

NONLOCAL FRACTIONAL $\mathbf{P}(Z, Y)$ -LAPLACIAN PROBLEMS WITH VARIABLE EXPONENTS: EXISTENCE RESULTS AND A NUMERICAL ILLUSTRATION

Omar Djidel^{1,2}, Salah Boulaaras^{3,†} and
Mokhtar Boumaaza⁴

Abstract In this work, we study the existence of weak solutions for a class of nonlocal problems driven by the fractional $\mathbf{p}(z, y)$ -Laplacian operator with spatially variable exponents. The considered model includes a nonlinear source term depending simultaneously on local interactions and a global integral over the domain, which introduces significant analytical difficulties. Specifically, we examine the problem:

$$(P) \begin{cases} \left(\int_{\Omega} h(z, w) dz \right)^{\beta(z)} (-\Delta_{\mathbf{p}(z, y)})^s w = (h(z, w))^{\alpha(z)}, & \text{in } \Omega, \\ w = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \\ w > 0, & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $(-\Delta_{\mathbf{p}(z, y)})^s$ denotes the fractional $\mathbf{p}(z, y)$ -Laplacian, and $\alpha(z)$, $\beta(z)$ are spatially varying exponents. By employing fixed-point arguments, specifically Tychonoff's and Schauder's fixed-point theorems, we establish two existence results for weak solutions. To complement the theoretical analysis, we also present a two-dimensional numerical example of a simplified fractional model ($\mathbf{p} = 2$) featuring a nonlinear and nonlocal source inspired by the analytical structure of the studied problem.

Keywords Nonlinear equations, fractional $\mathbf{p}(z, y)$ -Laplacian, fixed-point theorems, nonlinear equations, weak solutions.

MSC(2010) 35S15, 35R11, 35B09.

1. Introduction

Nonlocal partial differential equations (PDEs) have become a central topic in modern mathematical analysis because of their ability to describe complex and heterogeneous phenomena arising in physics, biology, finance, continuum mechanics, and image processing. Unlike classical local

[†]The corresponding authors.

¹Department of Mathematics and Computer Science, Faculty of Science and Technology, University of Ahmed Ben Yahia El Wancharissi of Tissemsilt, BP 182, 38000 Tissemsilt, Algeria

²Laboratory of Construction Engineering and Materials Sciences, University of Ahmed Ben Yahia El Wancharissi of Tissemsilt, BP 182, 38000 Tissemsilt, Algeria

³Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

⁴Department of Mathematics and Computer Science, University of Ghardaia, 47000 Ghardaia, Algeria

Email: djidel.omar@univ-tissemsilt.dz, omardjidel@yahoo.fr (O. Djidel),
s.boulaaras@qu.edu.sa (S. Boulaaras), boumaaza.mokhtar@yahoo.com (M. Boumaaza)

differential operators, nonlocal operators account for interactions between spatially separated points and can therefore represent long-range diffusion, heterogeneous constitutive laws, and memory-type effects. Among these models, equations involving the fractional $\mathbf{p}(z, y)$ -Laplacian have received considerable attention. They generalize the classical Laplacian, the fractional Laplacian, and the fractional p -Laplacian by allowing the growth exponent to depend on two spatial variables. This flexibility is particularly appropriate for processes with spatially dependent nonlinear behavior, including electrorheological fluids, heterogeneous porous-media flow, anomalous diffusion, and image reconstruction [18, 30, 32, 36].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let

$$\mathbf{p} : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, \infty)$$

be continuous and symmetric:

$$\mathbf{p}(z, y) = \mathbf{p}(y, z) \quad \text{for all } (z, y) \in \overline{\Omega} \times \overline{\Omega}. \quad (1.1)$$

We write

$$\mathbf{p}^- = \min_{(z, y) \in \overline{\Omega}^2} \mathbf{p}(z, y), \quad \mathbf{p}^+ = \max_{(z, y) \in \overline{\Omega}^2} \mathbf{p}(z, y), \quad (1.2)$$

and assume that

$$1 < \mathbf{p}^- \leq \mathbf{p}(z, y) \leq \mathbf{p}^+ < \infty. \quad (1.3)$$

For $s \in (0, 1)$, the fractional $\mathbf{p}(z, y)$ -Laplacian is formally defined by

$$(-\Delta)_{\mathbf{p}(\cdot, \cdot)}^s u(z) = 2 \text{ P. V. } \int_{\mathbb{R}^N} \frac{|u(z) - u(y)|^{\mathbf{p}(z, y)-2} (u(z) - u(y))}{|z - y|^{N+s\mathbf{p}(z, y)}} dy. \quad (1.4)$$

When $\mathbf{p}(z, y) \equiv 2$, this operator reduces, up to a normalization constant, to the usual fractional Laplacian. When $\mathbf{p}(z, y) \equiv p$, it becomes the fractional p -Laplacian. Thus, (1.4) provides a unified framework for both nonlocality and nonstandard growth [4, 7, 26, 30, 36]. The development of fractional and variable-exponent spaces has provided a robust functional framework for the analysis of such problems. Di Nezza, Palatucci, and Valdinoci [30] presented a comprehensive treatment of fractional Sobolev spaces and their fundamental embeddings. The variational and functional-analytic foundations of nonlocal fractional problems were also developed systematically by Bisci, Rădulescu, and Servadei [8]. Fan and Zhao [18], and subsequently Fan, Shen, and Zhao [17], established fundamental properties and Sobolev embeddings in variable-exponent spaces. Their contributions build upon the classical work of Kováčik and Rákosník [27], who developed the basic theory of $L^{p(z)}(\Omega)$ and $W^{k, p(z)}(\Omega)$. Diening, Harjulehto, Hästö, and Růžička [15] later presented a detailed treatment of variable-exponent Lebesgue and Sobolev spaces, including modular inequalities, density results, duality properties, and compact embeddings. Acerbi and Mingione [1] obtained important regularity results for stationary electrorheological fluids, demonstrating the relevance of variable-growth differential operators in continuum mechanics. The variable-exponent Lebesgue modular is defined by

$$\rho_{\mathbf{p}(\cdot)}(u) = \int_{\Omega} |u(z)|^{\mathbf{p}(z)} dz. \quad (1.5)$$

The associated Luxemburg norm is

$$\|u\|_{L^{p(z)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{\mathbf{p}(\cdot)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}. \quad (1.6)$$

For the fractional variable-exponent setting, the natural nonlocal modular is

$$\rho_{s,\mathbf{p}(\cdot,\cdot)}(u) = \int_{\mathbb{R}^{2N}} \frac{|u(z) - u(y)|^{\mathbf{p}(z,y)}}{\mathbf{p}(z,y)|z - y|^{N+s\mathbf{p}(z,y)}} dz dy. \quad (1.7)$$

The corresponding fractional space can be represented as

$$W^{s,\mathbf{p}(\cdot,\cdot)}(\mathbb{R}^N) = \left\{ u \in L^{\mathbf{p}(z)}(\mathbb{R}^N) : \rho_{s,\mathbf{p}(\cdot,\cdot)}(u) < \infty \right\}. \quad (1.8)$$

For problems with a homogeneous exterior condition, one introduces

$$W_0^{s,\mathbf{p}(\cdot,\cdot)}(\Omega) = \left\{ u \in W^{s,\mathbf{p}(\cdot,\cdot)}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}. \quad (1.9)$$

The modular–norm relations provide estimates of the form

$$\min \left\{ \|u\|^{\mathbf{p}^-}, \|u\|^{\mathbf{p}^+} \right\} \leq \rho_{s,\mathbf{p}(\cdot,\cdot)}(u) \leq \max \left\{ \|u\|^{\mathbf{p}^-}, \|u\|^{\mathbf{p}^+} \right\}, \quad (1.10)$$

where $\|\cdot\|$ denotes the norm of the relevant fractional variable-exponent space. These estimates replace the exact homogeneity available in constant-exponent spaces and play a central role in coercivity, compactness, and variational arguments [4, 7, 15, 26]. Bahrouni and Rădulescu [7] introduced a class of fractional Sobolev spaces suitable for nonlocal variational problems with variable growth. Azroul, Benkirane, and Shimi [4], together with Azroul et al. [5], extended this framework to fractional Kirchhoff-type problems and obtained several existence results. Kaufmann, Rossi, and Vidal [26] developed the concept of fractional Sobolev spaces with variable exponents and investigated the associated fractional $\mathbf{p}(z)$ -Laplacian operators.

A representative fractional Kirchhoff problem can be written as

$$\mathcal{M}(\rho_{s,\mathbf{p}(\cdot,\cdot)}(u)) (-\Delta)_{\mathbf{p}(\cdot,\cdot)}^s u = f(z, u) \quad \text{in } \Omega, \quad (1.11)$$

subject to

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \quad (1.12)$$

Here,

$$\mathcal{M} : [0, \infty) \longrightarrow (0, \infty)$$

is the Kirchhoff coefficient. Unlike a standard fractional equation, (1.11) contains two distinct nonlocal mechanisms. The operator $(-\Delta)_{\mathbf{p}(\cdot,\cdot)}^s$ is spatially nonlocal, whereas the coefficient

$$\mathcal{M}(\rho_{s,\mathbf{p}(\cdot,\cdot)}(u))$$

depends globally on the energy of u . Classical and fractional Kirchhoff problems have been investigated using variational methods, genus theory, fountain theorems, and critical-point arguments [12, 13, 24, 28, 32]. Existence results based on sub- and supersolutions were obtained for Kirchhoff systems with multiple and variable parameters [11, 21]. Fixed-point methods, which are also fundamental in nonvariational settings, are developed abstractly in [22, 33]. Comparison principles for degenerate quasilinear operators, which are useful for positivity, symmetry, and monotonicity arguments, were established in [14]. Nonlocal models may also contain a global reaction coefficient rather than, or in addition to, a Kirchhoff energy coefficient. A typical problem of this type has the form

$$\left(\int_{\Omega} h(z, u(z)) dz \right)^{\beta} (-\Delta)_{\mathbf{p}(\cdot,\cdot)}^s u(z) = h(z, u(z))^{\alpha} \quad \text{in } \Omega. \quad (1.13)$$

By defining

$$\Lambda(u) = \left(\int_{\Omega} h(z, u(z)) dz \right)^{-\beta}, \quad (1.14)$$

problem (1.13) becomes

$$(-\Delta)_{\mathbf{p}(\cdot, \cdot)}^s u(z) = \Lambda(u) h(z, u(z))^\alpha. \quad (1.15)$$

The value of $\Lambda(u)$ depends on the solution over the entire domain. Consequently, a local variation of u modifies the forcing at every point, which is the essential global-coupling mechanism in this class of problems. Semilinear and quasilinear nonlocal elliptic equations with global reaction terms have been studied using Bolzano-type arguments, approximation procedures, Schauder's fixed-point theorem, and sub-supersolution techniques [3, 19, 21, 35].

