

THE SHARP BOUNDS OF HANKEL DETERMINANTS AND HANKEL DETERMINANTS OF LOGARITHMIC COEFFICIENTS FOR STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH GREGORY COEFFICIENTS

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Abstract For some subclasses of Starlike and Convex functions associated with Gregory coefficient, the sharp upper bounds for the second Hankel determinant and the third-order Hankel determinant and Hankel determinant $\mathcal{H}_{2,1}(G_g/2)$ and $\mathcal{H}_{2,2}(G_g/2)$ with a logarithmic coefficient are computed.

Keywords Analytic functions, univalent functions, starlike functions, convex functions, Gregory coefficients, logarithmic coefficients.

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1. Introduction

Suppose that an analytic function g is expressed in the following series form:

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} \delta_j \zeta^j \quad (\zeta \in \Lambda) \quad (1.1)$$

where $\Lambda = \{\zeta : |\zeta| < 1\}$. We use $\mathcal{H}$ to represent the family of these functions. Also $\mathcal{S} \subset \mathcal{H}$ denote the family of one-to-one or univalent functions defined in Λ .

Let $\mathcal{S}^*$ be the subclass of $\mathcal{H}$ consisting of functions which satisfy the condition:

$$\operatorname{Re}\left\{\frac{\zeta g'(\zeta)}{g(\zeta)}\right\} > 0 \quad (\zeta \in \Lambda).$$

$\mathcal{S}^*$ is known as the subclass of starlike functions. Further a function $g(\zeta)$ is said to belong to the class $\mathcal{C}$ of convex function in Λ if $g(\zeta) \in \mathcal{C}$ and satisfy following inequality

$$\operatorname{Re}\left\{1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right\} > 0 \quad (\zeta \in \Lambda).$$

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Let $j \geq 1$ and $i \geq 1$. The Hankel determinants $\mathcal{H}_{i,j}(g)$ is defined as follows

$$\mathcal{H}_{i,j}(g) = \begin{vmatrix} \delta_j & \delta_{j+1} & \cdots & \delta_{j+i-1} \\ \delta_{j+1} & \delta_{j+2} & \cdots & \delta_{j+i} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{j+i-1} & \delta_{i+j} & \cdots & \delta_{j+2i-2} \end{vmatrix},$$

and the investigation of the Hankel determinants for various classes of analytic functions started in the 1960. It was Pommerenke [12, 13] who first studied Hankel determinant for the class $\mathcal{S}$ of the univalent functions given by (1.1), and he proved in [13] that for all the functions $g \in \mathcal{S}$ we have $|\mathcal{H}_{i,j}(g)| \leq Mj^{-(\frac{1}{2}+\alpha)i+\frac{3}{2}}$, where $j, i \in \mathbb{N}$, $i \geq 2$, $\alpha > \frac{1}{4000}$, and M depends only on i . For some special choices of j and i , we have

$$\mathcal{H}_{2,3}(g) = -\delta_4^2 + \delta_5\delta_3, \quad (1.2)$$

$$\mathcal{H}_{3,1}(g) = 2\delta_4\delta_3\delta_2 - \delta_3^3 + \delta_5\delta_3 - \delta_5\delta_2^2 - \delta_4^2. \quad (1.3)$$

Non-sharp estimate on third Hankel determinant for classes of starlike and convex functions were investigated by Zaprawa [23] as $|\mathcal{H}_{3,1}(g)| \leq 1$ and $|\mathcal{H}_{3,1}(g)| \leq 49/540$, respectively. Kowalczyk et al. [9], in 2018, proved that the bound $|\mathcal{H}_{3,1}(g)| \leq 4/135$ is sharp for the class of convex functions. However, the best known estimate for starlike functions is $|\mathcal{H}_{3,1}(g)| \leq 4/9$ due to Kowalczyk et al. [10]. The sharp bounds of Hankel determinant for univalent functions were found [2, 4, 11, 14–16, 18, 20–22].

The logarithmic coefficients λ_j of $g \in \mathcal{S}$ are defined by the formula

$$\log \frac{g(\zeta)}{\zeta} = 2 \sum_{j=1}^{\infty} \lambda_j(g) \zeta^j. \quad (1.4)$$

We use $\lambda_j(g) = \lambda_j$ when there is no confusion. These coefficients play an important role for various estimates in the theory of univalent functions.

In 2022, Kowalczyk and Lecko [7] studied the Hankel determinant, whose entries are logarithmic coefficients of $g \in \mathcal{S}$, that is

$$\mathcal{H}_{i,j}(G_g/2) = \begin{vmatrix} \lambda_j & \lambda_{j+1} & \cdots & \lambda_{j+i-1} \\ \lambda_{j+1} & \lambda_{j+2} & \cdots & \lambda_{j+i} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j+i-1} & \lambda_{i+j} & \cdots & \lambda_{j+2i-2} \end{vmatrix}.$$

Differentiating (1.4) and using (1.1), we have

$$\begin{cases} \lambda_1 = \frac{1}{2}\delta_2, \\ \lambda_2 = \frac{1}{2}(\delta_3 - \frac{1}{2}\delta_2^2), \\ \lambda_3 = \frac{1}{2}(\delta_4 - \delta_2\delta_3 + \frac{1}{3}\delta_2^3), \\ \lambda_4 = \frac{1}{2}(\delta_5 - \delta_4\delta_2 + \delta_3\delta_2^2 - \frac{1}{2}\delta_3^2 - \frac{1}{4}\delta_2^4). \end{cases} \quad (1.5)$$

Therefore,

$$\mathcal{H}_{2,1}(G_g/2) = \lambda_3\lambda_1 - \lambda_2^2 = \frac{1}{48}(12\delta_2\delta_4 + \delta_2^4 - 12\delta_3^2), \quad (1.6)$$

$$\begin{aligned} \mathcal{H}_{2,2}(G_g/2) &= \lambda_2\lambda_4 - \lambda_3^2 \\ &= \frac{1}{288}(72\delta_3\delta_5 + 18\delta_2^2\delta_3^2 + 72\delta_2\delta_3\delta_4 - 36\delta_3^3 - 6\delta_2^4\delta_3 - 36\delta_2^2\delta_5 - 12\delta_2^2\delta_4 + \delta_2^6 - 72\delta_4^2). \end{aligned} \quad (1.7)$$

In 2022, Shi et al. [17] introduced and studied the Hankel determinant of logarithmic coefficients for bounded turning functions connected to a three-leaf-shaped domain. In 2024, Allu and Shaji [1] studied second Hankel determinant for logarithmic inverse coefficients of convex and starlike functions. For more related results one may refer to the work [6, 8]

In 2024, Kazimoğlu [5] and co-authors introduced a subclass of starlike functions associated with Gregory coefficients as follows:

$$\mathcal{S}_G^* = \left\{ g : g \in \mathcal{H} \quad \text{and} \quad \frac{\zeta g'(\zeta)}{g(\zeta)} \prec \Phi(\zeta) = \frac{\zeta}{\ln(1+\zeta)} = \sum_{i=0}^{\infty} G_i \zeta^i \quad (\zeta \in \Lambda \quad \text{and} \quad \zeta \neq 0) \right\}. \quad (1.8)$$

Clearly, G_i is given, for some values of i , as follows:

$$G_0 = 1, G_1 = \frac{1}{2}, G_2 = -\frac{1}{12}, G_3 = \frac{1}{24}, G_4 = -\frac{79}{720}, G_5 = \frac{3}{160}, G_6 = -\frac{863}{60480}.$$

Srivastava. et al. [19] introduced and studied a class of novel convex functions associated with Gregory coefficients as follows:

$$\mathcal{C}_G = \left\{ g : g \in \mathcal{H} \quad \text{and} \quad 1 + \frac{\zeta g''(\zeta)}{g'(\zeta)} \prec \Phi(\zeta) = \frac{\zeta}{\ln(1+\zeta)} \quad (\zeta \in \Lambda) \right\}. \quad (1.9)$$

For the class $\mathcal{S}_G^*$ in [5] the authors showed that

$$|\mathcal{H}_{3,1}(g)| \leq \frac{43}{576}.$$

For the class $\mathcal{C}_G$ in [19] the authors obtained the bound of Hankel determinant

$$|\mathcal{H}_{3,1}(g)| \leq \frac{5461}{1241680}.$$

In the paper, we prove that $|\mathcal{H}_{3,1}(g)| \leq \frac{1}{36}$ for $g \in \mathcal{S}_G^*$, and so giving the sharp bound $|\mathcal{H}_{3,1}(g)| \leq \frac{1}{576}$ for $g \in \mathcal{C}_G$. Also, we obtain the sharp bounds of the Hankel determinant of logarithmic coefficients for the classes $\mathcal{S}_G^*$ and $\mathcal{C}_G$.

Lemma 1.1 (Lemma 1.1 [3]). *Let*

$$\varpi(\zeta) = c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \cdots \quad (1.10)$$

be a Schwarz function. Then

$$c_2 = (1 - |c_1|^2)d, \quad (1.11)$$

$$c_3 = -(1 - |c_1|^2)\bar{c}_1 d^2 + (1 - |c_1|^2)(1 - |d|^2)t, \quad (1.12)$$

$$\begin{aligned} c_4 &= (1 - |c_1|^2)\bar{c}_1^2 d^3 - (1 - |c_1|^2)(1 - |d|^2)(\bar{d}t^2 + 2t\bar{c}_1 d) \\ &\quad + (1 - |c_1|^2)(1 - |d|^2)(1 - |t|^2)s, \end{aligned} \quad (1.13)$$

for some d, t, s with $|d| \leq 1$, $|t| \leq 1$ and $|s| \leq 1$.

