

ON THE INVERSE PROBLEM FOR THE BURGERS' EQUATION WITH SMOOTH INITIAL DATA IN A BOUNDED DOMAIN

Lea Safetly^{1,†} and Toni Sayah¹

Abstract In this paper, we study the inverse problem of recovering smooth initial data for the one-dimensional Burgers' equation on a bounded spatial domain. Specifically, this problem focuses on recovering the initial condition from measurements at a part of the boundary over a given time interval. We therefore prove the uniqueness of the solution to the direct problem and establish, under certain assumptions, a stability estimate for the inverse problem based on a backward method in time and space, showing that the initial data can be reconstructed in this smooth case. To validate our theoretical results, we perform several numerical simulations and demonstrate the effectiveness of the proposed approaches.

Keywords Inverse problem, Burgers' equation, smooth initial data, stability estimate, numerical reconstruction.

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1. Introduction

Transport equations are partial differential equations [14], generally used to describe how a quantity is transported through space and time. They are the basis for the formulation of many physical problems, such as fluid flow, heat transfer, mass diffusion, and wave propagation.

In fact, a transport equation is a classical example of scalar conservation laws [23], which arise in the modeling of physical systems where quantities such as mass, energy, or momentum are conserved. Their solutions, often involving non-linear wave structures and discontinuities, pose significant analytical and computational challenges.

One important theory concerning the transport equation is the theory of renormalization developed by DiPerna, R. J., and Lions, P.-L. in [12] in 1989. It revolves around proving the existence and uniqueness of a solution to the transport equation with initial data in $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$ and the velocity field in $C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$.

This theory was extended by Ambrosio, L. in [2] in 2004 to BV vector fields with initial data in $L^p(\mathbb{R}^d)$, where the existence and uniqueness of a solution were also proved.

Numerically, accurately approximating solutions to conservation laws has long been a key area of interest. Considerable progress has been achieved through the development of reliable numerical techniques, including upwind schemes and finite volume methods [7, 8, 10, 11, 24, 29].

We focus in this work on the Burgers equation, a nonlinear transport equation known for shock formation. Initially, Bateman [5] introduced the Burgers' equation and solved its stationary solution. Later, Burgers [9] studied this equation for mathematical modeling of turbulence.

[†]The corresponding author.

¹Laboratoire de Mathématiques et Applications, Unité de recherche Mathématiques et Modélisation, CAR, Faculté des Sciences, Université Saint-Joseph de Beyrouth, B.P. 11-514 Riad El Solh, Beyrouth 1107 2050, Lebanon

Email: lea.safetly@net.usj.edu.lb (L. Safetly), toni.sayah@usj.edu.lb (T. Sayah)

And due to its interesting properties, it has been widely used for various fields such as modeling of gas dynamics, shock waves, traffic flows, etc. From a physical point of view, it is a special form of the momentum equation for irrotational and incompressible flows in which pressure gradients are neglected [35].

For $T > 0$, we recall the following Burgers' equation,

$$\begin{cases} \partial_t u(x, t) + \partial_x \left(\frac{u^2(x, t)}{2} \right) = 0, & (x, t) \in \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u_0 \in C^1(\mathbb{R})$.

However, problems involving non-smooth initial data, such as the Riemann problem, have also been widely studied. This problem involves solving the Burgers' equation with piecewise-constant initial data featuring a single discontinuity. For instance, authors in [15, 25] provided comprehensive theoretical analyses of the Riemann problem for scalar conservation laws, establishing the existence of weak solutions and introducing the concept of entropy conditions to ensure uniqueness. In [34], various approximate Riemann solvers were proposed and analyzed for their accuracy and robustness in handling discontinuities.

From a numerical perspective, several schemes have been developed and tested on such problems. For example, the work in [18] introduced schemes that are capable of capturing shocks and rarefaction waves. In addition, the upwind finite volume method, discussed in [24], has proven effective in approximating the solution of the Burgers' equation with discontinuous initial data, preserving the correct wave structure and satisfying the entropy condition.

In fact, the study of Burgers' equation with smooth initial data has played a fundamental role in understanding nonlinear wave propagation, shock formation, and the transition from smooth to singular solutions. While the classical method of characteristics yields local-in-time solutions for $u_0 \in C^1(\mathbb{R})$, the inevitable gradient blowup and subsequent shock development necessitate advanced analytical techniques, such as vanishing viscosity approximations, weak solution frameworks, and entropy conditions, to extend solutions beyond breakdown time. Previous works have extensively explored these aspects, both theoretically and numerically. For instance, Pereira et al. [30] studied a viscous Burgers-type equation with moving boundary conditions and provided both theoretical and numerical results. For the existence and uniqueness of entropy solutions for Burgers' equation, a rigorous framework for treating boundary conditions was first developed by Bardos, Leroux and Nédélec [4], who studied first-order Burgers' equations on the half-line $\mathbb{R}^+$ and introduced admissible boundary conditions ensuring well-posedness. This was extended by Dubois and Le Floch [13] to bounded domains via entropy-compatible boundary conditions, and further generalized to scalar conservation laws with general boundary conditions and continuous flux functions in [3]. These results are important for guaranteeing the existence and uniqueness of entropy solutions for the Burgers' equation on bounded domain.

In this paper, we consider the case where the initial data are smooth and on the bounded domain $[0, 1]$. Let $T > 0$, we consider the following Burgers' equation

$$\begin{cases} \partial_t u(x, t) + \partial_x \left(\frac{u^2(x, t)}{2} \right) = 0, & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \\ u(0, t) = \alpha(t), & t \in [0, T], \end{cases} \quad (1.2)$$

where $\Omega = (0, 1)$, $\alpha \in C^1([0, T])$, $u_0 \in C^1(\bar{\Omega})$ and for the compatibility of system (1.2), we must have $u_0(0) = \alpha(0)$.

We aim to address the inverse problem related to the Burgers' equation (1.2), by recovering the smooth initial data u_0 , from given boundary measurements (at $x = 1$), over a given time interval. In fact, in the formulation of inverse problems for partial differential equations, it is common to mention additional information of the solution on part of the boundary. Such as the inverse backward problem (also known as the retrospective inverse problem), where the objective is to determine the initial state of the system given the solution at the final time [1, 6]. And we note that such problems are known to be ill-posed [19, 21, 22, 26, 27]. Recently, the authors in [33] demonstrated the stability of the inverse problem for the one-dimensional Burgers' equation with Riemann initial data, addressing two cases: The recovery of initial data from boundary measurements, and the recovery from within the domain, and provided several numerical simulations to validate their results. Moreover, the authors in [31, 32] studied optimal control formulations for inverse problems associated with both classical and fractional Burgers equations, with applications to physical models such as wave propagation in fluids. In addition, Jenaliyev et al. [20] investigated the solvability of an inverse problem for the Burgers equation with an integral overdetermination condition in a degenerating domain.

From a numerical perspective, the authors of [28] approach to the numerical recovering of the initial condition in the inverse problem for a non-linear singularly perturbed reaction–diffusion–advection equation. Additionally, filtered gradient algorithms for inverse design problems of one-dimensional Burgers' equation were introduced in [17], where the authors present this inverse design for hyperbolic conservation laws through the one-dimensional Burgers' equation, motivated by the aircraft's sonic-boom minimization issues.

In this work, we begin by considering problem (1.2), and under suitable conditions on the time T , we establish a Theorem on the existence and uniqueness of classical solutions. Then, we solve the inverse problem for the one-dimensional Burgers' equation, by recovering the smooth initial data u_0 in $\bar{\Omega}$ from the boundary measurements at $x = 1$, over the time interval $[0, T]$, where we establish, under some assumptions, the stability of the recovered initial condition u_0 respectively as follows:

$$\|u_{01}(\cdot) - u_{02}(\cdot)\|_{L^\infty(\Omega)} \leq C \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)}, \quad (1.3)$$

where C is a positive constant depending on u_{01} or u_{02} .

Moreover, we perform corresponding numerical simulations of the inverse problem to validate our theoretical results concerning the stability of the inverse problem, using the backward method in space and time.

This paper is organized as follows. Section 2 presents the direct problem, including the existence and uniqueness of solutions on a bounded domain. In Section 3, we state the main theorem about the stability of the inverse problem, where we recover the smooth initial data from the boundary measurements at $x = 1$ over time. Finally, Section 4 validates our results through numerical simulations, analyzing the direct problem, the inverse problem, and the stability of the inverse problem for the one-dimensional Burgers' equation while applying a backward method in space and time.

2. Direct problem

In this section, we study the existence and uniqueness of classical solutions to the direct problem (1.2), on the bounded domain Ω . We consider smooth boundary condition $\alpha \in C^1([0, T])$ and

initial data $u_0 \in C^1(\overline{\Omega})$ and assume that

$$\alpha(t) \geq C_\alpha, \quad \forall t \in [0, T], \quad (2.1)$$

and

$$u_0(x) \geq C_0, \quad \forall x \in \overline{\Omega}, \quad (2.2)$$

where C_α and C_0 are two positive given constants.