For $u, v \in W_0^{s, \mathbf{p}(\cdot, \cdot)}(\Omega)$, the weak formulation corresponding to (1.13) is formally given by

$$\begin{aligned} & \left(\int_{\Omega} h(z, u(z)) dz \right)^\beta \int_{\mathbb{R}^{2N}} \frac{|u(z) - u(y)|^{\mathbf{p}(z, y) - 2} (u(z) - u(y)) (v(z) - v(y))}{|z - y|^{N + s\mathbf{p}(z, y)}} dz dy \\ &= \int_{\Omega} h(z, u(z))^\alpha v(z) dz. \end{aligned} \quad (1.16)$$

This formulation shows explicitly the interaction between the fractional variable-exponent energy and the nonlinear global source coefficient. More recently, Yahiaoui et al. [34] studied Kirchhoff-type problems involving a nonlocal fractional variable-exponent Laplacian and established the existence of weak solutions under suitable structural assumptions. Djidel et al. [16] investigated nonlocal fractional p -Laplacian elliptic problems through approximation methods and obtained existence and regularity results under weak conditions on the diffusion coefficients. Agarwal, Azroul, Kamali, and Shimi [2] considered a class of $p(z, \cdot)$ -integro-differential Kirchhoff-type problems with singular kernels in generalized fractional Sobolev spaces. Pei [31] studied ground-state solutions for a fractional p -Kirchhoff equation involving subcritical and critical exponential growth. El-Houari, Arhrrabi, and Moussa [23] investigated a Schrödinger–Choquard problem involving a variable-exponent $p(\cdot)$ -Laplacian. Azroul, Kamali, Ragusa, and Shimi [6] established multiplicity results for anisotropic nonlocal problems with variable exponents, while Louchaich and Saoudi [29] studied singular problems driven by fractional $p(\cdot)$ -Laplacian-type operators. Further existence results for fractional p -Laplacian boundary-value systems and fractional impulsive systems were established in [20, 25]. These contributions demonstrate that variational, approximation, fixed-point, and sub-supersolution techniques can be adapted to a broad class of fractional and nonlocal operators. Recent work by Boulaaras [9] addresses two-dimensional fractional heat equations driven by $(-\Delta)^s$ under homogeneous exterior conditions, with particular emphasis on long-time behavior, shape optimization, and Γ -convergence. Related analytical and numerical techniques for nonlinear p -Laplacian equations with nonstandard source terms were developed by Boulaaras and Guefaifia [10]. Together with the Kirchhoff-system results in [11], these studies emphasize the continuing importance of nonlocal diffusion, variable growth, nonlinear source terms, and computational illustration in the analysis of nonlinear PDEs. These developments confirm the growing importance of fractional nonlocal models with variable exponents. They also motivate the study of problems in which the variable-exponent fractional diffusion operator is coupled with a nonlinear global integral term. The main analytical difficulties arise from the simultaneous absence of locality and homogeneity, the nonlinear dependence of the operator on $\mathbf{p}(z, y)$, and the global dependence of the source coefficient on the unknown solution.

Motivated by these developments, in this paper we focus on the following nonlocal fractional problem:

$$\begin{cases} \left(\int_{\Omega} h(z, w) dz \right)^{\beta(z)} (-\Delta_{\mathbf{p}(z,y)})^s w = (h(z, w))^{\alpha(z)} + f(z, w), & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.17)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain, $s \in (0, 1)$, and $\mathbf{p} : \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}$ is a continuous function satisfying the following boundedness and symmetry conditions (see [7, 18]):

$$1 < \mathbf{p}^- \leq \mathbf{p}(z, y) \leq \mathbf{p}^+ < \infty, \quad (z, y) \in \overline{\Omega} \times \overline{\Omega}, \quad (1.18)$$

and

$$\mathbf{p}(z, y) = \mathbf{p}(y, z), \quad \forall (z, y) \in \overline{\Omega} \times \overline{\Omega}. \quad (1.19)$$

The fractional $\mathbf{p}(z, y)$ -Laplacian operator is defined by

$$(-\Delta_{\mathbf{p}(z,y)})^s w(z) := \text{P.V.} \int_{\mathbb{R}^N} \frac{|w(z) - w(y)|^{\mathbf{p}(z,y)-2} (w(z) - w(y))}{|z - y|^{N+s\mathbf{p}(z,y)}} dy, \quad z \in \mathbb{R}^N,$$

where P.V. stands for the Cauchy principal value.

We introduce two sets of structural hypotheses on $h(z, w)$ and $f(z, w)$.

First hypothesis. Let $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be such that:

- for a.e. $z \in \Omega$, the map $v \mapsto h(z, v)$ is continuous;
- for every $v \in \mathbb{R}$, the map $z \mapsto h(z, v)$ is measurable;
- there exists $f \in L^1(\Omega)$ such that

$$0 < h(z, v) \leq f(z), \quad \forall v \in \mathbb{R}, \text{ a.e. } z \in \Omega. \quad (1.20)$$

Moreover, the variable exponents $\alpha(z)$ and $\beta(z)$ satisfy one of the following:

$$\beta(z) \leq \alpha(z) \text{ and } \alpha(z) \in \left[0, \frac{1}{\mathbf{p}'(z)}\right], \quad (1.21)$$

or

$$\beta(z) \leq \frac{1}{\mathbf{p}'(z)} < \alpha(z), \quad f(z) \leq 1 \text{ a.e. } z \in \Omega. \quad (1.22)$$

Second hypothesis. Assume h satisfies:

$$h(z, v) > 0, \quad \forall v \in \mathbb{R}, \text{ a.e. } z \in \Omega, \quad (1.23)$$

and

$$|h(z, v_1) - h(z, v_2)| \leq \mathcal{H}_{\tau(z)}(z) |v_1 - v_2|^{\tau(z)}, \quad \forall v_1, v_2 \in \mathbb{R}, \text{ a.e. } z \in \Omega, \quad (1.24)$$

where $\mathcal{H}_{\tau(z)}$ satisfies

$$\begin{cases} \tau(z) \in (0, \mathbf{p}(z)), & \mathcal{H}_{\tau(z)} \in L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega), & \text{if } \tau(z) < \mathbf{p}(z), \\ \tau(z) = \mathbf{p}(z), & \mathcal{H}_{\mathbf{p}(z)} \in L^\infty(\Omega), \ 0 < \beta(z) \leq \alpha(z) \leq \frac{1}{\mathbf{p}'(z)}, & \text{if } \tau(z) = \mathbf{p}(z). \end{cases} \quad (1.25)$$

Under these assumptions, we establish the existence of at least two positive weak solutions for problem (1.17) using fixed-point theory. Specifically, we employ the Tychonoff and Schauder fixed-point theorems [33] to demonstrate the solvability of (1.17) in suitable fractional Sobolev spaces with variable exponents.

The existence and multiplicity of weak solutions for fractional and variable exponent problems have been extensively studied using both variational and topological methods. Bisci, Rădulescu, and Servadei [8] developed a comprehensive functional framework for fractional problems and established general existence results via critical point theory. Iannizzotto et al. [24] employed Morse theory to handle nonlocal equations, while Perera, Squassina, and Yang [32] analyzed bifurcation and multiplicity in critical fractional $\mathbf{p}$ -Laplacian settings. Complementary results for Kirchhoff-type and $\mathbf{p}$ -Laplacian problems were provided by Júlio, Corrêa, and Figueiredo [12], Corrêa and Figueiredo [13], and Liu [28], who used variational and topological arguments to demonstrate the existence of multiple solutions.

In addition to variational techniques, fixed-point theory has proven to be a powerful tool for nonlocal problems. Guo and Lakshmikantham [22] established general fixed-point theorems in ordered Banach spaces, while Smart [33] provided a unified presentation of Tychonoff and Schauder's results, which serve as the cornerstone of modern nonlinear analysis. These classical results were later applied by Zaouiche [35] to prove the existence of nontrivial weak solutions for nonlocal and nonhomogeneous elliptic problems. Related analytical techniques were also employed by Damascelli [14], who derived comparison principles for quasilinear degenerate elliptic operators, leading to symmetry and monotonicity results that are relevant to nonlocal frameworks.

In this work, we impose the hypotheses (1.20)–(1.22) on the nonlinear terms $h(z, w)$ and $f(z, w)$, as well as on the variable exponents $\alpha(z)$ and $\beta(z)$, ensuring the appropriate setting for the application of fixed-point theorems. By combining the Tychonoff and Schauder fixed-point principles, we establish the existence of positive weak solutions of problem (1.17) in suitable fractional Sobolev spaces with variable exponents.

The structure of the paper is as follows. Section 2 introduces the necessary mathematical preliminaries, including the definition and main properties of fractional Sobolev spaces with variable exponents, following the works of Diening et al. [15] and Azroul et al. [4]. Section 3 presents the main theoretical results, obtained through fixed-point arguments. Finally, a two-dimensional numerical example is provided to complement the theoretical analysis, illustrating a simplified fractional model with $\mathbf{p} = 2$ and a nonlinear, nonlocal source inspired by the analytical framework of the studied problem.