2. Main results

Theorem 2.1. *If $g \in \mathcal{S}_G^*$ is of the form (1.1), then*

$$|\mathcal{H}_{2,3}| = |\delta_5\delta_3 - \delta_4^2| \leq \frac{1}{36}.$$

The inequalities are sharp with the extremal function:

$$g_1(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\Phi(t^3) - 1}{t} dt\right) = \zeta + \frac{1}{6}\zeta^4 + \dots \quad (2.1)$$

Proof. Since $g \in \mathcal{S}_G^*$, it follows that there exists a Schwarz function ϖ , such that

$$\frac{\zeta g'(\zeta)}{g(\zeta)} = \frac{\varpi(\zeta)}{\ln(1 + \varpi(\zeta))}. \quad (2.2)$$

From (1.1), we have

$$\begin{aligned} \frac{\zeta g'(\zeta)}{g(\zeta)} &= 1 + \delta_2\zeta + (-\delta_2^2 + 2\delta_3)\zeta^2 + (-3\delta_2\delta_3 + \delta_2^3 + 3\delta_4)\zeta^3 \\ &\quad + (-4\delta_2\delta_4 - 2\delta_3^2 - \delta_2^4 + 4\delta_2^2\delta_3 + 4\delta_5)\zeta^4 + \dots \end{aligned} \quad (2.3)$$

Using (1.10) together with $\frac{\varpi(\zeta)}{\ln(1+\varpi(\zeta))}$, we have

$$\begin{aligned} \frac{\varpi(\zeta)}{\ln(1 + \varpi(\zeta))} &= 1 + \frac{1}{2}c_1\zeta + \frac{1}{12}(-c_1^2 + 6c_2)\zeta^2 + \frac{1}{24}(c_1^3 - 4c_1c_2 + 12c_3)\zeta^3 \\ &\quad + \frac{1}{720}(-19c_1^4 + 90c_2c_1^2 - 60c_2^2 - 120c_1c_3 + 360c_4)\zeta^4 + \dots \end{aligned} \quad (2.4)$$

Equating the coefficients of (2.3) and (2.4), we have

$$\delta_2 = \frac{1}{2}c_1, \quad (2.5)$$

$$\delta_3 = \frac{1}{12}(3c_2 + c_1^2), \quad (2.6)$$

$$\delta_4 = \frac{1}{72}(12c_3 + 5c_1c_2 + c_1^3), \quad (2.7)$$

$$\delta_5 = \frac{1}{1440}(180c_4 + 60c_1c_3 + 35c_1^2c_2 + 15c_2^2 - 2c_1^4). \quad (2.8)$$

Using (2.6) – (2.8) and (1.2), we obtain

$$\begin{aligned} \delta_3\delta_5 - \delta_4^2 &= \frac{1}{51840}(1620c_2c_4 - 660c_1c_2c_3 + 110c_1^2c_2^2 + 135c_2^3 - 13c_1^4c_2 + 540c_1^2c_4 - 60c_1^3c_3 \\ &\quad - 16c_1^6 - 1440c_3^2). \end{aligned} \quad (2.9)$$

From (2.9) and Lemma 1.1, we have

$$\begin{aligned} \mathcal{H}_{2,3}(g) &= \frac{1}{51840} \left[-16c_1^6 + 180\bar{c}_1^2(1 - |c_1|^2)^2d^4 - 1620d\bar{d}(1 - |d|^2)(1 - |c_1|^2)^2t^2 \right. \\ &\quad \left. + 660(1 - |c_1|^2)^2c_1\bar{c}_1d^3 - 360t(1 - |c_1|^2)^2(1 - |d|^2)\bar{c}_1d^2 \right] \end{aligned}$$

$$\begin{aligned}
& -660tdc_1(1-|d|^2)(1-|c_1|^2)^2 + 1620d(1-|d|^2)(1-|c_1|^2)^2(1-|t|^2)s \\
& + 110c_1^2(1-|c_1|^2)^2d^2 + 135(1-|c_1|^2)^3d^3 - 13(1-|c_1|^2)c_1^4d + 540(1-|c_1|^2)c_1^2\bar{c}_1^2d^3 \\
& + 60(1-|c_1|^2)c_1^3\bar{c}_1d^2 - 1440t^2(1-|c_1|^2)^2(1-|d|^2)^2 - 60(1-|c_1|^2) \\
& \times (1-|d|^2)c_1^3t + 540(1-|c_1|^2)(1-|t|^2)(1-|d|^2)c_1^2s - 540(1-|c_1|^2)(1-|d|^2)c_1^2\bar{d}t^2 \\
& - 1080d(1-|c_1|^2)(1-|d|^2)c_1^2\bar{c}_1t].
\end{aligned}$$

Thus, we yield

$$H_{2,3}(g) = \frac{1}{51840} [\alpha_1(c_1, d) + \alpha_2(c_1, d)t + \alpha_3(c_1, d)t^2 + \alpha_4(c_1, d, t)s],$$

where $c_1, d, t \in \bar{\Lambda}$ and

$$\begin{aligned}
\alpha_1(c_1, d) &= -16c_1^6 + (1-|c_1|^2) [(1-|c_1|^2)(525|c_1|^2d^3 + 180\bar{c}_1^2d^4 + 110c_1^2d^2 + 135d^3) \\
&\quad - 13c_1^4d + 540c_1^2\bar{c}_1^2d^3 + 60c_1^3\bar{c}_1d^2], \\
\alpha_2(c_1, d) &= 60(1-|c_1|^2)(1-|d|^2) [(-11c_1d - 6\bar{c}_1d^2)(1-|c_1|^2) - c_1^3 - 18c_1^2\bar{c}_1d], \\
\alpha_3(c_1, d) &= 180(1-|c_1|^2)(1-|d|^2) [(-|d|^2 - 8)(1-|c_1|^2) - 3\bar{d}c_1^2], \\
\alpha_4(c_1, d, t) &= 540(1-|t|^2)(1-|d|^2)(1-|c_1|^2) [3d(1-|c_1|^2) + c_1^2].
\end{aligned}$$

Setting $|c_1| = c$, $|d| = d$, $|t| = e$ and utilizing $|s| \leq 1$, we achieve

$$\begin{aligned}
|H_{2,3}(g)| &\leq \frac{1}{51840} (|\alpha_1(c_1, d)| + |\alpha_2(c_1, d)|e + |\alpha_3(c_1, d)|e^2 + |\alpha_4(c_1, d, t)|) \\
&\leq \frac{1}{51840} \Psi_1(c, d, e),
\end{aligned}$$

where

$$\Psi_1(c, d, e) = \beta_1(c, d) + \beta_2(c, d)e + \beta_3(c, d)e^2 + \beta_4(c, d)(1 - e^2),$$

with

$$\begin{aligned}
\beta_1(c, d) &= 16c^6 + (1-c^2) [(525c^2d^3 + 180c^2d^4 + 135d^3 + 110c^2d^2)(1-c^2) + 13c^4d \\
&\quad + 60c^4d^2 + 540c^4d^3], \\
\beta_2(c, d) &= 60(1-c^2)(1-d^2) [(11cd + 6cd^2)(1-c^2) + c^3 + 18c^3d], \\
\beta_3(c, d) &= 180(1-d^2)(1-c^2)[(8+d^2)(1-c^2) + 3dc^2], \\
\beta_4(c, d) &= 540(1-c^2)(1-d^2) [3d(1-c^2) + c^2].
\end{aligned}$$

We will maximize $\Psi_1(c, d, e)$ in the closed cuboid $\Theta : [0, 1] \times [0, 1] \times [0, 1]$.

By noting that

$$\beta_3(c, d) \leq 180(1-c^2)(1-d^2)[(1-c^2)(8+d^2) + 3c^2] = \theta_1(c, d).$$

Therefore, we get

$$\Psi_1(c, d, e) \leq \beta_1(c, d) + \beta_2(c, d)e + \theta_1(c, d)e^2 + \beta_4(c, d)(1 - e^2) = \chi_1(c, d, e).$$

Differentiating χ_1 with respect to e , we yield

$$\frac{\partial \chi_1}{\partial e} = \beta_2(c, d) + 2e[\theta_1(c, d) - \beta_4(c, d)].$$

In view of $\beta_2(c, d) \geq 0$ and

$$\theta_1(c, d) - \beta_4(c, d) = 180(1 - c^2)^2(1 - d^2)(d^2 - 9d + 8) \geq 0$$

on $[0, 1] \times [0, 1]$, we have $\frac{\partial \chi_1}{\partial e} \geq 0$ for all $e \in [0, 1]$. So, $\chi_1(c, d, e)$ attains its maximum value at $e = 1$. Thus, we achieve

$$\begin{aligned} \Psi_1(c, d, e) &\leq \chi_1(c, d, e) \\ &\leq \chi_1(c, d, 1) \\ &= \beta_1(c, d) + \beta_2(c, d) + \theta_1(c, d) \\ &= 16c^6 - 60c^5 + 900c^4 + 60c^3 - 2340c^2 + 1440 + (-13c^6 - 420c^5 + 13c^4 - 240c^3 \\ &\quad + 660c)d + (50c^6 + 420c^5 - 880c^4 - 780c^3 + 2090c^2 + 360c - 1260)d^2 + (-15c^6 \\ &\quad + 420c^5 - 375c^4 + 240c^3 + 255c^2 - 660c + 135)d^3 + (180c^6 - 360c^5 - 540c^4 \\ &\quad + 720c^3 + 540c^2 - 360c - 180)d^4 \\ &= \gamma_1(c, d). \end{aligned}$$

The critical points of $\gamma_1(c, d)$ satisfy the system of equations

$$\left\{ \begin{aligned} \frac{\partial \gamma_1}{\partial c} &= 96c^5 - 300c^4 + 3600c^3 + 180c^2 - 4680c + (-78c^5 - 2100c^4 + 52c^3 - 720c^2 + 660)d \\ &\quad + (300c^5 + 2100c^4 - 3520c^3 - 2340c^2 + 4180c + 360)d^2 + (-90c^5 + 2100c^4 - 1500c^3 \\ &\quad + 720c^2 + 510c - 660)d^3 + (1080c^5 - 1800c^4 - 2160c^3 + 2160c^2 + 1080c - 360)d^4 \\ &= 0, \\ \frac{\partial \gamma_1}{\partial d} &= 660c - 240c^3 + 13c^4 - 420c^5 - 13c^6 + (100c^6 + 840c^5 - 1760c^4 - 1560c^3 + 4180c^2 \\ &\quad + 720c - 2520)d + (-45c^6 + 1260c^5 - 1125c^4 + 720c^3 + 765c^2 - 1980c + 405)d^2 \\ &\quad + (720c^6 - 1440c^5 - 2160c^4 + 2880c^3 + 2160c^2 - 1440c - 720)d^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we get

$$\begin{aligned} &\left\{ \begin{aligned} c_{1,1} &\approx 1, \\ d_{1,1} &\approx 1.2571, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,2} &\approx 1, \\ d_{1,2} &\approx -1.8071, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,3} &\approx 1, \\ d_{1,3} &\approx -0.4500, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,4} &\approx -1, \\ d_{1,4} &\approx 0.6353, \end{aligned} \right. \\ &\left\{ \begin{aligned} c_{1,5} &\approx -1, \\ d_{1,5} &\approx -0.8626, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,6} &\approx -1, \\ d_{1,6} &\approx 0.4866, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,7} &= 0, \\ d_{1,7} &= 0, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,8} &\approx 1.3967, \\ d_{1,8} &\approx 2.6949, \end{aligned} \right. \\ &\left\{ \begin{aligned} c_{1,9} &\approx 1.5999, \\ d_{1,9} &\approx 0.7740, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,10} &\approx -0.6139, \\ d_{1,10} &\approx 1.0077, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,11} &\approx -0.6928, \\ d_{1,11} &\approx -9.1788, \end{aligned} \right. & \left\{ \begin{aligned} c_{1,12} &\approx -0.9967, \\ d_{1,12} &\approx 0.5649, \end{aligned} \right. \\ &\left\{ \begin{aligned} c_{1,13} &\approx -1.3017, \\ d_{1,13} &\approx -0.5042. \end{aligned} \right. \end{aligned}$$

Therefore, there are no critical points in $(0, 1) \times (0, 1)$.

