ll 1$

$$M_1(h, \delta) = M(\delta) + c_1(\delta)\, h \ln |h| + c_2(\delta)\, h + o(h), \tag{3.5}$$

where $M(\delta)$ *is given by* (3.3),

$$c_1(\delta) = -\frac{1}{\lambda_{10}}\,(f_x + g_y)(S_0, 0, \delta) = -\frac{1}{\lambda_{10}}\,\sigma(\delta),$$

$$c_2(\delta) = \oint_{L_0} \big[(f_x + g_y)(x, y, 0, \delta) - \sigma(\delta)\big]\mathrm{d}t + O(c_1),$$

and $\lambda_{10} > 0$ *denotes the positive eigenvalue of the matrix* $\frac{\partial(f_0,g_0)}{\partial(x,y)}(S_0)$.

The author [49] proved that the function M_1 has an expansion of the form for analytic system (3.1)

$$M_1(h, \delta) = \sum_{j\geq 0}(c_{2j} + c_{2j+1}h \ln|h|)h^j.$$

In fact, this is true if the functions $(f, g)|_{\varepsilon=0}$ and H are analytic. The authors [32] gave a simple proof of the above expansion and obtained a formula of the coefficient c_3. See [18,32] for deduction of formulas for c_0, c_1, c_2 and c_3. For a way to compute more coefficients, see [54].

From [49] we know that if

$$c_j(\delta_0) = 0, \quad j = 0, \cdots, l-1, \quad c_l(\delta_0) \neq 0,$$

for some $\delta_0 \in D$ and $l \geq 2$, then (3.1) has at most l limit cycles near L_0 for $|\varepsilon| + |\delta - \delta_0|$ small. The author [13] found a mistake in [49] and reproved the conclusion.

As in Hopf bifurcation, there are two ways to obtain limit cycles near L_0. One is to use the expansion of the first order Melnikov function M_1 in h to get zeros of M_1 and F respectively. For instance, one varies δ near some point δ_0 such that the coefficients $M(\delta), c_1(\delta)$ and $c_2(\delta)$ in (3.5) satisfy

$$0 < -M(\delta) \ll c_1(\delta) \ll c_2(\delta)$$

to get two simple zeros h_1 and h_2 of M_1 on J. Then for sufficiently small $\varepsilon > 0$ the function F has two simple zeros $h_j + O(\varepsilon)$, $j = 1, 2$, obtaining two limit cycles near L_0.

Another way includes the following steps:

1. Form a homoclinic loop of (3.1) near L_0 by solving the equation $d(\varepsilon, \delta) = 0$ using the implicit function theorem;

2. change its stability one or more times to produce one or more limit cycles;

3. break it by setting $d(\varepsilon, \delta) < 0$ or $d(\varepsilon, \delta) > 0$ to get a final limit cycle.

This method was originated in [15] and developed in [18, 50, 51, 59] to study limit cycle bifurcations near heteroclinic loops with two saddles and some compound cycles with two or more saddles.

4. Double homoclinic bifurcation

In this section, we continue to consider the C^k system (3.1) with $k \geq 1$. Suppose that when $\varepsilon = 0$, system (3.1) possesses a double homoclinic orbit L which consists of two homoclinic orbits L_1 and L_2 sharing the same hyperbolic saddle point S_0 as their limit set. The set $L \cup S_0$ is called a double homoclinic loop, sometimes L itself is also referred to as a double homoclinic loop. For definiteness, we still assume that L is oriented clockwise, see Figure 4.

Take points $A_i \in L_i$ for $i = 1, 2$, respectively. Make a transversal (or cross section) l_i to each L_i through A_i and take its positive direction as

$$n_i = \frac{1}{|f_0(A_i), g_0(A_i)|}\big(-g_0(A_i), f_0(A_i)\big), \quad i = 1, 2.$$

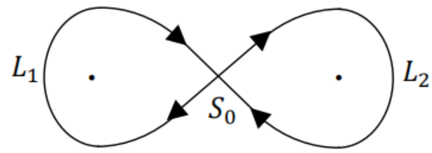


Figure 4. Clockwise-oriented double homoclinic loop.

When ε is sufficiently small, system (3.1) possesses a unique saddle point S near S_0 together with its four separatrices $L_i^s(\varepsilon,\delta)$ and $L_i^u(\varepsilon,\delta)$ near L_i for $i=1,2$, such that $L_1^s \cup L_2^s \cup S$ and $L_1^u \cup L_2^u \cup S$ constitute the stable and unstable manifolds of system (3.1) at S, respectively. Let A_i^s and A_i^u denote the intersection points of L_i^s and L_i^u with l_i, respectively. Then we can set

$$A_i^s = A_i + a_i^s(\varepsilon,\delta)n_i, \quad A_i^u = A_i + a_i^u(\varepsilon,\delta)n_i, \quad i=1,2,$$

as illustrated in Figure 5.

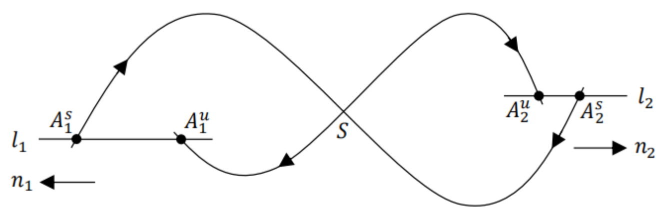


Figure 5. Separatrices near the double homoclinic loop.

As in Section 3, we can define two Poincaré maps $P_i: l_i \to l_i$ near L_i for $i=1,2$, together with the bifurcation functions near L_i

$$F_i(a,\varepsilon,\delta) = P_i(a,\varepsilon,\delta) - a, \quad i=1,2.$$

Clearly,

$$P_i(a_i^s-,\varepsilon,\delta) = a_i^u, \quad F_i(a_i^s-,\varepsilon,\delta) = a_i^u - a_i^s, \quad i=1,2.$$

Thus, the functions F_i and P_i are well defined on an interval $(-\varepsilon_0, a_i^s]$ in a for a small constant $\varepsilon_0 > 0$.

Now take a point $A = A_1 + an_1 \in l_1$ near A_1. Suppose the positive trajectory of system (3.1) passing through A intersects l_2 at $A^* = A_2 + a^*(a,\varepsilon,\delta)n_2$, and then intersects l_1 again at $B = A_1 + P(a,\varepsilon,\delta)n_1$. We thereby obtain a Poincaré map $P\colon l_1 \to l_1$ of system (3.1) near $L = L_1 \cup L_2$. It is easy to see that the map P is well defined at a near 0 if and only if it satisfies

$$a > a_1^s, \quad a^*(a,\varepsilon,\delta) > a_2^s.$$

Hence, there exists a unique function $a_0(\varepsilon,\delta)$ satisfying $a_0 \geq a_1^s$ such that the following limit exists finitely

$$\lim_{a\to a_0} P(a,\varepsilon,\delta) = P(a_0,\varepsilon,\delta).$$

Thus the domain of definition of P can be extended to $[a_0(\varepsilon,\delta),\varepsilon_0)$ in a. Define

$$F(a,\varepsilon,\delta) = P(a,\varepsilon,\delta) - a, \quad a \in [a_0,\varepsilon_0).$$

We call F as a bifurcation function of system (3.1) near L. Clearly, F has a root for $a > a_0$ if and only if system (3.1) possesses a limit cycle enclosing the saddle point S near L; such a cycle is referred to as a large cycle. The value a_0 is a root of F in a if and only if system (3.1) admits a double homoclinic loop or a single homoclinic loop near L. In the latter case, it is called a large homoclinic loop or large homoclinic orbit. The limit cycles of system (3.1) near L_i (corresponding to the roots of F_i) are called small cycles of system (3.1) near L. Let

$$\begin{aligned} d(\varepsilon,\delta,A_1) &= F\big(a_0(\varepsilon,\delta),\varepsilon,\delta\big),\\ d_i(\varepsilon,\delta,A_i) &= F_i(a_i^s(\varepsilon,\delta),\varepsilon,\delta), \quad i=1,2. \end{aligned}$$

Then it follows from Lemma 3.1 that

$$d_i(\varepsilon,\delta,A_i) = \frac{\varepsilon M_i(\delta)}{\big|\, f_0(A_i),\, g_0(A_i)\,\big|} + O(\varepsilon^2), \quad i=1,2,$$

where

$$M_i(\delta) = \oint_{L_i} \big(f_0 g - g_0 f\big)\big|_{\varepsilon=0} \exp\left(-\int_0^t (f_{0x}+g_{0y})\mathrm{d}\tau\right)\mathrm{d}t. \tag{4.1}$$

Obviously, system (3.1) possesses a homoclinic orbit near L_i if and only if $d_i = 0$, and admits a large homoclinic orbit near L if and only if $d = 0$ and $d_1 d_2 \neq 0$.

