

OPTIMAL CONTROL AND COST-EFFECTIVENESS ANALYSIS OF AN INTEGRATED INTERVENTION MODEL INCORPORATING VECTOR PREFERENCE FOR MOSO BAMBOO SOOTY MOLD DISEASE

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Abstract Understanding the transmission mechanisms of Moso bamboo sooty mold disease and the vector preference behavior of its vector insect (bamboo aphid) is of great significance for developing cost-effective integrated pest management strategies. This study constructs a compartmental model that incorporates vector preference to describe the transmission dynamics of the disease between bamboo plants and bamboo aphids. Through rigorous mathematical derivation, an explicit expression for the basic reproduction number $\mathcal{R}_0$ of the disease is obtained. Global sensitivity analysis identified the key parameters that most significantly influence the dynamics of disease transmission. Theoretical and numerical findings indicate: (i) When $\mathcal{R}_0 < 1$, the disease-free equilibrium of model (2.1) is globally asymptotically stable (GAS); when $\mathcal{R}_0 > 1$, the system is uniformly persistent and possesses a unique positive equilibrium, which is GAS. (ii) Under identical state variables and parameter values in this model, compared to the model ignoring vector preference ($\phi = 1$), the model incorporating vector preference ($\phi > 1$) overestimates the disease's potential transmission capacity while simultaneously underestimating the actual infection scale. (iii) Spraying insecticides and injecting nutrient solutions are long-term control measures that should be implemented throughout the entire disease management process. In contrast, using yellow adhesive traps to capture pests and removing bamboo plants exhibiting symptoms of sooty mold are short-term control strategies suitable for use during the early stages of an outbreak. Considering the cost-effectiveness of labor and materials, these measures can be discontinued in later stages. (iv) According to the optimal control model for the disease and related numerical simulation results, the integrated control measure combining insecticide spraying with nutrient solution injection is found to be the most cost-effective among all strategies. This study provides a theoretical basis and decision-making support for the prevention and control of the disease.

Keywords Moso bamboo sooty mold disease, vector preference, global stability, optimal control, sensitivity analysis, cost-effectiveness.

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1. Introduction

China has the world's largest bamboo resources, with over 770 species across 47 genera [3]. Based on the growth characteristics of their rhizomes, these are classified into three categories: Scattered, clumping, and mixed-growth bamboo [35]. Within scattered bamboos, the genus *Phyllostachys* holds the highest economic value, encompassing high-quality bamboo species such as Moso bamboo, Lei bamboo, Pale bamboo, and Gui bamboo. Among these, Moso bamboo is the core dominant bamboo species in China, featuring the widest distribution, the largest cultivation scale, and the optimal comprehensive utilization value [4]. According to the latest national forest and grassland ecological monitoring data, China's Moso bamboo forest area is approximately 5.2776 million hectares, accounting for nearly 70% of the country's total bamboo forest area [37]. It serves as the primary stand supporting the stability of China's bamboo forest ecosystem and the development of the bamboo industry [41]. Moso bamboo is a typical timber-shoot dual-purpose species with extremely high comprehensive utilization value, playing an irreplaceable core role in national economic development, ecological and environmental protection, rural revitalization, and the inheritance of bamboo culture [30]. It is worth noting that large-scale, contiguous, and intensive Moso bamboo plantations tend to create stable, suitable habitats, providing favorable conditions for the proliferation and spread of pests and diseases [15]. It has become a major challenge for the sustainable management and precision management of Moso bamboo forests, as well as for the production and operations of bamboo farmers and the maintenance of bamboo stands [5]. In view of the potential harm caused by sooty mold disease to the bamboo industry and forest ecosystems, it is crucial to gain an in-depth understanding of the transmission dynamics of the sooty mold pathogen between bamboo aphids and Moso bamboo, and to develop effective control strategies to curb its epidemic spread accordingly.

Mathematical modeling plays a crucial role in plant epidemiology and has become an important tool for understanding the patterns of plant disease spread, uncovering underlying mechanisms, and evaluating the effectiveness of control strategies [8, 22, 27, 38, 45–49]. In recent years, several studies have employed mathematical models to investigate the dynamics of vector-borne plant diseases. Al Basir et al. [9] developed a model of cassava mosaic virus transmission that incorporates developmental delays in the vector, revealing the stabilizing effect of these delays on epidemic dynamics and identifying the threshold conditions linking vector development duration and transmission rates to disease steady states and periodic fluctuations. Ding et al. [20] established a delayed diffusive model for pine wilt disease with saturation incidence and nonlocal effects, analyzed well-posedness, stability, and Hopf bifurcation, derived normal forms via the multiple time scales method, and identified spatially inhomogeneous periodic solutions and control thresholds. Andres et al. [19] proposed a SAIR-SI dynamic model describing the transmission of citrus huanglongbing between trees and psyllids. By deriving final-state expressions and combining them with genetic algorithms, they optimized the parameters for two control strategies: The removal of diseased trees and the eradication of psyllids. Recently, Gao et al. [18] developed a dynamic model of citrus huanglongbing that integrates intercropping patterns and vector preferences. They derived the basic reproduction number, analyzed the stability of equilibrium points and various branching behaviors, and, through numerical simulations, identified the optimal intercropping ratio and comprehensive control strategies. It is worth noting that each of the aforementioned dynamic models has its own unique strengths and can provide important theoretical insights and practical guidance for systems involving plant diseases transmitted by insect vectors.

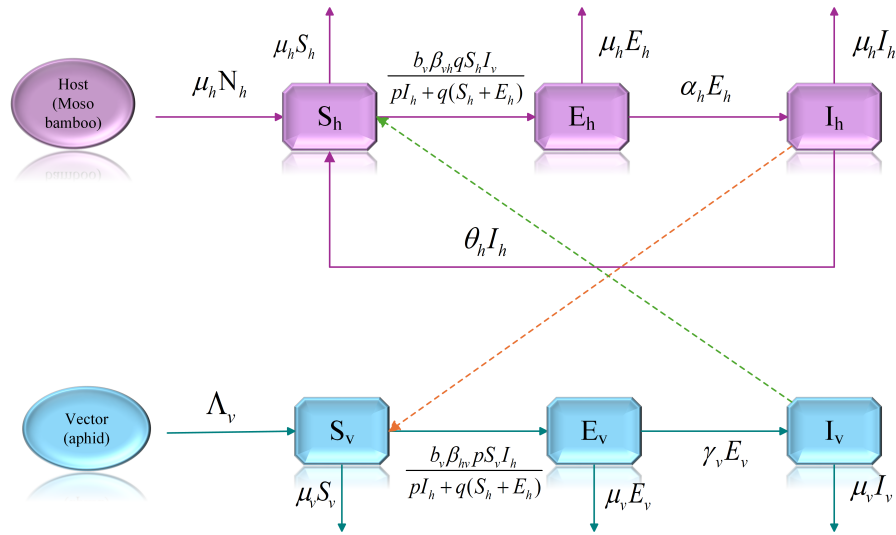


Figure 1. A schematic of system (2.1) showing transitions between different classes for Moso bamboo and bamboo aphids.

The aforementioned domestic and international studies indicate that the existing theoretical framework for modeling the dynamics of vector-borne plant diseases is becoming increasingly mature. These models can effectively elucidate the patterns of disease epidemics and provide crucial support for the theoretical assessment of control strategies. In contrast, existing research on Moso bamboo sooty mold disease has largely relied on biological experiments and field statistical analyses, focusing on the observable characteristics of disease onset and the severity of damage [6]. However, it has not yet been quantitatively characterized from a dynamical modeling perspective, nor has it elucidated the underlying epidemiological mechanisms of disease transmission. At the same time, most existing dynamic models of vector-borne diseases generally assume that the virus-transmission behavior of vector insects is homogeneous and uniform, overlooking the host-feeding and infection preferences that are common among insects in nature [11, 17, 18, 28, 32, 40, 43]. In the transmission system of bamboo sooty mold, the preferential behavior of vector insects significantly alters the efficiency of disease transmission and epidemiological trends, making it a key factor that aligns with field realities. Ignoring this characteristic leads to discrepancies between model simulations and actual disease outbreaks, making it challenging to support precise prevention and control measures. Furthermore, existing modeling studies on disease control have primarily focused on dynamic simulation and strategy design, generally lacking quantitative assessments of control costs and overall benefits. This makes it difficult to balance control efficacy with economic costs, thereby failing to provide precise theoretical foundations for the scientific and cost-effective integrated control of Moso bamboo sooty mold. Based on the current research progress and existing limitations in the modeling of Moso bamboo sooty mold, this paper incorporates the biological characteristics of vector host preference to construct a dynamic model for Moso bamboo sooty mold with integrated control measures, including insecticide spraying, diseased plant removal, nutrient solution injection, and yellow adhesive trap trapping [13]. Relying on optimal control theory to optimize control implementation strategies, this study quantifies the management effectiveness and economic investment of different control schemes, systematically conducts cost-effectiveness comparisons, and identifies an optimal control combination that balances control efficacy and economic investment. This research aims to clarify the epidemiological dynamics of bamboo sooty mold under

the regulation of vector preferences; the relevant conclusions can provide quantitative references and a theoretical basis for the refined control of bamboo sooty mold in bamboo plantations and the rational allocation of control resources.

The remaining sections of this paper are organized as follows: Section 2 constructs a dynamic model for integrated control of Moso bamboo sooty mold, incorporating vector preference, and elaborates on the disease's biological background and fundamental modeling assumptions. Section 3 investigates the properties of the dynamic model and calculates the basic reproduction number, which serves as the critical threshold determining disease persistence or extinction. This section also analyzes the local and global stability of all equilibria and explores the persistence of the proposed system. Section 4 conducts an optimal control study based on the deterministic model. Section 5 performs a global sensitivity analysis of model parameters and a cost-effectiveness analysis under multiple control scenarios to identify the optimal economic control strategy. Section 6 summarizes the research conclusions, explains the biological implications of the results, and outlines directions for future research.

2. Model formulation

Moso bamboo sooty mold is a fungal disease, and bamboo aphids act as primary insect vectors, facilitating disease outbreaks. Bamboo aphids have multiple generations per year and can cause continuous damage throughout the entire growth period of Moso bamboo, with the most severe infestation occurring from late March to late May, when Moso bamboo is shooting and renewing its leaves [23]. Adult aphids and nymphs cluster on tender branches and new leaves to suck sap, directly causing premature leaf senescence and abscission. Honeydew secreted by aphids adheres to bamboo tissues and acts as a nutritional substrate for sooty mold fungi, facilitating widespread fungal colonization and subsequent soot layer formation that substantially suppresses foliar photosynthesis [10]. The pathogens are dispersed by wind, rain, and insects. For some obligate parasitic sooty mold fungi, a time lag exists after epidermal colonization before typical black lesions emerge on bamboo leaves [3].

Based on the biological background described above, the Moso bamboo population is partitioned into three compartments at an arbitrary time t : Healthy hosts denoted as $S_h(t)$, infected asymptomatic hosts denoted as $E_h(t)$ (non-infectious at this stage), and symptomatic infectious hosts denoted as $I_h(t)$. The variable $N_h(t)$ stands for the total quantity of Moso bamboo within the studied woodland at time t , satisfying the identity $N_h(t) = S_h(t) + E_h(t) + I_h(t)$. Only mature individuals of the bamboo aphid vectors are incorporated into the established model. By the same grouping criteria, adult bamboo aphids are sorted into three status categories: Susceptible (S_v), exposed (E_v), and infectious (I_v). Exposed aphid individuals carry disease pathogens but remain incapable of pathogen transmission to the surrounding Moso bamboo. The total aphid population in the forest at time t is defined by $N_v(t)$, such that $N_v(t) = S_v(t) + E_v(t) + I_v(t)$.

Based on traditional management practices for Moso bamboo plantations, we have proposed several basic modeling assumptions, as described below. First, given that the prevention of bamboo sooty mold in Moso bamboo plantations primarily relies on standardized pre-planting management to curb fungal invasion, newly transplanted Moso bamboo seedlings are assumed to be pathogen-free following pre-planting inspections and sanitary treatments. Second, the natural mortality rates of Moso bamboo and bamboo aphids are denoted as μ_h and μ_v , respectively, while θ_h represents the additional mortality rate of infected bamboo hosts due to the disease. Finally, the bamboo stand is regulated through an immediate replanting scheme to maintain a constant

total population size: Individuals lost due to natural aging or pathogen infection are rapidly replenished at their respective constant rates $\mu_h N_h$ and $\theta_h I_h$.

Vector bias effect refers to the phenomenon where vectors exhibit different preferences when selecting hosts. To characterize the above phenomenon in the proposed model, vector bias parameters p and q are defined. These two parameters denote the probabilities that bamboo aphids randomly select infected and healthy moso bamboo plants, respectively. Sooty mold colonization enables epiphytic microorganisms on infected Moso bamboos to release volatile odors that attract bamboo aphids, which supports the modeling assumption that $p \geq q$. Assuming the biting rate of bamboo aphids is b_v , representing the number of bites per aphid per unit time, the number of bamboo plants bitten by infected aphids and the number of aphids infected by infected bamboo plants per unit time are $b_v I_v(t)$ and $b_v I_h(t)$, respectively. Following the approach in [25], the number of newly infected bamboo plants and the number of newly infected bamboo aphids per unit time are respectively: $\frac{b_v \beta_{vh} q S_h I_v}{p I_h(t) + q(S_h(t) + E_h(t))}$, $\frac{b_v \beta_{hv} p S_v I_h}{p I_h(t) + q(S_h(t) + E_h(t))}$, where β_{vh} denotes the probability per bite that an infected bamboo aphid transmits the infection to a healthy Moso bamboo, and β_{hv} denotes the probability per bite that a susceptible bamboo aphid acquires the infection from an infected Moso bamboo.

The compartment flow diagram of the Moso bamboo sooty mold mathematical model is provided in Figure 1. Referring to this schematic, the corresponding governing equations for disease spread are constructed as follows:

$$\begin{cases} \frac{dS_h(t)}{dt} = \mu_h N_h - \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - \mu_h S_h + \theta_h I_h, \\ \frac{dE_h(t)}{dt} = \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - (\mu_h + \alpha_h) E_h, \\ \frac{dI_h(t)}{dt} = \alpha_h E_h - (\mu_h + \theta_h) I_h, \\ \frac{dS_v(t)}{dt} = \Lambda_v - \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - \mu_v S_v, \\ \frac{dE_v(t)}{dt} = \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - (\gamma_v + \mu_v) E_v, \\ \frac{dI_v(t)}{dt} = \gamma_v E_v - \mu_v I_v, \end{cases} \quad (2.1)$$

where Λ_v is the recruitment rate of bamboo aphids per unit time, and α_h, γ_v represent the conversion rates. Assume that all parameters in system (2.1) are strictly positive, and their biological meanings are listed in Table 1.

It is easy to see that the total number of Moso bamboo and aphids satisfies the following linear equations:

$$\begin{cases} \frac{dN_h(t)}{dt} = 0, \\ \frac{dN_v(t)}{dt} = \Lambda_v - \mu_v N_v(t). \end{cases} \quad (2.2)$$

This means that for any $t > 0$, we have $N_h(t) = 1$, i.e., the population size remains constant. Clearly, it has a unique positive solution $(1, \frac{\Lambda_v}{\mu_v})$, which is globally asymptotically stable (GAS). Denote $M \triangleq \frac{\Lambda_v}{\mu_v}$. For simplicity, let $S_h(t) + E_h(t) + I_h(t) = 1$ and $S_v(t) + E_v(t) + I_v(t) = M$,

Table 1. Description of the variables and parameters for model (2.3).