2. The functional setting and tools

In this section, we recall some preliminary notions and results concerning variable exponent spaces, which provide the functional framework for our analysis.

Let $q \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(z) > 1, \forall z \in \overline{\Omega}\}.$$

We denote

$$q^- = \inf_{z \in \overline{\Omega}} q(z) \quad \text{and} \quad q^+ = \sup_{z \in \overline{\Omega}} q(z),$$

so that

$$1 < q^- \leq q(z) \leq q^+ < +\infty. \quad (2.1)$$

For $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{q(z)}(\Omega) = \left\{ w : \Omega \rightarrow \mathbb{R} \text{ measurable such that } \int_{\Omega} |w(z)|^{q(z)} dz < +\infty \right\}.$$

This is a vector space endowed with the *Luxemburg norm*

$$\|w\|_{L^{q(z)}(\Omega)} = \inf \left\{ \rho > 0 : \int_{\Omega} \left| \frac{w(z)}{\rho} \right|^{q(z)} dz < 1 \right\}.$$

It is well known that $(L^{q(\cdot)}(\Omega), \|\cdot\|_{L^{q(\cdot)}(\Omega)})$ is a separable and reflexive Banach space. The conjugate space of $L^{q(z)}(\Omega)$ is denoted by $L^{q'(z)}(\Omega)$, where $\frac{1}{q'(z)} + \frac{1}{q(z)} = 1$ (see [17, 18]).

Lemma 2.1 (Hölder inequality [17, 18]). *For all $\psi \in L^{q(z)}(\Omega)$ and $\phi \in L^{q'(z)}(\Omega)$, we have*

$$\left| \int_{\Omega} \psi(z)\phi(z) dz \right| \leq \left(\frac{1}{q^-} + \frac{1}{q^+} \right) \|\psi\|_{L^{q(z)}(\Omega)} \|\phi\|_{L^{q'(z)}(\Omega)} \leq 2 \|\psi\|_{L^{q(z)}(\Omega)} \|\phi\|_{L^{q'(z)}(\Omega)}.$$

Lemma 2.2 (Generalized product estimate [17, 18]). *Let $\mathbf{p}(\cdot), r(\cdot), t(\cdot) \in C_+(\overline{\Omega})$ such that*

$$\frac{1}{\mathbf{p}(z)} = \frac{1}{r(z)} + \frac{1}{t(z)}, \quad \forall z \in \overline{\Omega}.$$

Then there exists a constant $C > 0$ such that

$$\|\psi\phi\|_{L^{\mathbf{p}(z)}(\Omega)} \leq C \|\psi\|_{L^{t(z)}(\Omega)} \|\phi\|_{L^{r(z)}(\Omega)}$$

for all $\psi \in L^{t(\cdot)}(\Omega)$ and $\phi \in L^{r(\cdot)}(\Omega)$ satisfying $\psi\phi \in L^{\mathbf{p}(\cdot)}(\Omega)$.

A central notion in the study of generalized Lebesgue spaces is the *modular function* associated with $L^{r(z)}(\Omega)$, defined by

$$\sigma_{r(\cdot)}(w) = \int_{\Omega} |w(z)|^{r(z)} dz.$$

Lemma 2.3 (Properties of the modular [18]). *Let $w \in L^{r(z)}(\Omega)$ and $\{w_j\} \subset L^{r(z)}(\Omega)$. Then:*

- (a) $\|w\|_{L^{r(z)}(\Omega)} < 1$ (resp. $= 1, > 1$) $\iff \sigma_{r(z)}(w) < 1$ (resp. $= 1, > 1$).
- (b) If $\|w\|_{L^{r(z)}(\Omega)} < 1$, then $\|w\|_{L^{r(z)}(\Omega)}^{r^+} \leq \sigma_{r(z)}(w) \leq \|w\|_{L^{r(z)}(\Omega)}^{r^-}$.
- (c) If $\|w\|_{L^{r(z)}(\Omega)} > 1$, then $\|w\|_{L^{r(z)}(\Omega)}^{r^-} \leq \sigma_{r(z)}(w) \leq \|w\|_{L^{r(z)}(\Omega)}^{r^+}$.
- (d) $\lim_{j \rightarrow \infty} \sigma_{r(\cdot)}(w_j - w) = 0 \iff \lim_{j \rightarrow \infty} \|w_j - w\|_{L^{r(z)}(\Omega)} = 0$.

Lemma 2.4. *Let $r, t \in C_+(\overline{\Omega})$ with $r(z) \leq t(z)$ for all $z \in \overline{\Omega}$. Then for every $w \in L^{t(z)}(\Omega)$, we have $|w|^{r(z)} \in L^{\frac{t(z)}{r(z)}}(\Omega)$ and*

$$\| |w|^{r(z)} \|_{L^{\frac{t(z)}{r(z)}}(\Omega)} \leq \|w\|_{L^{t(z)}(\Omega)}^{r^-} + \|w\|_{L^{t(z)}(\Omega)}^{r^+},$$

and equivalently,

$$\| |w|^{r(z)} \|_{L^{\frac{t(z)}{r(z)}}(\Omega)} = \|w\|_{L^{t(z)}(\Omega)}^{\widehat{r}}, \quad \text{for some } \widehat{r} \in [r^-, r^+].$$

Lemma 2.5 (Continuous embedding). *Let $r, t \in C_+(\overline{\Omega})$ be such that $r(z) \leq t(z)$ for all $z \in \overline{\Omega}$. Then*

$$L^{t(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$$

continuously.

Lemma 2.6 (Norm of the constant function [15]). *Let $q \in C_+(\overline{\Omega})$. Then*

$$\min\{|\Omega|^{1/q^-}, |\Omega|^{1/q^+}\} \leq \|1\|_{L^{q(z)}(\Omega)} \leq \max\{|\Omega|^{1/q^-}, |\Omega|^{1/q^+}\},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω .

The results presented in this section are fundamental tools for the subsequent analysis. They provide the essential properties of variable exponent spaces that will be used to establish the functional framework and to prove the main existence results in Section 3, as well as to support the numerical study introduced in Section 3.2.

Fractional Sobolev spaces with variable exponent.

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$. Throughout this paper, we fix $s \in (0, 1)$, and assume that the function $\mathbf{p} : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty)$ is Lipschitz continuous and bounded, satisfying the following conditions:

$$1 < \min_{z, y \in \overline{\Omega}} \mathbf{p}(z, y) = \mathbf{p}^- \leq \mathbf{p}(z, y) \leq \max_{z, y \in \overline{\Omega}} \mathbf{p}(z, y) = \mathbf{p}^+ < \frac{N}{s}, \quad (2.2)$$

and

$$\mathbf{p}(z, y) = \mathbf{p}(y, z), \quad \overline{\mathbf{p}} := \mathbf{p}(z, z), \quad \forall (z, y) \in \overline{\Omega} \times \overline{\Omega}. \quad (2.3)$$

We now introduce the fractional Sobolev space with variable exponent via the Gagliardo approach:

$$\mathfrak{W}^{s, \mathbf{p}(z, y)}(\Omega) = \left\{ w \in L^{\overline{\mathbf{p}}}(\Omega) : \int_{\Omega \times \Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z, y)}}{\rho^{\mathbf{p}(z, y)} |z - y|^{N+s\mathbf{p}(z, y)}} dz dy < +\infty, \text{ for each } \rho > 0 \right\},$$

endowed with the norm

$$\|w\|_{\mathfrak{W}^{s, \mathbf{p}(z, y)}(\Omega)} = \|w\|_{L^{\overline{\mathbf{p}}}(\Omega)} + [w]_{\mathfrak{W}^{s, \mathbf{p}(z, y)}(\Omega)},$$

where $[w]_{\mathfrak{W}^{s, \mathbf{p}(z, y)}(\Omega)}$ is the Gagliardo seminorm with variable exponent, defined as

$$[w]_{\mathfrak{W}^{s, \mathbf{p}(z, y)}(\Omega)} = \inf \left\{ \rho > 0 : \int_{\Omega \times \Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z, y)}}{\rho^{\mathbf{p}(z, y)} |z - y|^{N+s\mathbf{p}(z, y)}} dz dy \leq 1 \right\}.$$

The space $(\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega), \|\cdot\|_{\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega)})$ is reflexive, separable Banach spaces (see [7, 26]).

We denote by $\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$ the subspace of $\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega)$ given by

$$\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega) = \{w \in \mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega) : w = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

Its associated norm is

$$[w]_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} = \|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} = \inf \left\{ \rho > 0 : \int_{D_\Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z,y)}}{\rho^{\mathbf{p}(z,y)} |z - y|^{N+s\mathbf{p}(z,y)}} dz dy \leq 1 \right\},$$

and this space is also reflexive, separable, and Banach (see [5]).

We define the modular function

$$\sigma_{\mathbf{p}(z,y)}(w) = \int_{D_\Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z,y)}}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy,$$

so that

$$[w]_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} = \|w\|_{\sigma_{\mathbf{p}(z,y)}(w)} = \inf \left\{ \rho > 0 : \sigma_{\mathbf{p}(z,y)}\left(\frac{w}{\rho}\right) \leq 1 \right\}.$$

Proposition 2.1 ([26]). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Assume that $0 < s < 1$ and $\mathbf{p} : \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}$ is continuous with $s\mathbf{p}(z,y) < N$ for all $(z,y) \in \overline{\Omega} \times \overline{\Omega}$, and that (2.2)–(2.3) hold. Let $r : \overline{\Omega} \rightarrow (1, +\infty)$ be continuous with*

$$1 < r^- = \min_{z \in \overline{\Omega}} r(z) \leq r(z) < \mathbf{p}_s^*(z) = \frac{N\overline{\mathbf{p}}(z)}{N - s\overline{\mathbf{p}}(z)}, \quad \forall z \in \overline{\Omega}.$$

Then there exists a constant $C = C(\Omega, N, r, s, \mathbf{p}) > 0$ such that

$$\|w\|_{L^{r(z)}(\Omega)} \leq C \|w\|_{\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega)}, \quad \forall w \in \mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega).$$

Hence, $\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega)$ is continuously embedded into $L^{r(z)}(\Omega)$ for all $r \in (1, \mathbf{p}_s^*)$, and the embedding is compact.