(1) For $c = 1$,

$$\gamma_1(1, d) = 16.$$

(2) For $c = 0$,

$$\gamma_1(0, d) = \tau_1(d) = 1440 - 1260d^2 + 135d^3 - 180d^4.$$

Upon calculations, we see that $\tau_1(d)$ is a decreasing function in $[0, 1]$ and hence attains its maximum value at $d = 0$. Hence

$$\gamma_1(0, d) = \tau_1(d) \leq \tau_1(0) \leq 1440.$$

(3) For $d = 0$

$$\gamma_1(c, 0) = \tau_2(c) = 1440 - 2340c^2 + 60c^3 + 900c^4 - 60c^5 + 16c^6.$$

Now, $\tau_2'(c) = 0$ when $c = \iota_0 = 0$ in the interval $[0, 1]$ with ι_0 as point of maxima. Hence,

$$\gamma_1(c, 0) = \tau_2(c) \leq \tau_2(0) = 1440.$$

(4) For $d = 1$,

$$\gamma_1(c, 1) = 135 + 545c^2 - 882c^4 + 218c^6.$$

Let

$$\tau_3(c) = \gamma_1(c, 1) = 135 + 545c^2 - 882c^4 + 218c^6, \quad (2.10)$$

$$\tau_3'(c) = 1090c - 3528c^3 + 1308c^5, \quad (2.11)$$

$$\tau_3''(c) = 1090 - 10584c^2 + 6540c^4. \quad (2.12)$$

For optimum value of $\tau_3(c)$, consider $\tau_3'(c) = 0$. From (2.11), we get $c = \iota_1 \approx 0.5966$ and $c = \iota_2 = 0$. Substituting the value of $c = \iota_1 \approx 0.5966$ in (2.12), we observe that $\tau_3'(\iota_1) \approx -1848.6 < 0$. Therefore, by the second derivative test, $\tau_3(c)$ has maximum value at $c = \iota_1$, from (2.10), it is obtained to be $\max\{\tau_3(c)\} \approx 227.0746$. Hence,

$$\gamma_1(c, 1) = \tau_3(c) \leq \tau_3(0.5966) \approx 227.0746.$$

Therefore, we yield

$$|\delta_5\delta_3 - \delta_4^2| \leq \frac{1440}{51840} = \frac{1}{36}.$$

□

Theorem 2.2. *If $g \in \mathcal{S}_G^*$, then*

$$|\mathcal{H}_{3,1}(g)| \leq \frac{1}{36}.$$

The bound is sharp and is given by function g_1 .

Proof. Let $g \in \mathcal{S}_G^*$. From (2.5) – (2.8) and (1.3), we have

$$\mathcal{H}_{3,1}(g) = \frac{1}{51840} (120c_3c_1^3 - 118c_2c_1^4 + 65c_2^2c_1^2 + 1500c_3c_2c_1 + 1620c_4c_2 - 1080c_4c_1^2 - 1440c_3^2 - 675c_2^3 + 32c_1^6). \quad (2.13)$$

From (2.13) and Lemma 1.1, we yield

$$\begin{aligned} \mathcal{H}_{3,1}(g) = & \frac{1}{51840} [32c_1^6 + 180(1 - |c_1|^2)^2\bar{c}_1^2d^4 - 1620(1 - |d|^2)(1 - |c_1|^2)^2d\bar{d}t^2 - 360(1 - |c_1|^2)^2 \\ & \times (1 - |d|^2)\bar{c}_1td^2 + 1620(1 - |c_1|^2)^2(1 - |d|^2)(1 - |t|^2)sd - 1500(1 - |c_1|^2)^2c_1\bar{c}_1d^3 \\ & + 65(1 - |c_1|^2)^2c_1^2d^2 + 1500(1 - |c_1|^2)^2(1 - |d|^2)c_1dt - 675(1 - |c_1|^2)^3d^3 \\ & - 118(1 - |c_1|^2)c_1^4d - 1080(1 - |c_1|^2)c_1^2\bar{c}_1^2d^3 + 1080(1 - |d|^2)(1 - |c_1|^2)c_1^2\bar{d}t^2 \\ & + 2160(1 - |c_1|^2)(1 - |d|^2)c_1^2\bar{c}_1td - 1080c_1^2(1 - |c_1|^2)(1 - |d|^2)s \\ & - 120c_1^3\bar{c}_1(1 - |c_1|^2)d^2 + 120c_1^3(1 - |c_1|^2)(1 - |d|^2)t - 1440(1 - |c_1|^2)^2(1 - |d|^2)^2t^2]. \end{aligned}$$

Thus, we have

$$\mathcal{H}_{3,1}(g) = \frac{1}{51840} [\alpha_5(c_1, d) + \alpha_6(c_1, d)t + \alpha_7(c_1, d)t^2 + \alpha_8(c_1, d, t)s],$$

where

$$\begin{aligned} \alpha_5(c_1, d) &= 32c_1^6 + (1 - |c_1|^2) [(180\bar{c}_1^2d^4 - 825|c_1|^2d^3 + 65c_1^2d^2 - 675d^3)(1 - |c_1|^2) \\ &\quad - 118c_1^4d - 1080c_1^2\bar{c}_1^2d^3 - 120c_1^3\bar{c}_1d^2], \\ \alpha_6(c_1, d) &= (1 - |d|^2)(1 - |c_1|^2)[(-360\bar{c}_1d^2 + 1500c_1d)(1 - |c_1|^2) + 2160c_1^2\bar{c}_1d + 120c_1^3], \\ \alpha_7(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2)[(-1440 - 180|d|^2)(1 - |c_1|^2) + 1080c_1^2\bar{d}], \\ \alpha_8(c_1, d, t) &= (1 - |c_1|^2)(1 - |t|^2)(1 - |d|^2)[1620d(1 - |c_1|^2) - 1080c_1^2]. \end{aligned}$$

Setting $c = |c_1|$, $d = |d|$, $e = |t|$ and upon using $|s| \leq 1$, we yield

$$\begin{aligned} |H_{3,1}(g)| &\leq \frac{1}{51840} (|\alpha_5(c_1, d)| + |\alpha_6(c_1, d)|e + |\alpha_7(c_1, d)|e^2 + |\alpha_8(c_1, d, t)|) \\ &\leq \frac{1}{51840} \Psi_2(c, d, e), \end{aligned}$$

where

$$\Psi_2(c, d, e) = \beta_5(c, d) + \beta_6(c, d)e + \beta_7(c, d)e^2 + \beta_8(c, d)(1 - e^2),$$

with

$$\begin{aligned} \beta_5(c, d) &= 32c^6 + (1 - c^2)[(675d^3 + 65c^2d^2 + 825c^2d^3 + 180c^2d^4)(1 - c^2) \\ &\quad + 118dc^4 + 120d^2c^4 + 1080d^3c^4], \\ \beta_6(c, d) &= (1 - c^2)(1 - d^2)[(360cd^2 + 1500dc)(1 - c^2) + 120c^3 + 2160c^3d], \\ \beta_7(c, d) &= (1 - d^2)(1 - c^2)[(1440 + 180d^2)(1 - c^2) + 1080dc^2], \\ \beta_8(c, d) &= (1 - c^2)(1 - d^2)[1080c^2 + 1620d(1 - c^2)]. \end{aligned}$$

Note that

$$\beta_7(c, d) \leq (1 - c^2)(1 - d^2)[(1 - c^2)(180d^2 + 1440) + 1080c^2] = \theta_2(c, d),$$

then, we have

$$\Psi_2(c, d, e) \leq \beta_5(c, d) + \beta_6(c, d)e + \theta_2(c, d)e^2 + \beta_8(c, d)(1 - e^2) = \chi_2(c, d, e).$$

Differentiating $\chi_2(c, d, e)$ with respect to e , we achieve

$$\frac{\partial \chi_2}{\partial e} = \beta_6(c, d) + 2[\theta_2(c, d) - \beta_8(c, d)]e.$$

Upon $\beta_6(c, d) \geq 0$ and

$$\theta_2(c, d) - \beta_8(c, d) = 180(1 - d^2)(1 - c^2)^2(d^2 - 9d + 8) \geq 0,$$

we have $\frac{\partial \chi_2}{\partial e} \geq 0$. So, $\chi_2(c, d, e)$ attains its maximum value at $e = 1$. Therefore, we have

$$\begin{aligned} \Psi_2(c, d, e) &\leq \chi_2(c, d, e) \\ &\leq \chi_2(c, d, 1) \\ &= \beta_5(c, d) + \beta_6(c, d) + \theta_2(c, d) \\ &= 32c^6 - 120c^5 + 360c^4 + 120c^3 - 1800c^2 + 1440 + (-118c^6 - 660c^5 + 118c^4 \\ &\quad - 840c^3 + 1500c)d + (-55c^6 + 480c^5 - 190c^4 - 840c^3 + 1505c^2 + 360c \\ &\quad - 1260)d^2 + (-255c^6 + 660c^5 + 105c^4 + 840c^3 - 525c^2 - 1500c + 675)d^3 \\ &\quad + (-180 - 360c + 540c^2 + 720c^3 - 540c^4 - 360c^5 + 180c^6)d^4 \\ &= \gamma_2(c, d). \end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \gamma_2}{\partial c} = 192c^5 - 600c^4 + 1440c^3 + 360c^2 - 3600c \\ \quad + (-708c^5 - 3300c^4 + 472c^3 - 2520c^2 + 1500)d \\ \quad + (-330c^5 + 2400c^4 - 760c^3 - 2520c^2 + 3010c + 360)d^2 \\ \quad + (-1530c^5 + 3300c^4 + 420c^3 + 2520c^2 - 1050c - 1500)d^3 \\ \quad + (1080c^5 - 1800c^4 - 2160c^3 + 2160c^2 + 1080c - 360)d^4 \\ \quad = 0, \\ \frac{\partial \gamma_2}{\partial d} = 1500c - 840c^3 + 118c^4 - 660c^5 - 118c^6 \\ \quad + (-110c^6 + 960c^5 - 380c^4 - 1680c^3 + 3010c^2 + 720c - 2520)d \\ \quad + (-765c^6 + 1980c^5 + 315c^4 + 2520c^3 - 1575c^2 - 4500c + 2025)d^2 \\ \quad + (-720 - 1440c + 2160c^2 + 2880c^3 - 2160c^4 - 1440c^5 + 720c^6)d^3 \\ \quad = 0. \end{array} \right.$$

By calculating, we achieve

$$\begin{aligned} & \begin{cases} c_{1,1} \approx 1, \\ d_{1,1} \approx 1.2775, \end{cases} \quad \begin{cases} c_{1,2} \approx 1, \\ d_{1,2} \approx -1.8434, \end{cases} \quad \begin{cases} c_{1,3} \approx 1, \\ d_{1,3} \approx -0.4341, \end{cases} \quad \begin{cases} c_{1,4} \approx -1, \\ d_{1,4} \approx -0.8505, \end{cases} \\ & \begin{cases} c_{1,5} = 0, \\ d_{1,5} = 0, \end{cases} \quad \begin{cases} c_{1,6} \approx 1.4857, \\ d_{1,6} \approx 4.4681, \end{cases} \quad \begin{cases} c_{1,7} \approx 0.2556, \\ d_{1,7} \approx 1.0307, \end{cases} \quad \begin{cases} c_{1,8} \approx -0.7075, \\ d_{1,8} \approx -19.5820, \end{cases} \\ & \begin{cases} c_{1,9} \approx -1.0078, \\ d_{1,9} \approx 0.5504, \end{cases} \quad \begin{cases} c_{1,10} \approx -1.8336, \\ d_{1,10} \approx -0.4557, \end{cases} \quad \begin{cases} c_{1,11} \approx -1.8676, \\ d_{1,11} \approx 3.1714. \end{cases} \end{aligned}$$

Thus, there are no critical points in $(0, 1) \times (0, 1)$.