Parameter	Description	Value	Source
b_v	Number of bites per bamboo aphid ($Week^{-1}$)	7	Estimated
β_{vh}, β_{hv}	Disease transmission rate ($Week^{-1}$)	0.015, 0.022	Estimated
μ_h, μ_v	Natural mortality rate of bamboo and bamboo aphid ($Week^{-1}$)	3.2×10^{-4} , 0.083	[31]
θ_h	Disease-induced death rate of infected bamboo ($Week^{-1}$)	7.931238×10^{-3}	[30]
α_h, γ_v	State transition rate ($Week^{-1}$)	0.035, 0.016	Estimated
Λ_v	Recruitment rate of bamboo aphid ($Week^{-1}$)	2.0×10^4	Estimated
p	Probability of aphid contacting an infected bamboo	0.8	Estimated
q	Probability of aphid contacting a healthy bamboo	0.2	Estimated
ρ_1	Efficacy of insecticide spraying for pest control	[0, 1]	-
ρ_2	Treatment efficacy of nutrient injections for moso bamboo	[0, 1]	-
ρ_3	Efficacy of culling diseased and symptomatic bamboo	[0, 1]	-
ρ_4	Trapping efficacy against pests using yellow adhesive traps	[0, 1]	-

system (2.1) is equivalent to the following system:

$$\begin{cases} \frac{dE_h(t)}{dt} = \frac{b_v \beta_{vh} q (1 - E_h(t) - I_h(t)) I_v}{q + (p - q) I_h} - (\mu_h + \alpha_h) E_h, \\ \frac{dI_h(t)}{dt} = \alpha_h E_h - (\mu_h + \theta_h) I_h, \\ \frac{dE_v(t)}{dt} = \frac{b_v \beta_{hv} p (M - E_v(t) - I_v(t)) I_h}{p I_h + q (S_h + E_h)} - (\gamma_v + \mu_v) E_v, \\ \frac{dI_v(t)}{dt} = \gamma_v E_v - \mu_v I_v. \end{cases} \quad (2.3)$$

From ecological perspectives, subsequent theoretical analysis for model (2.3) is restricted within the bounded domain $\Omega = \{(E_h(t), I_h(t), E_v(t), I_v(t)) \in \mathbb{R}_+^4 : E_h(t) \geq 0, I_h(t) \geq 0, E_v(t) \geq 0, I_v(t) \geq 0, E_h(t) + I_h(t) \leq 1, E_v + I_v \leq M\}$. Then it follows from [36] that all solutions of system (2.3) starting in Ω remain in Ω for all $t \geq 0$. Therefore, Ω is positively invariant, and it is sufficient to consider only the solutions in Ω . In this region, the usual existence, uniqueness, and continuation results hold for system (2.3). All initial states adopted for later computations are selected from elements belonging to Ω .

3. Model analysis

In model (2.3), the following composite parameters are defined to facilitate the analysis: $k_1 = \alpha_h + \mu_h$, $k_2 = \mu_h + \theta_h$, $k_3 = \gamma_v + \mu_v$.

3.1. Basic reproduction number and stability of the disease-free equilibrium

In this section, the dynamics of model (2.1) are mainly explored.

Next, the basic reproduction number $\mathcal{R}_0$ is derived. Model (2.1) always has the disease-free equilibrium (DFE), given by $E^0 = (S_h^0, E_h^0, I_h^0, S_v^0, E_v^0, I_v^0) = (1, 0, 0, 0, M, 0, 0)$, where $M = \frac{\Lambda_v}{\mu_v}$. Let $E_h(t)$, $I_h(t)$, $E_v(t)$ and $I_v(t)$ denote the infected compartments. Then, adopting the notation

from [12, 14], the matrices $\mathbb{F}$ and $\mathbb{V}$ corresponding to the new infection terms and the remaining transfer terms, respectively, are given by:

$$\mathbb{F} = \begin{pmatrix} 0 & 0 & 0 & b_v\beta_{vh} \\ 0 & 0 & 0 & 0 \\ 0 & b_v\beta_{hv}\frac{p}{q} & M & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{V} = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ -\alpha_h & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & -\gamma_v & \mu_v \end{pmatrix}.$$

Thus, the $\mathcal{R}_0$ of system (2.1) equals the spectral radius of $\mathbb{F}\mathbb{V}^{-1}$, i.e.

$$\mathcal{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \sqrt{\frac{b_v^2\beta_{vh}\beta_{hv}\gamma_v\alpha_h pM}{\mu_v k_1 k_2 k_3 q}}. \quad (3.1)$$

For biological interpretation, the above expression can be rewritten as

$$\mathcal{R}_0 = \sqrt{\mathcal{R}_{vh}\mathcal{R}_{hv}},$$

with

$$\mathcal{R}_{vh} = \frac{b_v\beta_{vh}\alpha_h}{\mu_v k_1} \quad \text{and} \quad \mathcal{R}_{hv} = \frac{b_v\beta_{hv}\gamma_v pM}{q k_2 k_3}.$$

For $\mathcal{R}_{vh}$, $\frac{1}{\mu_v}$ denotes the average lifespan of an infected vector (bamboo aphid), so $\mathcal{R}_{vh}$ is the average number of newly infected bamboo plants produced by one infectious vector during its entire lifetime; it is the reproduction number contributed by vector-to-host transmission. Similarly, for $\mathcal{R}_{hv}$, $\frac{1}{\mu_h + \theta_h}$ denotes the average infectious period of an infected bamboo plant, so $\mathcal{R}_{hv}$ is the average number of newly infected vectors generated by one infectious bamboo plant during its entire infectious period; it is the reproduction number contributed by host to vector transmission. Therefore, the basic reproduction number of the model is determined by the joint action of the two sub-reproduction numbers contributed by vector-borne transmission, and takes the form of the square root of their product.

Clearly, there always exists a DFE ($\bar{E}^0 = (0, 0, 0, 0, 0)$) for system (2.3). The following theorem establishes the local asymptotic stability (LAS) and GAS of the DFE ($\bar{E}^0$).

Remark 3.1. Through dimension reduction, we simplify the original six-dimensional system (system (2.1)) into an equivalent four-dimensional system (system (2.3)). To avoid symbolic confusion, E^0 and E^* denote the disease-free and endemic equilibria of the original system, whereas $\bar{E}^0$ and $\bar{E}^*$ represent the corresponding equilibria of the reduced system. Subsequent equilibrium stability analyses are conducted for the reduced system unless otherwise specified.

Theorem 3.1. *The DFE ($\bar{E}^0$) of system (2.3) is LAS if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix of system (2.3) at $\bar{E}_0$ takes the form:

$$J_{\bar{E}^0} = \begin{pmatrix} -k_1 & 0 & 0 & b_v\beta_{vh} \\ \alpha_h & -k_2 & 0 & 0 \\ 0 & \frac{b_v\beta_{hv}pM}{q} & -k_3 & 0 \\ 0 & 0 & \gamma_v & -\mu_v \end{pmatrix}.$$

Its characteristic equation of is as follows:

$$g(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (3.2)$$

where

$$\begin{aligned} a_1 &= \mu_v + k_1 + k_2 + k_3 > 0, \\ a_2 &= k_1k_2 + \mu_vk_3 + (k_1 + k_2)(\mu_v + k_3) > 0, \\ a_3 &= k_1k_2(\mu_v + k_3) + \mu_vk_3(k_1 + k_2) > 0, \\ a_4 &= \mu_vk_1k_2k_3(1 - \mathcal{R}_0^2). \end{aligned}$$

In the matrix $-J_{\bar{E}^0}$, all off-diagonal elements are non-positive, and all diagonal elements are non-negative. Furthermore, under the condition $\mathcal{R}_0 < 1$, all leading principal minors are positive. This implies that $-J_{\bar{E}^0}$ is an M-matrix. Therefore, $J_{\bar{E}^0}$ is a Hurwitz matrix, and all its eigenvalues have negative real parts. Consequently, the DFE is LAS when $\mathcal{R}_0 < 1$.

According to Descartes' Rule of Signs [7], if $\mathcal{R}_0 > 1$, $a_4 < 0$, the characteristic equation (3.2) has at least one positive root. According to the Routh-Hurwitz criterion, it follows that the DFE ($\bar{E}^0$) is unstable for $\mathcal{R}_0 > 1$. $\square$

Theorem 3.2. *If $\mathcal{R}_0 < 1$, then the DFE ($\bar{E}^0$) of system (2.3) is GAS.*

Proof. From [34], we find that $(E'_h, I'_h, E'_v, I'_v)^T = (F - V)(E_h, I_h, E_v, I_v)^T - f(E_h, I_h, E_v, I_v)$, where

$$\begin{aligned} f(E_h, I_h, E_v, I_v) &= \begin{pmatrix} \frac{b_v\beta_{vh}I_v(pI_h + qE_h)}{q + (p - q)I_h} \\ 0 \\ b_v\beta_{hvv}pI_h[\frac{M}{q} - \frac{M - E_v(t) - I_v(t)}{q + (p - q)I_h}] \\ 0 \end{pmatrix}, \\ F - V &= \begin{pmatrix} -k_1 & 0 & 0 & b_v\beta_{vh} \\ \alpha_h & -k_2 & 0 & 0 \\ 0 & \frac{b_v\beta_{hv}pN_v}{qN_h} & -k_3 & 0 \\ 0 & 0 & \gamma_v & -\mu_v \end{pmatrix}. \end{aligned}$$

Given that $F \geq 0$ and $V^{-1} \geq 0$, and noting that $\frac{M - E_v(t) - I_v(t)}{q + (p - q)I_h} \leq \frac{M}{q}$, we have $f(E_h, I_h, E_v, I_v) \geq 0$. The Lyapunov function is then defined as follows:

$$L = w^T V^{-1}(E_h, I_h, E_v, I_v)^T,$$

in which $w^T = (0, 1, 0, \frac{\alpha_h b_v \beta_{vh}}{\mathcal{R}_0(\alpha_h + \mu_h)(\mu_h + \theta_h)})$ is the left eigenvector of $V^{-1}F$ associated with the eigenvalue $\mathcal{R}_0 = \rho(V^{-1}F) = \rho(FV^{-1})$. Consequently, its orbital derivative along the trajectories of system (2.3) is

$$\begin{aligned} \left. \frac{dL_h(t)}{dt} \right|_{(3)} &= w^T V^{-1}(E'_h, I'_h, E'_v, I'_v)^T \\ &= w^T V^{-1}(F - V)(E_h, I_h, E_v, I_v)^T - w^T V^{-1}f(E_h, I_h, E_v, I_v) \\ &= (\mathcal{R}_0 - 1)w^T (E_h, I_h, E_v, I_v)^T - w^T V^{-1}f(E_h, I_h, E_v, I_v). \end{aligned} \quad (3.3)$$

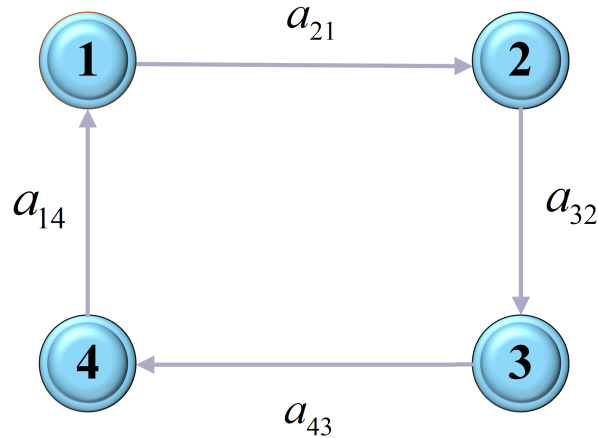


Figure 2. The weighted digraph $(\mathcal{G}, A)$ constructed for the relapse model (2.3).

Because $w^T \geq 0$, $V^{-1} \geq 0$, and $f(E_h, I_h, E_v, I_v) \geq 0$, it follows that whenever $\mathcal{R}_0 \leq 1$, we have $\frac{dL(t)}{dt} \leq 0$ within Ω . If $\mathcal{R}_0 < 1$, the condition $\frac{dL(t)}{dt} = 0$ forces $E_h(t) = I_h(t) = E_v(t) = I_v(t) = 0$. If $\mathcal{R}_0 = 1$, then $\frac{dL(t)}{dt} = -w^T V^{-1} f(E_h, I_h, E_v, I_v) = 0$ implies either $I_h(t) = I_v(t) = 0$ or $E_h(t) = I_h(t) = 0$. In the case $I_h(t) = I_v(t) = 0$, system (2.3) yields $\lim_{t \rightarrow \infty} E_h(t) = \lim_{t \rightarrow \infty} E_v(t) = 0$. In the case $E_h(t) = I_h(t) = 0$, system (2.3) gives $\lim_{t \rightarrow \infty} E_v(t) = \lim_{t \rightarrow \infty} I_v(t) = 0$. Consequently, if $\mathcal{R}_0 \leq 1$, the largest invariant set contained in $\{(E_h, I_h, E_v, I_v) \in \Omega : \frac{dL(t)}{dt} = 0\}$ is exactly the $\bar{E}^0 = (0, 0, 0, 0)$. By LaSalle's invariance principle, $\bar{E}^0$ is GAS in Ω when $\mathcal{R}_0 \leq 1$. $\square$

3.2. The existence and stability of endemic equilibrium

In this section, the existence and uniqueness of the endemic equilibrium (EE) for system (2.3) are firstly explored, followed by an analysis of its stability.

Let $\bar{E}^* = (E_h^*, I_h^*, E_v^*, I_v^*)$ be the EE of system (2.3), where $E_h^* > 0, I_h^* > 0, E_v^* > 0, I_v^* > 0$, then it satisfies the following equations:

$$\begin{cases} \frac{b_v \beta_{vh} q (1 - E_h^* - I_h^*) I_v^*}{q + (p - q) I_h^*} - \alpha_h E_h^* - \mu_h E_h^* = 0, \\ \alpha_h E_h^* - \mu_h I_h^* - \theta_h I_h^* = 0, \\ \frac{b_v \beta_{hv} p (M - E_v^* - I_v^*) I_h^*}{q + (p - q) I_h^*} - \gamma_v E_v^* - \mu_v E_v^* = 0, \\ \gamma_v E_v^* - \mu_v I_v^* = 0. \end{cases} \quad (3.4)$$

Clearly, $I_h^* = \frac{\alpha_h E_h^*}{\theta_h + \mu_h}$, $I_v^* = \frac{\gamma_v E_v^*}{\mu_v}$. Substitute I_h^* and I_v^* into the first equation:

$$E_v^* = \frac{\mu k_1 E_h^* (q + (p - q) \frac{\alpha_h}{k_2} E_h^*)}{b_v \beta_{vh} q \gamma_v (1 - (1 + \frac{\alpha_h}{k_2}) E_h^*)}.$$

From the third equation in (3.4), the following equation can be derived:

$$b_1 E_h^{*2} + b_2 E_h^* + b_3 = 0, \quad (3.5)$$

where

$$\begin{aligned} b_1 &= b_v \beta_{hv} p k_1 \frac{\alpha_h}{k_2} \left(1 + \frac{\gamma_v}{\mu_v}\right) \left[(p - q) \frac{\alpha_h}{k_2}\right] + k_1 k_3 \left[(p - q) \frac{\alpha_h}{k_2}\right]^2 > 0, \\ b_2 &= M b_v^2 \beta_{vh} \beta_{hv} p q \frac{\gamma_v}{\mu_v} \frac{\alpha_h}{k_2} \left(1 + \frac{\alpha_h}{k_2}\right) + k_1 q b_v \beta_{hv} p \frac{\alpha_h}{k_2} \left(1 + \frac{\gamma_v}{\mu_v}\right) + 2 k_1 k_3 q (p - q) \frac{\alpha_h}{k_2} > 0, \\ b_3 &= k_1 k_3 q^2 (1 - \mathcal{R}_0^2). \end{aligned}$$

According to Descartes' Rule of Signs [14], if $\mathcal{R}_0 > 1, b_3 < 0$, the characteristic equation (3.5) has exactly one positive root.