Remark 2.1 ([36]). 1. Proposition 2.1 remains valid when $\mathfrak{W}^{s,\mathbf{p}(z,y)}(\Omega)$ is replaced by $\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$, since their norms are equivalent.

2. The modular norm $\|\cdot\|_{\sigma_{\mathbf{p}(z,y)}}$ is equivalent to $\|\cdot\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)}$.

3. The modular $\sigma_{\mathbf{p}(z,y)}$ satisfies the properties stated in Lemma 2.3.

Proposition 2.2 ([5]). *For any $w \in \mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$, we have:*

1. *If $\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} > 1$, then*

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^-} \leq \sigma_{\mathbf{p}(z,y)}(w) \leq \|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^+}.$$

2. *If $\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} < 1$, then*

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^+} \leq \sigma_{\mathbf{p}(z,y)}(w) \leq \|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^-}.$$

Proposition 2.3 ([4, 7]). Let $s \in (0, 1)$ and $1 < \mathbf{p}^- \leq \mathbf{p}(z, y) \leq \mathbf{p}^+ < \infty$. Then:

- (i) The operator $(-\Delta_{\mathbf{p}(z, y)})^s : \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega) \rightarrow (\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega))^*$ is continuous, bounded, strictly monotone, and coercive.
- (ii) The inverse operator $[(-\Delta_{\mathbf{p}(z, y)})^s]^{-1} : (\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega))^* \rightarrow \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ is continuous.

Definition 2.1 (Weak solution). A function $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ is said to be a *weak solution* to problem (1.17) if

$$w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

and for every $\psi \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$,

$$\int_{\Omega} \psi (-\Delta_{\mathbf{p}(z, y)})^s w \, dz = \frac{1}{(\int_{\Omega} h(z, w) \, dz)^{\beta(z)}} \int_{\Omega} (h(z, w))^{\alpha(z)} \psi \, dz. \quad (2.4)$$

Lemma 2.7 ([14]). Let $p > 1$. There exists a constant $C = C(p) > 0$ such that, for every $v_1, v_2 \in \mathbb{R}$,

$$\langle |v_1 - v_2|^{p-2} (v_1 - v_2), (v_1 - v_2) \rangle \geq C \frac{|v_1 - v_2|^2}{(|v_1| + |v_2|)^{2-p}}. \quad (2.5)$$

Theorem 2.1. Let $s \in (0, 1)$ and let $\mathbf{p}(z, y)$ be continuous satisfying (2.2)–(2.3). If condition (1.20) and either (1.21) or (1.22) hold, then problem (1.17) admits a nonnegative weak solution $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$.

Theorem 2.2. Let $0 < s < 1$ and let $\mathbf{p}(z, y)$ satisfy (2.2)–(2.3). Assume that conditions (1.23)–(1.25) hold. Then problem (1.17) possesses a positive weak solution in $\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ according to Definition 2.1.

Remark 2.2. To the best of our knowledge, these results are new even in the particular case $\mathbf{p} = 2$. They establish the theoretical foundation for nonlocal problems with variable exponents and also support the numerical investigations presented in the final section, where a two-dimensional example with $\mathbf{p} = 2$ illustrates the analytical behavior of the model.

3. Existence of solutions

3.1. First existence theorem

We begin by proving Theorem 2.1 using Tychonoff's fixed-point theorem.

Proof of Theorem 2.1. The proof proceeds in three main steps.

Step 1. Consider the following auxiliary problem:

$$\begin{cases} (-\Delta_{\mathbf{p}(z, y)})^s w = \frac{(h(z, \xi))^{\alpha(z)}}{(\int_{\Omega} h(z, \xi) \, dz)^{\beta(z)}}, & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3.1)$$

for every $\xi \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$.

To obtain $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ satisfying (3.1), define

$$\Psi(\xi) = \frac{(h(z, \xi))^{\alpha(z)}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \in \left(\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)\right)^*. \quad (3.2)$$

For any $\psi \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ used as a test function in (3.1), we have

$$|\langle \Psi(\xi), \psi \rangle| \leq \frac{1}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \int_{\Omega} (h(z, \xi))^{\alpha(z)} |\psi| dz. \quad (3.3)$$

Assume now that $\alpha(z)$ and $\beta(z)$ satisfy (1.21). We analyze three cases separately. Applying condition (1.20), Hölder's inequality, Lemmas 2.4–2.6, and Proposition 2.1, we deduce the following estimates from (3.3).

• **Case 1.** $\alpha(z) \geq \beta(z)$ and $0 < \alpha(z) < \frac{1}{\mathbf{p}'(z)}$. Then,

$$\begin{aligned} |\langle \Psi(\xi), \psi \rangle| &\leq \frac{3 \|1\|_{L^{\pi(z)}(\Omega)} \|\psi\|_{L^{\mathbf{p}(z)}(\Omega)} \|(f(z, \xi))^{\alpha(z)}\|_{L^{\frac{1}{\alpha(z)}}(\Omega)}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \\ &\leq \frac{3 \|1\|_{L^{\pi(z)}(\Omega)} \|\psi\|_{L^{\mathbf{p}(z)}(\Omega)} \|f(z, \xi)\|_{L^1(\Omega)}^{\hat{\alpha}}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \\ &\leq c_1 C_{\Omega, \pi} \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \left(\int_{\Omega} f(z, \xi) dz\right)^{\hat{\alpha} - \beta^+} \\ &\leq c_1 C_{\Omega, \pi} \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \|f\|_{L^1(\Omega)}^{\hat{\alpha} - \beta^+}, \end{aligned} \quad (3.4)$$

where

$$\pi(z) = \frac{\mathbf{p}'(z)}{1 - \alpha(z)\mathbf{p}'(z)}, \quad \hat{\alpha} \in \left[\frac{1}{\alpha^-}, \frac{1}{\alpha^+}\right], \quad C_{\Omega, \pi} = 2 \max \left\{ |\Omega|^{\frac{1}{\pi^+}}, |\Omega|^{\frac{1}{\pi^-}} \right\}.$$

• **Case 2.** $\alpha(z) \geq \beta(z)$ and $\alpha(z) = 0$. Then

$$\begin{aligned} |\langle \Psi(\xi), \psi \rangle| &\leq \frac{2 \|\psi\|_{L^{\mathbf{p}(z)}(\Omega)} \|1\|_{L^{\mathbf{p}'(z)}(\Omega)}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \\ &\leq c_2 \|1\|_{L^{\mathbf{p}'(z)}(\Omega)} \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \|f\|_{L^1(\Omega)}^{-\beta^+} \\ &\leq c_2 C_{\Omega, \mathbf{p}'} \|f\|_{L^1(\Omega)}^{-\beta^+} \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)}, \end{aligned} \quad (3.5)$$

where

$$C_{\Omega, \mathbf{p}'} = 2 \max \left\{ |\Omega|^{\frac{1}{\mathbf{p}'^+}}, |\Omega|^{\frac{1}{\mathbf{p}'^-}} \right\}.$$

• **Case 3.** $\alpha(z) = \frac{1}{\mathbf{p}'(z)} \geq \beta(z)$. Then

$$|\langle \Psi(\xi), \psi \rangle| \leq 2 \|\psi\|_{L^{\mathbf{p}(z)}(\Omega)} \frac{\left(\int_{\Omega} h(z, \xi) dz\right)^{\frac{1}{\mathbf{p}'^-}}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}}$$

$$\begin{aligned}
&\leq c_3 \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \left(\int_{\Omega} h(z, \xi) dz \right)^{\frac{1}{\mathbf{p}'^-} - \beta^+} \\
&\leq c_3 \|f\|_{L^1(\Omega)}^{\frac{1}{\mathbf{p}'^-} - \beta^+} \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)}.
\end{aligned} \tag{3.6}$$

When (1.22) holds, using again Hölder's inequality and Lemma 2.5, we obtain

$$\begin{aligned}
|\langle \Psi(\xi), \psi \rangle| &\leq 2 \|\psi\|_{L^{\mathbf{p}(z)}(\Omega)} \frac{\left(\int_{\Omega} (h(z, \xi))^{\alpha(z) \mathbf{p}'(z)} dz \right)^{\frac{1}{\mathbf{p}'^-}}}{\left(\int_{\Omega} h(z, \xi) dz \right)^{\beta(z)}} \\
&\leq C \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \frac{\left(\int_{\Omega} h(z, \xi) dz \right)^{\frac{1}{\mathbf{p}'^-}}}{\left(\int_{\Omega} h(z, \xi) dz \right)^{\beta(z)}} \\
&\leq C \|\psi\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \|f\|_{L^1(\Omega)}^{\frac{1}{\mathbf{p}'^-} - \beta^+}.
\end{aligned} \tag{3.7}$$

Therefore, by combining estimates (3.4)–(3.7), the desired result follows. By applying Theorem 5.1 from [4], the problem admits a unique solution $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$.

By condition (1.20), $w = 0$ cannot be a solution of (3.1).

To prove that w is nonnegative, we choose $w^-(z) = \max(-w(z), 0)$ as a test function in (3.1). Using the inequality

$$(w(z) - w(y))(w^-(z) - w^-(y)) \leq -(w^-(z) - w^-(y))^2, \quad \text{for a.e. } (z, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

we deduce that

$$\begin{aligned}
\iint_{\Omega \times \Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z, y) - 2} (w(z) - w(y))(w^-(z) - w^-(y))}{|z - y|^{N + s \mathbf{p}(z, y)}} dz dy &= \frac{\int_{\Omega} (h(z, \xi))^{\alpha(z)} w^- dz}{\left(\int_{\Omega} h(z, \xi) dz \right)^{\beta(z)}} \\
&\leq 0.
\end{aligned}$$

Consequently,

$$\iint_{\Omega \times \Omega} \frac{|w(z) - w(y)|^{\mathbf{p}(z, y) - 2} (w^-(z) - w^-(y))^2}{|z - y|^{N + s \mathbf{p}(z, y)}} dz dy \leq 0,$$

which implies

$$\|w^-\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \leq 0.$$

Thus $w^- = 0$ and hence $w \geq 0$ in Ω . Since $w = 0$ on $\mathbb{R}^N \setminus \Omega$, it follows that w is nonnegative in $\mathbb{R}^N$.