Before starting the study of the inverse problem in the following section, it is crucial to verify whether the direct problem (1.2) admits a unique classical solution $u(x, t)$, for $x \in \overline{\Omega}$, using the Characteristics method.

We define the set

$$Y = \left\{ \bar{x} \in [0, 1] \mid u'_0(\bar{x}) = \min_{x \in [0, 1]} u'_0(x) \right\},$$

which will be used in the following theorem.

Theorem 2.1. *We consider problem (1.2) and we assume that T satisfies the following relations:*

1. *If $\min_{x \in \overline{\Omega}} u'_0(x) < 0$, then T satisfies one of these two cases:*

(a)

$$T \leq \frac{-1}{\min_{x \in \overline{\Omega}} u'_0(x)}, \quad (2.3)$$

or

(b)

$$T > \frac{-1}{\min_{x \in \overline{\Omega}} u'_0(x)} \quad \text{and} \quad T \geq \max_{\bar{x} \in Y} \frac{1 - \bar{x}}{u_0(\bar{x})}. \quad (2.4)$$

- 2.

$$\min_{t \in [0, T], \alpha'(t) > 0} \frac{\alpha(t)^2}{\alpha'(t)} \geq 1. \quad (2.5)$$

- 3.

$$\min_{\substack{x \in \overline{\Omega}, t \in [0, T], \\ \alpha(t) > u_0(x)}} \frac{(t + x)\alpha(t)u_0(x)}{\alpha(t) - u_0(x)} \geq 1. \quad (2.6)$$

Then, problem (1.2) admits a unique classical solution u .

Proof. To show the uniqueness of a classical solution u of problem (1.2), we must study all the possible cases where two characteristics intersect, and then deduce the largest real number T such that problem (1.2) admits a unique classical solution. Let $(X_i(t), t)$, $i = 1, 2$, be the characteristic curves satisfying:

$$\begin{cases} \partial_t X_i(t; 0, x_i) = u(X_i(t; 0, x_i), t), \\ X_i(0; 0, x_i) = x_i. \end{cases} \quad (2.7)$$

We can distinguish three cases:

1. First, we assume that $\min_{x \in \bar{\Omega}} u'_0(x) < 0$ and we consider the case where the feet of the two characteristics are in $\bar{\Omega}$ at time $t = 0$. Based on the Break time in the work of [29, Exercise 3.1], and in our case on the bounded domain $\bar{\Omega}$, the last time T satisfies

$$T \leq \frac{-1}{\min_{x \in \bar{\Omega}} u'_0(x)}. \quad (2.8)$$

In fact, this break time is calculated as follows: We consider two characteristic curves X_1 and X_2 from (2.7) and since the solution is constant along the characteristics, we equate these two characteristics, to get

$$u_0(x_1)t + x_1 = u_0(x_2)t + x_2.$$

Therefore, by re-arranging the terms, we obtain the following break time at which the characteristics intersect

$$t = -\frac{x_2 - x_1}{f'(u_0(x_2)) - f'(u_0(x_1))}.$$

Therefore, the break time is

$$t = \min_{x_1, x_2 \in \bar{\Omega}} \left(-\frac{x_2 - x_1}{u_0(x_2) - u_0(x_1)} \right) = \frac{-1}{\min_{x_1, x_2 \in \bar{\Omega}} \left(\frac{u_0(x_2) - u_0(x_1)}{x_2 - x_1} \right)} = \frac{-1}{\min_{x \in \bar{\Omega}} u'_0(x)}.$$

Furthermore, let

$$Y = \{\bar{x} \in \bar{\Omega} \setminus u'_0(\bar{x}) = \min_{x \in \bar{\Omega}} u'_0(x)\},$$

the set of all the points $\bar{x}$ minimizing u'_0 in $\bar{\Omega}$.

For each such $\bar{x} \in Y$, we compute the characteristic curve given by

$$\bar{X} = u_0(\bar{x})T + \bar{x}. \quad (2.9)$$

To ensure uniqueness, we verify that for all minimum points, the characteristic $\bar{X}$ lies outside the interval $\bar{\Omega}$ (in our case, we must have $\bar{X} \geq 1$). If this condition is satisfied, then the direct problem has a unique solution on $\bar{\Omega}$.

In practice and for this case, T must satisfy the following: For each $\bar{x} \in Y$ satisfying

$$T \geq \frac{-1}{u'_0(\bar{x})},$$

we have also from (2.9), for $\bar{X} = 1$,

$$T \geq \frac{1 - \bar{x}}{u_0(\bar{x})}, \quad \forall \bar{x} \in Y.$$

The above relations can be also written as follows:

$$T \geq \frac{-1}{\min_{x \in \bar{\Omega}} u'_0(x)} \quad \text{and} \quad T \geq \max_{\bar{x} \in Y} \frac{1 - \bar{x}}{u_0(\bar{x})}. \quad (2.10)$$

2. In this part, we consider (2.7) and we study the case where two characteristics X_1 and X_2 start at $x = 0$ at two different times t_i , $i = 1, 2$, meaning that for $i = 1, 2$ they satisfy $X_i(t_i) = 0$, and intersect respectively through $X_1(t)$ and $X_2(t)$ at time $t \geq t_i$. Since the solution is constant along the characteristics, we have

$$u(X_i(t; 0, x_i), t) = u(x_i, 0) = u_0(x_i) = u(0, t_i) = \alpha(t_i), \quad i = 1, 2.$$

Then, integrating the first line of (2.7) between t and t_i , $i = 1, 2$, we get

$$X_i(t) = \alpha(t_i)(t - t_i), \quad i = 1, 2.$$

We thus obtain by assuming $\alpha(t_1) \neq \alpha(t_2)$ and by fixing $X_1(t) = X_2(t) = X(t)$,

$$X(t) = \frac{(t_2 - t_1)\alpha(t_1)\alpha(t_2)}{\alpha(t_2) - \alpha(t_1)}.$$

We note that if $\alpha(t_1) = \alpha(t_2)$ for $t_1 \neq t_2$, the two characteristics X_1 and X_2 can never intersect at the same time t .

Therefore, to ensure the uniqueness of the initial data u_0 , we must have $X(t) \geq 1$ and then

$$\min_{\substack{t_1, t_2 \in [0, T], t_2 > t_1, \\ \alpha(t_2) > \alpha(t_1)}} \frac{(t_2 - t_1)\alpha(t_1)\alpha(t_2)}{\alpha(t_2) - \alpha(t_1)} \geq 1. \quad (2.11)$$

In fact, if we proceed as follows: Let $f(t_1, t_2) = \frac{(t_2 - t_1)\alpha(t_1)\alpha(t_2)}{\alpha(t_2) - \alpha(t_1)}$, then we have to minimize the following expression in terms of t_1 and t_2 in $[0, T]$:

$$f(t_1, t_2) = (t_2 - t_1)\alpha(t_1) + \frac{\alpha^2(t_1)(t_2 - t_1)}{\alpha(t_2) - \alpha(t_1)}.$$

Fixing t_1 and varying t_2 , we get that f reaches its minimum when t_2 tends to t_1 and that

$$\min_{\substack{t_2 \in [0, T], t_2 > t_1, \\ \alpha(t_2) > \alpha(t_1)}} f(t_1, t_2) = \min_{t_1 \in [0, T], \alpha'(t_1) > 0} \frac{\alpha^2(t_1)}{\alpha'(t_1)}.$$

Thus, by varying t_1 we get

$$\min_{\substack{t_1, t_2 \in [0, T], t_2 > t_1, \\ \alpha(t_2) > \alpha(t_1)}} f(t_1, t_2) = \min_{t \in [0, T], \alpha'(t) > 0} \frac{\alpha^2(t)}{\alpha'(t)}.$$

Therefore, we obtain

$$\min_{\substack{t_1, t_2 \in [0, T], t_2 > t_1, \\ \alpha(t_2) > \alpha(t_1)}} \frac{(t_2 - t_1)\alpha(t_1)\alpha(t_2)}{\alpha(t_2) - \alpha(t_1)} = \min_{t \in [0, T], \alpha'(t) > 0} \frac{\alpha(t)^2}{\alpha'(t)} \geq 1. \quad (2.12)$$

Thus, T must satisfy the last inequality (2.12).

3. In this case, we consider two characteristics, the first one starts at $x = 0$ at a time t_1 (denoted by $X_1(t_1) = 0$) and passes through $X_1(t)$ at time t , given as before by

$$X_1(t) = \alpha(t_1)(t - t_1),$$

the second one starts at $X_2(\tau) = x_2 \in \bar{\Omega}$ at time τ and passes through $X_2(t)$ at time t

$$X_2(t) = u_0(x_2)t + x_2.$$

We obtain by fixing $X_1(t) = X_2(t) = X(t)$,

$$X(t) = \frac{(t_1 + x_2)\alpha(t_1)u_0(x_2)}{\alpha(t_1) - u_0(x_2)}.$$

Therefore to ensure uniqueness of the initial data u_0 , we must have $X(t) \geq 1$ and then

$$\min_{\substack{x \in \bar{\Omega}, t \in [0, T], \\ \alpha(t) > u_0(x)}} \frac{(t + x)\alpha(t)u_0(x)}{\alpha(t) - u_0(x)} \geq 1. \quad (2.13)$$

In view of all the results, we deduce that to ensure the uniqueness of the classical solution of problem (1.2), T must satisfy the relations (2.10), (2.12) and (2.13). $\square$

Remark 2.1. We denote by T_{bu} the break time that is the largest value of T satisfying the relations (2.10), (2.12) and (2.13). So for $T \leq T_{bu}$, the classical solution of problem (1.2) is unique.