To summarize the above discussion, the following result is obtained.

Theorem 3.3. *When $\mathcal{R}_0 > 1$, system (2.3) has a unique EE $\bar{E}^* = (E_h^*, I_h^*, E_v^*, I_v^*)$, and there is no EE when $\mathcal{R}_0 \leq 1$.*

Theorem 3.4. *If $\mathcal{R}_0 > 1$, the EE $\bar{E}^* = (E_h^*, I_h^*, E_v^*, I_v^*)$ of model (2.3) is GAS in $\text{Int}(\mathbb{R}_+^4)$.*

Proof. To establish the uniqueness and global stability of $\bar{E}^*$, set $D_1 = E_h - E_h^* - E_h^* \ln \frac{E_h}{E_h^*}$, $D_2 = I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*}$, $D_3 = E_v - E_v^* - E_v^* \ln \frac{E_v}{E_v^*}$, and $D_4 = I_v - I_v^* - I_v^* \ln \frac{I_v}{I_v^*}$. Differentiation gives

$$\begin{aligned} \frac{dD_1(t)}{dt} &= \left(1 - \frac{E_h^*}{E_h}\right) \frac{dE_h(t)}{dt} \\ &= \left(1 - \frac{E_h^*}{E_h}\right) \left[b_v \beta_{vh} (1 - E_h - I_h) I_v - b_v \beta_{vh} \frac{(1 - E_h^* - I_h^*) I_v^*}{E_h^*} E_h \right] \\ &= b_v \beta_{vh} (1 - E_h^* - I_h^*) I_v^* \left(1 - \frac{E_h^*}{E_h}\right) \left(\frac{(1 - E_h - I_h) I_v}{(1 - E_h^* - I_h^*) I_v^*} - \frac{E_h}{E_h^*} \right) \\ &\leq b_v \beta_{vh} (1 - E_h^* - I_h^*) I_v^* \left(\frac{(1 - E_h - I_h) I_v}{(1 - E_h^* - I_h^*) I_v^*} - \ln \frac{(1 - E_h - I_h) I_v}{(1 - E_h^* - I_h^*) I_v^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*} \right) \\ &=: a_{14} G_{14}, \end{aligned}$$

where $a_{14} = b_v \beta_{vh} (1 - E_h^* - I_h^*) I_v^*$,

$$\begin{aligned} \frac{dD_2(t)}{dt} &= \left(1 - \frac{I_h^*}{I_h}\right) \frac{dI_h(t)}{dt} \\ &= \left(1 - \frac{I_h^*}{I_h}\right) (\alpha_h E_h - k_2 I_h) \\ &= \alpha_h E_h^* \left(1 - \frac{I_h^*}{I_h}\right) \left(\frac{E_h}{E_h^*} - \frac{I_h}{I_h^*} \right) \\ &\leq \alpha_h E_h^* \left(\frac{E_h}{E_h^*} - \ln \frac{E_h}{E_h^*} - \frac{I_h}{I_h^*} + \ln \frac{I_h}{I_h^*} \right) \\ &=: a_{21} G_{21}, \end{aligned}$$

where $a_{21} = \alpha_h E_h^*$,

$$\begin{aligned} \frac{dD_3(t)}{dt} &= \left(1 - \frac{E_v^*}{E_v}\right) \frac{dE_v(t)}{dt} \\ &= \left(1 - \frac{E_v^*}{E_v}\right) \left[b_v \beta_{hv} (M - E_v - I_v) I_h - b_v \beta_{hv} \frac{(M - E_v^* - I_v^*) I_h^*}{E_v^*} E_v \right] \end{aligned}$$

$$\begin{aligned}
&= b_v \beta_{hv} (M - E_v^* - I_v^*) I_h^* \left(1 - \frac{E_v^*}{E_v}\right) \left(\frac{(M - E_v - I_v) I_h}{(M - E_v^* - I_v^*) I_h^*} - \frac{E_v}{E_v^*}\right) \\
&\leq b_v \beta_{hv} (M - E_v^* - I_v^*) I_h^* \left(\frac{(M - E_v - I_v) I_h}{(M - E_v^* - I_v^*) I_h^*} - \ln \frac{(M - E_v - I_v) I_h}{(M - E_v^* - I_v^*) I_h^*} - \frac{E_v}{E_v^*} + \ln \frac{E_v}{E_v^*}\right) \\
&=: a_{32} G_{32},
\end{aligned}$$

where $a_{32} = b_v \beta_{hv} (M - E_v^* - I_v^*) I_h^*$,

$$\begin{aligned}
\frac{dD_4(t)}{dt} &= \left(1 - \frac{I_v^*}{I_v}\right) \frac{dI_v(t)}{dt} \\
&= \left(1 - \frac{I_v^*}{I_v}\right) \left(\gamma_v E_v - \frac{\gamma_v E_v^*}{I_v^*} I_v\right) \\
&= \gamma_v E_v^* \left(1 - \frac{I_v^*}{I_v}\right) \left(\frac{E_v}{E_v^*} - \frac{I_v}{I_v^*}\right) \\
&\leq \gamma_v E_v^* \left(\frac{E_v}{E_v^*} - \ln \frac{E_v}{E_v^*} - \frac{I_v}{I_v^*} + \ln \frac{I_v}{I_v^*}\right) \\
&=: a_{43} G_{43},
\end{aligned}$$

where $a_{43} = \gamma_v E_v^*$.

The corresponding weighted directed graph $(\mathcal{G}, A)$ consists of four vertices and contains a single cycle; see Figure 2. Along each cycle, $G_{14} + G_{21} + G_{32} + G_{43} = 0$. According to Theorem 3.5 in [34], there exist constants c_i for $i = 1, 2, 3, 4$ such that $V = \sum_{i=1}^4 c_i V_i$ serves as a Lyapunov function for system (2.3). The relationships among the constants c_i can be obtained from Theorems 3.3 and 3.4: $d^+(1) = 1$ implies $c_2 a_{21} = c_1 a_{14}$, $d^+(2) = 1$ implies $c_3 a_{32} = c_2 a_{21}$, $d^+(3) = 1$ implies $c_4 a_{43} = c_3 a_{32}$, and $d^-(4) = 1$ implies $c_1 a_{14} = c_4 a_{43}$. Hence, let $c_1 = 1$, $c_2 = \frac{\alpha_h + \mu_h}{\alpha_h}$, $c_3 = \frac{(\alpha_h + \mu_h) E_h^*}{(\gamma_v + \mu_v) E_v^*}$, $c_4 = \frac{(\alpha_h + \mu_h) E_h^*}{\gamma_v E_v^*}$.

Therefore, the Lyapunov function is:

$$\begin{aligned}
V &= c_1 D_1 + c_2 D_2 + c_3 D_3 + c_4 D_4 \\
&= D_1 + \frac{\alpha_h + \mu_h}{\alpha_h} D_2 + \frac{(\alpha_h + \mu_h) E_h^*}{(\gamma_v + \mu_v) E_v^*} D_3 + \frac{(\alpha_h + \mu_h) E_h^*}{\gamma_v E_v^*} D_4.
\end{aligned} \tag{3.6}$$

The inequality $\frac{dV(t)}{dt} \leq 0$ holds, and the equality $\frac{dV(t)}{dt} = 0$ holds if and only if $G_{14} = 0$, $G_{21} = 0$, $G_{32} = 0$, and $G_{43} = 0$, which yields $\frac{E_h}{E_h^*} = \frac{I_h}{I_h^*} = \frac{E_v}{E_v^*} = \frac{I_v}{I_v^*}$. According to the system equations, the only invariant set contained in the interior of $\mathbb{R}_+^4$ is $\bar{E}^*$. Therefore, the EE $\bar{E}^*$ is GAS within the interior of $\mathbb{R}_+^4$. $\square$

Theorem 3.5. *If $\mathcal{R}_0 > 1$, then model (2.1) is uniformly persistent; that is, there exists a positive constant ξ such that every positive solution $(S_h(t), E_h(t), I_h(t), S_v(t), E_v(t), I_v(t))$ of (2.1) satisfies*

$$\begin{aligned}
\liminf_{t \rightarrow \infty} S_h(t) &\geq \xi, & \liminf_{t \rightarrow \infty} E_h(t) &\geq \xi, & \liminf_{t \rightarrow \infty} I_h(t) &\geq \xi, \\
\liminf_{t \rightarrow \infty} S_v(t) &\geq \xi, & \liminf_{t \rightarrow \infty} E_v(t) &\geq \xi, & \liminf_{t \rightarrow \infty} I_v(t) &\geq \xi.
\end{aligned}$$

Proof. Define $\chi = \{(S_h, E_h, I_h, S_v, E_v, I_v) : S_h \geq 0, E_h \geq 0, I_h \geq 0, S_v \geq 0, E_v \geq 0, I_v \geq 0\}$, $\chi_0 = \{(S_h, E_h, I_h, S_v, E_v, I_v) : S_h \geq 0, E_h > 0, I_h > 0, S_v \geq 0, E_v > 0, I_v > 0\}$, $\partial\chi_0 = \chi \setminus \chi_0$. Let $u(t, t_0, x_0)$ denote the unique solution to model (2.1) corresponding to the initial condition $x_0 = (S_{h0}, E_{h0}, I_{h0}, S_{v0}, E_{v0}, I_{v0})$ at time t_0 .

Define the Poincaré map $f : \chi \rightarrow \chi$ by $f(x_0) = u(t_0 + 1, x_0)$ for all $x_0 \in \chi$. Then set $M_\partial = \{x_0 \in \partial\chi : f^m(x_0) \in \partial\chi \text{ for all } m \in \mathbb{Z}_+\}$. The following claim holds: $M_\partial = \{(S_h, 0, 0, S_v, 0, 0) : S_h \geq 0, S_v \geq 0\}$. Clearly, $\{(S_h, 0, 0, S_v, 0, 0) : S_h \geq 0, S_v \geq 0\} \subseteq M_\partial$. It remains to verify that

$$M_\partial \setminus \{(S_h, 0, 0, S_v, 0, 0) : S_h \geq 0, S_v \geq 0\} = \emptyset. \quad (3.7)$$

Suppose not, then there exists a point $(S_{h0}, E_{h0}, I_{h0}, S_{v0}, E_{v0}, I_{v0}) \in M_\partial \setminus \{(S_h, 0, 0, S_v, 0, 0) : S_h \geq 0, S_v \geq 0\}$. To show that (3.7) holds, it suffices to show that $E_h(t) \equiv 0$, $I_h(t) \equiv 0$, $E_v(t) \equiv 0$ or $I_v(t) \equiv 0$, for all $t \geq 0$. This result is proven by contradiction. Assume the contrary. Without loss of generality, take $E_{h0} = I_{h0} = E_{v0} = 0$ and $I_{v0} > 0$ (the remaining cases can be handled analogously). It can be deduced that $I_v(t) > 0$ and $S_h(t) > 0$ for any $t > t_0$. It follows from model (2.1) that $\left. \frac{dE_h(t)}{dt} \right|_{t=t_0} = b_v \beta_{vh} I_v(t_0) > 0$. Thus, $E_h(t) > 0$ for $0 < t - t_0 \ll 1$. Then, it follows from system (2.1) that $I_h(t) > 0$ and $I_v(t) > 0$ for $0 < t - t_0 \ll 1$. Hence, $(S_h, E_h, I_h, S_v, E_v, I_v) \notin \partial\chi_0$ for $0 < t - t_0 \ll 1$. This contradiction therefore implies that (3.7) holds. Therefore, $M_\partial = \{(S_h, 0, 0, S_v, 0, 0) : S_h \geq 0, S_v \geq 0\}$. In what follows, the existence of a constant $\eta > 0$ is established such that

$$\limsup_{m \rightarrow \infty} d(f^m(x_0), E^0) \geq \eta, \quad \forall x_0 \in \chi_0, \quad \forall m \in \mathbb{Z}_+, \quad (3.8)$$

where $E^0 = (S_h^0, 0, 0, S_v^0, 0, 0)$. According to Theorem 2 in [14],

$$\mathcal{R}_0 > 1 \Leftrightarrow \rho(\mathbb{F}\mathbb{V}^{-1}) > 1 \Leftrightarrow \rho(\exp(\mathbb{F} - \mathbb{V})) > 1.$$

Thus, when $\mathcal{R}_0 > 1$, there exists a sufficiently small $\varepsilon_0 > 0$ such that

$$\rho(\exp(\mathbb{F} - \mathbb{V} - \mathbb{E}_{\varepsilon_0})) > 1, \quad (3.9)$$

where

$$\mathbb{E}_{\varepsilon_0} = \begin{pmatrix} 0 & 0 & 0 & b_v \beta_{vh} \frac{\varepsilon_0}{S_h^0} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{b_v \beta_{hv} p \varepsilon_0}{q S_h^0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If (3.8) fails to hold, then there exists some $x_0 \in \chi_0$ such that $\limsup_{m \rightarrow \infty} d(f^m(x_0), E^0) < \zeta_0$ for any $\zeta_0 > 0$. Without loss of generality, suppose that $d(f^m(x_0), E^0) < \zeta_0$ for all $\zeta_0 > 0$ and $m \in \mathbb{Z}_+$. By the continuity of solutions with respect to initial conditions, there exists a sufficiently small $\varepsilon_0 > 0$ such that

$$\|u(t, f^m(x_0)) - u(t, E^0)\| \leq \varepsilon_0, \quad \forall t \in [t_0, t_0 + 1], \quad \forall m \in \mathbb{Z}_+. \quad (3.10)$$

For any $t \geq t_0$, there exists an integer $l_0 \in \mathbb{Z}_+$ such that $t - t_0 = l_0 + \tilde{t}$ with $\tilde{t} \in [0, 1)$. Then, from (3.10), it follows that

$$\|u(t, f^m(x_0)) - u(t, E^0)\| = \|u(t_0 + \tilde{t}, f^{m+l_0}(x_0)) - u(t_0 + \tilde{t}, E^0)\| \leq \varepsilon_0.$$

Hence, it follows that

$$S_h(t) \geq S_h^0 - \varepsilon_0, \quad S_v(t) \geq S_v^0 - \varepsilon_0, \quad \text{for all } t \geq t_0. \quad (3.11)$$

Therefore, from (2.1) and (3.11), we have

$$\begin{cases} \frac{dE_h(t)}{dt} \geq b_v \beta_{vh} \left(1 - \frac{\varepsilon_0}{S_h^0}\right) I_v - (\mu_h + \alpha_h) E_h, \\ \frac{dI_h(t)}{dt} = \alpha_h E_h - (\mu_h + \delta_h) I_h, \\ \frac{dE_v(t)}{dt} \geq \frac{b_v \beta_{hv} p(S_v^0 - \varepsilon_0)}{q S_h^0} I_h - (\gamma_v + \mu_v) E_v, \\ \frac{dI_v(t)}{dt} = \gamma_v E_v - \mu_v I_v. \end{cases} \quad (3.12)$$