Let $\sigma_0 = (f_{0x}+g_{0y})(S_0)$. When $\sigma_0 \neq 0$, the saddle S_0 is called rough. If $\sigma_0 = 0$, then

$$\sigma_1 = \oint_L (f_{0x}+g_{0y})\mathrm{d}t = \sum_{i=1}^{2}\oint_{L_i}(f_{0x}+g_{0y})\mathrm{d}t \equiv \sigma_{11}+\sigma_{12}$$

is finite. On the stability of L and the number of limit cycles of (3.1) near it we have the following two theorems by [17,20].

Theorem 4.1. ([17,26]) *Consider C^k system (3.1) with $k \geq 1$. Suppose $\sigma_0 = (f_{0x}+g_{0y})(S_0) \neq 0$. Then L is outer stable (outer unstable) when $\sigma_0 < 0$ (> 0). Moreover, there exist $\varepsilon_0 > 0$ and a neighbourhood U of L such that for all $0 < |\varepsilon| < \varepsilon_0$ and $\delta \in D$,*

- *(i) system (3.1) has at most one large cycle in U;*
- *(ii) system (3.1) has at most two limit cycles in U. If two limit cycles exist, they are either two disjoint small cycles or one large cycle enclosing one small cycle.*

Theorem 4.2. ([20]) *Consider C^k system (3.1) with $k \geq 3$. Suppose $\sigma_0 = 0$ and $\sigma_1 \neq 0$. Then*

- *(i) The double homoclinic loop L is outer stable (outer unstable) if $\sigma_1 < 0$ (> 0).*
- *(ii) There exist $\varepsilon_0 > 0$ and a neighbourhood U of L such that for $0 < |\varepsilon| < \varepsilon_0$ and $\delta \in D$, system (3.1) has at most two large cycles in U.*
- *(iii) If $\sigma_{11}\sigma_{12} \neq 0$, then for $0 < |\varepsilon| < \varepsilon_0$ and $\delta \in D$, system (3.1) has at most five limit cycles in U.*
- *(iv) If $\sigma_{11}\sigma_{12} = 0$ and the first saddle quantity of the C^k system (3.1) with $k \geq 7$ at the saddle point S_0 is nonzero, i.e., $R_1 \neq 0$, then for $0 < |\varepsilon| < \varepsilon_0$ and $\delta \in D$, system (3.1) has at most six limit cycles in U.*

The Theorem 4.1 above was obtained first in [26] and was improved on smoothness in [17]. The definition of the value R_1 in Theorem 4.2 and its formula can be found in [17, 18]. On determination of stability of L when $\sigma_0 = \sigma_1 = 0$, see [18, 23, 37].

Next, we consider limit cycle bifurcations near L in near-Hamiltonian systems. Introduce

$$M(\delta) = M_1(\delta) + M_2(\delta), \quad \sigma(\delta) = (f_x + g_y)(S_0, 0, \delta).$$

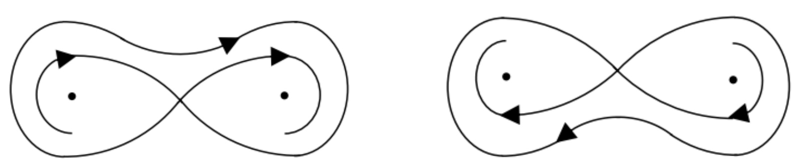
Then we have

Theorem 4.3. ([17, 26]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$, $H \in C^{k+1}$ *and* $k \geq 1$. *Then*

(i) When $|\varepsilon|$ *is sufficiently small and* $\delta \in D$, *a necessary condition for system* (3.1) *to admit a large cycle near* L *is* $M(\delta_0) = 0$ *for some* $\delta_0 \in D$.

(ii) Suppose $M(\delta_0) = 0$ *and* $\sigma(\delta_0) \neq 0$. *Then there exist* $\varepsilon_0 > 0$ *and a neighbourhood* U *of* L *such that for* $0 < |\varepsilon| < \varepsilon_0$ *and* $|\delta - \delta_0| < \varepsilon_0$, *system* (3.1) *has at most two limit cycles in* U. *If two limit cycles exist, then* $M_1(\delta_0) = M_2(\delta_0) = 0$, *and they are either two disjoint small cycles or one large cycle enclosing one small cycle.*

(iii) Let $M(\delta_0) = 0$ *and* $M'(\delta_0) \neq 0$ *with* $\delta_0 \in D \subset \mathbb{R}$. *Then there exists a continuously differentiable function* $\delta^*(\varepsilon) = \delta_0 + O(\varepsilon)$ *such that when* $\delta = \delta^*(\varepsilon)$, *system* (3.1) *possesses a large homoclinic orbit or a double homoclinic loop inside* U. *Moreover, if* $d_i(\varepsilon, \delta^*(\varepsilon), A_i) \neq 0 (i = 1, 2)$, *this homoclinic orbit is large as shown in Figure 6.*



(a) $d_1(\varepsilon, \delta^*(\varepsilon), A_1) > 0$ (b) $d_1(\varepsilon, \delta^*(\varepsilon), A_1) < 0$

Figure 6. Large homoclinic loop for $\delta = \delta^*(\varepsilon)$.

It is easy to see that the large cycle near L is generated from $L \cup S_0$ itself, while the small cycles are generated from $L_1 \cup S_0$ or $L_2 \cup S_0$. Under the conditions of Theorems 4.1 and 4.3, one can further give conditions for existence of two limit cycles. In fact, suppose $\delta \in \mathbb{R}^2$ and there exists $\delta_0 \in \mathbb{R}^2$ such that

$$M_1(\delta_0) = M_2(\delta_0) = 0, \quad \det \frac{\partial(M_1, M_2)}{\partial(\delta_1, \delta_2)}(\delta_0) \neq 0,$$

then two limit cycles will appear for some (ε, δ) near $(0, \delta_0)$. In this case, we say that the total cyclicity of L is 2. The condition $\sigma(\delta_0) \neq 0$ in Theorem 4.3 is usually called the nondegeneracy condition for nonisolated double homoclinic bifurcations.

If $\sigma(\delta_0) = 0$, the authors [20] further proved the following theorem.

Theorem 4.4. ([20]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$*,* $H \in C^{k+1}$ *and* $k \geq 3$*. Suppose there exists* $\delta_0 \in D$ *such that* $M(\delta_0) = 0$*,* $\sigma(\delta_0) = 0$*, and*

$$\begin{aligned}\sigma_1(\delta_0) &\equiv \sum_{i=1}^{2} \oint_{L_i} \big(f_x + g_y\big)(x, y, 0, \delta_0)\,\mathrm{d}t \\ &\equiv \sigma_{11}(\delta_0) + \sigma_{12}(\delta_0) \\ &\neq 0.\end{aligned}$$

Then

(i) There exist $\varepsilon_0 > 0$ *and a neighbourhood* U *of* L *such that for all* $0 < |\varepsilon| < \varepsilon_0$*,* $|\delta - \delta_0| < \varepsilon_0$*, system* (3.1) *has at most two large cycles in* U*. If* $\delta \in \mathbb{R}^2$ *and*

$$\det \frac{\partial(M, \sigma)}{\partial(\delta_1, \delta_2)}(\delta_0) \neq 0,$$

then system (3.1) *can admit two large cycles in* U*.*

(ii) If $\sigma_{11}(\delta_0)\sigma_{12}(\delta_0) \neq 0$*, then for all* $0 < |\varepsilon| < \varepsilon_0$*,* $|\delta - \delta_0| < \varepsilon_0$*, system* (3.1) *has at most five limit cycles in* U*. Moreover, if* $\delta \in \mathbb{R}^3$*, and*

$$M_1(\delta_0) = M_2(\delta_0) = 0, \quad \det \frac{\partial(M_1, M_2, \sigma)}{\partial(\delta_1, \delta_2, \delta_3)}(\delta_0) \neq 0,$$

then there exists (ε, δ) *near* $(0, \delta_0)$ *such that system* (3.1) *has* k *large cycles and* $5 - k$ *small cycles in* U *for* $k = 1$ *or* 2*.*