(5) For $c = 1$,

$$\gamma_2(1, d) = 32.$$

(6) For $c = 0$,

$$\gamma_2(0, d) = 1440 - 1260d^2 + 675d^3 - 180d^4 \leq 1440.$$

(7) For $d = 0$,

$$\gamma_2(c, 0) = 1440 - 1800c^2 + 120c^3 + 360c^4 - 120c^5 + 32c^6 \leq 1440.$$

(8) For $d = 1$

$$\gamma_2(c, 1) = 675 - 280c^2 - 147c^4 - 216c^6 \leq 675.$$

Thus, we have

$$|\mathcal{H}_{3,1}(g)| \leq \frac{1440}{51840} = \frac{1}{36}.$$

□

Theorem 2.3. *If $g \in \mathcal{C}_G$ is of the form (1.1), then*

$$|\mathcal{H}_{2,3}(g)| \leq \frac{1}{576}.$$

The inequality is sharp with the extremal function:

$$g_2(\zeta) = \int_0^\zeta \left(\exp \left(\int_0^x \frac{\Phi(t^3) - 1}{t} dt \right) \right) dx = \zeta + \frac{1}{24}\zeta^4 + \dots \quad (2.14)$$

Proof. If $g \in \mathcal{C}_G$, then there exists a Schwarz function $\varpi(\varepsilon)$ such that

$$1 + \frac{\zeta g''(\zeta)}{g'(\zeta)} = \frac{\varpi(\zeta)}{\ln(1 + \varpi(\zeta))}. \quad (2.15)$$

From (1.1) and some simple calculations, we achieve

$$1 + \frac{\zeta g''(\zeta)}{g'(\zeta)} = 1 + 2\delta_2\zeta + (-4\delta_2^2 + 6\delta_3)\zeta^2 + (8\delta_2^3 - 18\delta_2\delta_3 + 12\delta_4)\zeta^3 + (48\delta_2^2\delta_3 - 16\delta_2^4 + 20\delta_5 - 18\delta_3^2 - 32\delta_4\delta_2)\zeta^4 + \dots \quad (2.16)$$

From (2.16) and (2.4), we have

$$\delta_2 = \frac{1}{4}c_1, \quad (2.17)$$

$$\delta_3 = \frac{1}{36}(c_1^2 + 3c_2), \quad (2.18)$$

$$\delta_4 = \frac{1}{288}(c_1^3 + 5c_1c_2 + 12c_3), \quad (2.19)$$

$$\delta_5 = \frac{1}{7200}(-2c_1^4 + 15c_2^2 + 35c_1^2c_2 + 60c_1c_3 + 180c_4). \quad (2.20)$$

Using (2.18) – (2.20) and (1.2), we achieve

$$\mathcal{H}_{2,3}(g) = \frac{1}{2073600}(4320c_2c_4 - 1560c_1c_2c_3 + 335c_1^2c_2^2 + 360c_2^3 - 18c_1^4c_2 + 1440c_1^2c_4 - 120c_3c_1^3 - 3600c_3^2 - 41c_1^6). \quad (2.21)$$

From Lemma 1.1 and (2.21), we have

$$\begin{aligned} \mathcal{H}_{2,3}(g) = & \frac{1}{2073600} \left[-3600t^2(1 - |c_1|^2)^2(1 - |d|^2)^2 - 120c_1^3(1 - |c_1|^2)(1 - |d|^2)t + 120c_1^3\bar{c}_1 \right. \\ & \times (1 - |c_1|^2)d^2 + 1440c_1^2(1 - |c_1|^2)(1 - |t|^2)(1 - |d|^2)s \\ & - 1440c_1^2\bar{d}(1 - |c_1|^2)(1 - |d|^2)t^2 - 18c_1^4(1 - |c_1|^2)d + 1440c_1^2\bar{c}_1^2(1 - |c_1|^2)d^3 \\ & + 360(1 - |c_1|^2)^3d^3 + 335c_1^2(1 - |c_1|^2)^2d^2 - 1560c_1d(1 - |c_1|^2)^2(1 - |d|^2)t \\ & + 1560\bar{c}_1c_1(1 - |c_1|^2)^2d^3 + 4320d(1 - |c_1|^2)^2(1 - |t|^2)(1 - |d|^2)s \\ & - 1440\bar{c}_1d^2(1 - |c_1|^2)^2(1 - |d|^2)t - 4320d\bar{d}(1 - |c_1|^2)^2(1 - |d|^2)t^2 \\ & \left. + 720\bar{c}_1^2(1 - |c_1|^2)^2d^4 - 41c_1^6 - 2880tc_1^2\bar{c}_1(1 - |c_1|^2)(1 - |d|^2)d \right]. \end{aligned}$$

Thus, we yield

$$\mathcal{H}_{2,3}(g) = \frac{1}{2073600} [\alpha_9(c_1, d) + \alpha_{10}(c_1, d)t + \alpha_{11}(c_1, d)t^2 + \alpha_{12}(c_1, d, t)s],$$

where $c_1, d, t \in \bar{\Lambda}$ and

$$\begin{aligned} \alpha_9(c_1, d) &= -41c_1^6 + (1 - |c_1|^2) [(720\bar{c}_1^2d^4 + 1200|c_1|^2d^3 + 335c_1^2d^2 + 360d^3)(1 - |c_1|^2) \\ &\quad - 18c_1^4d + 1440c_1^2\bar{c}_1^2d^3 + 120c_1^3\bar{c}_1d^2], \\ \alpha_{10}(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2) [(-1440\bar{c}_1d^2 - 1560c_1d)(1 - |c_1|^2) - 120c_1^3 - 2880c_1^2\bar{c}_1d], \\ \alpha_{11}(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2) [(-720|d|^2 - 3600)(1 - |c_1|^2) - 1440\bar{d}c_1^2], \\ \alpha_{12}(c_1, d, t) &= (1 - |t|^2)(1 - |d|^2)(1 - |c_1|^2) [4320d(1 - |c_1|^2) + 1440c_1^2]. \end{aligned}$$

By putting $|c_1| = c$, $|t| = e$, $|d| = d$ and utilizing $|s| \leq 1$, we have

$$\begin{aligned} |\mathcal{H}_{2,3}(g)| &\leq \frac{1}{2073600} (|\alpha_9(c_1, d)| + |\alpha_{10}(c_1, d)|e + |\alpha_{11}(c_1, d)|e^2 + |\alpha_{12}(c_1, d, t)|) \\ &\leq \frac{1}{2073600} \Psi_3(c, d, e), \end{aligned}$$

where

$$\Psi_3(c, d, e) = \beta_9(c, d) + \beta_{10}(c, d)e + \beta_{11}(c, d)e^2 + \beta_{12}(c, d)(1 - e^2),$$

with

$$\begin{aligned} \beta_9(c, d) &= 41c^6 + (1 - c^2)[(1 - c^2)(360d^3 + 335c^2d^2 + 1200c^2d^3 + 720c^2d^4) + 18dc^4 \\ &\quad + 120c^4d^2 + 1440c^4d^3], \\ \beta_{10}(c, d) &= (1 - c^2)(1 - d^2)[(1560cd + 1440cd^2)(1 - c^2) + 2880c^3d + 120c^3], \\ \beta_{11}(c, d) &= (1 - d^2)(1 - c^2)[(3600 + 720d^2)(1 - c^2) + 1440dc^2], \\ \beta_{12}(c, d) &= (1 - c^2)(1 - d^2)[4320d(1 - c^2) + 1440c^2]. \end{aligned}$$

Note that

$$\beta_{11}(e, a) \leq (1 - c^2)(1 - d^2)[(1 - c^2)(720d^2 + 3600) + 1440c^2] = \theta_3(c, d),$$

then, we get

$$\Psi_3(c, d, e) \leq \beta_9(c, d) + \beta_{10}(c, d)e + \theta_3(c, d)e^2 + \beta_{12}(c, d)(1 - e^2) = \chi_3(c, d, e).$$

Differentiating $\chi_3(c, d, e)$ with respect to e , we yield

$$\frac{\partial \chi_3}{\partial e} = \beta_{10}(c, d) + 2e[\theta_3(c, d) - \beta_{12}(c, d)].$$

Upon taking $\beta_{10}(c, d) \geq 0$ and

$$\theta_3(c, d) - \beta_{12}(c, d) = 720(d^2 - 6d + 5)(1 - c^2)^2(1 - d^2) \geq 0,$$

we have $\frac{\partial \chi_3}{\partial e} \geq 0$. So, $\chi_3(c, d, e)$ attains its maximum value at $e = 1$. Therefore, we get

$$\begin{aligned} \Psi_3(c, d, e) &\leq \chi_3(c, d, e) \\ &\leq \chi_3(c, d, 1) \\ &= \beta_9(c, d) + \beta_{10}(c, d) + \theta_3(c, d) \\ &= 41c^6 - 120c^5 + 2160c^4 + 120c^3 - 5760c^2 + 3600 + (-18c^6 - 1320c^5 + 18c^4 \\ &\quad - 240c^3 + 1560c)d + (215c^6 + 1560c^5 - 1990c^4 - 3000c^3 + 4655c^2 + 1440c \\ &\quad - 2880)d^2 + (-240c^6 + 1320c^5 - 600c^4 + 240c^3 + 480c^2 - 1560c + 360)d^3 \\ &\quad + (720c^6 - 1440c^5 - 2160c^4 + 2880c^3 + 2160c^2 - 1440c - 720)d^4 \\ &= \gamma_3(c, d). \end{aligned}$$