3. Inverse problem: Recovering the initial conditions

In this section, we introduce the inverse problem, which consists in reconstructing the initial data $u_0(x)$, $x \in \bar{\Omega}$, given the measurements at the boundary $x = 1$ over time and establish the stability estimate for this inverse problem.

Let $s = 1 - x$, $\tau = T - t$ and $v(s, \tau) = u(1 - s, T - \tau) = u(x, t)$, such that the above problem (1.2) becomes:

$$\partial_t v(x, t) + \partial_x \left(\frac{v^2(x, t)}{2} \right) = 0 \quad \text{in } \Omega \times (0, T). \quad (3.1)$$

Consequently, the inverse problem is well defined if the conditions in Theorem 2.1 are satisfied. These conditions are expressed in terms of u_0 and T .

We assume that at the final time T , the solution becomes constant equal to its value at the boundary $x = 1$, denoted by $\alpha_0 > 0$. Hence, we set the initial data for Problem (3.1) as $v(x, 0) = u(1 - x, T) = \alpha_0$, $x \in \bar{\Omega}$. In practice, α_0 corresponds to the measurement of the solution at $x = 1$ at time T . For the boundary condition, we have $v(0, t) = u(1, T - t)$ in $[0, T]$. We assume that $u(1, T - \cdot) \in L^\infty(0, T)$.

Thus, the backward problem becomes the following reverse problem

$$\begin{cases} \partial_t v(x, t) + \partial_x \left(\frac{v^2(x, t)}{2} \right) = 0, & \text{in } \Omega \times (0, T), \\ v(x, 0) = \alpha_0, & \text{in } \bar{\Omega}, \\ v(0, t) = u(1, T - t), & \text{in } [0, T]. \end{cases} \quad (3.2)$$

Remark 3.1. In general, for the inverse problem, we consider directly the problem (3.2) with given measurements at $x = 1$ as follows:

$$\begin{cases} \partial_t v(x, t) + \partial_x \left(\frac{v^2(x, t)}{2} \right) = 0, & \text{in } \Omega \times (0, T), \\ v(x, 0) = \alpha_0, & \text{in } \bar{\Omega}, \\ v(0, t) = \beta(t), & \text{in } [0, T], \end{cases} \quad (3.3)$$

where $\beta \in L^\infty(0, T)$. The recovered initial data will be $u_0(x) = v(1 - x, T)$.

Remark 3.2. We note that, in practice, $\beta = u(1, T - \cdot)$ represents the discrete measurements at $x = 1$ over time. Then, we can interpolate it in order to obtain a continuous function on $[0, T]$, also denoted by β .

We now define the **inverse problem** associated to problem (3.3): Let $u(1, \cdot)$ be a given data on the boundary $x = 1$ of $\bar{\Omega}$. To recover the initial data u_0 of problem (1.2), it is sufficient to solve problem (3.3) and get $u_0(x) = v(1 - x, T)$, $x \in \bar{\Omega}$.

Theorem 3.1. *Let v be a classical solution of problem (3.3). We assume that $\beta \in C^1([0, T])$ and T satisfies the following inequality:*

$$\min_{\tau \in [0, T], v'(0, \tau) > 0} \frac{v(0, \tau)^2}{v'(0, \tau)} \geq 1, \quad (3.4)$$

and

$$\min_{\substack{\tau \in [0, T], x \in \bar{\Omega}, \\ \alpha_0 > v(0, \tau)}} \frac{v(0, \tau)(\alpha_0 \tau + x)}{\alpha_0 - v(0, \tau)} \geq 1. \quad (3.5)$$

Then the classical solution v is unique.

Proof. To show the uniqueness of the constructed initial data u_0 in $\bar{\Omega}$ of problem (1.2), given the measurements $u(1, t)$, $t \in [0, T]$ of problem (3.3), we must study all the possible cases where two characteristics intersect. In fact, let $(X_i(t), t)$, $i = 1, 2$, be the characteristic curves satisfying:

$$\begin{cases} \partial_t X_i(t) = v(X_i(t), t), \\ \partial_t v(X_i(t), t) = 0. \end{cases} \quad (3.6)$$

Then along the characteristics, v is constant: $v(X_i(t), t) = v(X_i(\tau), \tau)$, for any $\tau \in [0, T]$. We can distinguish two distinct cases:

1. First, we study the case where the two characteristics starting from point $X_i(\tau_i) = 0$ at time τ_i , and intersect respectively through $X_1(t)$ and $X_2(t)$ at time $t \geq \tau_i$. Having

$$v(x_i, t) = v(0, \tau_i) = u(1, t - \tau_i). \quad (3.7)$$

Integrating the first equation of (3.6) between τ_i , $i = 1, 2$ and t , and remarking that $X_i(\tau_i) = 0$, we obtain:

$$X_i(t) = \int_{\tau_i}^t v(X_i(\xi), \xi) d\xi,$$

and then

$$X_i(t) = v(0, \tau_i)(t - \tau_i). \quad (3.8)$$

Then considering

$$X_1(t) = v(0, \tau_1)(t - \tau_1)$$

and

$$X_2(t) = v(0, \tau_2)(t - \tau_2),$$

we get

$$\frac{X_1(t)}{v(0, \tau_1)} + \tau_1 = \frac{X_2(t)}{v(0, \tau_2)} + \tau_2.$$

For $X_1(t) = X_2(t) = X(t)$, we obtain

$$X(t) = \frac{v(0, \tau_2)v(0, \tau_1)}{v(0, \tau_2) - v(0, \tau_1)}(\tau_2 - \tau_1). \quad (3.9)$$

Therefore, the uniqueness of the constructed initial data $u_0(x) = v(1 - x, T)$ is ensured if for all $\tau_1, \tau_2 \in [0, T]$, with $\tau_1 < \tau_2$, the position $X(t)$ given by (3.9) remains outside the interval $\overline{\Omega}$ (more precisely if $X \geq 1$).

Alternatively, we can compute

$$X = \min_{\substack{\tau_1, \tau_2 \in [0, T], \\ v(0, \tau_2) > v(0, \tau_1)}} \frac{v(0, \tau_2)v(0, \tau_1)}{v(0, \tau_2) - v(0, \tau_1)}(\tau_2 - \tau_1), \quad (3.10)$$

which is equivalent to

$$X = \min_{\substack{\tau_1, \tau_2 \in [0, T], \\ u(1, T - \tau_2) > u(1, T - \tau_1)}} \frac{u(1, T - \tau_2)u(1, T - \tau_1)}{u(1, T - \tau_2) - u(1, T - \tau_1)}(\tau_2 - \tau_1), \quad (3.11)$$

and thus this value of X must be outside the interval $[0, 1]$ (more precisely if $X(t) \geq 1$). Furthermore, as $v(0, t) \in C^1([0, T])$ (i.e. $u(1, T - t) \in C^1([0, T])$), and following the same steps done previously in the proof of (2.12) in Lemma 2.1, we get that

$$X = \min_{\substack{\tau_1, \tau_2 \in [0, T], \\ v(0, \tau_2) > v(0, \tau_1)}} \frac{v(0, \tau_2)v(0, \tau_1)}{v(0, \tau_2) - v(0, \tau_1)}(\tau_2 - \tau_1) = \min_{\tau \in [0, T], v'(0, \tau) > 0} \frac{(v(0, \tau))^2}{v'(0, \tau)}, \quad (3.12)$$

which is equivalent to

$$X = \min_{\tau \in [0, T], u'(1, T - \tau) > 0} \frac{(u(1, T - \tau))^2}{u'(1, T - \tau)}, \quad (3.13)$$

and thus this value of X must also be outside the interval $\overline{\Omega}$ (more precisely if $X(t) \geq 1$).

- Here, we study the case where two characteristics, one start from point $X_0(t_0) = x_0$ at time $t_0 = 0$ and the other start from the point $X_1(\tau) = 0$ at time τ , intersect respectively through $X_0(t)$ and $X_1(t)$ at time $t \geq \tau$:

$$X_0(t) = v(x_0, 0)t + x_0,$$

and

$$X_1(t) = v(0, \tau)(t - \tau).$$

Fixing t , we obtain

$$\frac{X_0(t) + x_0}{v(x_0, 0)} = \frac{X_1(t)}{v(0, \tau)} + \tau.$$

Then for $X_0(t) = X_1(t) = X(t)$, we get

$$X(t) = \frac{v(0, \tau)(v(x_0, 0)\tau + x_0)}{v(x_0, 0) - v(0, \tau)}.$$

Therefore, to ensure the uniqueness of the initial data u_0 , $X(t)$ must remain outside $\overline{\Omega}$. Thus, we have the following

$$X = \min_{\substack{\tau \in [0, T], x \in \overline{\Omega}, \\ v(x, 0) > v(0, \tau)}} \frac{v(0, \tau)(v(x, 0)\tau + x)}{v(x, 0) - v(0, \tau)} \geq 1.$$

□

Remark 3.3. We denote by T_{bv} the break time that is the largest value of T satisfying the relations (3.4) and (3.5). So, for $T \leq T_{bv}$, the classical solution of problem (3.3) is unique.