Clearly, system (3.12) is an irreducible cooperative system. Consider the following auxiliary system

$$\frac{d\tilde{X}(t)}{dt} = (\mathbb{F} - \mathbb{V} - \mathbb{E}_{\varepsilon_0}) \tilde{X}(t), \quad (3.13)$$

where $\tilde{X}(t) = (\tilde{E}_h(t), \tilde{I}_h(t), \tilde{E}_v(t), \tilde{I}_v(t))$, and

$$\mathbb{F} - \mathbb{V} - \mathbb{E}_{\varepsilon_0} = \begin{pmatrix} -k_1 & 0 & 0 & b_v \beta_{vh} \left(1 - \frac{\varepsilon_0}{S_h^0}\right) \\ \alpha_h & -k_2 & 0 & 0 \\ 0 & \frac{b_v \beta_{hv} p(S_v^0 - \varepsilon_0)}{q S_h^0} & -k_3 & 0 \\ 0 & 0 & \gamma_v & -\mu_v \end{pmatrix}.$$

Based on the conclusion established in [44], there exists a positive vector v such that $\tilde{X}(t) = v \exp(\Pi t)$ is a solution of model (3.13), where $\Pi = \ln \rho(\exp(\mathbb{F} - \mathbb{V} - \mathbb{E}_{\varepsilon_0}))$. From (3.9), we have $\Pi > 0$, which implies

$$\lim_{t \rightarrow \infty} \tilde{E}_h(t) = \infty, \lim_{t \rightarrow \infty} \tilde{I}_h(t) = \infty, \lim_{t \rightarrow \infty} \tilde{E}_v(t) = \infty, \lim_{t \rightarrow \infty} \tilde{I}_v(t) = \infty.$$

By the comparison principle, it follows that

$$\lim_{t \rightarrow \infty} E_h(t) = \infty, \lim_{t \rightarrow \infty} I_h(t) = \infty, \lim_{t \rightarrow \infty} E_v(t) = \infty, \lim_{t \rightarrow \infty} I_v(t) = \infty,$$

which contradicts the boundedness of E^0 . Hence, (3.8) holds, and f is weakly uniformly persistent with respect to $(\chi_0, \partial\chi_0)$. Consequently, $W^s(E^0) \cap \chi_0 = \emptyset$. It is obvious that every forward orbit in M_{∂} converges to E^0 . By Theorem 4.6 in [50], system (2.1) admits uniform persistence with respect to $(\chi_0, \partial\chi_0)$. This completes the proof of Theorem 3.5. $\square$

4. Optimal control

This section develops control strategies for Moso bamboo sooty mold disease to reduce the number of infected plants while minimizing the overall control cost. Indeed, this problem is a typical optimal control problem. This section first introduces the control variables $U(t) = (u_1(t), u_2(t), u_3(t), u_4(t)) \in \Lambda$ associated with the state variables $(S_h, E_h, I_h, S_v, E_v, I_v)$. The control variables are bounded and satisfy

$$\Lambda = \{(u_1(t), u_2(t), u_3(t), u_4(t)) : u_i \text{ is Lebesgue measurable,}$$

$$u_i(t) \in [0, 1], \quad t \in [0, T_f], \quad i = 1, 2, 3, 4\},$$

where T_f represents the final time. Within the proposed control framework, the function $u_1(t)$ corresponds to an increase in the mortality rate of bamboo aphids (e.g., via insecticide spraying). $u_2(t)$ denotes a reduction in the transmission probability of Moso bamboo sooty mold from aphids to bamboo; this can be achieved, for example, by injecting nutrient solutions. $u_3(t)$ represents the removal of infected and symptomatic Moso bamboo by farmers. Finally, $u_4(t)$ indicates a decrease in the transmission probability of sooty mold from bamboo to aphids, for instance, using yellow adhesive traps.

For the optimal control problem of the Moso bamboo sooty mold model, the following objective function is considered:

$$\begin{aligned} & \mathcal{J}(u_1(t), u_2(t), u_3(t), u_4(t)) \\ &= \int_0^{T_f} [A_1 I_h(t) + A_2 (E_v(t) + I_v(t)) + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t)] dt, \end{aligned} \quad (4.1)$$

subject to

$$\begin{cases} \frac{dS_h(t)}{dt} = \mu_h N_h - (1 - \rho_2 u_2(t)) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - \mu_h S_h + (\theta_h + \rho_3 u_3(t)) I_h, \\ \frac{dE_h(t)}{dt} = (1 - \rho_2 u_2(t)) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - (\mu_h + \alpha_h) E_h, \\ \frac{dI_h(t)}{dt} = \alpha_h E_h - (\mu_h + \theta_h + \rho_3 u_3(t)) I_h, \\ \frac{dS_v(t)}{dt} = \Lambda_v - (1 - \rho_4 u_4(t)) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - \mu_v S_v - \rho_1 u_1(t) S_v, \\ \frac{dE_v(t)}{dt} = (1 - \rho_4 u_4(t)) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - (\gamma_v + \mu_v) E_v - \rho_1 u_1(t) E_v, \\ \frac{dI_v(t)}{dt} = \gamma_v E_v - \mu_v I_v - \rho_1 u_1(t) I_v, \end{cases} \quad (4.2)$$

where $S_h(0) \geq 0, E_h(0) \geq 0, I_h(0) \geq 0, S_v(0) \geq 0, E_v(0) \geq 0, I_v(0) \geq 0$. The coefficients A_i (for $i = 1, 2$) are positive constants. The constants B_i ($i = 1, 2, 3, 4$) serve as weight coefficients for the costs of the following interventions: Spraying insecticides, injecting nutrient solution, removing infected Moso bamboo, and using yellow adhesive traps, respectively. As in [21], the intervention costs are assumed to be nonlinear quadratic functions.

The primary objective is to maintain the health of Moso bamboo, while minimizing the number of infected bamboos and the cost of control measures with minimal control inputs $u_i(t)$ ($i = 1, 2, 3, 4$). That is, the optimal controls satisfy

$$\mathcal{J}(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t)) = \min_{(u_1(t), u_2(t), u_3(t), u_4(t)) \in \Lambda} \mathcal{J}(u_1(t), u_2(t), u_3(t), u_4(t)),$$

subject to the dynamics described by model (4.2). In the subsequent analysis, Pontryagin's maximum principle [29] is adopted to address this optimal control problem.

4.1. Existence of the optimal control problem

Drawing upon the research framework proposed in [16], the existence property for the controlled dynamical system is validated within the current subsection. To proceed with subsequent analysis, the Lagrangian functional $\mathcal{L}$ corresponding to the system (4.2) is formulated in the form

presented below:

$$\mathcal{L} = A_1 I_h(t) + A_2(E_v(t) + I_v(t)) + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t).$$

Theorem 4.1. *For the objective functional $\mathcal{J}(u_1(t), u_2(t), u_3(t), u_4(t)) = \int_0^T [A_1 I_h(t) + A_2(E_v(t) + I_v(t)) + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t)] dt$, associated with system (4.2) and defined over Λ , there exists an optimal control strategy $U^* = (u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t))$ such that*

$$\mathcal{J}(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t)) = \min_{(u_1(t), u_2(t), u_3(t), u_4(t)) \in \Lambda} \mathcal{J}(u_1(t), u_2(t), u_3(t), u_4(t)).$$

Proof. To demonstrate the existence of an optimal control policy U^* that minimizes $\mathcal{J}(U)$, we invoke Theorem III.4.1 from [16]. It is sufficient to verify the following assumptions:

- (H1) The collection comprising the control variables and their associated state variables is nonempty.
- (H2) The set of admissible controls is convex and closed.
- (H3) For the state system (4.2), each right-hand side is continuous, bounded above by a sum involving bounded control and state terms, and can be expressed as a linear function of U whose coefficients may depend on time and the state variables.
- (H4) The integrand of the objective functional is concave.
- (H5) There exist constants $C_1, C_2 > 0$ and $\beta^* > 1$ such that the integrand fulfills the inequality $\mathcal{L}(t, I_h, E_v, I_v, U) \geq C_1 \|U\|^{\beta^*} - C_2$.

Evidently, Λ is a nonempty family of Lebesgue-integrable functions defined on $[0, T_f]$ and mapping into $\mathbb{R}$. Note that all solutions stay bounded; thus, the admissible control set is both bounded and convex. Moreover, system (4.2) is bilinear with respect to the control variables, and the solutions of (4.2) are bounded. Consequently, condition (H3) is satisfied.

Let $\theta \in (0, 1)$, and $u(t), v(t) \in \Lambda$, we have

$$\begin{aligned} & \mathcal{L}(t, I_h, E_v, I_v, (1 - \theta)u(t) + \theta v(t)) - (1 - \theta)\mathcal{L}(t, I_h, E_v, I_v, u(t)) - \theta\mathcal{L}(t, I_h, E_v, I_v, v(t)) \\ &= \sum_{i=1}^4 B_i [(1 - \theta)^2 u_i^2 + \theta^2 v_i^2 + 2\theta(1 - \theta)u_i v_i - (1 - \theta)u_i^2 - \theta v_i^2] \\ &= - \sum_{i=1}^4 B_i \theta(1 - \theta)(u_i - v_i)^2 \\ &< 0. \end{aligned}$$

Thus, $\mathcal{L}(t, I_h, E_v, I_v, (1 - \theta)u(t) + \theta v(t)) < (1 - \theta)\mathcal{L}(t, I_h, E_v, I_v, u(t)) + \theta\mathcal{L}(t, I_h, E_v, I_v, v(t))$, which implies that $\mathcal{L}$ is convex in the control variable U . Accordingly, the assumption (H4) is verified.

Finally, notice that

$$A_1 I_h + A_2(E_v + I_v) + \sum_{i=1}^4 B_i u_i^2 \geq \sum_{i=1}^4 B_i u_i^2 \geq C_1 \|U\|^{\beta^*} - C_2,$$

where $C_1 = B_{\min}$, $\beta^* = 2$, and $C_2 > 0$. Accordingly, the assumption (H5) holds entirely. Outcomes documented inside the reference [24] further guarantee the presence of a minimising control profile U^* capable of reaching the minimal value of $\mathcal{J}(U)$. $\square$

Next, computation of minimizer expressions requires constructing the Hamiltonian functional corresponding to the controlled system (4.2). The Hamiltonian quantity marked $\mathcal{H}$ is formulated via the expression listed below:

$$\begin{aligned}\mathcal{H} = & A_1 I_h + A_2 E_v + A_2 I_v + B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2 + B_4 u_4^2 \\ & + \lambda_1 \left[\mu_h N_h - (1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - \mu_h S_h + \theta_h I_h + \rho_3 u_3 I_h \right] \\ & + \lambda_2 \left[(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - (\mu_h + \alpha_h) E_h \right] \\ & + \lambda_3 [\alpha_h E_h - (\mu_h + \theta_h + \rho_3 u_3) I_h] \\ & + \lambda_4 \left[\Lambda_v - (1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - (\mu_v + \rho_1 u_1) S_v \right] \\ & + \lambda_5 \left[(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - (\gamma_v + \mu_v + \rho_1 u_1) E_v \right] \\ & + \lambda_6 [\gamma_v E_v - (\mu_v + \rho_1 u_1) I_v],\end{aligned}$$

where

$$\begin{cases} X(t) = (S_h(t), E_h(t), I_h(t), S_v(t), E_v(t), I_v(t)), \\ U(t) = (u_1(t), u_2(t), u_3(t), u_4(t)), \\ \lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t)). \end{cases}$$

4.2. Optical control solution

The subsequent analysis applies the Pontryagin optimality conditions to the previously defined Hamiltonian $\mathcal{H}(t, X(t), U(t), \lambda(t))$. Drawing upon the findings presented in [21], the corresponding results are derived as follows:

Theorem 4.2. *Given optimal controls $u_1^*, u_2^*, u_3^*, u_4^*$ and solutions $S_h^*, E_h^*, I_h^*, S_v^*, E_v^*, I_v^*$ of the corresponding state system (4.2), there exist adjoint variables $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t))$ satisfying the equations as follows:*

$$\begin{aligned}\frac{d\lambda_1(t)}{dt} &= \lambda_1 \mu_h + (\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q I_v (p I_h + q E_h)}{[p I_h + q(S_h + E_h)]^2} \\ &\quad - (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h q}{[p I_h + q(S_h + E_h)]^2}, \\ \frac{d\lambda_2(t)}{dt} &= -(\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v q}{[p I_h + q(S_h + E_h)]^2} \\ &\quad - (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h q}{[p I_h + q(S_h + E_h)]^2} + \lambda_2(\mu_h + \alpha_h) - \lambda_3 \alpha_h, \\ \frac{d\lambda_3(t)}{dt} &= -A_1 - \lambda_1(\theta_h + \rho_3 u_3) + \lambda_3(\mu_h + \theta_h + \rho_3 u_3) \\ &\quad - (\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v p}{[p I_h + q(S_h + E_h)]^2}\end{aligned}\tag{4.3}$$

$$\begin{aligned}
& + (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v q (S_h + E_h)}{[p I_h + q (S_h + E_h)]^2}, \\
\frac{d\lambda_4(t)}{dt} &= \lambda_4 \left[(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p I_h}{p I_h + q (S_h + E_h)} + \mu_v + \rho_1 u_1 \right] \\
& - \lambda_5 \left[(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p I_h}{p I_h + q (S_h + E_h)} \right], \\
\frac{d\lambda_5(t)}{dt} &= -A_2 + \lambda_5 (\gamma_v + \mu_v + \rho_1 u_1) - \lambda_6 \gamma_v, \\
\frac{d\lambda_6(t)}{dt} &= -A_2 + (\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h}{p I_h + q (S_h + E_h)} + \lambda_6 (\mu_v + \rho_1 u_1),
\end{aligned}$$

with transversality conditions $\lambda_i(t) = 0$ ($i = 1, 2, \dots, 6$). Furthermore, the optimal control are given by

$$\begin{cases} u_1^* = \max \left\{ 0, \min \left\{ 1, \frac{\rho_1 (\lambda_4 S_v + \lambda_5 E_v + \lambda_6 I_v)}{2B_1} \right\} \right\}, \\ u_2^* = \max \left\{ 0, \min \left\{ 1, \frac{\rho_2 b_v \beta_{vh} q S_h I_v (\lambda_2 - \lambda_1)}{2B_2 [p I_h + q (S_h + E_h)]} \right\} \right\}, \\ u_3^* = \max \left\{ 0, \min \left\{ 1, \frac{\rho_3 I_h (\lambda_3 - \lambda_1)}{2B_3} \right\} \right\}, \\ u_4^* = \max \left\{ 0, \min \left\{ 1, \frac{\rho_4 b_v \beta_{hv} p S_v I_h (\lambda_5 - \lambda_4)}{2B_4 [p I_h + q (S_h + E_h)]} \right\} \right\}. \end{cases} \quad (4.4)$$

Proof. Based on Pontryagin's Maximum Principle, the following relations hold:

$$\begin{cases} \frac{\partial \mathcal{H}(t, X(t), U(t), \lambda(t))}{\partial U} = 0, \\ \frac{d\lambda(t)}{dt} = - \frac{\partial \mathcal{H}(t, X(t), U(t), \lambda(t))}{\partial X(t)}, \\ \lambda(T_f) = 0. \end{cases}$$

Applying the adjoint conditions to $\mathcal{H}(t, X(t), U(t), \lambda(t))$ with $X = X^*$, that is

$$\begin{aligned}
\frac{d\lambda_1(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial S_h(t)}, & \frac{d\lambda_2(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial E_h(t)}, \\
\frac{d\lambda_3(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial I_h(t)}, & \frac{d\lambda_4(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial S_v(t)}, \\
\frac{d\lambda_5(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial E_v(t)}, & \frac{d\lambda_6(t)}{dt} &= - \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial I_v(t)}.
\end{aligned}$$

The equation (4.1) are obtained. The optimal conditions at U^* can be calculated as follows:

$$\begin{aligned}
\frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial u_1} &= 0, & \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial u_2} &= 0, \\
\frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial u_3} &= 0, & \frac{\partial \mathcal{H}(t, X^*(t), U^*(t), \lambda(t))}{\partial u_4} &= 0,
\end{aligned}$$

that is

$$\begin{cases} 2B_1u_1 - \rho_1(\lambda_4S_v + \lambda_5E_v + \lambda_6I_v) = 0, \\ 2B_2u_2 + (\lambda_1 - \lambda_2)\rho_2 \frac{b_v\beta_{vh}qS_hI_v}{pI_h + q(S_h + E_h)} = 0, \\ 2B_3u_3 + (\lambda_1 - \lambda_3)\rho_3I_h = 0, \\ 2B_4u_4 + (\lambda_4 - \lambda_5)\rho_4 \frac{b_v\beta_{hv}pS_vI_h}{pI_h + q(S_h + E_h)} = 0. \end{cases}$$

Utilizing the predefined lower and upper bounds of the control variables, the optimal solutions to the system in (4.2) are derived. This completes the proof of Theorem 4.2. $\square$

4.3. Uniqueness of optimal control problem

In this subsection, we establish the uniqueness of the optimal control strategy for the proposed Moso bamboo sooty mold model. The main result is stated in the following theorem.