The critical points of $\gamma_3(c, d)$ satisfy the system of equations

$$\left\{ \begin{array}{l} \frac{\partial \gamma_3}{\partial c} = 246c^5 - 600c^4 + 8640c^3 + 360c^2 - 11520c + (-108c^5 - 6600c^4 + 72c^3 - 720c^2 \\ \quad + 1560)d + (1290c^5 + 7800c^4 - 7960c^3 - 9000c^2 + 9310c + 1440)d^2 \\ \quad + (-1440c^5 + 6600c^4 - 2400c^3 + 720c^2 + 960c - 1560)d^3 \\ \quad + (4320c^5 - 7200c^4 - 8640c^3 + 8640c^2 + 4320c - 1440)d^4 \\ \quad = 0, \\ \frac{\partial \gamma_3}{\partial d} = 1560c - 240c^3 + 18c^4 - 1320c^5 - 18c^6 + (430c^6 + 3120c^5 - 3980c^4 - 6000c^3 \\ \quad + 9310c^2 + 2880c - 5760)d + (-720c^6 + 3960c^5 - 1800c^4 + 720c^3 + 1440c^2 - 4680c \\ \quad + 1080)d^2 + (2880c^6 - 5760c^5 - 8640c^4 + 11520c^3 + 8640c^2 - 5760c - 2880)d^3 \\ \quad = 0. \end{array} \right.$$

After calculation, we get

$$\left\{ \begin{array}{l} c_{1,1} \approx 1, \\ d_{1,1} \approx 1.2496, \end{array} \right. \left\{ \begin{array}{l} c_{1,2} \approx 1, \\ d_{1,2} \approx -1.8079, \end{array} \right. \left\{ \begin{array}{l} c_{1,3} \approx 1, \\ d_{1,3} \approx -0.4417, \end{array} \right. \left\{ \begin{array}{l} c_{1,4} \approx -1, \\ d_{1,4} \approx 0.6327, \end{array} \right.$$

$$\left\{ \begin{array}{l} c_{1,5} \approx -1, \\ d_{1,5} \approx -0.8626, \end{array} \right. \left\{ \begin{array}{l} c_{1,6} \approx -1, \\ d_{1,6} = 0.5077, \end{array} \right. \left\{ \begin{array}{l} c_{1,7} = 0, \\ d_{1,7} = 0, \end{array} \right. \left\{ \begin{array}{l} c_{1,8} \approx 1.4022, \\ d_{1,8} \approx 1.8883, \end{array} \right.$$

$$\left\{ \begin{array}{l} c_{1,9} \approx 1.5127, \\ d_{1,9} \approx 0.6837, \end{array} \right. \left\{ \begin{array}{l} c_{1,10} \approx -0.6000, \\ d_{1,10} \approx 1.0039, \end{array} \right. \left\{ \begin{array}{l} c_{1,11} \approx -0.6947, \\ d_{1,11} \approx -6.0383, \end{array} \right. \left\{ \begin{array}{l} c_{1,12} \approx -0.9975, \\ d_{1,12} \approx 0.5730, \end{array} \right.$$

$$\left\{ \begin{array}{l} c_{1,13} \approx -1.3950, \\ d_{1,13} \approx -0.4969. \end{array} \right.$$

Thus, there are no critical points in $(0, 1) \times (0, 1)$.

(9) For $c = 1$,

$$\gamma_3(1, d) = 41.$$

(10) For $c = 0$,

$$\gamma_3(0, d) = 3600 - 2880d^2 + 360d^3 - 720d^4 \leq 3600.$$

(11) For $d = 0$,

$$\gamma_3(c, 0) = 3600 - 5760c^2 + 120c^3 + 2160c^4 - 120c^5 + 41c^6 \leq 3600.$$

(12) For $d = 1$

$$\gamma_3(c, 1) = 360 + 1535c^2 - 2572c^4 + 718c^6.$$

Let

$$\tau_4(c) = \gamma_3(c, 1) = 360 + 1535c^2 - 2572c^4 + 718c^6, \quad (2.22)$$

$$\tau_4'(c) = 3070c - 10288c^3 + 4308c^5, \quad (2.23)$$

$$\tau_4''(c) = 3070 - 30864c^2 + 21540c^4. \quad (2.24)$$

For optimum value of $\tau_4(c)$, consider $\tau_4'(c) = 0$. From (2.23), we get $c = \iota_3 = 0$ and $c = \iota_4 \approx 0.5913$. Substituting the value of $c = \iota_4$ in (2.24), we observe that $\tau_4'(\iota_4) \approx -5088 < 0$. Therefore, by the second derivative test, $\tau_4(c)$ has maximum value at $c = \iota_4$, from (2.22), it is obtained to be $\max\{\tau_4(c)\} \approx 612.9646$. Hence,

$$\gamma_3(c, 1) = \tau_4(c) \leq \tau_4(0.5913) \approx 612.9646.$$

Thus, we have

$$|\mathcal{H}_{2,3}(g)| \leq \frac{3600}{2073600} = \frac{1}{576}.$$

□

Theorem 2.4. *Let $g \in \mathcal{C}_G$. Then*

$$|\mathcal{H}_{3,1}(g)| \leq \frac{1}{576}.$$

The bound is sharp and is given by function g_2 .

Proof. Let $g \in \mathcal{C}_G$. From (2.17) – (2.20) and (1.3), we have

$$\begin{aligned} \mathcal{H}_{3,1}(g) = & \frac{1}{18662400} (-7560c_2^3 + 3285c_1^2c_2^2 - 2232c_1^4c_2 + 455c_1^6 + 38880c_2c_4 \\ & + 18360c_1c_2c_3 - 16200c_1^2c_4 - 32400c_3^2). \end{aligned} \quad (2.25)$$

From (2.25) and Lemma 1.1, we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(g) = & \frac{1}{18662400} \left[455c_1^6 + 6480d^4\bar{c}_1^2(1 - |c_1|^2)^2 - 38880t^2d\bar{d}(1 - |d|^2)(1 - |c_1|^2)^2 \right. \\ & - 12960t\bar{c}_1d^2(1 - |c_1|^2)^2(1 - |d|^2) + 38880sd(1 - |t|^2)(1 - |d|^2)(1 - |c_1|^2)^2 \\ & - 18360c_1\bar{c}_1d^3(1 - |c_1|^2)^2 + 18360tdc_1(1 - |d|^2)(1 - |c_1|^2)^2 + 3285c_1^2d^2(1 - |c_1|^2)^2 \\ & - 7560(1 - |c_1|^2)^3d^3 - 2232c_1^4(1 - |c_1|^2)t - 16200c_1^2\bar{c}_1^2d^3(1 - |c_1|^2) + 16200c_1^2 \\ & \times (1 - |d|^2)(1 - |c_1|^2)\bar{d}t^2 + 32400c_1^2\bar{c}_1d(1 - |c_1|^2)(1 - |d|^2)t - 16200c_1^2s(1 - |d|^2) \\ & \left. \times (1 - |c_1|^2)(1 - |t|^2) - 32400t^2(1 - |d|^2)^2(1 - |c_1|^2)^2 \right]. \end{aligned}$$

Thus, we have

$$\mathcal{H}_{3,1}(g) = \frac{1}{18662400} [\alpha_{13}(c_1, d) + \alpha_{14}(c_1, d)t + \alpha_{15}(c_1, d)t^2 + \alpha_{16}(c_1, d, t)s],$$

where

$$\begin{aligned} \alpha_{13}(c_1, d) = & 455c_1^6 + (1 - |c_1|^2) [(6480\bar{c}_1^2d^4 - 10800|c_1|^2d^3 + 3285c_1^2d^2 - 7560d^3)(1 - |c_1|^2) \\ & - 2232c_1^4d - 16200c_1^2\bar{c}_1^2d^3], \end{aligned}$$

$$\begin{aligned}\alpha_{14}(c_1, d) &= (1 - |d|^2)(1 - |c_1|^2) [(-12960\bar{c}_1 d^2 + 18360c_1 d)(1 - |c_1|^2) + 32400c_1^2 \bar{c}_1 d], \\ \alpha_{15}(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2)[(1 - |c_1|^2)(-32400 - 6480|d|^2) + 16200c_1^2 \bar{d}], \\ \alpha_{16}(c_1, d, t) &= (1 - |c_1|^2)(1 - |t|^2)(1 - |d|^2) [38880d(1 - |c_1|^2) - 16200c_1^2].\end{aligned}$$

Writing $|c_1| = c$, $|d| = d$, $|t| = e$ and using $|s| \leq 1$, and using the triangle inequality, we achieve

$$\begin{aligned}|H_{3,1}(g)| &\leq \frac{1}{18662400} (|\alpha_{13}(c_1, d)| + |\alpha_{14}(c_1, d)|e + |\alpha_{15}(c_1, d)|e^2 + |\alpha_{16}(c_1, d, t)|) \\ &\leq \frac{1}{18662400} \Psi_4(c, d, e),\end{aligned}$$

where

$$\Psi_4(c, d, e) = \beta_{13}(c, d) + \beta_{14}(c, d)e + \beta_{15}(c, d)e^2 + \beta_{16}(c, d)(1 - e^2),$$

with

$$\begin{aligned}\beta_{13}(c, d) &= 455c^6 + (1 - c^2)[(7560d^3 + 3285c^2d^2 + 10800c^2d^3 + 6480c^2d^4)(1 - c^2) \\ &\quad + 2232dc^4 + 16200d^3c^4], \\ \beta_{14}(c, d) &= (1 - c^2)(1 - d^2) [(12960cd^2 + 18360dc)(1 - c^2) + 32400c^3d], \\ \beta_{15}(c, d) &= (1 - d^2)(1 - c^2) [(6480d^2 + 32400)(1 - c^2) + 16200dc^2], \\ \beta_{16}(c, d) &= (1 - c^2)(1 - d^2) [16200c^2 + (1 - c^2)38880d].\end{aligned}$$

It is clear that for $d \in [0, 1]$,

$$\beta_{15}(c, d) \leq (1 - c^2)(1 - d^2) [(32400 + 6480d^2)(1 - c^2) + 16200c^2] = \theta_4(c, d).$$

Then, we get

$$\Psi_4(c, d, e) \leq \beta_{13}(c, d) + \beta_{14}(c, d)e + \theta_4(c, d)e^2 + \beta_{16}(c, d)(1 - e^2) = \chi_4(c, d, e).$$

Differentiating $\chi_4(c, d, e)$ with respect to e , we achieve

$$\frac{\partial \chi_4}{\partial e} = \beta_{14}(c, d) + 2e[\theta_4(c, d) - \beta_{16}(c, d)].$$