Remark 3.4. In the last theorem, we don't need to study the case where two characteristics start from the point $x = 0$ at different times, since the initial condition on $v(x, 0)$ is assumed to be constant, for $x \in \overline{\Omega}$.

Remark 3.5. In the numerical simulations performed in the next section, we also compare the two break times T_{bu} and T_{bv} in the case where the data satisfy $\alpha(t) = \alpha_0$.

Let us now study the stability of the above inverse problem:

Theorem 3.2. Let v_1 , and v_2 be two distinct solutions of problem (3.3), associated respectively to u_1 and u_2 , where $u_{0i}(\cdot) = v_i(1 - \cdot, T)$, $i = 1, 2$, such that $\sup u'_{0i}$ is bounded by a positive constant C_0 . Under the assumptions of Theorem 3.1 satisfied by $v_i(0, \cdot) = u_i(1, T - \cdot)$, $i = 1, 2$, the following stability inequality holds:

$$\|u_{01}(\cdot) - u_{02}(\cdot)\|_{L^\infty(\Omega)} \leq C \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)}, \quad (3.14)$$

where C is a positive constant depending on u_{01} or u_{02} .

Proof. Let u_1 and u_2 be two distinct solutions of (1.2) and let v_1 and v_2 be the corresponding solutions of (3.3) associated respectively to u_1 and u_2 . Then

$$\partial_t v_1(x, t) + \partial_x \left(\frac{v_1^2(x, t)}{2} \right) = 0, \quad (3.15)$$

$$\partial_t v_2(x, t) + \partial_x \left(\frac{v_2^2(x, t)}{2} \right) = 0. \quad (3.16)$$

For the Burgers' equation (3.3), let $(X_i(t), t)$, $i = 1, 2$, be the characteristic curves satisfying:

$$\begin{cases} \frac{d}{dt} X_i(t) = v_i(X_i(t), t), \\ \frac{d}{dt} v_i(X_i(t), t) = 0. \end{cases} \quad (3.17)$$

Then along the characteristics, v_i is constant: $v_i(X_i(t), t) = v_i(X_i(\tau), \tau)$, $i = 1, 2$. Consider the characteristic starting from point $X_i(\tau) = 0$ at time τ , and passing through $X_i(T) = x_i$ at time $t = T$, we get

$$v_i(x_i, T) = v_i(0, \tau) = u_i(1, T - \tau). \quad (3.18)$$

Integrating the first equation of (3.17) between τ and T , and remarking that $X_i(\tau) = 0$ and $X_i(T) = x_i$, we obtain:

$$x_i = X_i(T) = \int_{\tau}^T v_i(X_i(\xi), \xi) d\xi,$$

and then

$$x_i = v_i(0, \tau)(T - \tau). \quad (3.19)$$

Equation (3.18) gives

$$u_{0i}(1 - x_i) = u_i(1 - x_i, 0) = v_i(x_i, T) = u_i(1, T - \tau).$$

This last equation allows us to obtain the following:

$$\begin{aligned} |u_{01}(1 - x_1) - u_{02}(1 - x_1)| &\leq |u_1(1, T - \tau) - u_2(1, T - \tau)| + |u_{02}(1 - x_2) - u_{02}(1 - x_1)| \\ &\leq \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)} + \sup_{x \in \Omega} |u'_{02}(x)| |x_1 - x_2|. \end{aligned} \quad (3.20)$$

However, equation (3.19) gives

$$\begin{aligned} |x_1 - x_2| &= |T - \tau| |v_1(0, \tau) - v_2(0, \tau)| \\ &\leq T |u_1(1, T - \tau) - u_2(1, T - \tau)| \\ &\leq T \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)}. \end{aligned} \quad (3.21)$$

Therefore from (3.20) and (3.21), we obtain

$$|u_{01}(1 - x_1) - u_{02}(1 - x_1)| \leq (1 + T \sup_{x \in \Omega} |u'_{02}(x)|) \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)},$$

and then

$$\|u_{01}(\cdot) - u_{02}(\cdot)\|_{L^\infty(\Omega)} \leq (1 + T \sup_{x \in \Omega} |u'_{02}(x)|) \|u_1(1, \cdot) - u_2(1, \cdot)\|_{L^\infty(0, T)}.$$

□

4. Numerical simulations

In this section, we show numerical simulations performed using MATLAB software to validate the above theoretical results. We consider the one-dimensional Burgers' equation and the problem is given as follows:

$$\partial_t u(x, t) + \partial_x \left(\frac{u^2(x, t)}{2} \right) = 0 \quad \text{in } (0, L) \times (0, T), \quad (4.1)$$

where we consider, in both cases, the space domain $(0, L)$ and the time domain $(0, T)$ where L and T are positive constants.

We begin by showing numerical simulations corresponding to the direct problem, where, for a given smooth initial data u_0 , we compute the solution over time and compare the obtained result with the exact solution. Next, we treat the inverse problem, which consists in reconstructing the smooth initial condition u_0 , from given measurements at $x = 1$ over the time interval $[0, T]$, compare it with the exact one and study the stability of this inverse problem.

At the discrete level, we discretize the space and time domains using a uniform grid.

- The domain $[0, L]$ is split into M segments $[x_i, x_{i+1}]$ such that $x_i = i\Delta x$, for $i = 0, 1, \dots, M$, where $\Delta x = \frac{L}{M}$ is the space step.
- The time domain $[0, T]$ is divided into segments $[t_n, t_{n+1}]$ such that $t_n = n\Delta t$, for $n = 0, 1, \dots, N$, where $\Delta t = \frac{T}{N}$ is the time step size.

We consider $L = 1$, $T = 2$ and the following initial condition:

$$u(x, 0) = a + b \sin^2 \left(\frac{m\pi x}{L} \right), \quad (4.2)$$

where a and b are positive constants and m is an integer determining the frequency of the function. Moreover, we consider the boundary condition $u(0, t) = \alpha(t)$, for $t \in [0, T]$ (taking into account the compatibility condition $u(0, 0) = \alpha(0) = \alpha_0 = a$ between the boundary and initial conditions). The positive constant a of the initial data u_0 allows the signal u to advance over time and the positive term $b \sin^2 \left(\frac{m\pi x}{L} \right)$ constitutes the fluctuation of u_0 with the space. For simplicity, we consider $M = 1000$, $N = 1000T$ and $m = 1$.

Godunov scheme

The numerical scheme used in this section is the upwind method, the Godunov scheme, introduced in [16] and given by the following: We have $u_i^0 = u_0(x_i)$, $i = 0, \dots, M$,

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)), \quad i = 1, \dots, M, \quad (4.3)$$

where $F(u_i^n, u_{i+1}^n)$ and $F(u_{i-1}^n, u_i^n)$ are the fluxes given by

$$F(u_i^n, u_{i+1}^n) = \max \left(\frac{(\max(u_i^n, 0))^2}{2}, \frac{(\min(u_{i+1}^n, 0))^2}{2} \right) \quad (4.4)$$

and

$$F(u_{i-1}^n, u_i^n) = \max \left(\frac{(\max(u_{i-1}^n, 0))^2}{2}, \frac{(\min(u_i^n, 0))^2}{2} \right). \quad (4.5)$$

Moreover, we have the following CFL condition, as referred in [29]:

$$\max_i |f'(u_i^n)| \frac{\Delta t}{\Delta x} \leq \frac{1}{2}, \quad (4.6)$$

where $f'(u_i^n) = u_i^n$ (for the Burgers' equation), maintaining the stability of the numerical scheme.

4.1. Direct and inverse problems: With boundary conditions $u(0, t) = a$

In this section, we show numerical simulations for the direct and inverse problems and the stability of the inverse problem, while considering the boundary condition $u(0, t) = \alpha_0 = a$.

4.1.1. Direct problem: With boundary conditions $u(0, t) = a$

For the numerical simulations, we use the above upwind scheme (4.3), where the boundary condition is given by $u_0^{n+1} = u(0, t_{n+1}) = \alpha(0) = \alpha_0 = a$, with $a = 0.5$ and $b = 0.14$.

The algorithm updates the solution at each spatial point for each time step and the exact solution is calculated using the characteristics method.

Figures 1, 2, 3 and 4 show comparisons between the exact and numerical solutions at $t = 0.1$, $t = 0.5$, $t = 1$ and $t = 1.5$, respectively. These curves show that the numerical solution is very close to the exact one over time.