Therefore, the operator

$$\mathcal{G} : \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega) \rightarrow \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega), \quad \xi \mapsto w = \mathcal{G}(\xi),$$

is well defined.

Step 2. We now show that

$$\mathcal{G}(\overline{\mathcal{B}}_{\mathcal{R}}(0)) \subset \overline{\mathcal{B}}_{\mathcal{R}}(0),$$

where $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ denotes the closed ball in $\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ centered at the origin with radius $\mathcal{R} > 0$.

It suffices to prove that if w solves problem (3.1), then

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} \leq \mathcal{R},$$

for a suitable constant $\mathcal{R}$ to be determined. Taking w as a test function in (3.1) yields

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(z) - w(y)|^{\mathbf{p}(z,y)}}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy = \frac{\int_{\Omega} (h(z, \xi))^{\alpha(z)} w(z) dz}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}}. \quad (3.8)$$

As in Step 1, assume that $\alpha(z)$ and $\beta(z)$ satisfy (1.21) and distinguish three cases. Using Hölder's inequality together with Lemmas 2.4–2.6 and Propositions 2.1–2.2, for any $w \in \mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$ with $\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} > 1$, we derive from (3.8):

- If $\alpha(z) \geq \beta(z)$ and $0 < \alpha(z) < 1/\mathbf{p}'(z)$, then

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} \leq [c_1 C_{\Omega,\pi} \|f\|_{L^1(\Omega)}^{\hat{\alpha}-\beta^+}]^{\frac{1}{\mathbf{p}^--1}}.$$

- If $\alpha(z) \geq \beta(z)$ and $\alpha(z) = 0$, then

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} \leq [c_2 C_{\Omega,\mathbf{p}'} \|f\|_{L^1(\Omega)}^{-\beta^+}]^{\frac{1}{\mathbf{p}^--1}}.$$

- If $\alpha(z) = 1/\mathbf{p}'(z) \geq \beta(z)$, then

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} \leq [c_3 \|f\|_{L^1(\Omega)}^{\frac{1}{\mathbf{p}'} - \beta^+}]^{\frac{1}{\mathbf{p}^--1}}.$$

Furthermore, if condition (1.22) holds, one obtains

$$\|w\|_{\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)} \leq [C \|f\|_{L^1(\Omega)}^{\frac{1}{\mathbf{p}'} - \beta^+}]^{\frac{1}{\mathbf{p}^--1}}.$$

Hence, from (3.1)–(3.1), we conclude that if conditions (1.20) and either (1.21) or (1.22) are fulfilled, there exists $\mathcal{R} > 0$ such that

$$\mathcal{G}(\overline{\mathcal{B}}_{\mathcal{R}}(0)) \subset \overline{\mathcal{B}}_{\mathcal{R}}(0).$$

Step 3. We now establish that $\mathcal{G} : \overline{\mathcal{B}}_{\mathcal{R}}(0) \rightarrow \overline{\mathcal{B}}_{\mathcal{R}}(0)$ is weakly continuous.

Let $\{\xi_k\}_{k \in I}$ be a sequence in $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ such that $\xi_k \rightharpoonup \xi$ weakly in $\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$. Set $\mathcal{G}(\xi_k) = w_k$ and $\mathcal{G}(\xi) = w$. Since $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ is weakly compact in $\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega)$, it suffices to show that $w_k \rightharpoonup w$ weakly in this space.

Assume, by contradiction, that

$$w_k \rightharpoonup \tilde{w} \quad \text{weakly in } \mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega). \quad (3.9)$$

By Theorem 2.1, the embedding $\mathfrak{W}_0^{s,\mathbf{p}(z,y)}(\Omega) \hookrightarrow L^{\overline{\mathbf{p}}(z)}(\Omega)$ is compact, hence up to a subsequence,

$$\xi_k(x) \rightarrow \xi(x) \quad \text{a.e. in } \Omega. \quad (3.10)$$

Since w_k and w are positive solutions of (3.1), we take $\psi = (w_k - w)$ as a test function in (3.1). Subtracting the corresponding identities, we get

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_k(z) - w_k(y)|^{\mathbf{p}(z,y)-2} (w_k(z) - w_k(y)) (\psi(z) - \psi(y))}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy$$

$$\begin{aligned}
& - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\mathbf{w}(z) - \mathbf{w}(y)|^{\mathbf{p}(z,y)-2} (\mathbf{w}(z) - \mathbf{w}(y)) (\psi(z) - \psi(y))}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy \\
& = \frac{\int_{\Omega} (h(z, \xi_k))^{\alpha(z)} \psi dz}{\left(\int_{\Omega} h(z, \xi_k) dz\right)^{\beta(z)}} - \frac{\int_{\Omega} (h(z, \xi))^{\alpha(z)} \psi dz}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}}.
\end{aligned} \tag{3.11}$$

Combining relations (3.9)–(3.11), and applying the weak lower semicontinuity of the norm, we conclude that $\tilde{\mathbf{w}} = \mathbf{w}$. Therefore, $\mathcal{G}$ is weakly continuous. $\square$

Conclusion

Since $\mathcal{G}$ maps the closed convex bounded set $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ into itself and is weakly continuous, Tychonoff's fixed-point theorem guarantees the existence of a fixed point $\mathbf{w} \in \mathfrak{W}_0^{s, \mathbf{p}(z,y)}(\Omega)$ satisfying $\mathcal{G}(\mathbf{w}) = \mathbf{w}$. Hence, problem (3.1) admits at least one nonnegative weak solution. Hence, by applying Lemma 2.5, for $\mathbf{p}^- \geq 2$ it follows that

$$\begin{aligned}
& c \|\mathbf{w}_k - \mathbf{w}\|_{\mathfrak{W}_0^{s, \mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^-} \\
& \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\mathbf{w}_k(z) - \mathbf{w}_k(y)|^{\mathbf{p}(z,y)-2} (\mathbf{w}_k(z) - \mathbf{w}_k(y)) (\psi(z) - \psi(y))}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy \\
& \quad - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\mathbf{w}(z) - \mathbf{w}(y)|^{\mathbf{p}(z,y)-2} (\mathbf{w}(z) - \mathbf{w}(y)) (\psi(z) - \psi(y))}{|z - y|^{N+s\mathbf{p}(z,y)}} dz dy.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\begin{aligned}
& c \|\mathbf{w}_k - \mathbf{w}\|_{\mathfrak{W}_0^{s, \mathbf{p}(z,y)}(\Omega)}^{\mathbf{p}^-} \\
& \leq \frac{1}{\left(\int_{\Omega} h(z, \xi_k) dz\right)^{\beta(z)}} \int_{\Omega} \left[(h(z, \xi_k))^{\alpha(z)} - (h(z, \xi))^{\alpha(z)} \right] (\mathbf{w}_k - \mathbf{w}) dz \\
& \quad + \left[\frac{1}{\left(\int_{\Omega} h(z, \xi_k) dz\right)^{\beta(z)}} - \frac{1}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \right] \int_{\Omega} (h(z, \xi))^{\alpha(z)} (\mathbf{w}_k - \mathbf{w}) dz.
\end{aligned} \tag{3.13}$$

Furthermore, from condition (1.20), we have

$$\begin{cases} \int_{\Omega} h(z, \xi_k) dz > 0, & \text{for every } k \in \mathbb{N}, \\ \int_{\Omega} h(z, \xi) dz > 0. \end{cases} \tag{3.14}$$

Hence,

$$\left(\int_{\Omega} h(z, \xi_k) dz \right)^{\beta(z)} \longrightarrow \left(\int_{\Omega} h(z, \xi) dz \right)^{\beta(z)} \quad \text{strongly in } \mathbb{R}, \tag{3.15}$$

and

$$(h(z, \xi_k))^{\alpha(z)} \longrightarrow (h(z, \xi))^{\alpha(z)} \quad \text{strongly in } L^{\mathbf{p}'(z)}(\Omega). \tag{3.16}$$

Indeed, relations (3.15)–(3.16) follow directly from (1.20) and (3.10), together with the dominated convergence theorem.

Since $w_k \in \bar{\mathcal{B}}_{\mathcal{R}}(0)$, by passing to the limit as $k \rightarrow +\infty$ in (3.13), we deduce that

$$w_k \longrightarrow w \quad \text{strongly in } \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega).$$

Returning to (3.9), we conclude that $w = \tilde{w}$. Therefore, w is the unique weak limit point of the sequence $\{w_k\}_{k \in I}$, and consequently,

$$w_k = \mathcal{G}(\xi_k) \rightharpoonup w = \mathcal{G}(\xi) \quad \text{weakly in } \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega).$$

The case $1 < \mathbf{p}^- \leq \mathbf{p}(z, \cdot) \leq \mathbf{p}^+ < 2$ can be treated using analogous arguments, and is therefore omitted.

Hence, by invoking the Tychonoff fixed-point theorem, the operator $\mathcal{G}$ admits at least one fixed point $w \in \bar{\mathcal{B}}_{\mathcal{R}}(0)$ such that $w = \mathcal{G}(w)$, which constitutes a weak solution of problem (1.17).

3.2. Second existence Theorem

Under the hypotheses imposed on h , $\alpha(z)$, and $\beta(z)$, we establish in this subsection an existence result for weak solutions of problem (1.17) by applying the Schauder fixed-point theorem.

Proof of Theorem 2.2. The proof is divided into several steps.