Upon $\beta_{14}(c, d) \geq 0$ and

$$2e[\theta_4(c, d) - \beta_{16}(c, d)] = 12960e(1 - d^2)(d^2 - 6d + 5)(1 - c^2)^2 \geq 0,$$

we have $\frac{\partial \chi_4}{\partial e} \geq 0$. Therefore, $\chi_4(c, d, e)$ attains its maximum value at $e = 1$. Hence, we have

$$\begin{aligned}\Psi_4(c, d, e) &\leq \chi_4(c, d, e) \\ &\leq \chi_4(c, d, 1) \\ &= \beta_{13}(c, d) + \beta_{14}(c, d) + \theta_4(c, d) \\ &= 455c^6 + 16200c^4 - 48600c^2 + 32400 + (-2232c^6 - 14040c^5 + 2232c^4 - 4320c^3 \\ &\quad + 18360c)d + (3285c^6 + 12960c^5 - 16290c^4 - 25920c^3 + 38925c^2 + 12960c2 \\ &\quad - 25920)d^2 + (-5400c^6 + 14040c^5 + 2160c^4 + 4320c^3 - 4320c^2 - 18360c + 7560)d^3 \\ &\quad + (6480c^6 - 12960c^5 - 19440c^4 + 25920c^3 + 19440c^2 - 12960c - 6480)d^4\end{aligned}$$

$$= \gamma_4(c, d).$$

The optimal points of γ_4 have to satisfy

$$\left\{ \begin{array}{l} \frac{\partial \gamma_4}{\partial c} = 2730c^5 + 64800c^3 - 97200c + (-13392c^5 - 70200c^4 + 8928c^3 - 12960c^2 + 18360) \\ \quad d + (19710c^5 + 64800c^4 - 65160c^3 - 77760c^2 + 77850c + 12960)d^2 \\ \quad + (-32400c^5 + 70200c^4 + 8640c^3 + 12960c^2 - 8640c - 18360)d^3 \\ \quad + (38880c^5 - 64800c^4 - 77760c^3 + 77760c^2 + 38880c - 12960)d^4 \\ \quad = 0, \\ \frac{\partial \gamma_4}{\partial d} = 18360c - 4320c^3 + 2232c^4 - 14040c^5 - 2232c^6 + (6570c^6 + 25920c^5 \\ \quad - 32580c^4 - 51840c^3 + 77850c^2 + 25920c - 51840)d + (-16200c^6 + 42120c^5 \\ \quad + 6480c^4 + 12960c^3 - 12960c^2 - 55080c + 22680)d^2 + (25920c^6 - 51840c^5 \\ \quad - 77760c^4 + 103680c^3 + 77760c^2 - 51840c - 25920)d^3 \\ \quad = 0. \end{array} \right.$$

By calculating, we have

$$\left\{ \begin{array}{l} c_{1,1} \approx 1, \\ d_{1,1} \approx 1.2641, \end{array} \right. \left\{ \begin{array}{l} c_{1,2} \approx 1, \\ d_{1,2} \approx -1.8785, \end{array} \right. \left\{ \begin{array}{l} c_{1,3} \approx 1, \\ d_{1,3} \approx -0.3856, \end{array} \right. \left\{ \begin{array}{l} c_{1,4} \approx -1, \\ d_{1,4} \approx -0.8394, \end{array} \right.$$

$$\left\{ \begin{array}{l} c_{1,5} = 0, \\ d_{1,5} = 0, \end{array} \right. \left\{ \begin{array}{l} c_{1,6} \approx 1.5453, \\ d_{1,6} \approx 1.9646, \end{array} \right. \left\{ \begin{array}{l} c_{1,7} \approx -0.5077, \\ d_{1,7} \approx 1.0168, \end{array} \right. \left\{ \begin{array}{l} c_{1,8} \approx -0.6990, \\ d_{1,8} \approx -7.6912, \end{array} \right.$$

$$\left\{ \begin{array}{l} c_{1,9} \approx -1.0179, \\ d_{1,9} \approx 0.5713, \end{array} \right. \left\{ \begin{array}{l} c_{1,10} \approx -1.4949, \\ d_{1,10} \approx -0.4220, \end{array} \right. \left\{ \begin{array}{l} c_{1,11} \approx -24.4533, \\ d_{1,11} \approx 0.4896. \end{array} \right.$$

Thus, there is no solution of this system in $(0, 1) \times (0, 1)$.

(13) For $c = 1$,

$$\gamma_4(1, d) = 455.$$

(14) For $c = 0$,

$$\gamma_4(0, d) = 32400 - 25920d^2 + 7560d^3 - 6480d^4 \leq 32400.$$

(15) For $d = 0$,

$$\gamma_4(c, 0) = 32400 - 48600c^2 + 16200c^4 + 455c^6 \leq 32400.$$

(16) For $d = 1$

$$\gamma_4(c, 1) = \tau_5(c) = 7560 + 5445c^2 - 15138c^4 + 2588c^6.$$

It is easy to observe that $\tau'_5(c) = 0$ for $c = \iota_5 \approx 0.4347$ and $c = \iota_6 = 0$ in the interval $[0, 1]$ as points of maxima and minima respectively. The maximum value of $\tau_5(c) \approx 8065.8$.

Hence,

$$\gamma_4(c, 1) = \tau_5(c) \leq 8065.8.$$

Thus, we yield

$$|\mathcal{H}_{3,1}(g)| \leq \frac{32400}{18662400} = \frac{1}{576}.$$

□

Theorem 2.5. *If $g \in \mathcal{S}_G^*$ is of the form (1.1), then*

$$|\mathcal{H}_{2,1}(G_g/2)| = |\lambda_1\lambda_3 - \lambda_2^2| \leq \frac{1}{64}.$$

The inequality is sharp with the extremal function:

$$g_3(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\Phi(t^3) - 1}{t} dt\right) = \zeta + \frac{1}{4}\zeta^3 + \cdots \quad (2.26)$$

Proof. Let $g \in \mathcal{S}_G^*$. From (1.6), (2.5), (2.6) and (2.7), we have

$$\mathcal{H}_{2,1}(G_g/2) = \frac{1}{2304}(48c_1c_3 - 4c_1^2c_2 + 3c_1^4 - 36c_2^2). \quad (2.27)$$

From Lemma 1.1 and (2.27), we achieve

$$\begin{aligned} \mathcal{H}_{2,1}(G_g/2) &= \frac{1}{2304}[3c_1^4 - 36(1 - |c_1|^2)^2d^2 - 4c_1^2d(1 - |c_1|^2) + 48c_1(1 - |d|^2)(1 - |c_1|^2)t \\ &\quad - 48d^2(1 - |c_1|^2)|c_1|^2]. \end{aligned}$$

By using the triangle inequality, and setting $|c_1| = c$, $|d| = d$, and using $|t| \leq 1$, we have

$$\begin{aligned} |\mathcal{H}_{2,1}(G_g/2)| &\leq \frac{1}{2304}[3c^4 + 36(1 - c^2)^2d^2 + 4c^2d(1 - c^2) + 48c(1 - d^2)(1 - c^2) + 48d^2(1 - c^2)c^2] \\ &= \frac{1}{2304}\Upsilon(c, d). \end{aligned}$$

By differentiating $\Upsilon(c, d)$ with respect to d , we get

$$\frac{\partial \Upsilon}{\partial d} = 4(1 - c^2)c^2 + 24(1 - c^2)(c^2 - 4c + 3)d \geq 0.$$

So, $\Upsilon(c, d)$ attains its maximum value at $d = 1$. Therefore, we have

$$|\mathcal{H}_{2,1}(G_g/2)| \leq \frac{1}{2304}\Upsilon(c, 1) = \frac{1}{2304}(36 - 20c^2 - 16c^4) \leq \frac{36}{2304} = \frac{1}{64}.$$

□

Theorem 2.6. *If $g \in \mathcal{S}_G^*$, then*

$$|\mathcal{H}_{2,2}(G_g/2)| = |\lambda_2\lambda_4 - \lambda_3^2| \leq \frac{1}{144}.$$

The bound is sharp and is given by function g_1 .

Proof. Let $g \in \mathcal{S}_G^*$. From (2.5) – (2.8) and (1.7), we have

$$\begin{aligned} \mathcal{H}_{2,2}(G_g/2) = \frac{1}{829440} (6480c_2c_4 + 1680c_1c_2c_3 + 1160c_1^2c_2^2 - 1080c_2^3 - 292c_1^4c_2 - 1080c_1^2c_4 \\ - 600c_1^3c_3 + 17c_1^6 - 5760c_3^2). \end{aligned} \quad (2.28)$$

Using (2.28) and Lemma 1.1, we yield

$$\begin{aligned} \mathcal{H}_{2,2}(G_g/2) = \frac{1}{829440} \Big[17c_1^6 + 720\bar{c}_1^2(1 - |c_1|^2)^2d^4 - 6480|d|^2(1 - |d|^2)(1 - |c_1|^2)^2t^2 \\ - 1440\bar{c}_1d^2(1 - |c_1|^2)^2(1 - |d|^2)t + 6480d(1 - |d|^2)(1 - |t|^2)(1 - |c_1|^2)^2s \\ - 1680|c_1|^2(1 - |c_1|^2)^2d^3 + 1160c_1^2(1 - |c_1|^2)^2d^2 + 1680c_1d(1 - |d|^2)(1 - |c_1|^2)^2t \\ - 1080(1 - |c_1|^2)^3d^3 - 292c_1^4(1 - |c_1|^2)d - 1080c_1^2\bar{c}_1^2(1 - |c_1|^2)d^3 \\ + 1080c_1^2\bar{d}(1 - |c_1|^2)(1 - |d|^2)t^2 + 2160d\bar{c}_1c_1^2(1 - |c_1|^2)(1 - |d|^2)t \\ - 1080c_1^2(1 - |t|^2)(1 - |d|^2)(1 - |c_1|^2)s + 600c_1^3\bar{c}_1(1 - |c_1|^2)d^2 \\ - 600c_1^3(1 - |d|^2)(1 - |c_1|^2)t - 5760(1 - |d|^2)^2(1 - |c_1|^2)^2t^2 \Big]. \end{aligned}$$

Thus, we get

$$\mathcal{H}_{2,2}(G_g/2) = \frac{1}{829440} [\alpha_{17}(c_1, d) + \alpha_{18}(c_1, d)t + \alpha_{19}(c_1, d)t^2 + \alpha_{20}(c_1, d, t)s],$$

where

$$\begin{aligned} \alpha_{17}(c_1, d) &= 17c_1^6 + (1 - |c_1|^2)[(1 - |c_1|^2)(720\bar{c}_1^2d^4 - 600|c_1|^2d^3 + 1160c_1^2d^2 - 1080d^3) \\ &\quad - 292c_1^4d - 1080c_1^2\bar{c}_1^2d^3 + 600\bar{c}_1c_1^3d^2], \\ \alpha_{18}(c_1, d) &= (1 - |d|^2)(1 - |c_1|^2) [2160c_1^2\bar{c}_1d - 600c_1^3 + (-1440\bar{c}_1d^2 + 1680c_1d)(1 - |c_1|^2)], \\ \alpha_{19}(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2)[1080c_1^2\bar{d} + (1 - |c_1|^2)(-5760 - 720|d|^2)], \\ \alpha_{20}(c_1, d, t) &= (1 - |c_1|^2)(1 - |t|^2)(1 - |d|^2) [-1080c_1^2 + 6480d(1 - |c_1|^2)]. \end{aligned}$$