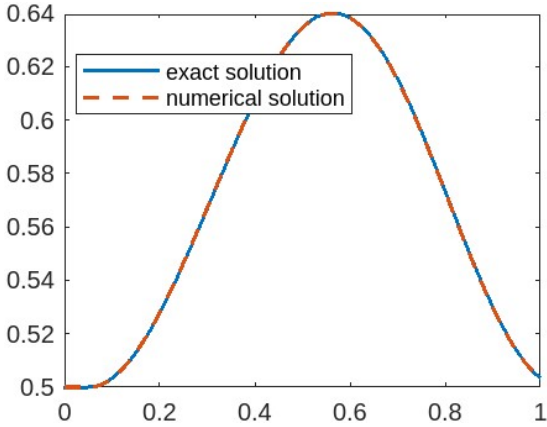


Figure 1. Comparison of the exact and numerical solution for $t = 0.1$.

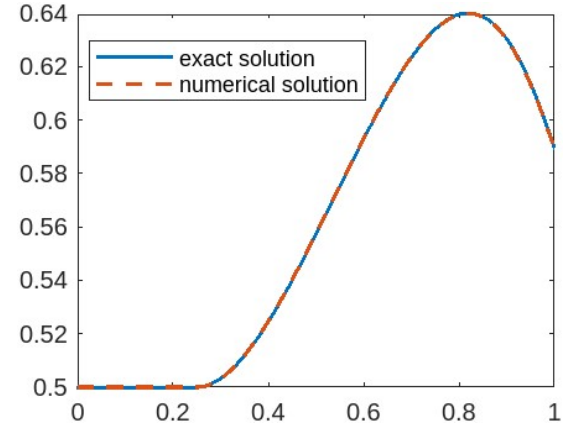


Figure 2. Comparison of the exact and numerical solution for $t = 0.5$.

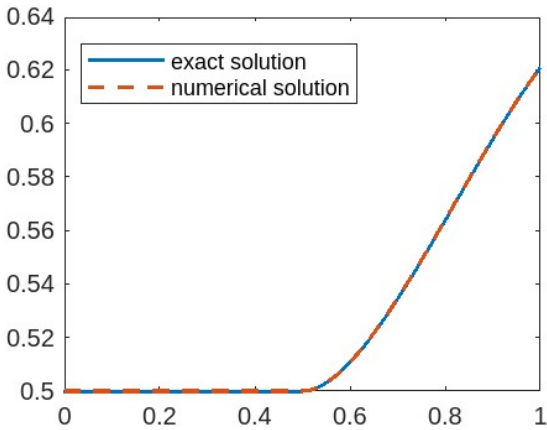


Figure 3. Comparison of the exact and numerical solution for $t = 1$.

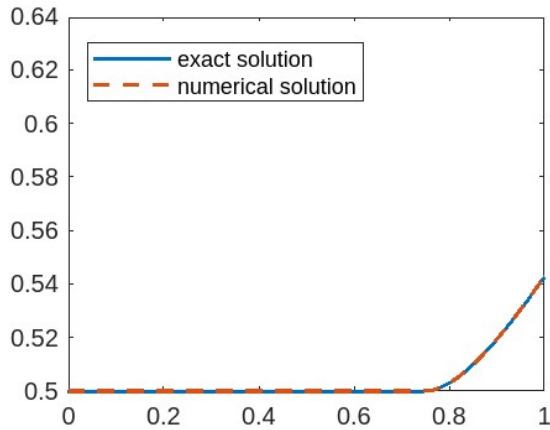


Figure 4. Comparison of the exact and numerical solution for $t = 1.5$.

4.1.2. Inverse problem: Measurements on $x = 1$ and $u(0, t) = a$

The inverse problem involves reconstructing the initial condition $u_0(x)$ given the data at the boundary $x = L = 1$ over time. In this part, we consider problem (3.1) and $a = 0.5$.

We assume that T is sufficiently big such that the variable $u(x, s)$ becomes constant (equal to a) in the domain $[0, L]$ at the time T . Consequently, the initial data associated to problem

(3.1), $v(x, 0) = u(1 - x, T)$, becomes equal to this same constant. Concerning the boundary condition, we have $v(1, t) = u(0, s) = u(0, T - t) = a$ in $[0, T]$.

Problem (3.1) can be discretized, using the Godunov method as follows: $v_i^0 = u_i^N$, $i = 0, \dots, M$,

$$\begin{cases} \frac{1}{\Delta t}(v_i^{n+1} - v_i^n) - \frac{1}{\Delta x}(F(v_i^n, v_{i+1}^n) - F(v_{i-1}^n, v_i^n)) = 0, & i = 0, \dots, M-1, \\ v_M^n = u_M^{N-n}, & n = 1, \dots, N, \end{cases} \quad (4.7)$$

where $F(v_i^n, v_{i+1}^n)$ and $F(v_{i-1}^n, v_i^n)$ are the numerical fluxes defined by (4.4) and (4.5) and v_i^n is the numerical approximation of $v(x_i, t_n)$.

The numerical method used here works backward in space and time, starting from the final condition and propagating the solution to $t = 0$.

To perform numerical simulations corresponding to the recovery of the initial data u_0 , we begin by solving the direct discrete problem (4.3) with the initial data (4.2), and we compute u_M^n , $n = 0, \dots, N$, which will be the boundary condition v_M^n of the inverse problem (3.2). Then we solve the problem (3.2), to recover the constructed initial data u_i^0 $i = 0, \dots, M$, which will be compared to the exact initial data $u(x_i, 0)$ $i = 0, \dots, M$.

Uniqueness of the solution of the inverse problem

The fluctuations of the initial data given by formula (4.2) depends directly on the values of b .

Figures 5 and 6 show the comparison between the exact initial data and the reconstructed one for $b = 0.14$ and $b = 10$.

We remark that for the case $b = 0.14$, the exact and reconstructed solutions are very close, where the case corresponding to $b = 10$ shows that the exact and reconstructed initial data are far away in some regions.

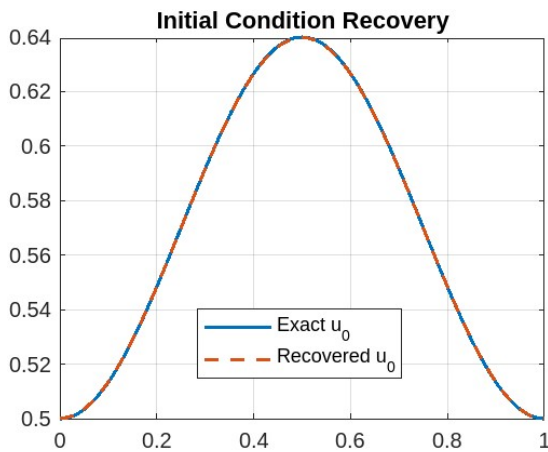


Figure 5. Comparison between exact and recovered initial data for $b = 0.14$.

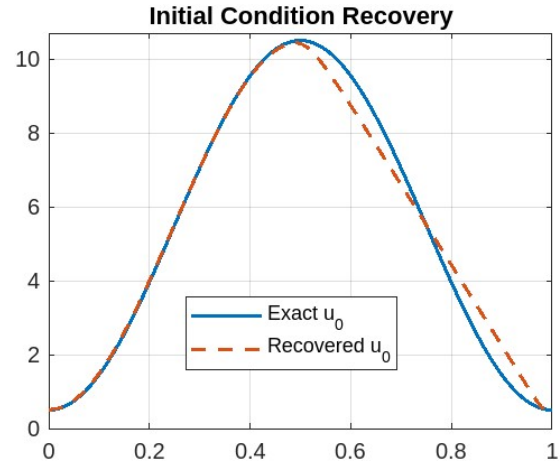


Figure 6. Comparison between exact and recovered initial data for $b = 10$.

In the following we determine the admissible values of b for which the uniqueness is satisfied. To ensure the uniqueness of the reconstructed initial data u_0 (of the inverse problem), we refer

to Theorem 3.1 and check for which values of b the following relation is verified

$$\min_{\tau \in [0, T], v'(0, \tau) > 0} \frac{v(0, \tau)^2}{v'(0, \tau)} \geq 1. \quad (4.8)$$

Numerically, we check if it is satisfied at each time t_n , $1 \leq n \leq N$, by taking $v(0, t_n) \approx v_0^n$ and by approximating $v'(0, t_n)$, with $(v_0^n)' \approx \frac{v_0^n - v_0^{n-1}}{\Delta t}$. Then, relation (4.8) is replaced numerically by

$$\min_{1 \leq n \leq N, v_0^{n-1} < v_0^n} \frac{(v_0^n)^2}{(v_0^n)'} \geq 1. \quad (4.9)$$

As for relation (3.5) in Theorem 3.1, we also check numerically, if it is satisfied at each time t_n , $1 \leq n \leq N$, by taking $v(0, t_n) \approx v_0^n$ and by that we can replace it by

$$\min_{1 \leq n \leq N, 1 \leq i \leq M, \frac{v_0^n(\alpha_0 t_n + x_i)}{\alpha_0 - v_0^n} \geq 1. \quad (4.10)$$

Applying this approach with $T = 2$, we determine the admissible values of b in the initial condition (4.2), where for $m = 1$, the maximum value of b is given by $b = 1.75$, and for $m = 3$, the maximum value of b is given by $b = 0.07$.