Step 1. For $\xi \in L^{\mathbf{p}(z)}(\Omega)$, following the same reasoning as in the proof of Theorem 2.1, we define

$$\Psi(\xi) := \frac{(h(z, \xi))^{\alpha(z)}}{(\int_{\Omega} h(z, \xi) dz)^{\beta(z)}} \in \left(\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega) \right)^*. \quad (3.17)$$

We make the following observations:

- If $\tau(z) \in (0, \mathbf{p}(z))$ and $\mathcal{H}_{\tau} \in L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z) - \tau(z)}}(\Omega)$, then, under assumptions (1.24)–(1.25) and by applying Hölder's inequality, Lemmas 2.4–2.6, and Proposition 2.1, one deduces that

$$h(\cdot, \xi) \in L^1(\Omega) \quad \text{and} \quad (h(\cdot, \xi))^{\alpha(z)} \in L^{\mathbf{p}'(z)}(\Omega).$$

Consequently, $\Psi(\xi) \in \left(\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega) \right)^*$.

- Similarly, if $\tau(z) = \mathbf{p}(z)$ and $\mathcal{H}_{\mathbf{p}(z)} \in L^{\infty}(\Omega)$, we again obtain $\Psi(\xi) \in \left(\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega) \right)^*$.

Therefore, by Theorem 5.1 of [4], problem (3.1) admits a unique weak solution $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$. Since $h(z, \xi) > 0$ for all $\xi \in \mathbb{R}$ and almost every $z \in \Omega$, the same arguments as in Theorem 2.1 show that $w > 0$ in Ω .

Hence, we define the nonlinear operator

$$\begin{aligned} \mathcal{N} : L^{\mathbf{p}(z)}(\Omega) &\rightarrow \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega), \\ \xi &\mapsto w = \mathcal{N}(\xi), \end{aligned}$$

which is well-defined. If w is a fixed point of $\mathcal{N}$, then w is a weak solution of (1.17). Thus, it suffices to prove that $\mathcal{N}$ admits a fixed point.

Step 2. We now prove the existence of $\mathcal{R} > 0$ such that

$$\mathcal{N}(\bar{\mathcal{B}}_{\mathcal{R}}(0)) \subset \bar{\mathcal{B}}_{\mathcal{R}}(0),$$

where $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ denotes the closed ball of radius $\mathcal{R}$ centered at the origin in $\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$. To this end, we take w as a test function in (3.1). Using Propositions 2.1–2.2, for any $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$ with $\|w\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} > 1$, we deduce

$$\|w\|_{L^{\mathbf{p}(z)}(\Omega)}^{\mathbf{p}^-} \leq c^{\mathbf{p}^-} \frac{\int_{\Omega} (h(z, \xi))^{\alpha(z)} |w| dz}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}}. \quad (3.18)$$

Using assumptions (1.23)–(1.25) and applying again Lemmas 2.4–2.6, we distinguish two main cases.

- If $0 < \beta(z) < \alpha(z)$ and $\alpha(z) = \frac{1}{\mathbf{p}'(z)}$, then

$$\begin{aligned} \|w\|_{L^{\mathbf{p}(z)}(\Omega)}^{\mathbf{p}^-} &\leq 2c^{\mathbf{p}^-} \|w\|_{L^{\mathbf{p}(z)}(\Omega)} \frac{\left(\int_{\Omega} h(z, \xi) dz\right)^{\frac{1}{\mathbf{p}'^-}}}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \\ &\leq C \|w\|_{L^{\mathbf{p}(z)}(\Omega)} \left(\int_{\Omega} h(z, \xi) dz\right)^{\frac{1}{\mathbf{p}'^-} - \beta(z)} \\ &\leq C \|w\|_{L^{\mathbf{p}(z)}(\Omega)} \left(\int_{\Omega} (\mathcal{H}_{\tau(z)}(z) |\xi|^{\tau(z)} + h(z, 0)) dz\right)^{\frac{1}{\mathbf{p}'^-} - \beta^+} \\ &\leq C \|w\|_{L^{\mathbf{p}(z)}(\Omega)} \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z) - \tau(z)}}(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\tau}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\frac{1}{\mathbf{p}'^-} - \beta^+}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\|w\|_{L^{\mathbf{p}(z)}(\Omega)}^{\mathbf{p}^- - 1} \\ &\leq C' \begin{cases} \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z) - \tau(z)}}(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\tau}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\frac{1}{\mathbf{p}'^-} - \beta^+}, & 0 < \tau(z) < \mathbf{p}(z), \\ \left(2 \|\mathcal{H}_{\mathbf{p}(z)}\|_{L^\infty(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\mathbf{p}}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\frac{1}{\mathbf{p}'^-} - \beta^+}, & \tau(z) = \mathbf{p}(z). \end{cases} \end{aligned}$$

Since $0 < \widehat{\tau}^{\frac{1 - \beta^+ \mathbf{p}^-}{\mathbf{p}^-}} < 1$ for all $0 < \tau(z) \leq \mathbf{p}(z)$, there exists $\mathcal{R}_1 > 0$ depending on $\mathbf{p}^-$, $\tau(z)$, $\alpha(z)$, and $\beta(z)$ such that

$$\|w\|_{L^{\mathbf{p}(z)}(\Omega)} \leq \mathcal{R}_1. \quad (3.19)$$

- If $0 < \beta(z) < \alpha(z) < \frac{1}{\mathbf{p}'(z)}$, we similarly obtain

$$\|w\|_{L^{\mathbf{p}(z)}(\Omega)}^{\mathbf{p}^-} \leq C_2 C_{\Omega, \pi} \|w\|_{L^{\mathbf{p}(z)}(\Omega)} \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z) - \tau(z)}}(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\tau}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\widehat{\alpha} - \beta^+},$$

where $\pi(z) := \frac{\mathbf{p}'(z)}{1 - \alpha(z)\mathbf{p}'(z)}$ and $C_{\Omega, \pi} := 2 \max\{|\Omega|^{1/\pi^+}, |\Omega|^{1/\pi^-}\}$. Thus,

$$\|w\|_{L^{\mathbf{p}(z)}(\Omega)} \leq C'' \begin{cases} \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z) - \tau(z)}}(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\tau}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\frac{\widehat{\alpha} - \beta^+}{\mathbf{p}^- - 1}}, & 0 < \tau(z) < \mathbf{p}(z), \\ \left(2 \|\mathcal{H}_{\mathbf{p}(z)}\|_{L^\infty(\Omega)} \|\xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\mathbf{p}}} + \|h(z, 0)\|_{L^1(\Omega)}\right)^{\frac{\widehat{\alpha} - \beta^+}{\mathbf{p}^- - 1}}, & \tau(z) = \mathbf{p}(z). \end{cases}$$

Since

$$0 < \frac{\hat{\tau}(\hat{\alpha} - \beta^+)}{\mathbf{p}^- - 1} < 1,$$

there exists a positive constant $\mathcal{R}_2$ depending on $\mathbf{p}^-$, $\hat{\tau}$, $\hat{\alpha}$, and β^+ such that

$$\|w\|_{L^{\mathbf{p}}(\Omega)} \leq \mathcal{R}_2. \quad (3.20)$$

• If $0 < \beta(z) \leq \frac{1}{\mathbf{p}'(z)}$ and $\alpha(z) = \beta(z)$, we obtain

$$\|w\|_{L^{\mathbf{p}}(\Omega)} \leq \mathcal{R}_3 := C_3 C_{\pi, \Omega}, \quad (3.21)$$

where

$$\pi(z) := \frac{\mathbf{p}'(z)}{1 - \alpha(z)\mathbf{p}'(z)}, \quad \hat{\pi} \in \left[\frac{1}{\pi^-}, \frac{1}{\pi^+} \right], \quad C_{\Omega, \pi} := 2 \max\{|\Omega|^{\frac{1}{\pi^+}}, |\Omega|^{\frac{1}{\pi^-}}\}.$$

Combining (3.19)–(3.21), we deduce that for all $\tau(z) \in (0, \mathbf{p}(z)]$, there exists a constant $\mathcal{R} > 0$ such that

$$\mathcal{N}(\bar{\mathcal{B}}_{\mathcal{R}}(0)) \subset \bar{\mathcal{B}}_{\mathcal{R}}(0).$$

Step 3. The operator $\mathcal{N}$ is continuous and compact on $\bar{\mathcal{B}}_{\mathcal{R}}(0)$.

(i) *Continuity of $\mathcal{N}$.* Let $\{\xi_i\}_i \subset L^{\mathbf{p}(z)}(\Omega)$ be such that

$$\xi_i \rightarrow \xi \quad \text{strongly in } L^{\mathbf{p}(z)}(\Omega). \quad (3.22)$$

Let $w_i = \mathcal{N}(\xi_i)$ and $w = \mathcal{N}(\xi)$ be the corresponding positive weak solutions of (3.1). Using $(w_i - w)$ as a test function in their respective equations and following the same computations as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} & c^{-\mathbf{p}^-} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}^{\mathbf{p}^-} \\ & \leq \|w_i - w\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)}^{\mathbf{p}^-} \\ & \leq \frac{\int_{\Omega} (h(z, \xi_i))^{\alpha(z)} (w_i - w) dz}{\left(\int_{\Omega} h(z, \xi_i) dz\right)^{\beta(z)}} - \frac{\int_{\Omega} (h(z, \xi))^{\alpha(z)} (w_i - w) dz}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \\ & \leq \frac{\int_{\Omega} \left\{ (h(z, \xi_i))^{\alpha(z)} - (h(z, \xi))^{\alpha(z)} \right\} (w_i - w) dz}{\left(\int_{\Omega} h(z, \xi_i) dz\right)^{\beta(z)}} \\ & \quad + \left\{ \frac{1}{\left(\int_{\Omega} h(z, \xi_i) dz\right)^{\beta(z)}} - \frac{1}{\left(\int_{\Omega} h(z, \xi) dz\right)^{\beta(z)}} \right\} \int_{\Omega} (h(z, \xi))^{\alpha(z)} (w_i - w) dz. \end{aligned} \quad (3.23)$$

Since $0 < \alpha(z) \leq \frac{1}{\mathbf{p}'(z)}$, an application of Hölder's inequality yields

$$\begin{aligned} & \left| \int_{\Omega} \left\{ (h(z, \xi_i))^{\alpha(z)} - (h(z, \xi))^{\alpha(z)} \right\} (w_i - w) dz \right| \\ & \leq \int_{\Omega} |h(z, \xi_i) - h(z, \xi)|^{\alpha(z)} |w_i - w| dz \end{aligned}$$

$$\leq \left\| |h(z, \xi_i) - h(z, \xi)|^{\alpha(z)} \right\|_{L^{\mathbf{p}'(z)}(\Omega)} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}. \quad (3.24)$$

Using assumptions (1.24)–(1.25) and Lemma 2.4, we consider two cases.