Theorem 4.3. *For the optimal control problem defined by the state system (4.2) and the objective functional (4.1), suppose the final time T_f is sufficiently small. Then the optimal control strategy $U^*(t) = (u_1^*, u_2^*, u_3^*, u_4^*)$ is unique.*

Proof. To establish uniqueness, we reformulate the optimality system (4.5) comprising the state equations (4.2), the adjoint equations (4.3), and the optimality conditions (4.4) as a fixed-point problem on a suitable Banach space

$$\begin{aligned} \frac{dS_h(t)}{dt} &= \mu_h N_h - (1 - \rho_2 u_2(t)) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - \mu_h S_h + (\theta_h + \rho_3 u_3(t)) I_h, \\ \frac{dE_h(t)}{dt} &= (1 - \rho_2 u_2(t)) \frac{b_v \beta_{vh} q S_h I_v}{p I_h + q(S_h + E_h)} - (\mu_h + \alpha_h) E_h, \\ \frac{dI_h(t)}{dt} &= \alpha_h E_h - (\mu_h + \theta_h + \rho_3 u_3(t)) I_h, \\ \frac{dS_v(t)}{dt} &= \Lambda_v - (1 - \rho_4 u_4(t)) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - \mu_v S_v - \rho_1 u_1(t) S_v, \\ \frac{dE_v(t)}{dt} &= (1 - \rho_4 u_4(t)) \frac{b_v \beta_{hv} p S_v I_h}{p I_h + q(S_h + E_h)} - (\gamma_v + \mu_v) E_v - \rho_1 u_1(t) E_v, \\ \frac{dI_v(t)}{dt} &= \gamma_v E_v - \mu_v I_v - \rho_1 u_1(t) I_v, \\ \frac{d\lambda_1(t)}{dt} &= \lambda_1 \mu_h + (\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q I_v (p I_h + q E_h)}{[p I_h + q(S_h + E_h)]^2} \\ &\quad - (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h q}{[p I_h + q(S_h + E_h)]^2}, \\ \frac{d\lambda_2(t)}{dt} &= -(\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v q}{[p I_h + q(S_h + E_h)]^2} \\ &\quad - (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v I_h q}{[p I_h + q(S_h + E_h)]^2} + \lambda_2(\mu_h + \alpha_h) - \lambda_3 \alpha_h, \\ \frac{d\lambda_3(t)}{dt} &= -A_1 - \lambda_1(\theta_h + \rho_3 u_3) + \lambda_3(\mu_h + \theta_h + \rho_3 u_3) \end{aligned} \tag{4.5}$$

$$\begin{aligned}
& -(\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h I_v p}{[p I_h + q(S_h + E_h)]^2} \\
& + (\lambda_4 - \lambda_5)(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p S_v q(S_h + E_h)}{[p I_h + q(S_h + E_h)]^2}, \\
\frac{d\lambda_4(t)}{dt} &= \lambda_4 \left[(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p I_h}{p I_h + q(S_h + E_h)} + \mu_v + \rho_1 u_1 \right] \\
& - \lambda_5 \left[(1 - \rho_4 u_4) \frac{b_v \beta_{hv} p I_h}{p I_h + q(S_h + E_h)} \right], \\
\frac{d\lambda_5(t)}{dt} &= -A_2 + \lambda_5(\gamma_v + \mu_v + \rho_1 u_1) - \lambda_6 \gamma_v, \\
\frac{d\lambda_6(t)}{dt} &= -A_2 + (\lambda_1 - \lambda_2)(1 - \rho_2 u_2) \frac{b_v \beta_{vh} q S_h}{p I_h + q(S_h + E_h)} + \lambda_6(\mu_v + \rho_1 u_1).
\end{aligned}$$

Let $Y(t) = (X(t), \lambda(t))^T \in \mathbb{R}^{12}$ denote the concatenated state and adjoint variables, where $X = (S_h, E_h, I_h, S_v, E_v, I_v)$ and $\lambda = (\lambda_1, \dots, \lambda_6)$. From the invariant region Ω and the linearity of the adjoint system, there exist positive constants M_X and M_λ such that $\|X(t)\| \leq M_X$ and $\|\lambda(t)\| \leq M_\lambda$ for all $t \in [0, T_f]$. Consequently, the optimality system (4.5) can be expressed compactly as:

$$\dot{Y}(t) = \mathcal{F}(Y(t)),$$

where $\mathcal{F}$ is a smooth function composed of the state dynamics and the adjoint dynamics with the control variables substituted by (4.4). Since the state variables remain in a compact set, $\mathcal{F}$ is globally Lipschitz continuous with respect to Y . That is, there exists a constant $L > 0$ such that for any two admissible trajectories Y^1 and Y^2 ,

$$\|\mathcal{F}(Y^1) - \mathcal{F}(Y^2)\| \leq L\|Y^1 - Y^2\|.$$

Specifically, there exists a constant $L_\Phi > 0$ such that for the corresponding controls $U^i = \Phi(X^i, \lambda^i)$ (for $i = 1, 2$),

$$\|U^1(t) - U^2(t)\| \leq L_\Phi \|Y^1(t) - Y^2(t)\|. \quad (4.6)$$

Define the Banach space $\mathcal{B} = C([0, T_f], \mathbb{R}^{12})$. For any given $Y = (X, \lambda) \in \mathcal{B}$, let $U(t) = \Phi(X(t), \lambda(t))$. Consider the forward initial-value problem

$$\dot{\tilde{X}} = f(\tilde{X}, U), \quad \tilde{X}(0) = X_0,$$

and the backward terminal-value problem

$$\dot{\tilde{\lambda}} = \Psi(\tilde{X}, U, \tilde{\lambda}), \quad \tilde{\lambda}(T_f) = 0.$$

By the Lipschitz continuity of f and Ψ , each of these problems admits a unique solution in the respective time direction. Therefore, the mapping $\mathcal{T}(Y) := (\tilde{X}, \tilde{\lambda})$ is well-defined. It is immediate that fixed points of $\mathcal{T}$ are in one-to-one correspondence with solutions of the optimality system (4.5). Now, take two arbitrary elements $Y^1 = (X^1, \lambda^1)$ and $Y^2 = (X^2, \lambda^2)$ in $\mathcal{B}$. Let $\delta_X(t) = \tilde{X}^1(t) - \tilde{X}^2(t)$, $\delta_\lambda(t) = \tilde{\lambda}^1(t) - \tilde{\lambda}^2(t)$, and $\delta_U(t) = U^1(t) - U^2(t)$. By applying the Lipschitz estimate (4.6) to the state difference, we have:

$$\|\delta_X(t)\| \leq \int_0^t L_f (\|\delta_X(s)\| + \|\delta_U(s)\|) ds \leq L_f(1 + L_\Phi) T_f \cdot \|Y^1 - Y^2\|_{\mathcal{B}},$$

where L_f is the Lipschitz constant of the state function f . Taking the supremum over $[0, T_f]$, we obtain:

$$\|\delta_X\|_\infty \leq K_X T_f \cdot \|Y^1 - Y^2\|_{\mathcal{B}}, \quad \text{with } K_X = L_f(1 + L_\Phi). \quad (4.7)$$

Similarly, for the adjoint variables, since $\lambda^i(T_f) = 0$, we integrate the difference of the adjoint equations backward and use the Lipschitz constant L_Ψ of the adjoint system. This yields:

$$\|\delta_\lambda(t)\| \leq \int_t^{T_f} L_\Psi (\|\delta_X(s)\| + \|\delta_U(s)\| + \|\delta_\lambda(s)\|) ds.$$

Substituting (4.6) and (4.7) and taking the supremum over t , there exists a constant $K_\lambda > 0$ such that:

$$\|\delta_\lambda\|_\infty \leq K_\lambda T_f (K_X T_f + L_\Phi) \cdot \|Y^1 - Y^2\|_{\mathcal{B}}. \quad (4.8)$$

Combining inequalities (4.7) and (4.8), we arrive at:

$$\|\mathcal{T}(Y^1) - \mathcal{T}(Y^2)\|_{\mathcal{B}} \leq \|\delta_X\|_\infty + \|\delta_\lambda\|_\infty \leq \Gamma(T_f) \cdot \|Y^1 - Y^2\|_{\mathcal{B}},$$

where

$$\Gamma(T_f) = K_X T_f + K_\lambda T_f (K_X T_f + L_\Phi).$$

It is easy to check that $\Gamma(0) = 0$ and $\Gamma(T_f)$ is a continuous function of T_f . Therefore, one can always choose a sufficiently small terminal time $T_f^* > 0$ such that for any $0 < T_f < T_f^*$, we have $\Gamma(T_f) < 1$. Under this condition, $\mathcal{T}$ is a strict contraction on the complete Banach space $\mathcal{B}$. By the Banach fixed-point theorem, $\mathcal{T}$ admits a unique fixed point $Y^* = (X^*, \lambda^*)$ in $\mathcal{B}$. Consequently, the optimal control $U^* = \Phi(X^*, \lambda^*)$ is uniquely determined. This completes the proof. $\square$

5. Numerical simulation

This section performs numerical simulations of system (2.1) to validate the theoretical results obtained in Section 3. Subsequently, a global sensitivity analysis is conducted on the basic reproduction number $\mathcal{R}_0$ in system (2.1). Next, the impact of vector preference ($\phi = \frac{p}{q}$) on the transmission dynamics of sooty mold disease in Moso bamboo is discussed. Finally, using cost-effectiveness analysis, the incremental cost-effectiveness ratio (ICER) is calculated to compare the merits of different intervention strategies.

5.1. Long-term behavior

Section 3 demonstrated that $\mathcal{R}_0$ acts as the threshold parameter controlling the persistence or extinction of the disease. Subsequent numerical simulations validate the theoretical findings for system (2.1).

Take $b_v = 12$, $\beta_{vh} = 1.55$, $\beta_{hv} = 2.22$, while the remaining parameters are kept unchanged. The following initial conditions are adopted: $S_h(0) = 0.8$, $E_h(0) = 0.2$, $I_h(0) = 0$, $S_v(0) = 800$, $E_v(0) = 150$, $I_v(0) = 50$. Figure 3 displays the simulation outcomes, consistent with the conclusion of Theorem 3.2. System (2.1) admits a unique DFE (E^0), and the disease goes extinct. If one selects the parameters listed in Table 1, then an EE (E^*) exists, and the disease persists (see Figure 4).

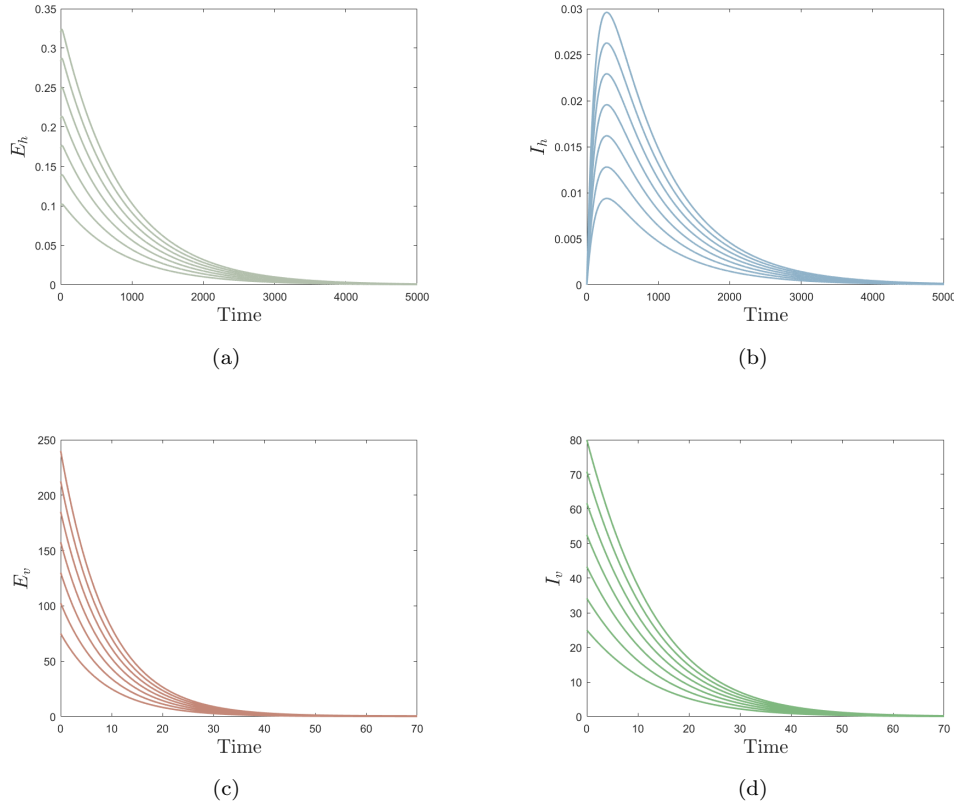


Figure 3. The long-term behavior of $E_h(t)$, $I_h(t)$, $E_v(t)$ and $I_v(t)$ when $\mathcal{R}_0 = 0.0079 < 1$. It shows that the solution approaches E^0 .