Figures 7, 8, 9 and 10 show comparisons between the reconstructed and exact initial condition for $m = 1$ and $b = 1.5$, $b = 1.75$, $b = 2$ and $b = 3$, respectively.

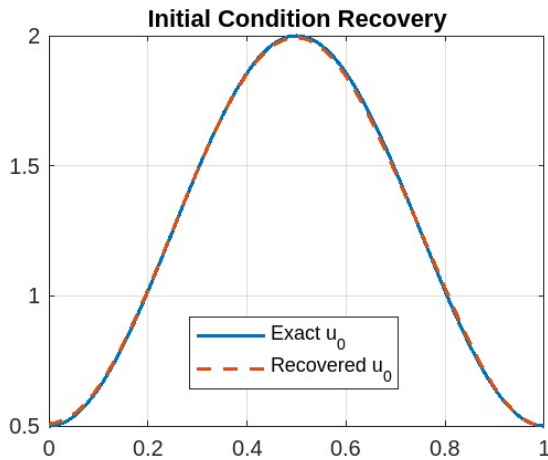


Figure 7. Comparison between reconstructed and exact initial condition for $m = 1$ and $b = 1.5$.

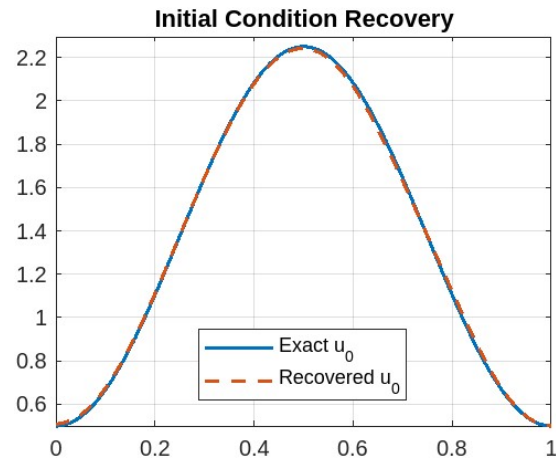


Figure 8. Comparison between reconstructed and exact initial condition for $m = 1$ and $b = 1.75$.

Figures 11, 12 and 13 show the comparison between the reconstructed and exact initial condition for $m = 3$ and $b = 0.01$, $b = 0.07$ and $b = 0.15$, respectively.

Remark 4.1. We note that in the direct problem (4.1), the value of b influences u_0' and leads to issues with the non-uniqueness of the corresponding direct solution. Numerically, we find that the admissible values of b related to Theorem 2.1, are similar to those obtained above.

White noise and relative error

In this part, we validate our algorithm by computing the data u_M^n , $n = 0, \dots, N$, using the direct iterative scheme (4.3), and then we study the stability of the scheme, taking $\hat{u}_M^n = u_M^n + \varepsilon_n$,

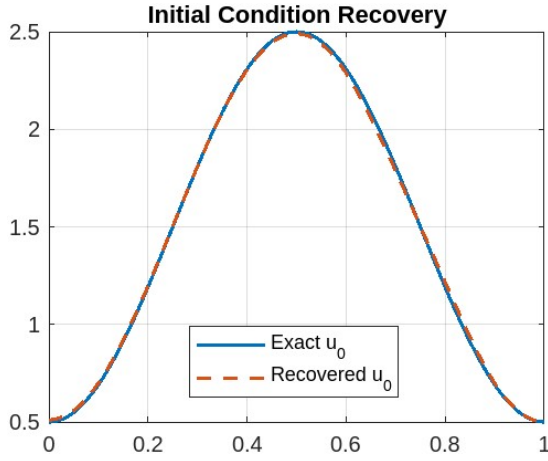


Figure 9. Comparison between reconstructed and exact initial condition for $m = 1$ and $b = 2$.

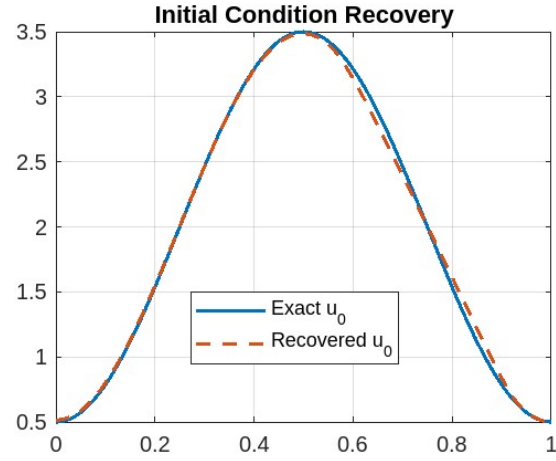


Figure 10. Comparison between reconstructed and exact initial condition for $m = 1$ and $b = 3$.

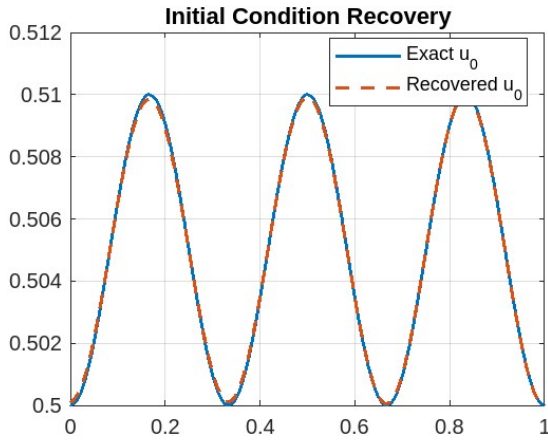


Figure 11. Comparison between reconstructed and exact initial condition for $m = 3$ and $b = 0.01$.

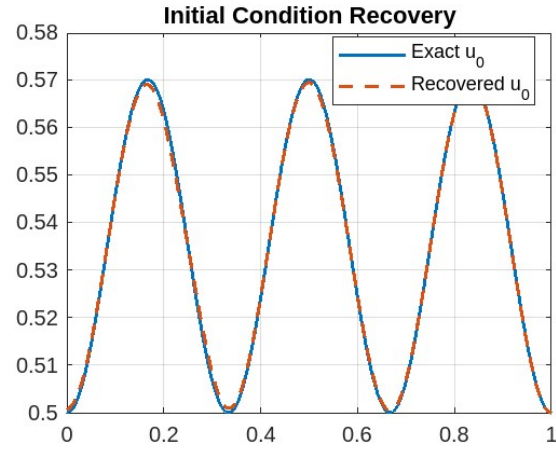


Figure 12. Comparison between reconstructed and exact initial condition for $m = 3$ and $b = 0.07$.

with the white noise represented in the interval $[t_n, t_{n+1}]$ by $\varepsilon_n = \alpha \cdot u_M^n \cdot X_n$, where α is the percentage error (e.g. $\alpha = 0.1$ means an error of 10%) with respect to the maximum value of the signal u_M^n , and X_n follows a standard normal distribution.

Figures 14 and 15 show comparisons between the exact initial condition u_0 and the numerical approximation u_i^0 , with white noise at $x = 1$ with $\alpha = 0.1$, for $(m, b) = (1, 1)$ and $(m, b) = (3, 0.04)$, respectively.

Stability

To numerically validate the stability relation (3.14) of solutions of this problem, expressed as:

$$\|\hat{u}^{0j} - u^0\|_{L^\infty(\Omega)} \leq C \|\hat{u}_M^{nj} - u_M^n\|_{L^\infty(0,T)}, \quad (4.11)$$

where C is a positive constant. We consider $T = 2$, $M = 1000$ and $N = MT$.

In order to validate the stability relation (4.11), we present the following example:

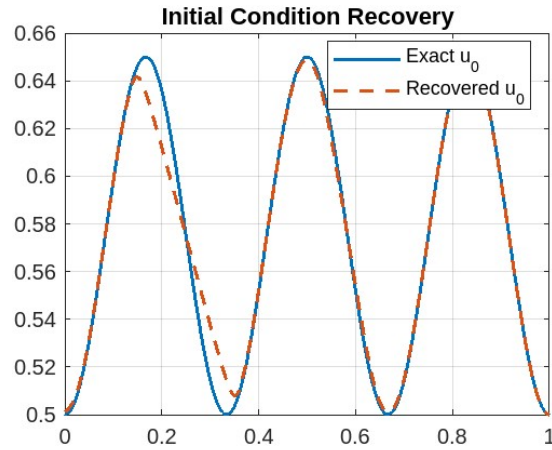


Figure 13. Comparison between reconstructed and exact initial condition for $m = 3$ and $b = 0.15$.