Case 1. If $\alpha(z) = \frac{1}{\mathbf{p}'(z)}$, then

$$\begin{aligned} & \left| \int_{\Omega} \left\{ (h(z, \xi_i))^{\frac{1}{\mathbf{p}'(z)}} - (h(z, \xi))^{\frac{1}{\mathbf{p}'(z)}} \right\} (w_i - w) dz \right| \\ & \leq 2 \left\| |h(z, \xi_i) - h(z, \xi)|^{\frac{1}{\mathbf{p}'(z)}} \right\|_{L^{\mathbf{p}'(z)}(\Omega)} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_1 \left\| |h(z, \xi_i) - h(z, \xi)| \right\|_{L^1(\Omega)}^{\frac{1}{\mathbf{p}'}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_1 \left(\int_{\Omega} \mathcal{H}_{\tau(z)}(z) |\xi_i - \xi|^{\tau(z)} dz \right)^{\frac{1}{\mathbf{p}'}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}. \end{aligned} \quad (3.25)$$

Then, applying Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned} & \left| \int_{\Omega} \left\{ (h(z, \xi_i))^{\frac{1}{\mathbf{p}'(z)}} - (h(z, \xi))^{\frac{1}{\mathbf{p}'(z)}} \right\} (w_i - w) dz \right| \\ & \leq C_1 \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}-\tau(z)}}(\Omega)} \|\xi_i - \xi\|_{L^{\frac{\mathbf{p}(z)}{\tau(z)}}(\Omega)}^{\tau(z)} \right)^{\frac{1}{\mathbf{p}'}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_1 \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}-\tau(z)}}(\Omega)} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\widehat{\tau}} \right)^{\frac{1}{\mathbf{p}'}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq \begin{cases} C_1 \left(c \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}-\tau(z)}}(\Omega)}^{\frac{1}{\mathbf{p}'}} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\frac{\widehat{\tau}}{\mathbf{p}'}} \right) \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}, & \text{if } \tau(z) \in (0, \mathbf{p}(z)), \\ C_1 \left(c \|\mathcal{H}_{\tau(z)}\|_{L^{\infty}(\Omega)}^{\frac{1}{\mathbf{p}'}} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\frac{\widehat{\mathbf{p}}}{\mathbf{p}'}} \right) \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}, & \text{if } \tau(z) = \mathbf{p}(z). \end{cases} \end{aligned} \quad (3.26)$$

Case 2. And if $\alpha(z) \in (0, \frac{1}{\mathbf{p}'(z)})$,

$$\begin{aligned} & \left| \int_{\Omega} \left\{ (h(z, \xi_i))^{\alpha(z)} - (h(z, \xi))^{\alpha(z)} \right\} (w_i - w) dz \right| \\ & \leq 3 \|1\|_{L^{\pi(z)}(\Omega)} \left\| |h(z, \xi_i) - h(z, \xi)|^{\alpha(z)} \right\|_{L^{\frac{1}{\alpha(z)}}(\Omega)} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq 3 \|1\|_{L^{\pi(z)}(\Omega)} \left\| |h(z, \xi_i) - h(z, \xi)|^{\alpha(z)} \right\|_{L^{\frac{1}{\alpha(z)}}(\Omega)} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_2 \| |h(z, \xi_i) - h(z, \xi)| \|_{L^1(\Omega)}^{\widehat{\alpha}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_2 \left(\int_{\Omega} |h(z, \xi_i) - h(z, \xi)| dz \right)^{\widehat{\alpha}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \end{aligned}$$

$$\leq C_2 \left(\int_{\Omega} \mathcal{H}_{\tau(z)}(z) |\xi_i - \xi|^{\tau(z)} dz \right)^{\hat{\alpha}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}.$$

Then, applying Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned} & \left| \int_{\Omega} \left\{ (h(z, \xi_i))^{\alpha(z)} - (h(z, \xi))^{\alpha(z)} \right\} (w_i - w) dz \right| \\ & \leq C_2 \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)} \|\xi_i - \xi\|^{\tau(z)}_{L^{\frac{\mathbf{p}(z)}{\tau(z)}}(\Omega)} \right)^{\hat{\alpha}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq C_2 \left(2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)} \|\xi_i - \xi\|_{L^{\mathbf{p}(\Omega)}}^{\hat{\tau}} \right)^{\hat{\alpha}} \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)} \\ & \leq \begin{cases} C_2 \left(c_2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)}^{\hat{\alpha}} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\hat{\tau}\hat{\alpha}} \right) \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}, & \text{if } \tau(z) \in (0, \mathbf{p}(z)), \\ C_2 \left(c_2 \|\mathcal{H}_{\tau(z)}\|_{L^{\infty}(\Omega)}^{\hat{\alpha}} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\hat{\alpha}\hat{\mathbf{p}}} \right) \|w_i - w\|_{L^{\mathbf{p}(z)}(\Omega)}, & \text{if } \tau(z) = \mathbf{p}(z). \end{cases} \end{aligned} \quad (3.27)$$

On the other hand, using (1.23)-(1.25), by Hölder's inequality and Lemma 2.4, we deduce that

$$\begin{aligned} & \left| \int_{\Omega} h(z, \xi_i) dz - \int_{\Omega} h(z, \xi) dz \right| \\ & \leq \int_{\Omega} |h(z, \xi_i) - h(z, \xi)| dz \\ & \leq \int_{\Omega} \mathcal{H}_{\tau(z)}(z) |\xi_i - \xi|^{\tau(z)} dz \\ & \leq 2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)} \|\xi_i - \xi\|^{\tau(z)}_{L^{\frac{\mathbf{p}(z)}{\tau(z)}}(\Omega)} \\ & \leq 2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\hat{\tau}} \\ & \leq \begin{cases} 2 \|\mathcal{H}_{\tau(z)}\|_{L^{\frac{\mathbf{p}(z)}{\mathbf{p}(z)-\tau(z)}}(\Omega)} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\hat{\tau}} & \text{if } \tau(z) \in (0, \mathbf{p}(z)), \\ 2 \|\mathcal{H}_{\mathbf{p}(z)}\|_{L^{\infty}(\Omega)} \|\xi_i - \xi\|_{L^{\mathbf{p}(z)}(\Omega)}^{\hat{\mathbf{p}}} & \text{if } \tau(z) = \mathbf{p}(z). \end{cases} \end{aligned} \quad (3.28)$$

Now, by combining inequalities (3.24)-(3.28), and taking the limit as $i \rightarrow +\infty$ in (3.23), we obtain:

$$w_i \rightarrow w \text{ strongly in } L^{\mathbf{p}(z)}(\Omega).$$

This establishes the continuity of the operator $\mathcal{N}$. Applying the same arguments with Lemma 2.7, we deduce the continuity of $\mathcal{N}$ for the case $1 < \mathbf{p}^- \leq \mathbf{p}(z, y) \leq \mathbf{p}^+ < 2$. The proof follows similarly and is omitted for brevity.

(ii) *Compactness of $\mathcal{N}$* . Let $w = \mathcal{N}(\xi)$ for some function $\xi \in \overline{\mathcal{B}_{\mathcal{R}}}(0)$. Following the same reasoning as in Step 2, we derive the existence of a positive constant C , depending only on the data, such that:

$$\|w\|_{\mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)} \leq C.$$

Since, by Proposition 2.1, the set $\mathcal{N}(\overline{\mathcal{B}_{\mathcal{R}}}(0))$ is relatively compact in $L^{\mathbf{p}(z)}(\Omega)$, the compactness of $\mathcal{N}$ follows.

To establish the existence of a solution, we invoke the Schauder fixed-point theorem. Thus, we obtain the existence of $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$, which is a solution to Problem (1.17). $\square$

Remark 3.1. (H_1) If assumptions (1.23)–(1.24) hold and one of the following conditions is satisfied:

- $\alpha(z) \in \left[0, \frac{1}{\mathbf{p}'(z)}\right]$, $\beta(z) \leq 0$, $\tau(z) \in (0, \mathbf{p}(z)]$, and $0 < \frac{\widehat{\tau}(\widehat{\alpha} - \beta^+)}{\mathbf{p}^- - 1} < 1$;
- $\alpha(z) \in \left(0, \frac{1}{\mathbf{p}'(z)}\right]$, $\beta(z) \leq 0$, $\tau(z) \in (0, \mathbf{p}(z))$, $\frac{\widehat{\tau}(\widehat{\alpha} - \beta^+)}{\mathbf{p}^- - 1} = 1$, and $CC_{\pi, \Omega}^{\mathbf{p}^- - 1} \|\mathcal{H}_{\tau}\|_{L^{\frac{\widehat{\alpha} - \beta^+}{\mathbf{p}^- - 1}}(\Omega)} < 1$;
- $\alpha(z) \in \left(0, \frac{1}{\mathbf{p}'(z)}\right]$, $\beta(z) \leq 0$, $\tau(z) = \mathbf{p}(z)$, $\frac{\widehat{\tau}(\widehat{\alpha} - \beta^+)}{\mathbf{p}^- - 1} = 1$, and $CC_{\pi, \Omega}^{\mathbf{p}^- - 1} \|\mathcal{H}_{\tau}\|_{L^{\frac{\widehat{\alpha} - \beta^+}{\mathbf{p}^- - 1}}(\Omega)} < 1$;
- $\alpha(z) = 0$, $\beta(z) = -\frac{\mathbf{p}(z) - 1}{\tau(z)}$, $\tau(z) \in (0, \mathbf{p}(z))$, and $CC_{\pi, \Omega}^{\mathbf{p}^- - 1} \|\mathcal{H}_{\tau}\|_{L^{\frac{1}{\mathbf{p}(z) - \tau(z)}}(\Omega)}^{\frac{1}{\tau}} < 1$;
- $\alpha(z) = 0$, $\beta(z) = -\frac{\mathbf{p}(z) - 1}{\mathbf{p}(z)}$, $\tau(z) = \mathbf{p}(z)$, and $CC_{\pi, \Omega}^{\mathbf{p}^- - 1} \|\mathcal{H}_{\tau}\|_{L^{\frac{1}{\mathbf{p}}}} < 1$.