5.2. Global sensitivity analysis

The dynamics of a model are often governed by its input parameters, and the choice of these parameters introduces some degree of uncertainty. This part applies Latin Hypercube Sampling (LHS) combined with Partial Rank Correlation Coefficients (PRCC) to assess the impacts of uncertainty and the parameter sensitivity of the model. LHS, a stratified sampling technique carried out without replacement, facilitates an efficient exploration of parameter variability across the uncertain range of each parameter. PRCC, in turn, quantifies the strength of association between the model output and the parameters, thereby indicating the degree to which each parameter influences the results [26]. Hence, through sensitivity analysis, the parameters that substantially affect the model's outcomes can be identified. Generating the LHS matrices starts with the assumption that all model parameters follow a uniform distribution. For each LHS run, 1000 model simulations are then executed, drawing parameter values from Table 1 and setting ranges to 10% above or below each value.

Based on the above parameters of model (2.3), a global sensitivity analysis is conducted on the basic reproductive number $\mathcal{R}_0$. The sensitivity analysis results (Figure 5) reveal that $\mathcal{R}_0$ is positively associated with the parameters $b_v, \alpha_h, \gamma_v, \beta_{vh}, \beta_{hv}$, and ϕ , and negatively associated with the other parameters μ_h, μ_v, θ_h . In particular, among all parameters, b_v exerts the strongest influence on $\mathcal{R}_0$, followed by β_{vh} , and then β_{hv} , whereas $\mu_h, \theta_h, \alpha_h$ have a comparatively minor effect on $\mathcal{R}_0$.

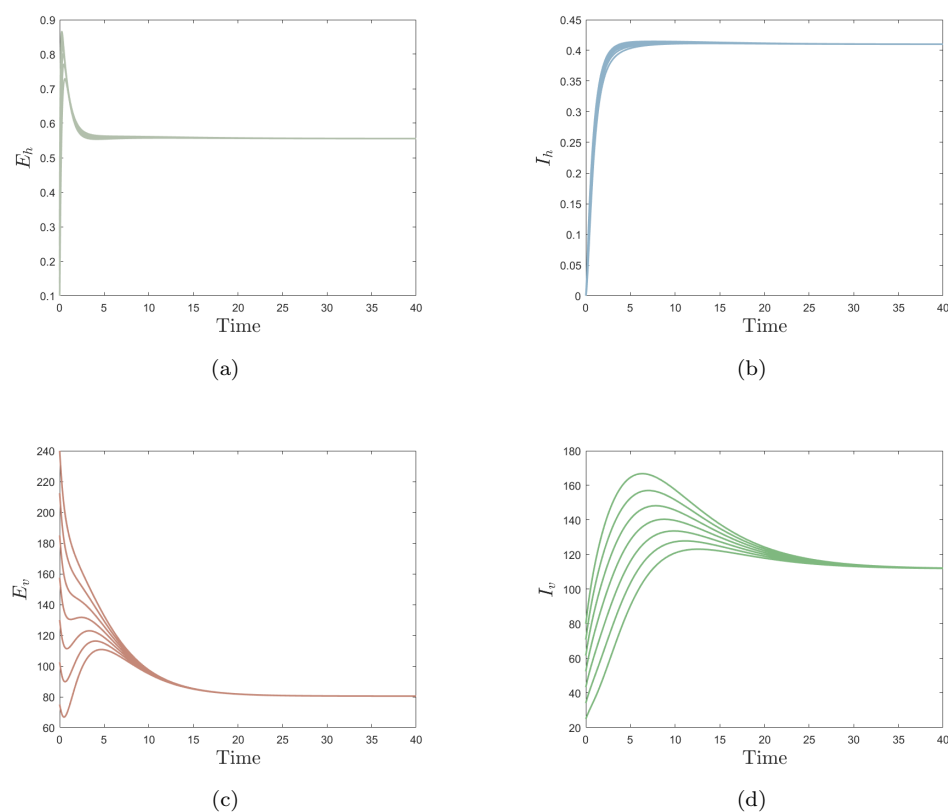


Figure 4. The long-term behavior of $S_h(t)$, $E_h(t)$, $I_h(t)$, $S_v(t)$, $E_v(t)$ and $I_v(t)$ when $\mathcal{R}_0 = 1.3869 > 1$. It shows that the solution approaches E^* .

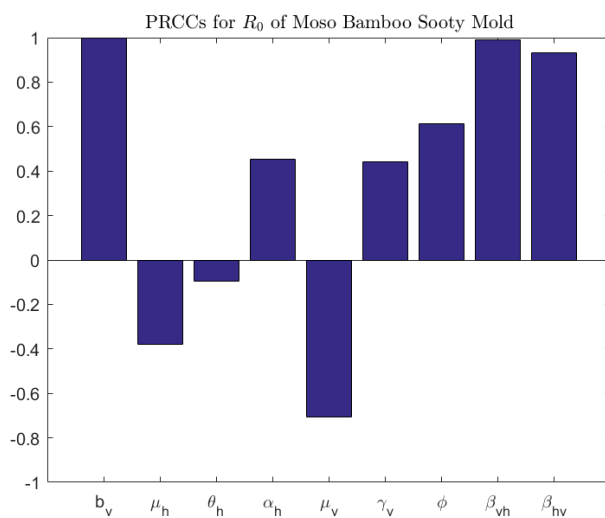


Figure 5. Global sensitivity analysis of $\mathcal{R}_0$ for model (2.1).

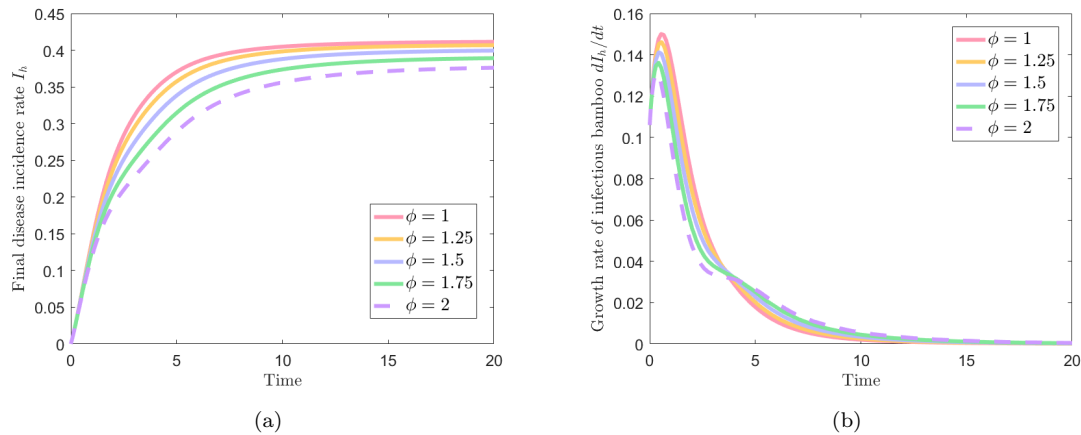


Figure 6. Effect of vector preference ϕ on (a) final infected number and (b) growth rate dynamics.

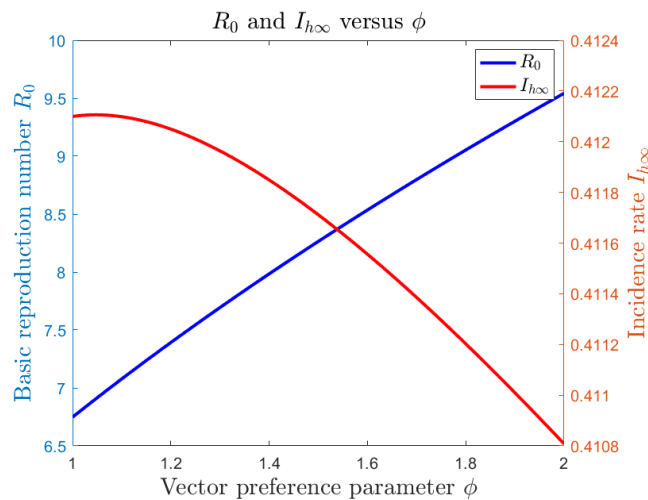


Figure 7. Effect of ϕ on R_0 (blue curve) and $I_{h\infty}$ (red curve).

5.3. Effects of vector preference (ϕ) on the transmission of Moso bamboo sooty mold disease

From a mathematical standpoint, based on the time-series patterns displayed in Figure 6, an increase in the parameter ϕ produces two primary effects. First, it lowers the ultimate cumulative number of infected bamboo plants I_h (Figure 6(a)). Second, it diminishes the growth momentum of the infected bamboo population, resulting in a reduced peak in the growth rate $\frac{dI_h(t)}{dt}$ (Figure 6(b)). Moreover, it allows the system to reach a steady state more quickly (i.e., the growth rate approaches zero). Consequently, the preference level ϕ of vector insects for infected bamboo exhibits a negative correlation with the transmission intensity of Moso bamboo sooty mold disease. A larger ϕ value leads to a lower final infection rate among bamboo plants, a smaller maximum growth rate of infected plants, and a faster attainment of dynamic equilibrium in the system, thus markedly suppressing the spread of Moso bamboo sooty mold disease.

From a mathematical standpoint: On one hand, the parameter ϕ exhibits a positive corre-

lation with the basic reproduction number $\mathcal{R}_0$ (reflected by the monotonic upward trend of the blue curve in Figure 7), meaning that an increase in ϕ leads to a corresponding rise in $\mathcal{R}_0$. On the other hand, ϕ displays a negative correlation with the eventual number of infected plants $I_{h\infty}$ (as shown by the monotonic downward trend of the red curve in Figure 7).

Specifically, as ϕ grows, the spread is concentrated around the infected plants, thereby restricting the spread of the pathogen to healthy areas and leading to a final reduction in scale. Hence, given identical state variables and parameter values in model (2.1), when compared with a model that ignores vector preference ($\phi = 0$), the model incorporating vector preference ($\phi > 0$) tends to overestimate the potential transmission capacity of the disease while underestimating the actual scale of infection. This discrepancy arises from the dual regulatory influence of vector preference on both transmission efficiency and the likelihood of susceptible bamboo plants being exposed. Therefore, integrating these two critical aspects is essential for a complete and accurate evaluation of the disease transmission risk.

5.4. Cost-effectiveness analysis

In the next subsection, we employ the numerical techniques described in [21] to examine the optimal solution of the optimal system (4.2), the associated optimal control, and provide a plausible interpretation for each scenario. The primary aim of this paper is to investigate the impact of four control measures (namely, injecting nutrient solution, removing infected bamboos, applying insecticides, and deploying yellow adhesive traps) on the transmission of Moso bamboo sooty mold disease. To demonstrate the optimal control strategies, the initial conditions are assigned the following values: $S_h(0) = 0.8$, $E_h(0) = 0.2$, $I_h(0) = 0$, $S_v(0) = 500$, $E_v(0) = 10$ and $I_v(0) = 5$. In the simulations, the remaining parameter values are taken from Table 1. Selecting appropriate cost weights for each variable in the main objective is a challenging task, yet it is not the central focus of this study [33]. Therefore, all cost weights ($A_1, A_2, B_1, B_2, B_3, B_4$) in this paper are set to 1 and remain unchanged in the subsequent analysis. It should be noted that the cost weights used in the simulations here are solely for theoretical purposes to illustrate the control strategies proposed in this article [2].

We perform an iterative procedure [21] to numerically solve the boundary value problem of the optimality system, which includes the state and adjoint equations together with the four control variables. A fourth-order Runge–Kutta scheme is employed to compute the optimal control and state values. Initially, we make an initial estimate of the control variables U_i ($i = 1, 2, 3, 4$) over the interval $[0, T_f]$. Then, the state variables are calculated forward in time using system (4.2). The resulting state variable values are substituted into the adjoint equation (4.3). Next, employing the backward fourth-order Runge–Kutta method, the adjoint equations are solved backward in time under the given final condition (4.4). At the end of each iteration, the controls are updated using the optimal control formulas. The iterations continue until convergence.

The computational outcomes of the optimal control for the Moso bamboo sooty mold disease model (2.1) are shown in Figure 8. From Figure 8, it is evident that the total number of E_h , I_h , E_v , and I_v is significantly decreased when applying the optimal controls, relative to the total number of infected individuals in the absence of any control measures. A greater number of infected bamboos and bamboo aphids occur without controls compared to the total numbers of infected bamboos and bamboo aphids when the optimal controls are applied. The associated time-varying control variables ($u_1(t), u_2(t), u_3(t), u_4(t)$) are displayed in Figure 9. During the initial phase of the simulation, the control variables $u_1(t)$ and $u_2(t)$ first drop sharply and then

Table 2. The PRCC of the parameters in model (4.2).

Parameter	p-value	PRCC	Parameter	p-value	PRCC
β_{vh}	< 0.0001	0.9921	μ_h	< 0.0001	-0.3996
β_{hv}	< 0.0001	0.9279	μ_v	< 0.0001	-0.7025
b_v	< 0.0001	0.9980	θ_h	0.0024	-0.1769
γ_v	< 0.0001	0.4723	α_h	< 0.0001	0.4244
ϕ	< 0.0001	0.6024			

steadily decline to zero until the end of the simulation. In contrast, the control variables $u_3(t)$ and $u_4(t)$ rise gently and then gradually approach zero. These findings indicate that $u_1(t)$ and $u_2(t)$ represent long-acting control measures, and it is advisable to apply them throughout the entire disease management process. These results also imply that $u_3(t)$ and $u_4(t)$ are short-acting control approaches, which can be applied during the early phase of a disease outbreak and halted in the later stage due to cost considerations.

To sum up, computational simulations of the Moso bamboo sooty mold disease control model (4.2) demonstrate that the disease can be suppressed within the orchard through the application of time-dependent control variables ($u_1(t), u_2(t), u_3(t), u_4(t)$).

To assess the efficacy of the control strategies, various intervention schemes are compared, including paired, triple, and quadruple control combinations, as described in [42]. Following the actual management practices for Moso bamboo sooty mold disease, we do not examine individual control measures in this study. Paired, triple, and quadruple controls are employed to investigate the joint implementation of two, three, and four control strategies, respectively. The scenarios for the combined control strategies are proposed as follows:

- **Scenario 1: Coupled control strategies**

Strategy A: Pesticide + Nutrient solution ($u_1(t), u_2(t)$).

Strategy B: Pesticide + Removal of Moso bamboo with sooty mold symptoms ($u_1(t), u_3(t)$).

Strategy C: Pesticide + Yellow adhesive trap ($u_1(t), u_4(t)$).

Strategy D: Nutrient solution + Removal of Moso bamboo with sooty mold symptoms ($u_2(t), u_3(t)$).

Strategy E: Nutrient solution + Yellow adhesive trap ($u_2(t), u_4(t)$).

Strategy F: Removal of Moso bamboo with sooty mold symptoms + Yellow adhesive trap ($u_3(t), u_4(t)$).

- **Scenario 2: Threefold control strategies**

Strategy G: Pesticide + Nutrient solution + Removal of Moso bamboo with sooty mold symptoms ($u_1(t), u_2(t), u_3(t)$).

Strategy H: Pesticide + Nutrient solution + Yellow adhesive trap ($u_1(t), u_2(t), u_4(t)$).

Strategy I: Pesticide + Removal of Moso bamboo with sooty mold symptoms + Yellow adhesive trap ($u_1(t), u_3(t), u_4(t)$).

Strategy J: Nutrient solution + Removal of Moso bamboo with sooty mold symptoms + Yellow adhesive trap ($u_2(t), u_3(t), u_4(t)$).