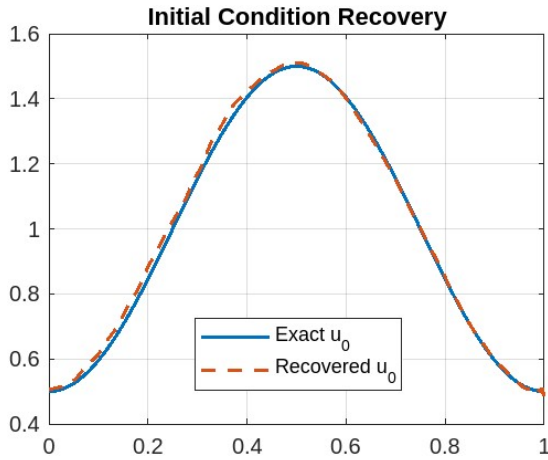


Figure 14. Comparison of the exact and reconstructed initial conditions u_0 with white noise on u at $x = 1$, for $\alpha = 0.1$, $m = 1$ and $b = 1$.

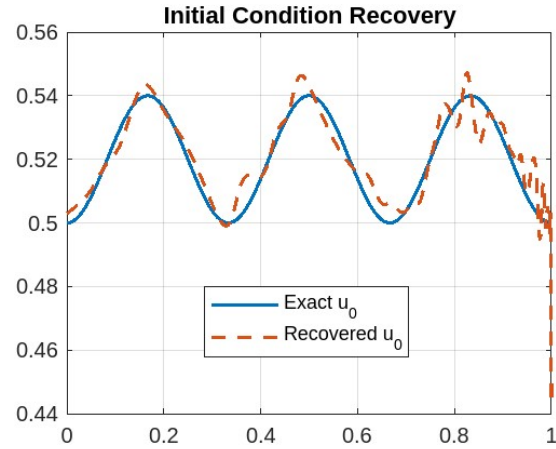


Figure 15. Comparison of the exact and reconstructed initial conditions u_0 with white noise on u at $x = 1$, for $\alpha = 0.1$, $m = 3$ and $b = 0.04$.

We solve the direct problem (4.3) to obtain the solution u_M^n , $n = 1, \dots, N$, which will be perturbed by an error e_u . In the numerical simulations, this perturbation will be given by $e_u^j = 0.01j$, $j = 1, \dots, 10$. Hence, the perturbed data at the boundary $x = 1$ will be denoted as $\hat{u}_M^{nj} = u_M^n + e_u^j$, $j = 1, \dots, 10$, which replaces the measured solution at $x = 1$ over time steps $t_n = n\Delta t$, $n = 0, \dots, N$. Next, we solve the inverse problem to recover the initial values, denoted by $\hat{u}^{0j}$. We then calculate the following errors:

$$e_1^j = \|\hat{u}_M^{nj} - u_M^n\|_{L^\infty(0,T)} \quad \text{and} \quad e_2^j = \|\hat{u}^{0j} - u^0\|_{L^\infty(\Omega)}.$$

We treat the following: For $(m, b) = (1, 1)$ and $(m, b) = (3, 0.04)$, to analyze the relationship between the previously defined norms, Figures 16 and 17 present the corresponding curves on a logarithmic scale ($\log_{10}(e_2^j)$ with respect to $\log_{10}(e_1^j)$), which are both of slope 0.99, very close to the expected theoretical value of 1. Note that the slope here is calculated using the least-squares method.

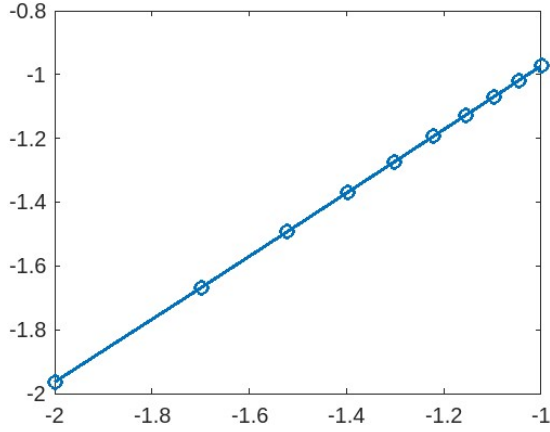


Figure 16. Variation of $\log_{10}(e_2^j)$ with respect to $\log_{10}(e_1^j)$, for $m = 1$ and $b = 1$.

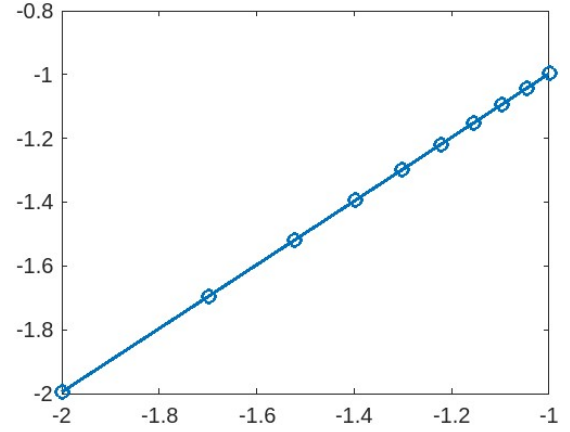


Figure 17. Variation of $\log_{10}(e_2^j)$ with respect to $\log_{10}(e_1^j)$, for $m = 3$ and $b = 0.04$.

4.2. Direct and inverse problem: With boundary conditions $u(0, t) = \alpha(t)$

In this section, we show numerical simulations for the direct and inverse problems, and the stability of the inverse problem, while considering the boundary condition $u(0, t) = \alpha(t)$, $t \in [0, T]$.

4.2.1. Direct problem: With boundary conditions $u(0, t) = \alpha(t)$

Using the same Godunov scheme (4.3) applied in the case where $\alpha(0) = a$, we study here the direct problem where the boundary condition is given by: For $t \in [0, T]$,

$$u(0, t) = \alpha(t) = \begin{cases} 0.5 + 3t^2 \left(t - \frac{T}{8} \right) & \text{if } t < \frac{T}{8}, \\ 0.5, & \text{if } t \geq \frac{T}{8}. \end{cases} \quad (4.12)$$

We consider $T = 2$. Figures 18 and 19 show comparisons between the exact and numerical solution at $t = 0.1$ and $t = 0.5$, respectively.

4.2.2. Inverse problem: Measurements on $x = 1$ and $u(0, t) = \alpha(t)$

As done in the previous section, we study the inverse problem using the backward method in time and space, with the initial data (4.2) and the boundary condition (4.12). Moreover, to ensure the uniqueness of the solution of the inverse problem, we follow the same approaches done before, and after several numerical simulations we get that the maximum value of b for $m = 1$ is $b = 1.75$ as well.

Figures 20 and 21 show comparisons between the reconstructed and exact initial data for $b = 1$ and $b = 1.75$, respectively, where for these values of b , the conditions are well verified and the uniqueness of the solution of the inverse problem is well satisfied.

However, Figures 22 and 23 show comparisons between the exact and reconstructed initial data for $b = 2$ and $b = 3$, respectively, which are greater than the maximum value of $b = 1.75$.

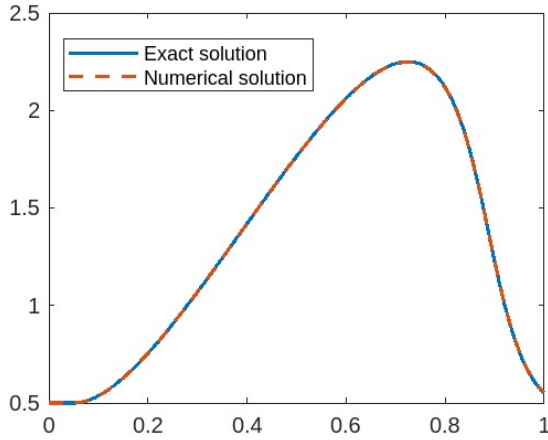


Figure 18. Comparison between the exact and numerical solution for $m = 1$ and $b = 1.75$.

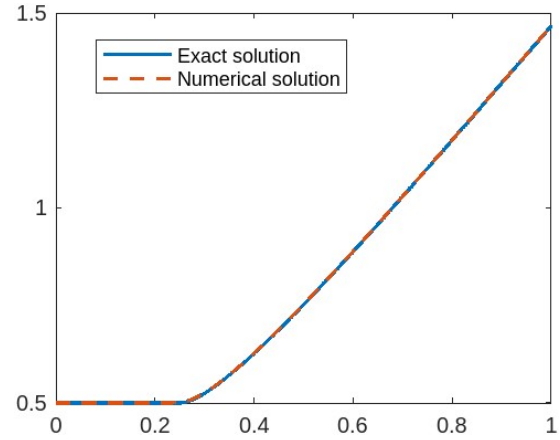


Figure 19. Comparison between the exact and numerical solution for $m = 1$ and $b = 1.75$.

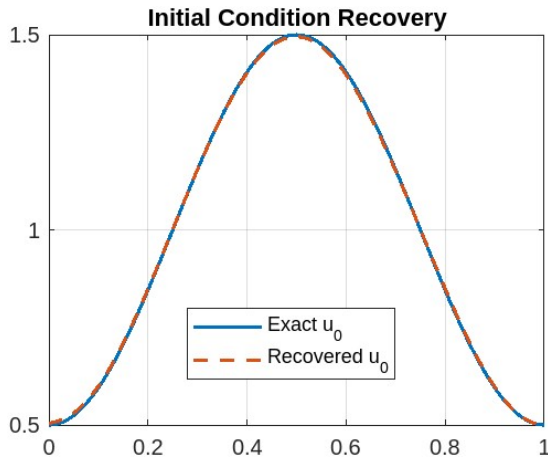


Figure 20. Comparison between the recovered and exact initial data for $m = 1$ and $b = 1$.