(iii) If $\sigma_{11}(\delta_0)\sigma_{12}(\delta_0) = 0$ *and* $\dfrac{\partial R_1}{\partial \varepsilon}(0, \delta_0) \neq 0$*, where* $R_1(\varepsilon, \delta)$ *denotes the first order saddle quantity of system* (3.1) *at the saddle point* S*, then for* $0 < |\varepsilon| < \varepsilon_0$*,* $|\delta - \delta_0| < \varepsilon_0$*, system* (3.1) *has at most six limit cycles in* U*. If* $\delta \in \mathbb{R}^4$*, and*

$$M_1(\delta_0) = M_2(\delta_0) = \sigma_{0i}(\delta_0) = 0, \quad \det \frac{\partial(M_1, M_2, \sigma, \sigma_{0i})}{\partial(\delta_1, \delta_2, \delta_3, \delta_4)}(\delta_0) \neq 0,$$

with $i = 1$ *or* 2*, then system* (3.1) *can have* k *large cycles and* $6 - k$ *small cycles in* U *for* $k = 1$ *or* 2*.*

When condition (3.4) holds, system (3.1) possesses three families of closed orbits near L. Without loss of generality, suppose that S_0 is at the origin with $H(0, 0) = 0$. Then the periodic orbits can be expressed as

$$\begin{aligned}&L_h : H(x, y) = h, \quad 0 < h \ll 1, \\ &L_{1h} : H(x, y) = h, \quad x < 0, \quad 0 < -h \ll 1, \\ &L_{2h} : H(x, y) = h, \quad x > 0, \quad 0 < -h \ll 1\end{aligned}$$

with the propeties $L_h \to L$, $L_{ih} \to L_i$ as $h \to 0$, for $i = 1, 2$. Correspondingly, we have three first order Melnikov functions

$$\begin{aligned}M_1(h, \delta) &= \oint_{L_h} \big(f_0 g - g_0 f\big)\big|_{\varepsilon=0} \mathrm{d}t, \quad 0 < h \ll 1, \\ M_{1i}(h, \delta) &= \oint_{L_{ih}} \big(f_0 g - g_0 f\big)\big|_{\varepsilon=0} \mathrm{d}t, \quad 0 < -h \ll 1.\end{aligned}$$

Similar to Theorem 3.5, if system (3.1) is of class C^2, then M_1 is a C^2 function for $0 < h \ll 1$, and

$$M_1(h,\delta) = M(\delta) + c_1(\delta)h\ln h + c_2(\delta)h + o(h), \quad 0 < h \ll 1,$$

where

$$c_1(\delta) = -\frac{2\sigma(\delta)}{\lambda_1(0,\delta)} = -2\left(\frac{\lambda_2}{\lambda_1}\right)'_{\varepsilon}(0,\delta),$$

$$c_2(\delta) = \sum_{i=1}^{2} \oint_{L_i} \big[\big(f_x + g_y\big)(x,y,0,\delta) - \sigma(\delta)\big]\mathrm{d}t + O(c_1).$$

If (3.1) is of class C^∞, then as in the case of homoclinic loop we have from [18]

$$M_1(h,\delta) = \sum_{j\ge 0}(c_{2j} + c_{2j+1}h\ln|h|)h^j.$$

The authors [20] posed the following conjecture:

Conjecture 4.1. *If*

$$c_j(\delta_0) = 0, \quad j = 0,\cdots,l-1, \quad c_l(\delta_0) \neq 0,$$

for some $\delta_0 \in D$ and $l \ge 3$, then (3.1) has at most l large limit cycles near L for $|\varepsilon| + |\delta - \delta_0|$ small.

Remark 4.1. If system (3.1) is centrally symmetric with respect to the origin, then $M_{11}(h,\delta) = M_{12}(h,\delta)$, and the phase portrait of (3.1) near L is symmetric with respect to the origin. Hence, in this case, no large homoclinic loop can appear near L since $d(\varepsilon,\delta,A_1) = 0$ if and only if $d_i(\varepsilon,\delta,A_i) = 0$ for $i = 1,2$. Also, the above conjecture is true for centrally symmetric systems by [13].

Now we present a simple application of the above theorems together with Remark 4.1 to investigate limit cycle bifurcations of a cubic system of the form

$$\dot{x} = y, \quad \dot{y} = x - x^3 - \varepsilon(x^2 + a_1x + a_0)y. \tag{4.2}$$

Without loss of generality, we assume the small parameter ε is nonnegative. It is easy to see that when $\varepsilon = 0$, system (4.2) possesses the Hamiltonian function

$$H(x,y) = -\frac{1}{2}(x^2 - y^2) + \frac{1}{4}x^4.$$

From (4.1), we have

$$M_1 = -2\big(I_3 - a_1I_2 + a_0I_1\big), \quad M_2 = -2\big(I_3 + a_1I_2 + a_0I_1\big),$$

where

$$I_j = \int_0^{\sqrt{2}} x^j\sqrt{1 - \frac{1}{2}x^2}\,\mathrm{d}x, \quad j = 1,2,3.$$

Direct computation yields $I_1 = \frac{2}{3}$, $I_2 = \frac{1}{4}\sqrt{2}\,\pi$, $I_3 = \frac{32}{15}$. Consequently,

$$M_i = -\frac{64}{15} + (-1)^{i-1}\frac{\pi}{\sqrt{2}}a_1 - \frac{4}{3}a_0, \quad i = 1,2,$$

and

$$M_1 + M_2 = -\frac{8}{15}\big(16 + 5a_0\big).$$

It is also easy to see that $\sigma = -a_0$. Therefore, by Theorem 4.3, system (4.2) has at most one large cycle near the double homoclinic loop $L : H = 0$, and there exists a homoclinic bifurcation curve in the (a_0, a_1) plane:

$$a_0 = a_0^*(\varepsilon, a_1) = -\frac{16}{5} + O(\varepsilon),$$

such that the corresponding homoclinic loop tends to L as $\varepsilon \to 0$. We remark that system (4.2) may have two large cycles, of which the bigger one is not near L. When $a_0 = a_0^*$, we have

$$M_i = (-1)^{i-1}\frac{\pi}{\sqrt{2}}a_1 + O(\varepsilon), \quad i = 1, 2.$$

Noting that system (4.2) is centrally symmetric when $a_1 = 0$, we have $d_1 = d_2 = 0$ for $a_1 = 0$ and $a_0 = a_0^*$. Therefore, for $a_0 = a_0^*$,

$$(-1)^{i-1}\varepsilon a_1 d_i \geq 0, \quad i = 1, 2.$$

Then for $\varepsilon > 0$, by Figure 3 one can see that there are three cases according to the value of a_1 as follows:

(i) Upper homoclinic bifurcation curve L_u^H: $a_0 = a_0^*(\varepsilon, a_1)$, $a_1 > 0$.

(ii) Lower homoclinic bifurcation curve L_d^H: $a_0 = a_0^*(\varepsilon, a_1)$, $a_1 < 0$.

(iii) Double homoclinic bifurcation point $a_0 = a_0^*(\varepsilon, 0)$, $a_1 = 0$.

In addition, by Theorems 3.3 and 3.4, system (4.2) also possesses a left homoclinic bifurcation curve

$$L_l^H : a_0 = \frac{3\pi}{4\sqrt{2}}a_1 - \frac{16}{5} + O(\varepsilon),$$

and a right homoclinic bifurcation curve

$$L_r^H : a_0 = -\frac{3\pi}{4\sqrt{2}}a_1 - \frac{16}{5} + O(\varepsilon).$$

Further, by Theorem 3.4 there exist left and right saddle-node bifurcation curves of small cycles L_l^{2M} and L_r^{2M}, whose endpoints are the intersection points of L_l^H, L_r^H with the straight line $a_0 = 0$. Note that the double homoclinic loop of system (4.2) is unstable for $(a_0, a_1) = \big(a_0^*(\varepsilon, 0), 0\big) \equiv A^*$. Taking a sufficiently small neighbourhood V of A^*, one can obtain the bifurcation diagram for $(a_0, a_1) \in V$ as shown in Figure 7. We remind that the saddle-node bifurcation curves L_l^{2M} and L_r^{2M} do not intersect V.

Jebrane and Zoladek [38] studied the Poincaré bifurcations of system (4.2) and proved that the system admits at most three limit cycles. If three limit cycles exist, they are either two large cycles enclosing one small cycle, or one large cycle enclosing two small cycles. The above discussion can be regarded as a supplement to [38].