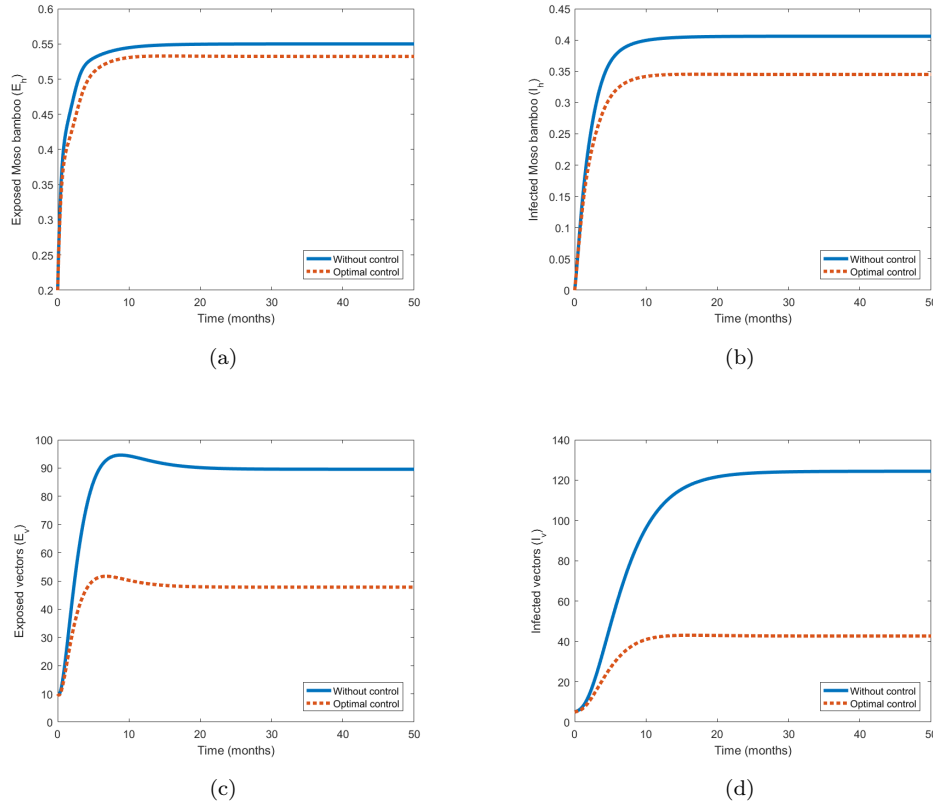


Figure 8. Simulation results of the Moso bamboo sooty mold model (2.1) as a function of time without control and with optimal control for: (a) Total number of E_h ; (b) total number of I_h ; (c) total number of E_v ; (d) total number of I_v . In the simulations, we assume $\rho_1 = 0.5$, $\rho_2 = 0.5$, $\rho_3 = 0.6$, $\rho_4 = 0.5$, and $T_f = 50$ weeks, and the other parameter values are derived from Table 1.

• Scenario 3: Fourfold control strategies

Strategy K: Pesticide + Nutrient solution + Removal of Moso bamboo with sooty mold symptoms + Yellow adhesive trap ($u_1(t), u_2(t), u_3(t), u_4(t)$).

Controlling or eliminating Moso bamboo sooty mold in a bamboo forest requires both substantial labor and high expenses. Thus, to identify the most economically viable strategy, a cost-effectiveness analysis must be conducted. To accomplish this, the incremental cost-effectiveness ratio (ICER) is calculated to compare the various costs and health outcomes associated with implementing these different interventions. The ICER represents the additional cost incurred per additional unit of health benefit gained. We assume that the costs of the various control interventions are proportional to the number of controls applied. To progressively compare competing intervention strategies (usually two or more), one intervention is compared against another that is less effective [1]. In the following, to quantify the cost-effectiveness of the control measures, we consider the ICER of one intervention strategy X relative to another strategy Y :

$$\text{ICER} = \frac{\Delta C}{\Delta E} = \frac{\text{Cost}_{(X)} - \text{Cost}_{(Y)}}{\text{Effectiveness}_{(X)} - \text{Effectiveness}_{(Y)}}.$$

The Cost is defined by the expense associated with the intervention strategies, namely: $\text{Cost} = \int_0^{T_f} [A_1 I_h(t) + A_2 (E_v(t) + I_v(t)) + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t)] dt$. The

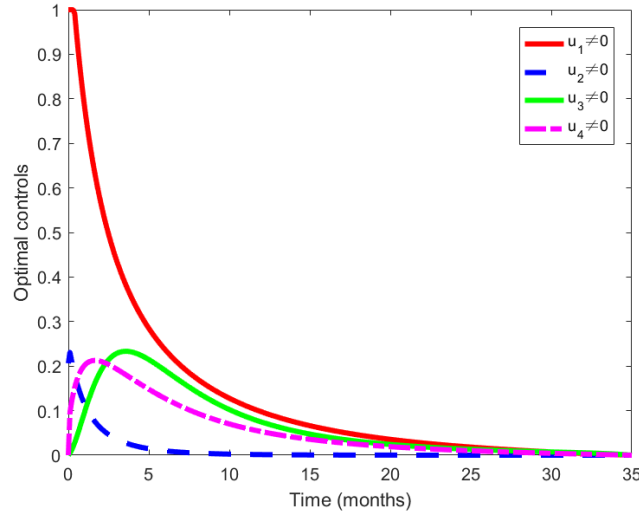


Figure 9. The optimal controls of the Moso bamboo sooty mold model (2.1) for: (a) Insecticide treatment (u_1); (b) nutrient solution treatment (u_2); (c) removal of Moso bamboo with sooty mold symptoms (u_3); (d) yellow adhesive trap control (u_4). In the simulations, we assume $\rho_1 = 0.5$, $\rho_2 = 0.5$, $\rho_3 = 0.6$, $\rho_4 = 0.5$, and $T_f = 50$ weeks, and the other parameter values are derived from Table 1.

Effectiveness is defined by the total number of infectious individuals averted, i.e., $\text{Effectiveness} = \int_0^{T_f} S_h(t)dt$. In this paper, we take the cost-effectiveness threshold as $\lambda = 32$. Then, according to [39], (1) when $\Delta E > 0$, $\text{ICER} = \frac{\Delta C}{\Delta E} < \lambda$, or (2) when $\Delta E < 0$, $\text{ICER} = \frac{\Delta C}{\Delta E} > \lambda$, strategy X is more cost-effective than strategy Y .

To determine the most cost-effective strategy, paired control approaches (Scenario 1) are first employed in the model simulation. The outcomes are then sorted in ascending order of effectiveness at preventing infection, and the control strategies are ranked accordingly. Figure 10 presents the time-dependent simulation results of the Moso bamboo sooty mold model (2.1) using strategies A, B, C, D, E, and F for the total number of E_h , I_h , E_v , and I_v . The corresponding optimal control results for insecticide application, nutrient solution treatment, removal of bamboo with sooty mold symptoms, and yellow adhesive trap deployment are displayed in Figure 11. The simulation outcomes are ordered by increasing effectiveness in averting infection, and the ranking of the control strategies based on the ICER is shown in Table 3. The ICER values are then computed as follows: $\text{ICER}_{(F)} = 6.9435$, $\text{ICER}_{(A)} = -10.8870$, $\text{ICER}_{(B)} = -22.6886$, $\text{ICER}_{(C)} = -4.1426$, $\text{ICER}_{(D)} = 10.3487$, $\text{ICER}_{(E)} = 11.6895$.

Now, upon comparing $\text{ICER}_{(F)}$ and $\text{ICER}_{(A)}$ in Table 3, it is evident that Strategy A results in a cost saving of 10.8870 relative to Strategy F. In other words, Strategy A is both more economical and more effective than Strategy F. Likewise, Strategies B and C are also superior to Strategy F. For Strategies D and F, given that $\Delta E > 0$ and $\text{ICER}_{(D)} = 10.3487 > \lambda = 32$, and $\text{ICER}_{(E)} = 11.6895 > \lambda = 32$, one can infer that Strategy F outperforms both Strategy D and Strategy E. Consequently, Strategies D, E, and F are removed from the set of viable control options. Similarly, from Table 4, it follows that Strategy A is more cost-effective and efficient than Strategies C and B. Hence, we can conclude that Strategy A possesses the lowest ICER value and is more cost-effective than all other strategies in Scenario 1.

Next, the triple controls (namely, Strategy G, Strategy H, Strategy I, and Strategy J), which consist of combinations of three measures, are concurrently implemented in the model simulation

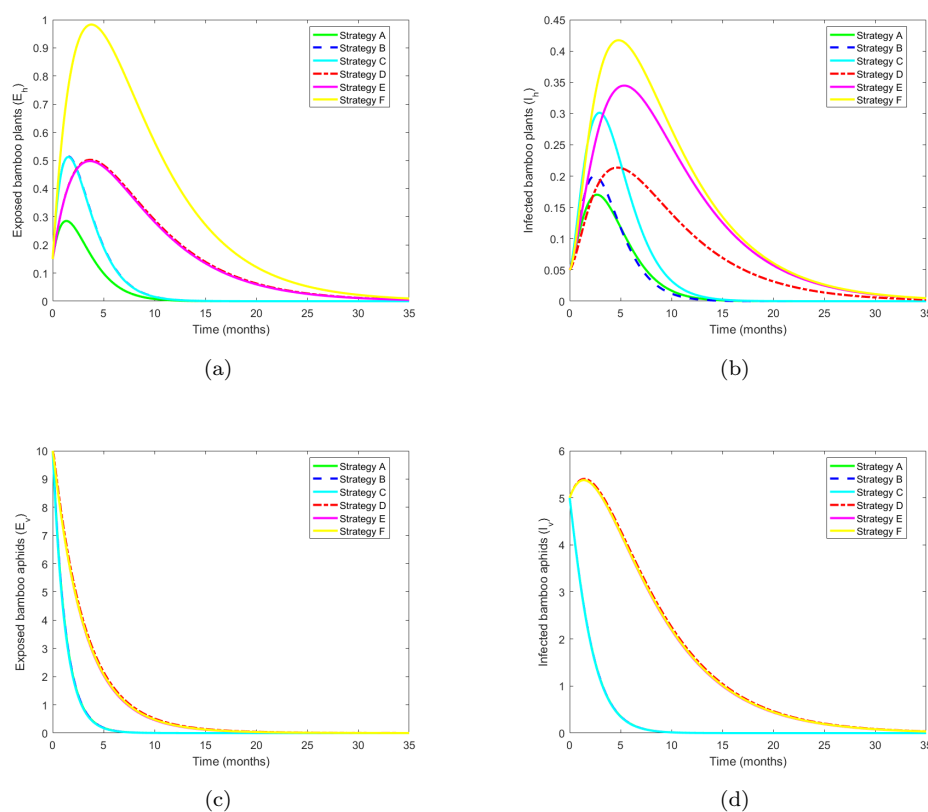


Figure 10. Numerical simulation of the Moso bamboo sooty mold model (2.1) over time, employing plans A, B, C, D, E, and F for: (a) Total number of E_h ; (b) total number of I_h ; (c) total number of E_v ; (d) total number of I_v .

Table 3. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
F	272.31	1.8906×10^3	6.9435
A	289.68	1.7015×10^3	-10.8870
B	279.60	1.7252×10^3	-22.6886
C	284.72	1.8392×10^3	-4.1426
D	267.42	1.8400×10^3	10.3487
E	270.12	1.8650×10^3	11.6895

to determine the most cost-effective approach for preventing and managing bamboo sooty mold in forest plantations. Figure 12 displays the time-dependent simulation outcomes of the bamboo sooty mold model (2.1) using strategies G, H, I, and J for the total number of E_h , I_h , E_v and I_v . The corresponding optimal control outcomes for pesticide application, nutrient solution application, removal of bamboo exhibiting sooty mold symptoms, and deployment of yellow adhesive traps are presented in Figure 13.

Tables 5–7 list the sorting results of simulation outputs sorted by ascending effectiveness

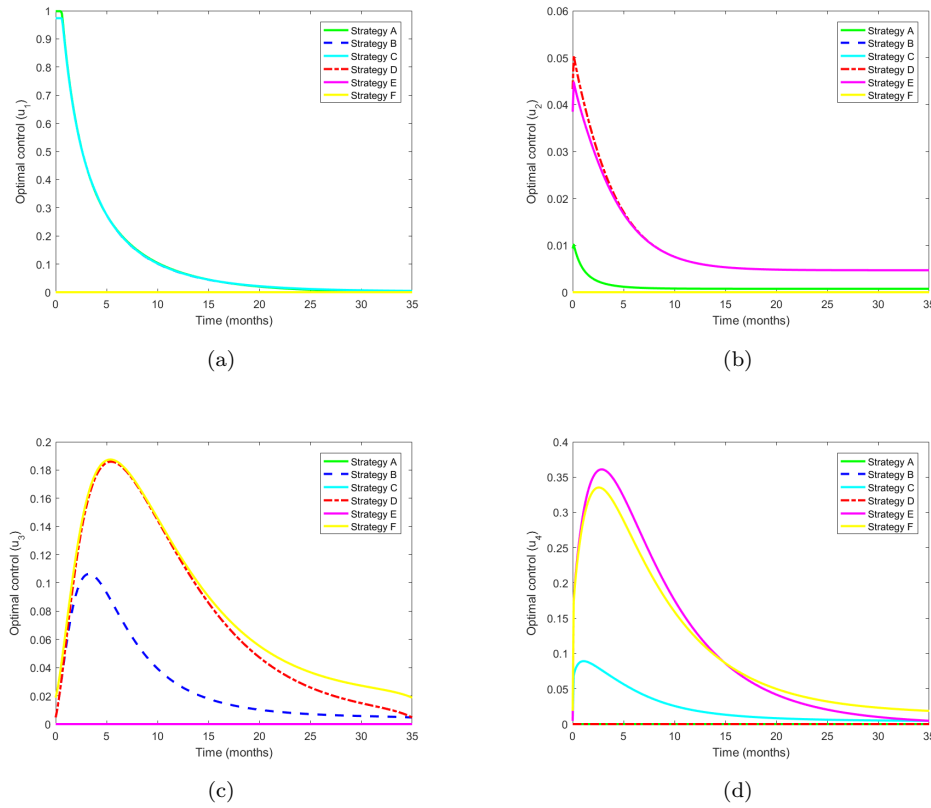


Figure 11. The optimal control measures for the Moso bamboo sooty mold model (2.1) employing strategies A, B, C, D, E, and F for: (a) Pesticide application; (b) nutrient solution application; (c) removal of Moso bamboo with sooty mold symptoms; (d) yellow adhesive trap management.

Table 4. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
C	284.72	1.8392×10^3	6.4605
A	289.68	1.7015×10^3	-27.76
B	279.60	1.7252×10^3	22.27

in averting infection, alongside the ICER-based ranking of all control intervention strategies. Consequently, following the ranking approach used for the paired control strategies, Strategy H, which exhibits the lowest ICER value, emerges as the most cost-effective strategy in Scenario 2.

Subsequently, the most cost-effective strategy obtained from the coupled control Strategy A, the threefold control Strategy H, and the fourfold control Strategy K is compared. Figure 14 presents the time-dependent simulation results of the Moso bamboo sooty mold model (2.1) under Strategies A, H, and K for the total population of E_h , I_h , E_v , and I_v . The corresponding optimal control results for pesticide treatment, nutrient solution injection, elimination of Moso bamboo individuals with sooty mold symptoms, and yellow sticky trap deployment are illustrated in Figure 15.