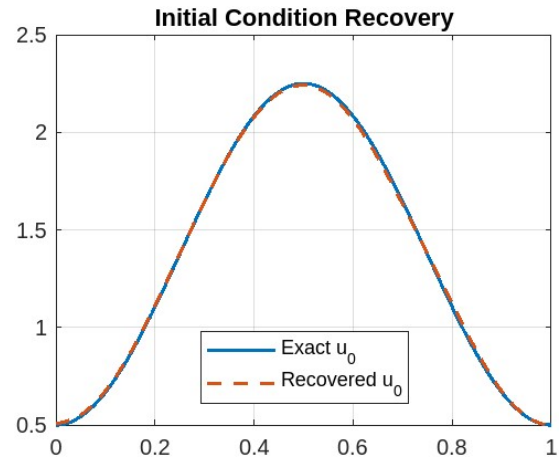


Figure 21. Comparison between the recovered and exact initial data for $m = 1$ and $b = 1.75$.

Stability

To numerically validate the stability relation (3.14) of solutions of this problem, expressed as:

$$\|\hat{u}^{0j} - u^0\|_{L^\infty(\Omega)} \leq C \|\hat{u}_M^{nj} - u_M^n\|_{L^\infty(0,T)}, \quad (4.13)$$

where C is a positive constant. We consider $T = 2$, $M = 1000$ and $N = MT$.

In order to validate the stability relation (4.13), we present the same example as in the previous part.

For $m = 1$ and $b = 1$, Figure 24 presents the corresponding curve on a logarithmic scale ($\log_{10}(e_2^j)$ with respect to $\log_{10}(e_1^j)$), which is of slope 0.99 very close to the expected theoretical value of 1. Note that the slope here is calculated using the least-squares method.

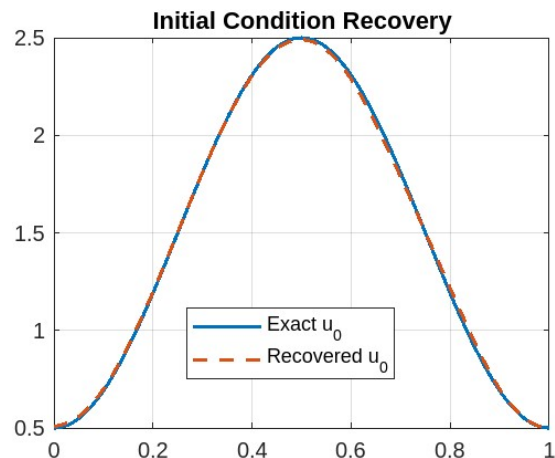


Figure 22. Comparison between the recovered and exact initial data for $m = 1$ and $b = 2$.

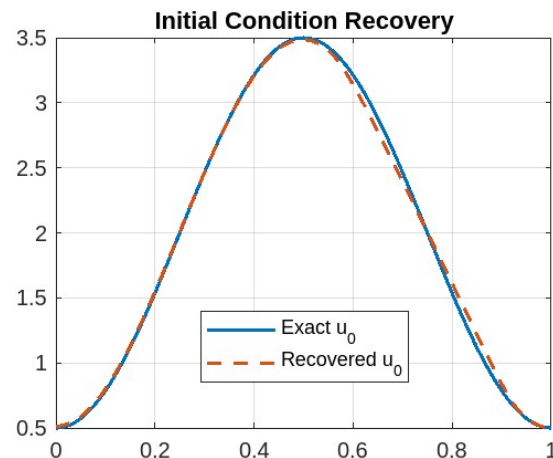


Figure 23. Comparison between the recovered and exact initial data for $m = 1$ and $b = 3$.

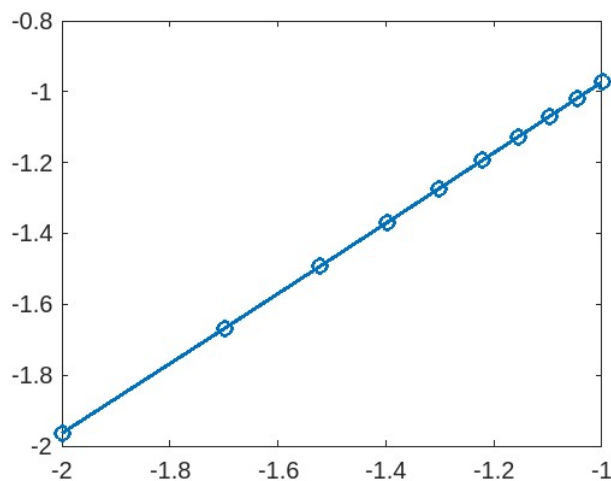


Figure 24. Variation of $\log_{10}(e_2^j)$ with respect to $\log_{10}(e_1^j)$, for $m = 1$ and $b = 1$.

4.3. Inverse problem: Non-smooth initial data and measurements on the boundary $x = 1$

In this section, although we have not studied this case theoretically, we reconstruct non-smooth initial data u_0 from the boundary data at $x = 1$ over time. We use the same backward method in space and time as in the previous case, starting from the final condition (the boundary condition) $u(1, t)$, $t \in [0, T]$, and propagating the solution backward to $t = 0$. We therefore consider the following non-smooth initial data

$$u_0(x) = a + b \cdot \begin{cases} x & \text{if } x \leq 0.5, \\ 1 - x & \text{if } x > 0.5, \end{cases}$$

where $a = 0.5$, $b = 1.75$ and we consider $M = 4000$.

Figure 25 shows the comparison between the reconstructed and initial data u_0 , for $b = 1.75$.

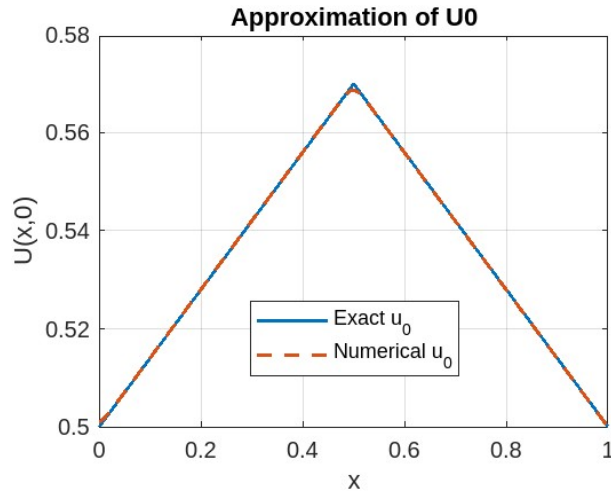


Figure 25. Comparison between exact and numerical initial data for $b = 1.75$.

We repeat the same work for the following non-smooth initial condition:

$$u_0(x) = a + b \cdot \begin{cases} 4x & \text{if } x \leq 0.25, \\ -4x + 2 & \text{if } 0.25 < x \leq 0.5, \\ 4x - 2 & \text{if } 0.5 < x \leq 0.75, \\ -4x + 4 & \text{if } x > 0.75, \end{cases}$$

where $a = 0.5$, $b = 0.05$ and we consider $M = 4000$. Figure 26 shows the comparison between exact and reconstructed initial data u_0 for $b = 0.05$.

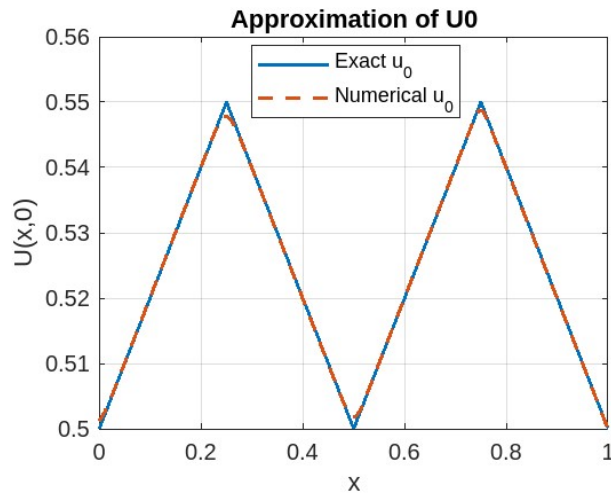


Figure 26. Comparison between exact and numerical initial data for $b = 0.05$.

Remark 4.2. Using the same algorithm and numerical methods as before, we observe that they perform well even for non-smooth initial data.

5. Conclusion

In this work, we studied the inverse problem for Burgers' equation with smooth initial data in the bounded domain $(0, 1)$, where the objective is to recover the initial data from boundary measurements at $x = 1$.

We established theoretical results on the existence and uniqueness of classical solutions for both the direct and inverse problems, and derived a stability estimate for the inverse problem. Then these theoretical results were validated through numerical simulations based on the Godunov scheme.

All data supporting the findings of this study are generated numerically and can be reproduced from the methodology described in the paper.

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