In 2023, the authors [31] considered analytic system (3.1) and obtained an important relation of the three functions M_{11}, M_{12} and M_1 as stated in the theorem below. Here, the statement is improved.

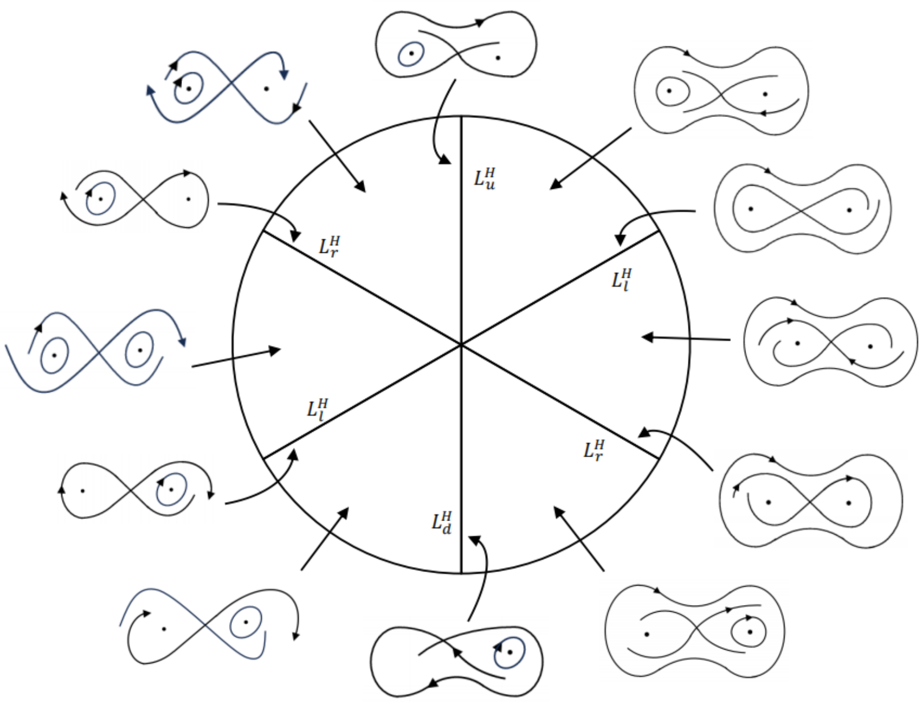


Figure 7. Bifurcation diagram of system (4.2) near A^*.

Theorem 4.5. ([31]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$, $H \in C^{k+1}$ *and* $k \geq 2$. *If the functions* H, $f|_{\varepsilon=0}$ *and* $g|_{\varepsilon=0}$ *are analytic in a neighborhood of the double homoclinic loop* L, *then there exist analytic functions* $\psi_1(h,\delta)$, $\psi_2(h,\delta)$ *and* $\varphi(h,\delta)$ *for* $|h| < \varepsilon_0$ *with* ε_0 *a positive constant such that*

$$M_{1i}(h,\delta) = \varphi(h,\delta)h\ln|h| + \psi_i(h,\delta),\ \ h \in (-\varepsilon_0, 0), \quad i = 1,2,$$

and

$$M_1(h,\delta) = 2\varphi(h,\delta)h\ln h + \psi(h,\delta), \quad h \in (0,\varepsilon_0),$$

where $\psi(h,\delta) = \psi_1(h,\delta) + \psi_2(h,\delta)$.

Further, let system (3.1) *be centrally symmetric, and*

$$\varphi(h,\delta) = \sum_{i\geq 0} c_{2i+1}(\delta)h^i, \quad \psi(h,\delta) = \sum_{i\geq 0} c_{2i}(\delta)h^i.$$

If there exist $\delta_0 \in D$ *and* $k \geq 1$ *such that*

$$c_k(\delta_0) \neq 0, \quad c_j(\delta_0) = 0, \quad j = 0, \cdots, k-1$$

and

$$\operatorname{rank} \frac{\partial(c_0, c_1, \cdots, c_{k-1})}{\partial(\delta_1, \cdots, \delta_m)}(\delta_0) = k,$$

then for any given neighborhood U *of* L *there exists* δ *near* δ_0 *such that for* $0 < \varepsilon \ll 1$ *system* (3.1) *has at least* $\left[\frac{5k}{2}\right]$ *limit cycles in* U.

This result can be used to study the number of limit cycles of polynomial systems. For example, the system

$$\dot{x} = y(y^2 - 1) + \varepsilon \sum_{i=0}^{n} a_i x^{2i+1}, \quad \dot{y} = -x,$$

can have $\left[\frac{5n}{2}\right]$ limit cycles near the curve defined by $2(x^2 - y^2) + y^4 = 0$ for $1 \le n \le 56$ and $0 < \varepsilon \ll 1$. See [31] for details.

5. Heteroclinic bifurcation

In this section, we continue to investigate the C^k system (3.1) with $k \ge 2$. Suppose that system $(3.1)|_{\varepsilon=0}$ has a heteroclinic loop $L_0 = L_{10} \cup L_{20} \cup S_{10} \cup S_{20}$ consisting of two hyperbolic saddles S_{10} and S_{20} and two heteroclinic orbits L_{10} and L_{20} connecting them. For definiteness, we assume that L_0 is oriented clockwise and that the Poincaré map is well defined inside it with $\omega(L_{20}) = \alpha(L_{10}) = S_{10}$ and $\omega(L_{10}) = \alpha(L_{20}) = S_{20}$ as shown in Figure 8.

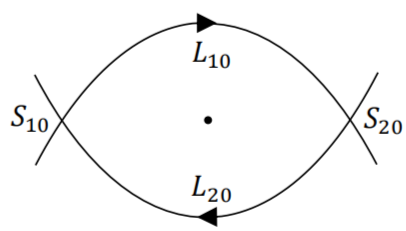


Figure 8. Clockwise-oriented heteroclinic loop.

The main task is, as before, to discuss the following issues

(1) the stability of L_0,

(2) the existence of a homoclinic or heteroclinic loop L_ε satisfying $\lim\limits_{\varepsilon \to 0} L_\varepsilon = L_0$, and

(3) the number of limit cycles near it for ε small.

We first introduce some notations. For ε small (3.1) has hyperbolic saddles $S_{1\varepsilon}$ and $S_{2\varepsilon}$ near S_{10} and S_{20} respectively. Take points $A_j \in L_{j0}$, $j = 1, 2$, and make cross sections l_j passing through A_j and having the direction n_j with

$$n_j = \frac{1}{|(f_0(A_j), g_0(A_j))|}(-g_0(A_j), f_0(A_j)).$$

Then for small $\varepsilon \ne 0$, system (3.1) has four separatrices $L_j^{u,s}$ which intersect l_j at $A_j^{u,s}$ and satisfy

$$\alpha(A_1^u) = \omega(A_2^s) = S_{1\varepsilon}, \quad \alpha(A_2^u) = \omega(A_1^s) = S_{2\varepsilon}.$$

Let $d_j(\varepsilon, \delta, A_j)$ denote, as before, the distance from A_j^u to A_j^s. Then similar to Lemma 3.1 we have

Lemma 5.1. *For C^k system (3.1) the functions $d_1(\varepsilon, \delta, A_1)$ and $d_2(\varepsilon, \delta, A_2)$ are C^k in (ε, δ) satisfy*

$$d_i(\varepsilon, \delta, A_i) = \frac{\varepsilon M_{i0}(\delta)}{\big|\, f_0(A_i),\, g_0(A_i) \,\big|} + O(\varepsilon^2), \quad i = 1, 2,$$

where

$$M_{i0}(\delta) = \int_{L_{i0}} (f_0 g - g_0 f)\big|_{\varepsilon=0} \exp\left(-\int_0^t (f_{0x} + g_{0y}) \mathrm{d}\tau\right) \mathrm{d}t.$$

Obviously, system (3.1) possesses a heteroclinic orbit near L_{i0} if and only if $d_i = 0$. The sign of d_i determines the relative position of the separatrices L_i^s and L_i^u, similar to Figure 3.

We now introduce three fundamental results concerning the stability of the unperturbed heteroclinic loop L_0 and the limit cycle bifurcation under small perturbations.

Let $r_{i0} = |\lambda_{i2}^0|/\lambda_{i1}^0$ denote the hyperbolic ratio at saddle S_{i0}, where $\lambda_{i1}^0 > 0$, $\lambda_{i2}^0 < 0$ are eigenvalues of the linearized system of $(3.1)|_{\varepsilon=0}$ at S_{i0}, $i = 1, 2$.