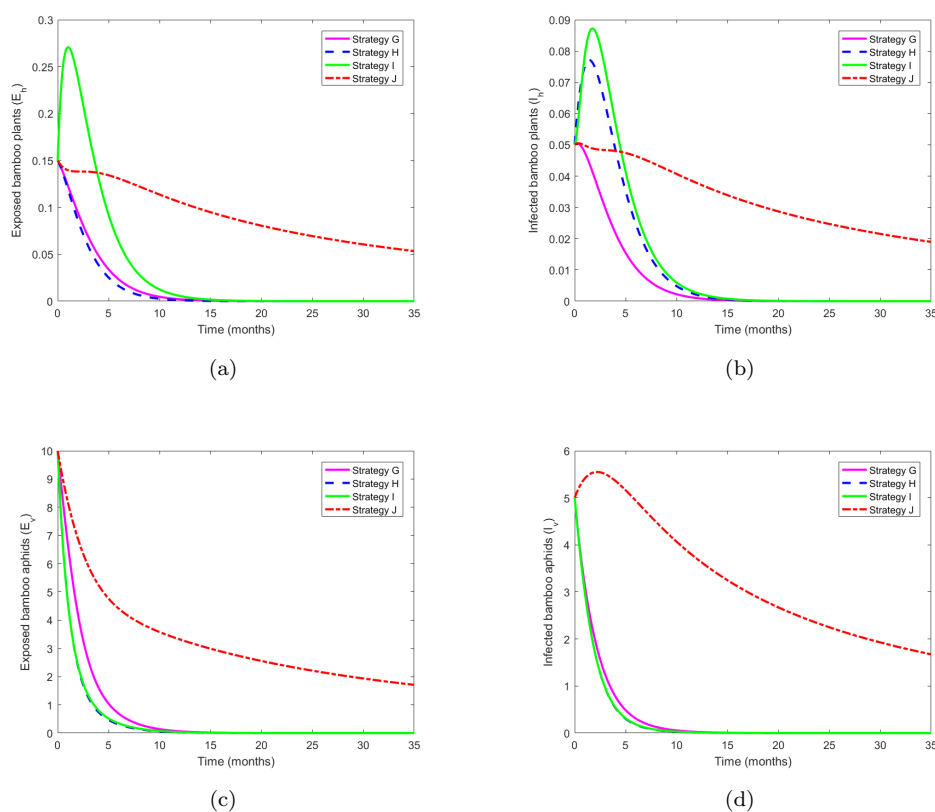


Figure 12. Numerical simulation of the Moso bamboo sooty mold model (2.1) over time employing plans G, H, I, and J for: (a) Total number of E_h ; (b) total number of I_h ; (c) total number of E_v ; (d) total number of I_v .

Table 5. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
I	284.28	2.3720×10^3	8.3439
G	289.16	2.3591×10^3	-2.6434
H	289.20	2.3544×10^3	-3.5772
J	289.02	2.3600×10^3	-2.5316

Table 6. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
J	289.02	2.3600×10^3	8.1655
G	289.16	2.3591×10^3	-6.4286
H	289.20	2.3544×10^3	-31.1111

Clearly, based on Table 8, Strategy A exhibits the lowest ICER value, followed by Strategy K, and then Strategy H. Hence, Strategy A is the most cost-effective strategy. In other words,

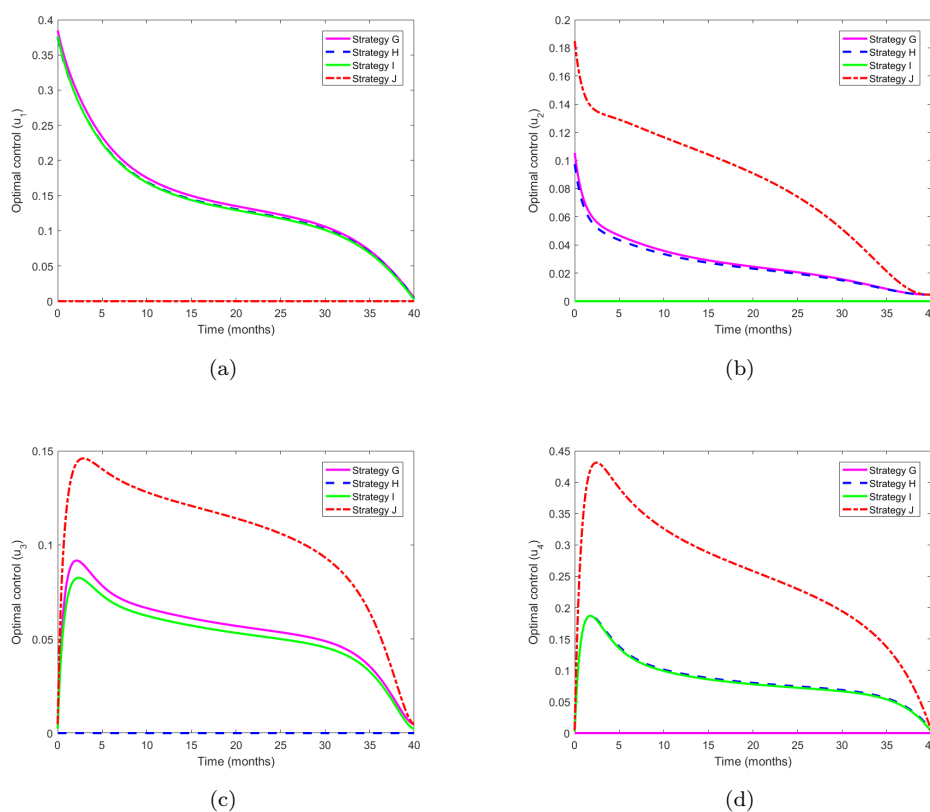


Figure 13. The optimal control measures for the Moso bamboo sooty mold model (2.1) employing plans G, H, I, and J for: (a) Pesticide application; (b) nutrient solution application; (c) removal of Moso bamboo with sooty mold symptoms; (d) yellow adhesive trap management.

Table 7. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
G	289.16	2.3591×10^3	8.1585
H	289.20	2.3544×10^3	-117.5000

when compared to threefold and fourfold control strategies, the coupled control strategy proves to be the most economically efficient.

To summarize, in Scenario 1, Strategy A yields the lowest ICER value and is therefore the most cost-effective among all coupled control strategies. In Scenario 2, Strategy H has the smallest ICER value and is the most cost-effective among all three control strategies. Nevertheless, when the control strategies of Scenario 1 and Scenario 2 are compared against the integrated control schemes from Scenario 3, Strategy A exhibits the optimal overall cost-effectiveness (see Fig. 14 and Table 8).

Furthermore, in terms of cost-effectiveness, the simulation results reveal that twofold combined control strategies incur lower operational costs than threefold and fourfold control schemes. Additionally, the results demonstrate that the coupled control strategy performs as effectively as the threefold control strategy in terms of infections averted. These control interventions

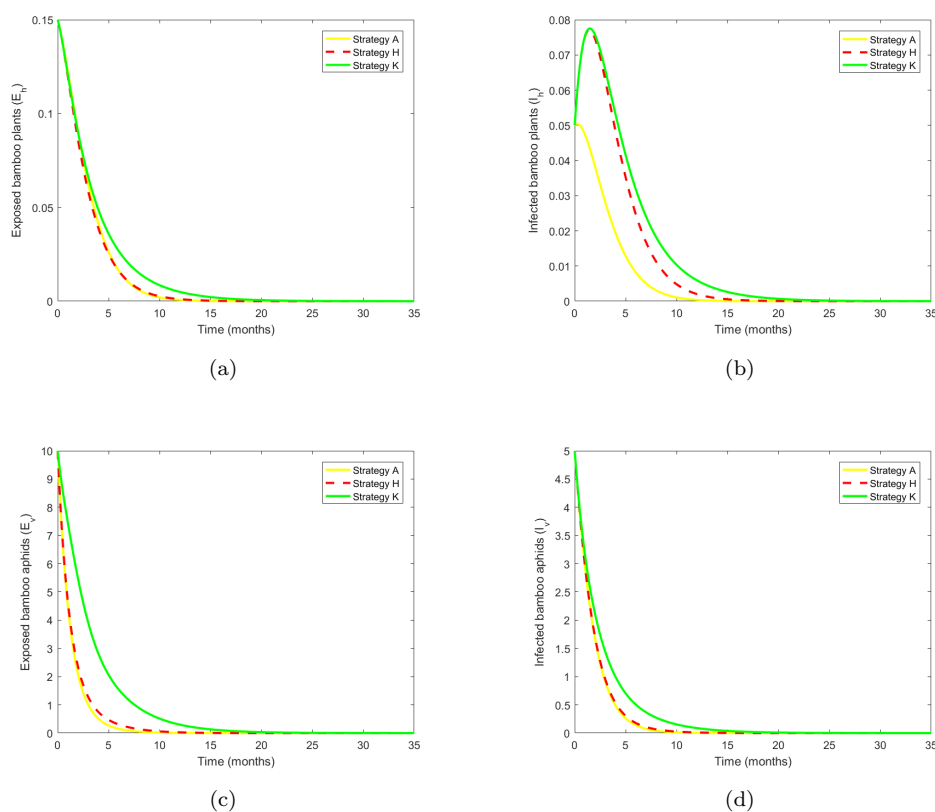


Figure 14. Numerical simulation of the Moso bamboo sooty mold model (2.1) over time, employing plans A, H, and K for: (a) Total number of E_h ; (b) total number of I_h ; (c) total number of E_v ; (d) total number of I_v .

highlight the necessity of eliminating symptomatic Moso bamboo and suppressing populations of bamboo aphids.

Table 8. ICER in increasing order of total infection averted.

Strategies	Total infection averted	Total cost	ICER
A	289.68	1.7015×10^3	5.8737
H	289.20	2.3544×10^3	-1360.2
K	289.88	3.4044×10^3	8514.5

6. Conclusion

In this study, a compartmental model incorporating vector preferences is developed to describe the transmission dynamics of sooty mold disease in Moso bamboo between bamboo plants and their insect vectors (such as bamboo aphids). By applying next-generation matrix methods, we derived an explicit expression for the basic reproduction number $\mathcal{R}_0$ and provided a corresponding biological interpretation. Building on established research, this work employs dynamical

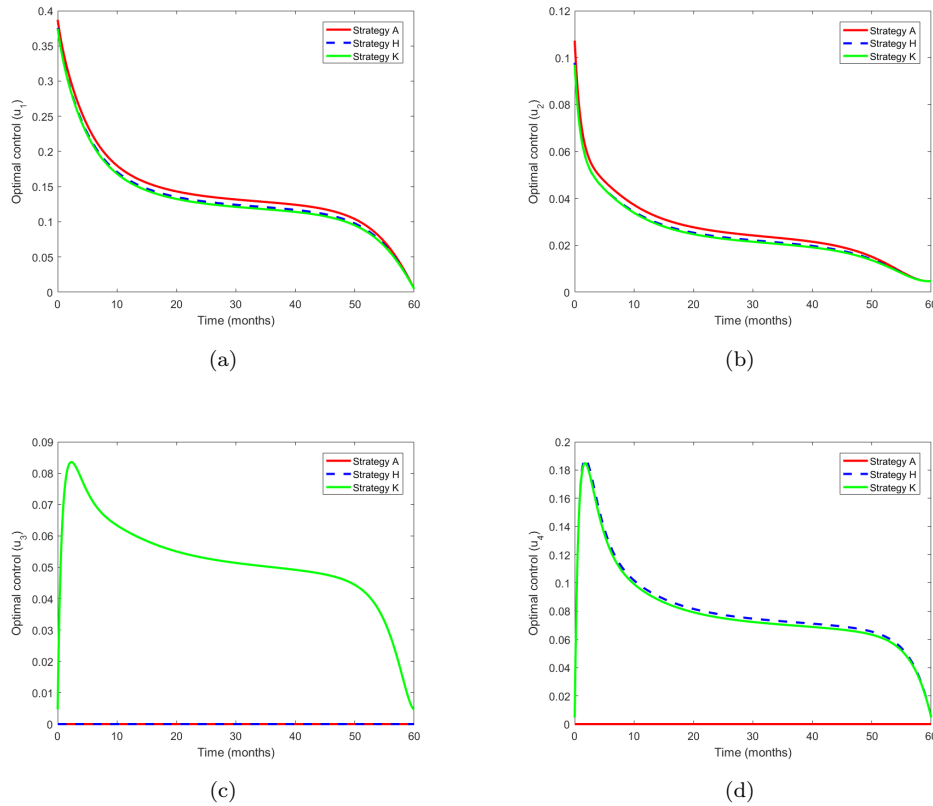


Figure 15. The optimal control measures for the Moso bamboo sooty mold model (2.1) employing plans A, H, and K for: (a) Pesticide application; (b) nutrient solution application; (c) removal of Moso bamboo with sooty mold symptoms; (d) yellow adhesive trap management.

systems theory to systematically investigate the dynamical characteristics of the six-dimensional nonlinear system representing Moso bamboo sooty mold. Theoretical results indicate that when $\mathcal{R}_0 < 1$, the disease-free equilibrium (DFE) of system (2.1) is locally asymptotically stable (LAS) and globally asymptotically stable (GAS). Conversely, when $\mathcal{R}_0 > 1$, the system possesses a unique positive equilibrium (EE), which is GAS. Further analysis confirms that when $\mathcal{R}_0 > 1$, the system is uniformly persistent.

Additionally, optimal control theory is applied to the model, and the associated optimal control problem is analyzed. By employing Pontryagin's maximum principle, the necessary conditions for the existence of an optimal solution to this problem are derived. Numerical simulations validate the theoretical results and contrast the effects of diverse control measures on Moso bamboo sooty mold disease transmission. Based on the study results, the main conclusions are summarized as follows:

- In bamboo forests, infected vector insects play a critical role in the transmission of Moso bamboo sooty mold. Effectively suppressing vector populations helps interrupt the spread of the disease.
- Under identical state variables and parameter values in this system, compared to the model ignoring vector preference ($\phi = 1$), the model incorporating vector preference ($\phi > 1$) overestimates Moso bamboo sooty mold disease's potential transmission capacity while

simultaneously underestimating the actual infection scale.

- The study findings indicate that spraying insecticides and injecting nutrient solutions are long-term control measures that should be implemented throughout the entire disease management process. In contrast, using yellow sticky traps to capture pests and removing bamboo plants exhibiting symptoms of sooty mold are short-term control strategies suitable for use during the early stages of an outbreak. Considering the cost-effectiveness of labor and materials, these measures can be discontinued in later stages.
- Based on the severity of the disease and the results of the numerical simulations presented in this paper, the application of insecticides is the core control strategy for managing sooty mold disease in Moso bamboo.
- According to the optimal control model (4.2) for bamboo sooty mold and related numerical simulation results, the integrated control measure combining insecticide spraying with nutrient solution injection (Strategy A) is found to be the most cost-effective among all strategies and therefore considered the preferred approach for controlling bamboo sooty mold.

This research establishes a deterministic mathematical model to investigate the transmission dynamics of Moso bamboo sooty mold, incorporating vector preference within an isolated forest stand. Based on optimal control theory, the corresponding optimal control problem is analyzed, and the most cost-effective disease intervention strategy is subsequently derived. However, the current model does not incorporate insecticide resistance in vector insects. Existing studies indicate that vector insect resistance can substantially influence disease transmission. Therefore, future work will integrate insecticide resistance into the model to better reflect real-world control scenarios. The findings of this study offer a theoretical foundation and decision-making reference for forestry management authorities when designing integrated control strategies against Moso bamboo sooty mold disease.

Declaration

Competing interest. The authors declare that they have no conflict of interest.

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