Using the triangle inequality and setting $|t| = e$, $|d| = d$, $|c_1| = c$, and using $|s| \leq 1$, we yield

$$\begin{aligned} |H_{2,2}(G_g/2)| &\leq \frac{1}{829440} (|\alpha_{17}(c_1, d)| + |\alpha_{18}(c_1, d)|e + |\alpha_{19}(c_1, d)|e^2 + |\alpha_{20}(c_1, d, t)|) \\ &\leq \frac{1}{829440} \Psi_5(c, d, e), \end{aligned}$$

where

$$\Psi_5(c, d, e) = \beta_{17}(c, d) + \beta_{18}(c, d)e + \beta_{19}(c, d)e^2 + \beta_{20}(c, d)(1 - e^2),$$

with

$$\begin{aligned} \beta_{17}(c, d) &= 17c^6 + (1 - c^2)[(1080d^3 + 1160c^2d^2 + 600c^2d^3 + 720c^2d^4)(1 - c^2) \\ &\quad + 292dc^4 + 1080d^3c^4 + 600c^4d^2], \\ \beta_{18}(c, d) &= (1 - c^2)(1 - d^2) [(1440cd^2 + 1680dc)(1 - c^2) + 2160c^3d + 600c^3], \\ \beta_{19}(c, d) &= (1 - d^2)(1 - c^2) [(720d^2 + 5760)(1 - c^2) + 1080dc^2], \end{aligned}$$

$$\beta_{20}(c, d) = (1 - d^2)(1 - c^2)[(1 - c^2)6480d + 1080c^2].$$

It is clear that for $d \in [0, 1]$,

$$\beta_{19}(c, d) \leq (1 - c^2)(1 - d^2)[(5760 + 720d^2)(1 - c^2) + 1080c^2] = \theta_5(c, d).$$

Therefore, we achieve

$$\Psi_5(c, d, e) \leq \beta_{17}(c, d) + \beta_{18}(c, d)e + \theta_5(c, d)e^2 + \beta_{20}(c, d)(1 - e^2) = \chi_5(c, d, e).$$

It is easy to see that

$$\begin{aligned} \frac{\partial \chi_5}{\partial e} &= \beta_{18}(c, d) + 2e[\theta_5(c, d) - \beta_{20}(c, d)] \\ &= \beta_{18}(c, d) + 1440e(1 - c^2)^2(1 - d^2)(d^2 - 9d + 8) \\ &\geq 0. \end{aligned}$$

Thus, we get

$$\begin{aligned} \Psi_5(c, d, e) &\leq \chi_5(c, d, e) \\ &\leq \chi_5(c, d, 1) \\ &= \beta_{17}(c, d) + \beta_{18}(c, d) + \theta_5(c, d) \\ &= 17c^6 - 600c^5 + 4680c^4 + 600c^3 - 10440c^2 + 5760 + (-292c^6 - 480c^5 + 292c^4 \\ &\quad - 1200c^3 + 1680c)d + (560c^6 + 2040c^5 - 5680c^4 - 3480c^3 + 10160c^2 + 1440c \\ &\quad - 5040)d^2 + (-480c^6 + 480c^5 + 960c^4 + 1200c^3 - 1560c^2 - 1680c + 1080)d^3 \\ &\quad + (720c^6 - 1440c^5 - 2160c^4 + 2880c^3 + 2160c^2 - 1440c - 720)d^4 \\ &= \gamma_5(c, d). \end{aligned}$$

The optimal points of γ_5 has to satisfy

$$\left\{ \begin{aligned} \frac{\partial \gamma_5}{\partial c} &= 102c^5 - 3000c^4 + 18720c^3 + 1800c^2 - 20880c + (-1752c^5 - 2400c^4 + 1168c^3 \\ &\quad - 3600c^2 + 1680)d + (3360c^5 + 10200c^4 - 22720c^3 - 10440c^2 + 20320c + 1440)d^2 \\ &\quad + (-2880c^5 + 2400c^4 + 3840c^3 + 3600c^2 - 3120c - 1680)d^3 \\ &\quad + (4320c^5 - 7200c^4 - 8640c^3 + 8640c^2 + 4320c - 1440)d^4 \\ &= 0, \\ \frac{\partial \gamma_5}{\partial d} &= 1680c - 1200c^3 + 292c^4 - 480c^5 - 292c^6 + (1120c^6 + 4080c^5 \\ &\quad - 11360c^4 - 6960c^3 + 20320c^2 + 2880c - 10080)d + (-1440c^6 + 1440c^5 \\ &\quad + 2880c^4 + 3600c^3 - 4680c^2 - 5040c + 3240)d^2 \\ &\quad + (2880c^6 - 5760c^5 - 8640c^4 + 11520c^3 + 8640c^2 - 5760c - 2880)d^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we get

$$\begin{aligned} & \begin{cases} c_{1,1} \approx 1, \\ d_{1,1} \approx 1.3973, \end{cases} \quad \begin{cases} c_{1,2} \approx 1, \\ d_{1,2} \approx -1.7964, \end{cases} \quad \begin{cases} c_{1,3} \approx 1, \\ d_{1,3} \approx -0.6009, \end{cases} \quad \begin{cases} c_{1,4} \approx -1, \\ d_{1,4} \approx 0.5677, \end{cases} \\ & \begin{cases} c_{1,5} \approx -1, \\ d_{1,5} \approx -0.8722, \end{cases} \quad \begin{cases} c_{1,6} \approx -1, \\ d_{1,6} \approx 0.2674, \end{cases} \quad \begin{cases} c_{1,7} = 0, \\ d_{1,7} = 0, \end{cases} \quad \begin{cases} c_{1,8} \approx 4.5615, \\ d_{1,8} \approx 0.3102, \end{cases} \\ & \begin{cases} c_{1,9} \approx -0.6201, \\ d_{1,9} \approx 1.0782, \end{cases} \quad \begin{cases} c_{1,10} \approx -0.7785, \\ d_{1,10} \approx -6.2161, \end{cases} \quad \begin{cases} c_{1,11} \approx -0.9950, \\ d_{1,11} \approx 0.4322, \end{cases} \quad \begin{cases} c_{1,12} \approx -1.0609, \\ d_{1,12} \approx -0.5187, \end{cases} \\ & \begin{cases} c_{1,13} \approx -21.2751, \\ d_{1,13} \approx 0.3520. \end{cases} \end{aligned}$$

Thus, there is no solution of this system in $(0, 1) \times (0, 1)$.

(17) For $c = 1$,

$$\gamma_5(1, d) = 17.$$

(18) For $c = 0$,

$$\gamma_5(0, d) = 5760 - 5040d^2 + 1080d^3 - 720d^4 \leq 5760.$$

(19) For $d = 0$,

$$\gamma_5(c, 0) = 5760 - 10440c^2 + 600c^3 + 4680c^4 - 600c^5 + 17c^6 \leq 5760.$$

(20) For $d = 1$

$$\gamma_5(c, 1) = \tau_6(c) = 1080 + 320c^2 - 1908c^4 + 525c^6.$$

Now, $\tau_6'(c) = 0$ when $c = \iota_7 = 0$ and $c = \iota_8 \approx 0.2949$ in the interval $[0, 1]$ with ι_7 and ι_8 as points of minima and maxima respectively. Hence,

$$\gamma_5(c, 0) = \tau_6(c) \leq \tau_6(0.2949) \approx 1093.7.$$

Thus, we yield

$$|\mathcal{H}_{2,2}(G_g/2)| \leq \frac{5760}{829440} = \frac{1}{144}.$$

□

Theorem 2.7. *If $g \in \mathcal{C}_G$, then*

$$|\mathcal{H}_{2,1}(G_g/2)| = |\lambda_1\lambda_3 - \lambda_2^2| \leq \frac{1}{576}.$$

The bound is sharp and is given by function:

$$g_4(\zeta) = \int_0^\zeta \left(\exp \left(\int_0^x \frac{\Phi(t^3) - 1}{t} dt \right) \right) dx = \zeta + \frac{1}{12}\zeta^3 + \dots \quad (2.29)$$

Proof. Let $g \in \mathcal{C}_G$. From (1.6), (2.17), (2.18) and (2.19), we have

$$\mathcal{H}_{2,1}(G_g/2) = \frac{1}{331776}(-576c_2^2 - 24c_1^2c_2 + 864c_1c_3 + 35c_1^4). \quad (2.30)$$

Using (2.30) and Lemma 1.1, we get

$$\begin{aligned} \mathcal{H}_{2,1}(G_g/2) &= \frac{1}{331776}[-576(1 - |c_1|^2)^2d^2 - 24c_1^2(1 - |c_1|^2)d - 864d^2(1 - |c_1|^2)|c_1|^2 \\ &\quad + 864c_1(1 - |d|^2)(1 - |c_1|^2)t + 35c_1^4]. \end{aligned}$$

By setting $|d| = d$, $|c_1| = c$ and using $|t| \leq 1$, and using the triangle inequality, we yield

$$\begin{aligned} |\mathcal{H}_{2,1}(G_g/2)| &\leq \frac{1}{331776}[576d^2(1 - c^2)^2 + 24c^2(1 - c^2)d + 864d^2(1 - c^2)c^2 \\ &\quad + 864c(1 - c^2)(1 - d^2) + 35c^4] \\ &= \frac{1}{331776}\Gamma(c, d). \end{aligned}$$

Differentiating $\Gamma(c, d)$ with respect to d , we have

$$\frac{\partial \Gamma}{\partial d} = 24c^2(1 - c^2) + 576d(1 - c^2)(c^2 - 3c + 2) \geq 0,$$

which shows that $\Gamma(c, d)$ is an increasing function on the closed interval $[0, 1]$ and the maximum value occurs at $d = 1$. Thus, we obtain

$$\Gamma(c, d) \leq \Gamma(c, 1) = -312c^4 - 264c^2 + 576 \leq 576.$$

Therefore, we have

$$|\mathcal{H}_{2,1}(G_g/2)| \leq \frac{576}{331776} = \frac{1}{576}.$$

□

Theorem 2.8. If $g \in \mathcal{C}_G$, then

$$|\mathcal{H}_{2,2}(G_g/2)| = |\lambda_2\lambda_4 - \lambda_3^2| \leq \frac{1}{2304}.$$

The bound is sharp and is given by function g_2 .