Theorem 5.1. *Suppose that system* (3.1) *is of class* C^2. *Then*

(i) If $r_{10}r_{20} > 1$ (< 1), *then* L_0 *is internally stable (unstable).*

(ii) If $r_{10}r_{20} = 1$, *then*

$$\sigma_1 = \lim_{a\to 0} \int_{\widehat{AB}} (f_{0x} + g_{0y})\, \mathrm{d}t$$

exists and is finite, where $\widehat{AB}$ *denotes the orbital arc of* $(3.1)|_{\varepsilon=0}$ *starting from* $A \in l_1$ *and arriving at* $B \in l_1$ *for the first time. Moreover,* L_0 *is stable (unstable) when* $\sigma_1 < 0$ (> 0). *In particular, if* $r_{10} = r_{20} = 1$, *we have*

$$\sigma_1 = \sum_{i=1}^{2} \int_{L_i} (f_{0x} + g_{0y})\, \mathrm{d}t \equiv \sigma_{11} + \sigma_{12}.$$

The first conclusion of the above theorem was obtained by [4], and the second one by [27]. A proof of this theorem can be found in [17]. The following theorem was obtained by [23], and its proof was given in [17].

Theorem 5.2. ([23]) *Let system* (3.1) *be a* C^3 *system, and suppose that*

$$r_{10} = r_{20} = 1, \quad \sigma_1 = 0, \quad \sigma_2 \equiv R_{11} + \frac{\lambda_{11}^0}{\lambda_{21}^0} R_{12} e^{\sigma_{11}} \neq 0,$$

where R_{11} *and* R_{12} *denote the first order saddle quantities of system* $(3.1)|_\varepsilon = 0$ *at* S_{10} *and* S_{20} *respectively.*

(i) If L_0 *is oriented clockwise, then* L_0 *is internally stable (unstable) when* $\sigma_2 > 0$ (< 0).

(ii) If L_0 *is oriented counterclockwise, then* L_0 *is internally unstable (stable) when* $\sigma_2 > 0$ (< 0).

Theorem 5.3. ([17]) *Assume that system* (3.1) *is of class at least* C^3.

(i) If $(r_{10} - 1)(r_{20} - 1) > 0$, *then there exist* $\varepsilon_0 > 0$ *and a neighbourhood* U *of* L_0 *such that for all* $0 < |\varepsilon| < \varepsilon_0$ *and* $\delta \in D$, *system* (3.1) *has at most one cycle in* U.

(ii) If $(r_{10} - 1)(r_{20} - 1) \leq 0$ *and* $r_{10}r_{20} \neq 1$, *then for all* $0 < |\varepsilon| < \varepsilon_0$ *and* $\delta \in D$, *system* (3.1) *has at most two cycles in* U.

The first (second, resp.) conclusion of Theorem 5.3 was first obtained by [48] ([46], resp.) for C^∞ systems.

Next, we present some results on bifurcation phenomenon near L_0 for near-Hamiltonian systems. Let system (3.1) satisfy (3.4) with $f, g \in C^k$, $H \in C^{k+1}$ and $k \geq 3$.

We first show how to get a homoclinic loop L_ε passing through the saddle point $S_{1\varepsilon}$ near S_{10} or $S_{2\varepsilon}$ near S_{20}. For definiteness, let us study the existence condition of L_ε passing through $S_{1\varepsilon}$.

Take $A_j \in l_j$, both close to S_{20}. Let $\tilde{A}_2 = P_1(A_1^u) \in l_2$, where $P_1 : l_1 \to l_2$ is a Dulac map of system (3.1) near S_{20}. Then we can write

$$A_2^{u,s} = A_2 + a^{u,s}(\varepsilon, \delta)n_2, \quad \tilde{A}_2 = A_2 + \tilde{a}(\varepsilon, \delta)n_2.$$

Then for $A = A_2 + an_2 \in l_2$ with $a < a^s(\varepsilon, \delta)$, the positive orbit of system (3.1) starting at A intersects l_2 at $B = A_2 + P(a, \varepsilon, \delta)n_2$ for the first time. This yields a map $P : l_2 \to l_2$, called a Poincaré map of system (3.1) near L_0. See Figure 9.

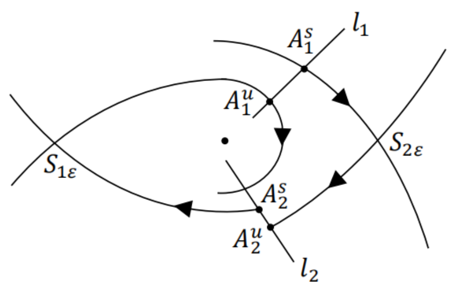


Figure 9. The case with $d_1 > 0$.

As before, the limit $P(a^s, \varepsilon, \delta) = \lim_{a \to a^s} P(a, \varepsilon, \delta)$ exists. Let

$$d(\varepsilon, \delta, A_2) = P(a^s, \varepsilon, \delta) - a^s.$$

Then for $\varepsilon \neq 0$ with $d_1 > 0$, there exists a homoclinic loop passing through $S_{1\varepsilon}$ if and only if $d(\varepsilon, \delta, A_2) = 0$.

By [18, 27] we have

$$d(\varepsilon, \delta, A_2) = \varepsilon \frac{M_0(\delta) + O(\varepsilon \ln |\varepsilon|)}{|(H_x(A_2), H_y(A_2))|},$$

where

$$\begin{aligned} &M_0(\delta) = M_{10}(\delta) + M_{20}(\delta), \\ &M_{j0}(\delta) = \int_{L_{j0}} \big(g\mathrm{d}x - f\mathrm{d}y\big)\big|_{\varepsilon=0}, \quad j = 1, 2. \end{aligned} \tag{5.1}$$

Introduce

$$\begin{aligned} &\Delta_j(\varepsilon, \delta) = \varepsilon(f_x + g_y)(S_{j\varepsilon}, \varepsilon, \delta), \quad j = 1, 2, \\ &\bar{\sigma}_j(\delta) = (f_x + g_y)(S_{j0}, 0, \delta), \quad j = 1, 2, \\ &\bar{\sigma}_3(\delta) = \sum_{j=1}^{2} \int_{L_{j0}} (f_x + g_y)\big|_{\bar{\sigma}_1 = \bar{\sigma}_2 = 0} \,\mathrm{d}t, \\ &\bar{\sigma}_4(\delta) = \frac{\partial R_1(S_{1\varepsilon})}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \end{aligned}$$

where $R_1(S_{1\varepsilon})$ denotes the first order saddle quantity of system (3.1) at $S_{1\varepsilon}$. Clearly, $\Delta_j(\varepsilon,\delta) = \varepsilon(\bar{\sigma}_j(\delta)+O(\varepsilon))$.

The authors [27] obtained the following result.

Theorem 5.4. ([27]) *Let system (3.1) satisfy (3.4) with $f,g \in C^k$, $H \in C^{k+1}$ and $k \geq 3$. Then*

(1) If $M_0(\delta) \neq 0$ for all $\delta \in D$, then (3.1) has neither limit cycle, homoclinic loop nor heteroclinic loop near L_0 for all $\varepsilon \neq 0$ small, where M_0 is defined in (5.1).

(2) If

$$M_0(\delta_0)=0, \quad \frac{\partial M_0}{\partial \delta_1}(\delta_0) \neq 0, \quad \delta_0=(\delta_{10},\cdots,\delta_{m0}) \in D,$$

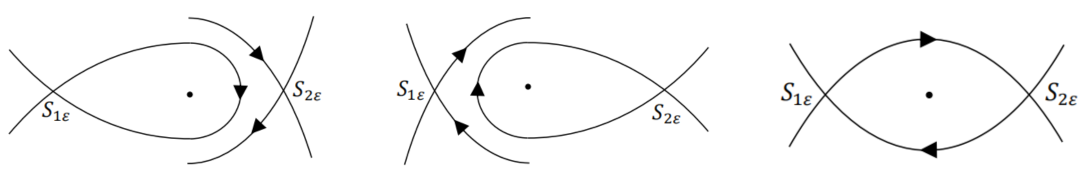
then there exists a differentiable function $\delta^*(\varepsilon,\delta_2,\cdots,\delta_m) = \varphi(\delta_2,\cdots,\delta_m)+O(\varepsilon)$, where $M_0(\varphi(\delta_2,\cdots,\delta_m),\delta_2,\cdots,\delta_m)=0$, such that (3.1) has a homoclinic or heteroclinic loop L_ε near L_0 for $|\varepsilon|+|\delta-\delta_0|$ small if and only if $\delta=\delta^*(\varepsilon,\delta_2,\cdots,\delta_m)$. The loop is a homoclinic loop through S_1 (S_2, resp.) if $d_1>0$ (<0, resp.), and it is a heteroclinic loop if $d_1=0$.

(3) If

$$M_0(\delta_0)=0, \quad \bar{\sigma}_1(\delta_0)+\bar{\sigma}_2(\delta_0) \neq 0, \quad \delta_0 \in D,$$

and $\Delta_1(\varepsilon,\delta)\Delta_2(\varepsilon,\delta) \geq 0$ for all $|\varepsilon|+|\delta-\delta_0|$ small, then (3.1) has at most one limit cycle near L_0 for all $|\varepsilon|+|\delta-\delta_0|$ small. If further $\delta \in \mathbb{R}$ and $M_0'(\delta_0) \neq 0$, then there exists a differentiable function $\delta^*(\varepsilon)=\delta_0+O(\varepsilon)$ such that (3.1) has a unique limit cycle near L_0 if and only if $\mu(\delta-\delta^*(\varepsilon))>0$ for $|\varepsilon|+|\delta-\delta_0|$ small, and that system (3.1) has a homoclinic or heteroclinic loop L_ε near L_0 if and only if $\delta=\delta^*(\varepsilon)$, where $\mu=M_0'(\delta_0)(\bar{\sigma}_1(\delta_0)+\bar{\sigma}_2(\delta_0))$.

The phase portrait for the loop L_ε, when it exists, is illustrated in Figure 10.



(a) $d_1>0$ (b) $d_1<0$ (c) $d_1=0$

Figure 10. Phase portrait of the loop L_ε ($d=0$).

From [18, 59], we have the following theorem.

Theorem 5.5. ([18, 59]) *Let system (3.1) satisfy (3.4) with $f,g \in C^k$, $H \in C^{k+1}$ and $k \geq 3$. Suppose there exist $\delta_0 \in \mathbb{R}^m$ and an integer $l=1,2,3$ or 4 such that*

$$M_0(\delta_0)=0, \quad M_{10}(\delta_0) \neq 0, \quad \bar{\sigma}_j(\delta_0)=0, \quad j=1,\cdots,l-1,$$
$$\bar{\sigma}_l(\delta_0) \neq 0, \quad \operatorname{rank}\frac{\partial(M_0,\bar{\sigma}_1,\cdots,\bar{\sigma}_{l-1})}{\partial(\delta_1,\cdots,\delta_m)}(\delta_0)=l.$$

Then system (3.1) has at least l limit cycles near L_0 for some (ε,δ) near $(0,\delta_0)$.

We note that under (3.4) system $(3.1)|_{\varepsilon=0}$ has a family of periodic orbits inside L_0 L_h : $H(x,y)=h$, $h \in J$. We can suppose that $H(L_0)=0$ with $J=(\alpha,0)$. In this case, similar to Theorem 3.5, one can prove that the first order Melnikov function of (3.1)

$$M_1(h,\delta)=\oint_{L_h}(g\mathrm{d}x-f\mathrm{d}y)\big|_{\varepsilon=0}, \quad h \in J$$

has an expansion of the form (3.5)

$$M_1(h,\delta) = M_0(\delta) + c_1(\delta)\, h \ln|h| + c_2(\delta)\, h + c_3(\delta)\, h^2 \ln|h| + o(h^2 \ln|h|).$$

For formulas of c_1, c_2 and c_3, see [18, 32]. The authors [35] proved that if

(i) $M_0(\delta_0) = 0,\ c_1(\delta_0) \neq 0,\ \delta_0 \in D$,

(ii) system (3.1) is of class C^∞,

then (3.1) has at most two limit cycles near L_0 for all $|\varepsilon| + |\delta - \delta_0|$ small. Moreover, two limit cycles can appear near L_0 under additional conditions.

Similar to the proof of Theorem 5.3 it is possible to prove that the above conclusion is true if the condition (ii) is changed into

(ii)′ system (3.1) is of class C^3.

Next we suppose system (3.1) satisfies (3.4) and

$$\begin{aligned} &H(-x,-y) = H(x,y),\\ &f(-x,-y,\varepsilon,\delta) = -f(x,y,\varepsilon,\delta),\\ &g(-x,-y,\varepsilon,\delta) = -g(x,y,\varepsilon,\delta). \end{aligned} \tag{5.2}$$

That is, it is centrally symmetric. Then by Theorem 5.4, under conditions (i) and (ii)′ (3.1) has at most one limit cycle near L_0 for all $|\varepsilon| + |\delta - \delta_0|$ small.

Similar to Theorem 3.4, by [13], one can prove that under (5.2) if

$$M_0(\delta_0) = c_1(\delta_0) = 0, \quad c_2(\delta_0) \neq 0, \quad \delta_0 \in D,$$

then C^3 system (3.1) has at most two limit cycles near L_0 for all $|\varepsilon| + |\delta - \delta_0|$ small. Moreover, two limit cycles can appear near L_0 under certain additional conditions.

Now we suppose that (5.2) holds and that $(3.1)|_{\varepsilon=0}$ has a double heteroclinic loop $L_0 = L_{10} \cup L_{20} \cup L'_{10} \cup L'_{20}$ passing through three hyperbolic saddles O (the origin), S_{10} and S'_{10}, where L_{10}, L_{20} and S_{10} lie in the region $x \geq 0$, and L'_{i0} and L_{i0} are centrally symmetric. For definiteness, let L_0 be oriented clockwise, see Figure 11.

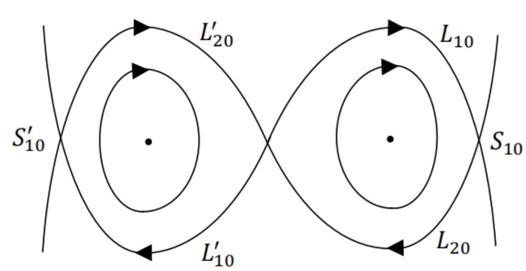


Figure 11. A double heteroclinic loop with clockwise orientation.

We can suppose that L_0 is defined by the equation $H(x,y) = 0$. Introduce

$$\begin{aligned} \mu_1(\delta) &= (f_x + g_y)(O, 0, \delta),\\ \mu_2(\delta) &= (f_x + g_y)(S_{10}, 0, \delta),\\ \mu_3(\delta) &= \sum_{j=1}^{2} \int_{L_{j0}} (f_x + g_y)|_{\mu_1=\mu_2=0}\, \mathrm{d}t,\\ \mu_4(\delta) &= \left.\frac{\partial R_1}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad R_1 = R_1(O). \end{aligned}$$

Then we have the following theorem.

Theorem 5.6. ([18, 59]) *Let system* (3.1) *satisfy* (3.4) *with* $f, g \in C^k$, $H \in C^{k+1}$ *and* $k \geq 3$. *Suppose* (5.2) *holds, and* $(3.4)|_{\varepsilon=0}$ *has a double heteroclinic loop* L_0 *as shown in Figure 11. Let there exist* $\delta_0 \in \mathbb{R}^m$ *and integer* $l = 1, 2, 3$ *or* 4 *such that*

$$M_0(\delta_0) = 0, \quad M_{10}(\delta_0) \neq 0, \quad \mu_j(\delta_0) = 0, \quad j = 1, \cdots, l-1,$$
$$\mu_l(\delta_0) \neq 0, \quad \operatorname{rank} \frac{\partial(M_0, \mu_1, \cdots, \mu_{l-1})}{\partial(\delta_1, \cdots, \delta_m)}(\delta_0) = l.$$

Then system (3.1) *can have* $\left[\frac{8l}{3}\right]$ *limit cycles near* L_0 *for some* (ε, δ) *near* $(0, \delta_0)$.

We mention that under the conditions of Theorem 5.5 or 5.6 at least one alien limit cycle appears, which can not be obtained by any zero of the first order Melnikov functions.

6. Bifurcations of other types

In this section we introduce two types of limit cycle bifurcations with a nilpotent singular point considered in [24, 34], giving two theorems whose proof can also be found in [18].