Proof. Let $g \in \mathcal{C}_G$. From (2.17) – (2.20) and (1.7), we achieve

$$\begin{aligned} \mathcal{H}_{2,2}(G_g/2) &= \frac{1}{2388787200}(1244160c_2c_4 + 69120c_1c_2c_3 + 165600c_1^2c_2^2 - 69120c_2^3 - 38304c_1^4c_2 \\ &\quad - 51840c_1^2c_4 - 82080c_1^3c_3 - 199c_1^6 - 1036800c_3^2). \end{aligned} \quad (2.31)$$

Using (2.31) and Lemma 1.1, we have

$$\begin{aligned} \mathcal{H}_{2,2}(G_g/2) &= \frac{1}{2388787200} \left[-199c_1^6 - 1036800(1 - |c_1|^2)^2(1 - |d|^2)^2t^2 - 414720\bar{c}_1d^2 \right. \\ &\quad \times (1 - |c_1|^2)^2(1 - |d|^2)t + 207360\bar{c}_1^2(1 - |c_1|^2)^2d^4 - 82080c_1^3(1 - |c_1|^2)(1 - |d|^2)t \\ &\quad \left. + 82080(1 - |c_1|^2)\bar{c}_1c_1^3d^2 - 51840c_1^2(1 - |d|^2)(1 - |t|^2)(1 - |c_1|^2)s + 103680c_1^2\bar{c}_1d \right] \end{aligned}$$

$$\begin{aligned} & \times (1 - |d|^2)(1 - |c_1|^2)t + 51840c_1^2\bar{d}(1 - |c_1|^2)(1 - |d|^2)t^2 - 51840\bar{c}_1^2c_1^2d^3(1 - |c_1|^2) \\ & - 38304c_1^4(1 - |c_1|^2)d - 69120d^3(1 - |c_1|^2)^3 + 165600c_1^2d^2(1 - |c_1|^2)^2 + 69120c_1d \\ & \times (1 - |c_1|^2)^2(1 - |d|^2)t + 1244160d(1 - |c_1|^2)^2(1 - |t|^2)(1 - |d|^2)s - 69120c_1\bar{c}_1 \\ & \times (1 - |c_1|^2)^2d^3 - 1244160\bar{d}d(1 - |c_1|^2)^2(1 - |d|^2)t^2 \Big]. \end{aligned}$$

Therefore, we get

$$\mathcal{H}_{2,2}(G_g/2) = \frac{1}{2388787200} [\alpha_{21}(c_1, d) + \alpha_{22}(c_1, d)t + \alpha_{23}(c_1, d)t^2 + \alpha_{24}(c_1, d, t)s],$$

where

$$\begin{aligned} \alpha_{21}(c_1, d) &= -199c_1^6 + (1 - |c_1|^2)[(207360\bar{c}_1^2d^4 + 165600c_1^2d^2 - 69120d^3)(1 - |c_1|^2) - 38304c_1^4d \\ & \quad - 51840c_1^2\bar{c}_1^2d^3 + 82080c_1^3\bar{c}_1d^2], \\ \alpha_{22}(c_1, d) &= (1 - |d|^2)(1 - |c_1|^2) [103680\bar{c}_1c_1^2d - 82080c_1^3 \\ & \quad + (-414720\bar{c}_1d^2 + 69120c_1d)(1 - |c_1|^2)], \\ \alpha_{23}(c_1, d) &= (1 - |c_1|^2)(1 - |d|^2)[51840\bar{d}c_1^2 + (1 - |c_1|^2)(-1036800 - 207360|d|^2)], \\ \alpha_{24}(c_1, d, t) &= (1 - |c_1|^2)(1 - |t|^2)(1 - |d|^2)[-51840c_1^2 + 1244160d(1 - |c_1|^2)]. \end{aligned}$$

By setting $|d| = d$, $|c_1| = c$, $|t| = e$ and using $|s| \leq 1$, and using the triangle inequality, we achieve

$$\begin{aligned} |H_{2,2}(G_g/2)| &\leq \frac{1}{2388787200} (|\alpha_{21}(c_1, d)| + |\alpha_{22}(c_1, d)|e + |\alpha_{23}(c_1, d)|e^2 + |\alpha_{24}(c_1, d, t)|) \\ &\leq \frac{1}{2388787200} \Psi_6(c, d, e), \end{aligned} \quad (2.32)$$

where

$$\Psi_6(c, d, e) = \beta_{21}(c, d) + \beta_{22}(c, d)e + \beta_{23}(c, d)e^2 + \beta_{24}(c, d)(1 - e^2),$$

with

$$\begin{aligned} \beta_{21}(c, d) &= 199c^6 + (1 - c^2)[(69120d^3 + 165600c^2d^2 + 207360c^2d^4)(1 - c^2) + 38304dc^4 \\ & \quad + 51840d^3c^4 + 82080c^4d^2], \\ \beta_{22}(c, d) &= (1 - c^2)(1 - d^2) [(414720cd^2 + 69120dc)(1 - c^2) + 103680c^3d + 82080c^3], \\ \beta_{23}(c, d) &= (1 - d^2)(1 - c^2) [(207360d^2 + 1036800)(1 - c^2) + 51840dc^2], \\ \beta_{24}(c, d) &= (1 - d^2)(1 - c^2) [1244160d(1 - c^2) + 51840c^2]. \end{aligned}$$

It is clear that for $d \in [0, 1]$,

$$\beta_{23}(c, d) \leq (1 - c^2)(1 - d^2) [(103680 + 207360d^2)(1 - c^2) + 51840c^2] = \theta_6(c, d).$$

Thus, we have

$$\Psi_6(c, d, e) \leq \beta_{21}(c, d) + \beta_{22}(c, d)e + \theta_6(c, d)e^2 + \beta_{24}(c, d)(1 - e^2) = \chi_6(c, d, e).$$

It is easy to see that

$$\begin{aligned}\frac{\partial \chi_6}{\partial e} &= \beta_{22}(c, d) + 2e[\theta_6(c, d) - \beta_{24}(c, d)] \\ &= \beta_{22}(c, d) + 414720e(d^2 - 6d + 5)(1 - c^2)^2(1 - d^2) \\ &\geq 0.\end{aligned}$$

So, $\chi_6(c, d, e)$ attains its maximum value at $e = 1$. Therefore, we achieve

$$\begin{aligned}\Psi_6(c, d, e) &\leq \chi_6(c, d, e) \\ &\leq \chi_6(c, d, 1) \\ &= \beta_{21}(c, d) + \beta_{22}(c, d) + \theta_6(c, d) \\ &= 199c^6 - 82080c^5 + 984960c^4 + 82080c^3 - 2021760c^2 + 1036800 + (-38304c^6 \\ &\quad - 34560c^5 + 38304c^4 - 34560c^3 + 69120c)d + (83520c^6 + 496800c^5 - 1026720c^4 \\ &\quad - 911520c^3 + 1772640c^2 + 414720c - 829440)d^2 + (-51840c^6 + 34560c^5 \\ &\quad + 120960c^4 + 34560c^3 - 138240c^2 - 69120c + 69120)d^3 + (207360c^6 \\ &\quad - 414720c^5 - 622080c^4 + 829440c^3 + 622080c^2 - 414720c - 207360)d^4 \\ &= \gamma_6(c, d).\end{aligned}$$

The optimal points of γ_6 has to satisfy

$$\left\{ \begin{aligned} \frac{\partial \gamma_6}{\partial c} &= 1194c^5 - 410400c^4 + 3939840c^3 + 246240c^2 - 4043520c \\ &\quad + (-229824c^5 - 172800c^4 + 153216c^3 - 103680c^2 + 69120)d \\ &\quad + (501120c^5 + 2484000c^4 - 4106880c^3 - 2734560c^2 + 3545280c + 414720)d^2 \\ &\quad + (-311040c^5 + 172800c^4 + 483840c^3 + 103680c^2 - 276480c - 69120)d^3 \\ &\quad + (1244160c^5 - 2073600c^4 - 2488320c^3 + 2488320c^2 + 1244160c - 414720)d^4 \\ &= 0, \\ \frac{\partial \gamma_6}{\partial d} &= 69120c - 34560c^3 + 38304c^4 - 34560c^5 - 38304c^6 + (167040c^6 + 993600c^5 \\ &\quad - 2053440c^4 - 1823040c^3 + 3545280c^2 + 829440c - 1658880)d + (-155520c^6 \\ &\quad + 103680c^5 + 362880c^4 + 103680c^3 - 414720c^2 - 207360c + 207360)d^2 + (829440c^6 \\ &\quad - 1658880c^5 - 2488320c^4 + 3317760c^3 + 2488320c^2 - 1658880c - 829440)d^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we get

$$\begin{aligned} &\begin{cases} c_{1,1} \approx 1, \\ d_{1,1} \approx 1.6357, \end{cases} & \begin{cases} c_{1,2} \approx 1, \\ d_{1,2} \approx -1.7233, \end{cases} & \begin{cases} c_{1,3} \approx 1, \\ d_{1,3} \approx -0.9124, \end{cases} & \begin{cases} c_{1,4} \approx -1, \\ d_{1,4} \approx 0.5828, \end{cases} \\ &\begin{cases} c_{1,5} \approx -1, \\ d_{1,5} \approx -0.9450, \end{cases} & \begin{cases} c_{1,6} \approx -1, \\ d_{1,6} \approx -0.3600, \end{cases} & \begin{cases} c_{1,7} = 0, \\ d_{1,7} = 0, \end{cases} & \begin{cases} c_{1,8} \approx 7.0535, \\ d_{1,8} \approx 0.1661, \end{cases} \end{aligned}$$

$$\begin{cases} c_{1,9} \approx -0.7583, \\ d_{1,9} \approx 1.1265, \\ c_{1,13} \approx -19.7079, \\ d_{1,13} \approx 0.2705. \end{cases} \quad \begin{cases} c_{1,10} \approx -0.9107, \\ d_{1,10} \approx -2.1250, \end{cases} \quad \begin{cases} c_{1,11} \approx -0.9908, \\ d_{1,11} \approx 0.2296, \end{cases} \quad \begin{cases} c_{1,12} \approx -1.0071, \\ d_{1,12} \approx -0.7215, \end{cases}$$

Thus, there is no solution of this system in $(0, 1) \times (0, 1)$.

(21) For $c = 1$,

$$\gamma_6(1, d) = 199.$$

(22) For $c = 0$,

$$\gamma_6(0, d) = 1036800 - 829440d^2 + 69120d^3 - 207360d^4 \leq 1036800.$$

(23) For $d = 0$,

$$\gamma_6(c, 0) = 1036800 - 2021760c^2 + 82080c^3 + 984960c^4 - 82080c^5 + 199c^6 \leq 1036800.$$

(24) For $d = 1$

$$\gamma_6(c, 1) = \tau_7(c) = 69120 + 234720c^2 - 504576c^4 + 200935c^6.$$

Now, $\tau_7'(c) = 0$ when $c = \iota_9 = 0$ and $c = \iota_{10} \approx 0.05283$ in the interval $[0, 1]$ with ι_9 and ι_{10} as points of minima and maxima respectively. Hence,

$$\gamma_6(c, 0) = \tau_7(c) \leq \tau_7(0.5283) \approx 99694.$$

Thus, we yield

$$|\mathcal{H}_{2,2}(G_g/2)| \leq \frac{1036800}{2388787200} = \frac{1}{2304}.$$

□

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