Consider the following C^r system

$$\begin{aligned} \dot{x} &= H_y(x, y) + \varepsilon f(x, y, \varepsilon, \delta), \\ \dot{y} &= -H_x(x, y) + \varepsilon g(x, y, \varepsilon, \delta), \end{aligned} \tag{6.1}$$

where $H \in C^\infty$, $f, g \in C^r$ with $2 \leq r \leq \infty$, $\varepsilon \geq 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. Suppose that unperturbed system $(6.1)|_{\varepsilon=0}$ has a nilpotent singular point at the origin. That is, the function $H(x, y)$ satisfies $H_x(0,0) = H_y(0,0) = 0$ and

$$\frac{\partial\left(H_y, -H_x\right)}{\partial(x, y)}(0,0) \neq 0, \quad \det \frac{\partial\left(H_y, -H_x\right)}{\partial(x, y)}(0,0) = 0.$$

Then, without loss of generality, we may assume

$$H_{yy}(0,0) = 1, \quad H_{xy}(0,0) = H_{xx}(0,0) = 0.$$

In this case, the expansion of H at the origin has the form

$$H(x, y) = \frac{1}{2} y^2 + \sum_{i+j \geq 3} h_{ij} x^i y^j. \tag{6.2}$$

By the implicit function theorem, there exists a unique C^∞ function

$$E(x) = \sum_{j \geqslant 2} e_j x^j$$

such that $H_y(x, E(x)) = 0$ for $0 < |x| \ll 1$. Then one has

$$H(x, E(x)) = \sum_{j \geqslant 3} h_j x^j, \quad 0 < |x| \ll 1. \tag{6.3}$$

On the classification of nilpotent singular points, we have the following definition (see [18]).

Definition 6.1. Suppose that the system $(6.1)|_{\varepsilon=0}$ has a nilpotent singular point at the origin and satisfies (6.2) and (6.3). Let $k \geq 3$ be an integer such that

$$h_j = 0 \quad \text{for} \quad j < k, \quad h_k \neq 0. \tag{6.4}$$

The origin is called a nilpotent cusp of order $m-1$ if $k = 2m-1$; the origin is called a nilpotent center of order $m-1$ (a nilpotent saddle of order $m-1$, respectively) if $k = 2m$ and $h_k > 0$ (if $k = 2m$ and $h_k < 0$, respectively).

Han et al. [24] investigated properties of the first order Melnikov function

$$M_1(h,\delta) = \oint_{H(x,y)=h} (g\mathrm{d}x - f\mathrm{d}y)\big|_{\varepsilon=0}$$

of (6.1), and studied limit cycle bifurcations near a nilpotent center by using the expansion of the function at $h = 0$ as follows.

Theorem 6.1. ([24]) *Consider the C^r system (6.1) with $r \geq 2$. Let (6.2) hold and the Hamiltonian system $(6.1)|_{\varepsilon=0}$ have a nilpotent center at the origin with $h_k > 0$ and $k = 2m$. Suppose that the functions $f(x,y,0,\delta)$ and $g(x,y,0,\delta)$ are both C^∞. Then*

(i) The function $M_1(h,\delta)$ is C^∞ for $h > 0$ small and has an expansion of the form

$$M_1(h,\delta) = h^{\frac{1+m}{2m}} \sum_{j\geq 0} b_j(\delta) h^{\frac{j}{m}}, \quad 0 < h \ll 1. \tag{6.5}$$

(ii) If there exist $l \geq 1$ and $\delta_0 \in D$ such that the coefficients $\{b_j\}$ in (6.5) satisfy

$$b_l(\delta_0) \neq 0, \quad b_j(\delta_0) = 0, \quad 0 \leq j \leq l-1$$

and

$$\operatorname{rank} \frac{\partial(b_0, \cdots, b_{l-1})}{\partial \delta}(\delta_0) = l,$$

then system (6.1) has at least l limit cycles near the origin for some (ε,δ) near $(0,\delta_0)$.

We remark that the conditions in Theorem 6.1 are a little weaker than those in the paper [24] since the paper there required $f, g \in C^\infty$. Recently, the authors [21] studied the number of limit cycles near a degenerate center for a class of planar near-Hamiltonian systems.

We now suppose that the system $(6.1)|_{\varepsilon=0}$ satisfies (6.2) and (6.4) with $k = 2m+1$, $m \geq 1$ so that the origin is a cusp of order m. Assume that $(6.1)|_{\varepsilon=0}$ has a cuspidal loop L_0 satisfying $H(x,y) = 0$. Then the equation $H(x,y) = h$ defines two families of periodic orbits L_h^+ and L_h^- for $h \in (0,h^+)$ and $h \in (h^-,0)$, respectively, where $h^+ > 0$, $h^- < 0$ are constants. Clearly,

$$\lim_{h\to 0-} L_h^- = \lim_{h\to 0+} L_h^+ = L_0.$$

Then correspondingly, the C^r system (6.1) has two Melnikov functions as given below:

$$M^+(h,\delta) = \oint_{L_h^+} (g\mathrm{d}x - f\mathrm{d}y)\big|_{\varepsilon=0}, \quad h \in (0,h^+),$$

and

$$M^-(h,\delta) = \oint_{L_h^-} (g\mathrm{d}x - f\mathrm{d}y)\big|_{\varepsilon=0}, \quad h \in (h^-,0).$$

The authors [34] proved the following theorem.

Theorem 6.2. ([34]) *Under the above assumptions if the functions $f(x, y, 0, \delta)$ and $g(x, y, 0, \delta)$ are both C^∞, then the first order Melnikov functions $M^\pm$ have expansions of the forms*

$$M^-(h, \delta) = |h|^{\frac{1}{2}+\frac{1}{k}} \sum_{l \geq 0} B_l(\delta)|h|^{\frac{l}{k}} + \varphi(h, \delta), \quad 0 \leq -h \ll 1,$$

and

$$M^+(h, \delta) = h^{\frac{1}{2}+\frac{1}{k}} \sum_{l \geq 0} B_l^*(\delta)h^{\frac{l}{k}} + \psi(h, \delta), \quad 0 \leq h \ll 1,$$

respectively, where φ and ψ are C^∞ functions in h for $|h|$ small.

The authors [57] further found useful relations between the coefficients $B_l(\delta)$ and $B_l^*(\delta)$, and proved $\varphi = \psi$, improving the above theorem. These facts can be used to obtain limit cycles of (6.1) near L_0 for ε small.

For more results and some recent advances on limit cycle bifurcations of planar systems, readers are advised to consult [1, 10, 18, 19, 21, 40, 52, 55, 58].

References

[1] M. Cai and M. Han, *Finding limit cycles near a double heteroclinic loop on the cylinder*, Qual. Theory Dyn. Syst., 2026, 25, 137.

[2] A. Chen, L. Guo and X. Deng, *Existence of solitary waves and periodic waves for a perturbed generalized BBM equation*, J. Differ. Equ., 2016, 261, 5324–5349.

[3] X. Chen and M. Han, *Further study on Horozov-Ilievs method of estimating the number of limit cycles*, Sci. China Math., 2022, 65, 2255–2270.

[4] L. A. Cherkas, *On the stability of singular cycles*, Differ. Equ., 1968, 4, 1012–1017.

[5] S. N. Chow, C. Li and D. Wang, *Normal Form and Bifurcations of Planar Vector Fields*, Cambridge University Press, Cambridge, 1994.

[6] J. Christopher, *Geometric Singular Perturbation Theory*, Lecture Notes Math., vol. 1609, Springer-Verlag, 1995.

[7] C. Christopher, C. Li and J. Torregrosa, *Limit Cycles of Differential Equations*, Birkhäuser-Verlag, 2024.

[8] Z. Du and J. Li, *Geometric singular perturbation analysis to Camassa-Holm-Kuramoto-Sivashinsky equation*, J. Differ. Equ., 2022, 306, 418–438.

[9] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Differ. Equ., 1979, 31, 53–98.

[10] W. Geng, M. Han, Y. Tian and A. Ke, *Heteroclinic bifurcation of limit cycles in perturbed cubic Hamiltonian systems by higher-order analysis*, J. Differ. Equ., 2023, 357, 412–435.

[11] M. Grau, F. Mansas and J. Villadelprat, *A Chebyshev criterion for Abelian integrals*, Trans. Am. Math. Soc., 2011, 363, 109–129.

[12] J. Guckenheimer and P. Holmes, *Nonlinear Oscillation Dynamical Systems and Bifurcation of Vector Fields*, Springer-Verlag, 1983.

[13] M. Han, *On the number of limit cycles bifurcating from a homoclinic or heteroclinic loop* (In Chinese), Sci. China Ser. A, 1993, 36(2), 113–132.

[14] M. Han, *Bifurcations of invariant tori and subharmonic solutions for periodic perturbed systems*, Sci. China Ser. A, 1994, 37(11), 1325–1336.

[15] M. Han, *Cyclicity of planar homoclinic loops and quadratic integrable systems*, Sci. China (Series A), 1997, 40(12), 1247–1258.

[16] M. Han, *On Hopf cyclicity of planar systems*, J. Math. Anal. Appl., 2000, 245, 404–422.

[17] M. Han, *Periodic Solutions and Bifurcation Theory of Dynamical Systems* (In Chinese), Science Press, Beijing, 2002.

[18] M. Han, *Bifurcation Theory of Limit Cycles*, Science Press, Beijing, 2013.

[19] M. Han, *On the uniqueness of limit cycles in codimension two bifurcations*, J. Nonlinear Model. Anal., 2024, 6, 514–525.

[20] M. Han and J. Chen, *On the number of limit cycles in double homoclinic bifurcations*, Sci. China Ser. A, 2000, 43(9), 401–414.

[21] M. Han and J. Chen, *Bifurcation of limit cycles of a class of near-Hamiltonian systems near a degenerate center*, Chaos, Solitons and Fractals, 2026, 208, 118367.

[22] M. Han, W. Hou and W. Liu, *Global dynamics of a polynomial Liénard system*, J. Appl. Anal. Comput., 2025, 16(2), 937–965.

[23] M. Han, S. Hu and X. Liu, *On the stability of double homoclinic and heteroclinic cycles*, Nonlinear Anal., 2003, 53, 701–713.

[24] M. Han, J. Jiang and H. Zhu, *Limit cycle bifurcations in near-Hamiltonian systems by perturbing a nilpotent center*, Int. J. Bifurcation Chaos, 2008, 18(10), 3013–3027.

[25] M. Han and A. Ke, *On symmetry property of center manifolds of differential systems*, J. Appl. Anal. Comput., 2025, 15(1), 1–8.

[26] M. Han, D. Luo and D. Zhu, *The uniqueness of limit cycles bifurcated from a separatrix cycle (II)* (in Chinese), Acta Math. Sin., 1992, 35(4), 523–530.

[27] M. Han, D. Luo and D. Zhu, *The uniqueness of limit cycles bifurcated from a separatrix cycle (III)* (in Chinese), Acta Math. Sin., 1992, 35(5), 673–684.

[28] M. Han, L. Sheng and X. Zhang, *Bifurcation theory for finitely smooth planar autonomous differential systems*, J. Differ. Equ., 2018, 264, 3596–3618.

[29] M. Han and J. Yang, *The maximum number of zeros of functions with parameters and application to differential equations*, J. Nonlinear Model. Anal., 2021, 3, 13–34.

[30] M. Han and J. Yang, *Foundations of Qualitative Theory of Ordinary Differential Equations* (In Chinese), Science Press, Beijing, 2023.

[31] M. Han, J. Yang and J. Li, *General study on limit cycle bifurcation near a double homoclinic loop*, J. Differ. Equ., 2023, 347, 1–23.

[32] M. Han, J. Yang, A. A. Tarta and Y. Gao, *Limit cycles near homoclinic and hetero-clinic loops*, J. Dyn. Differ. Equ., 2008, 20(4), 923–944.

[33] M. Han and Y. Ye, *On the coefficients appearing in the expansion of Melnikov function in homoclinic bifurcations*, Ann. Differ. Equ., 1998, 14, 156–162.

[34] M. Han, H. Zang and J. Yang, *Limit cycle bifurcations by perturbing a cuspidal loop in a Hamiltonian system*, J. Differ. Equ., 2009, 246(1), 129–163.

[35] M. Han and Z. Zhang, *Cyclicity* 1 *and* 2 *conditions for a* 2*-polycycle of integrable systems on the plane*, J. Differ. Equ., 1999, 155, 245–261.

[36] M. Han and D. Zhu, *Bifurcation Theory of Differential Equations* (In Chinese), Coal Industry Publishing House, Beijing, 1994.

[37] M. Han and H. Zhu, *The loop quantities and bifurcations of homoclinic loops*, J. Differ. Equ., 2007, 234, 339–359.

[38] J. M. Jebrane and H. Zoladek, *Abelian integrals in nonsymmetric perturbations of symmetric Hamiltonian vector fields*, Adv. Appl. Math., 1994, 15, 1–12.

[39] A. Ke and M. Han, *Hopf bifurcation of three-dimensional systems with parameters*, J. Differ. Equ., 2025, 440, 113463.

[40] F. Liang and M. Han, *Expansion coefficients and their relation for Melnikov functions near polycycles*, J. Differ. Equ., 2015, 435, 113312.

[41] C. Liu and D. Xiao, *The monotonicity of the ratio of two Abelian integrals*, Trans. Am. Math. Soc., 2013, 365, 5525–5544.

[42] S. Liu and M. Han, *Bifurcation theory of limit cycles by higher order Melnikov functions and applications*, J. Differ. Equ., 2024, 403, 29–66.

[43] W. Liu and M. Han, *Bifurcations of traveling wave solutions of a generalized Burgers-Fisher equation*, J. Math. Anal. Appl., 2024, 533(2), 128012.

[44] D. Luo, M. Han and D. Zhu, *The uniqueness of limit cycles bifurcated from a separatrix cycle (I)* (in Chinese), Acta Math. Sin., 1992, 35(3), 407–417.

[45] V. K. Melnikov, *On the stability of the center for time periodic perturbations*, Trans. Moscow Math. Soc., 1963, 12, 1–57.

[46] A. Mourtada, *Degenerate and nontrivial hyperbolic polycaycles with two vertices*, J. Differ. Equ., 1994, 113, 68–83.

[47] K. Patra and C. S. Rao, *Periodic traveling wave solutions of a singularly perturbed generalized mKdV equation with higher-degree nonlinearities*, Int. J. Bifurcation Chaos, 2026, 36(4), 2650039.

[48] J. W. Reyn, *Generating of limit cycles from separatrix polygons in the phase plane*, Lecture Note in Math., 1980, 810, 264–289.

[49] R. Roussarie, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*, Bol. Soc. Bras. Mat., 1986, 17, 67–101.

[50] L. Sheng and M. Han, *Bifurcation of limit cycles from a compound loop with five saddles*, J. Appl. Anal. Comput., 2019, 9(6), 2482–2495.

[51] L. Sheng, M. Han and Y. Tian, *On the number of limit cycles bifurcating from a compound polycycle*, Int. J. Bifurcation Chaos, 2020, 30(7), 2050099.

[52] X. Sun, Y. Tian and M. Han, *Homoclinic bifurcation near a loop tangent to an invariant straight line*, J. Differ. Equ., 2025, 425, 157–189.

[53] X. Sun and M. Zhang, *Dynamics of a quartic Korteweg-de Vries equation with multiple dissipations via an Abelian integral approach*, Chaos, 2025, 35(8), 083118.

[54] Y. Tian and M. Han, *Hopf and homoclinic bifurcations for near-Hamiltonian systems*, J. Differ. Equ., 2017, 262, 3214–3234.

[55] Y. Tian and P. Yu, *Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis*, J. Differ. Equ., 2018, 264, 5950–5976.

[56] W. Yan, R. Yu and L. Wang, *Existence of solitary waves and periodic waves in a perturbed generalized Gardner equation*, Qual. Theory Dyn. Syst., 2026, 25, 106.

[57] J. Yang and M. Han, *Some properties of Melnikov functions near a cuspidal loop*, Sci. China Math., 2024, 67(4), 767–786.

[58] J. Yang and M. Han, *Limit cycles near a compound cycle in a near-Hamiltonian system with smooth perturbations*, Chaos, Solitons and Fractals, 2024, 184, 114963.

[59] J. Yang, Y. Xiong and M. Han, *Limit cycle bifurcations near a 2-polycycle or double 2-polycycle of planar systems*, Nonlinear Anal., 2014, 95, 756–773.

[60] L. Zhang, M. Han, M. Zhang and C. M. Khalique, *A new type of solitary wave solution of the mKdV equation under singular perturbations*, Int. J. Bifurcation Chaos, 2020, 30, 1–14.

[61] L. Zhang and W. Yang, *Persistence of the traveling wave solutions for the modified equal width equation with singular perturbation*, Qual. Theory Dyn. Syst., 2026, 25(2), 50.

Received August 2026; Accepted September 2026; Available online September 2026.