Then problem (1.17) admits a nonnegative weak solution $w \in \mathfrak{W}_0^{s, \mathbf{p}(z, y)}(\Omega)$.

(H_2) All the existence results stated in (H_1) can equivalently be obtained following the approach developed in Theorem 2.2.

4. Numerical illustration and simulation

To complement the theoretical analysis, we present a two-dimensional numerical experiment illustrating the qualitative behavior of the solution. The computation concerns a simplified constant-exponent case, namely $\mathbf{p} = 2$, while retaining the nonlinear and nonlocal structure of the source term. This experiment is intended as an illustration of the analytical model rather than a complete numerical validation of the general variable-exponent problem.

4.1. Mathematical formulation

Let $\Omega = (0, 1)^2 \subset \mathbb{R}^2$. We consider the fractional problem

$$\left(\int_{\Omega} h(z, w(z)) \, dz \right)^{\beta} (-\Delta)^s w(z) = [h(z, w(z))]^{\alpha}, \quad z \in \Omega, \quad (4.1)$$

subject to the homogeneous exterior condition

$$w(z) = 0, \quad z \in \mathbb{R}^2 \setminus \Omega. \quad (4.2)$$

For the numerical experiment, we take

$$s = 0.45, \quad \alpha = 0.8, \quad \beta = 0.5.$$

The nonlinear auxiliary function is defined by

$$h(z, w) = 1 + \gamma e\left(-\frac{\|z - z_c\|^2}{2\sigma^2}\right) + \eta w, \quad (4.3)$$

where

$$\gamma = 2, \quad \sigma = 0.15, \quad \eta = 0.3, \quad z_c = (0.5, 0.5).$$

The Gaussian term describes a spatially localized excitation centered at z_c , whereas the term ηw introduces a weak linear feedback. The constant term helps maintain the positivity of $h(z, w)$ over the range of solution values encountered in the computation.

For sufficiently regular functions, the fractional Laplacian is formally defined by

$$(-\Delta)^s w(z) = C_{2,s} \text{P.V.} \int_{\mathbb{R}^2} \frac{w(z) - w(\zeta)}{\|z - \zeta\|^{2+2s}} d\zeta, \quad (4.4)$$

where $C_{2,s} > 0$ is a normalization constant and P.V. denotes the Cauchy principal value.

For the numerical illustration, the integral operator is replaced by a finite-horizon discrete approximation. Let $\{z_i\}_{i=1}^{N^2}$ denote the nodes of a uniform Cartesian grid on Ω , and let ΔV be the area associated with each grid cell. At an interior grid point z_i , we set

$$(A_R w)_i = C_{2,s} \sum_{\substack{j \neq i \\ 0 < \|z_i - z_j\| \leq R}} \frac{w_i - w_j}{\|z_i - z_j\|^{2+2s}} \Delta V, \quad (4.5)$$

where $R > 0$ is the prescribed interaction radius. The singular diagonal contribution $j = i$ is excluded in accordance with the principal-value interpretation of the fractional operator.

The operator A_R is a truncated, finite-horizon approximation of $(-\Delta)^s$. It retains interactions between grid points separated by distances not exceeding R , thereby reducing the computational cost of evaluating the full integral operator. Consequently, the interaction radius R influences both the numerical cost and the accuracy with which the discrete operator represents the fractional Laplacian.

4.2. Discrete formulation and iterative solution

4.2.1. Spatial grid and nodal representation

Let

$$\Omega = (0, 1)^2$$

and denote a generic spatial point by

$$\xi = (z, y) \in \Omega.$$

The domain is partitioned using a uniform Cartesian grid containing N nodes in each coordinate direction. The grid spacing is

$$h = \frac{1}{N-1}, \quad (4.6)$$

and the nodal coordinates are

$$z_i = ih, \quad y_j = jh, \quad i, j = 0, \dots, N-1. \quad (4.7)$$

The corresponding grid points are

$$\xi_{i,j} = (z_i, y_j) = (ih, jh). \quad (4.8)$$

The area associated with a grid cell is

$$\Delta V = h^2. \quad (4.9)$$

For computational purposes, the pair of indices (i, j) is mapped to a single global index

$$k = i + Nj + 1, \quad k = 1, \dots, N^2. \quad (4.10)$$

The inverse mapping associates each global index k with a unique pair (i, j) . After this reindexing, the spatial node $\xi_{i,j}$ is denoted by ξ_k , and the discrete solution vector is written as

$$\mathbf{w} = (w_1, \dots, w_{N^2})^\top \in \mathbb{R}^{N^2}, \quad w_k \approx \mathbf{w}(\xi_k). \quad (4.11)$$

The homogeneous exterior Dirichlet condition is imposed by extending the numerical solution by zero outside Ω :

$$\mathbf{w}(\xi) = 0, \quad \xi \in \mathbb{R}^2 \setminus \Omega. \quad (4.12)$$

Unlike a classical boundary condition prescribed only on $\partial\Omega$, condition (4.12) is specified on the entire complement of Ω . This distinction is essential for nonlocal operators because an interior point interacts with points lying outside the computational domain.

4.2.2. Finite-horizon approximation of the fractional operator

For a sufficiently regular function, the fractional Laplacian in $\mathbb{R}^2$ is formally defined by

$$(-\Delta)^s \mathbf{w}(\xi) = C_{2,s} \text{P.V.} \int_{\mathbb{R}^2} \frac{\mathbf{w}(\xi) - \mathbf{w}(\zeta)}{\|\xi - \zeta\|^{2+2s}} d\zeta, \quad (4.13)$$

where $s \in (0, 1)$, P.V. denotes the Cauchy principal value, and $C_{2,s} > 0$ is the normalization constant

$$C_{2,s} = \frac{4^s \Gamma(1+s)}{\pi |\Gamma(-s)|}. \quad (4.14)$$

To reduce the computational cost, the integral in (4.13) is truncated to interactions whose distance does not exceed a prescribed radius $R > 0$. The resulting finite-horizon operator is

$$(-\Delta)_R^s \mathbf{w}(\xi) = C_{2,s} \text{P.V.} \int_{\|\xi - \zeta\| \leq R} \frac{\mathbf{w}(\xi) - \mathbf{w}(\zeta)}{\|\xi - \zeta\|^{2+2s}} d\zeta. \quad (4.15)$$

The interaction radius determines the neighborhood

$$\mathcal{B}_R(\xi) = \{\zeta \in \mathbb{R}^2 : 0 < \|\xi - \zeta\| \leq R\}. \quad (4.16)$$

For two distinct interior grid points ξ_k and ξ_ℓ , define the discrete interaction coefficient

$$K_{k\ell} = \begin{cases} C_{2,s} \frac{\Delta V}{\|\xi_k - \xi_\ell\|^{2+2s}}, & 0 < \|\xi_k - \xi_\ell\| \leq R, \\ 0, & \|\xi_k - \xi_\ell\| > R. \end{cases} \quad (4.17)$$

The singular term corresponding to $\ell = k$ is excluded, consistently with the principal-value interpretation of the continuous operator.

The discrete interaction neighborhood of ξ_k is

$$\mathcal{N}_R(k) = \{\ell \in \{1, \dots, N^2\} : \ell \neq k, \|\xi_k - \xi_\ell\| \leq R\}. \quad (4.18)$$

Using this notation, the interior contribution to the operator becomes

$$\sum_{\ell \in \mathcal{N}_R(k)} K_{k\ell}(w_k - w_\ell). \quad (4.19)$$

Because of the exterior condition (4.12), interactions between ξ_k and points in $\mathbb{R}^2 \setminus \Omega$ produce an additional diagonal contribution. Formally, this contribution is

$$B_k = C_{2,s} \int_{\substack{\zeta \in \mathbb{R}^2 \setminus \Omega \\ \|\xi_k - \zeta\| \leq R}} \frac{1}{\|\xi_k - \zeta\|^{2+2s}} d\zeta. \quad (4.20)$$

Indeed, since $w(\zeta) = 0$ outside Ω , the exterior part of the truncated integral is

$$C_{2,s} \int_{\substack{\zeta \in \mathbb{R}^2 \setminus \Omega \\ \|\xi_k - \zeta\| \leq R}} \frac{w_k - w(\zeta)}{\|\xi_k - \zeta\|^{2+2s}} d\zeta = B_k w_k. \quad (4.21)$$

In the numerical implementation, B_k may be approximated by introducing zero-valued exterior grid points:

$$B_k^h = C_{2,s} \Delta V \sum_{\substack{\xi_\ell^{\text{ext}} \notin \Omega \\ 0 < \|\xi_k - \xi_\ell^{\text{ext}}\| \leq R}} \frac{1}{\|\xi_k - \xi_\ell^{\text{ext}}\|^{2+2s}}. \quad (4.22)$$

For simplicity, the same notation B_k is used below for the numerical approximation of the exterior coefficient.

The complete finite-horizon discrete operator is therefore

$$(A_R \mathbf{w})_k = \sum_{\ell \in \mathcal{N}_R(k)} K_{k\ell}(w_k - w_\ell) + B_k w_k. \quad (4.23)$$

Equivalently,

$$(A_R \mathbf{w})_k = \left(\sum_{\ell \in \mathcal{N}_R(k)} K_{k\ell} + B_k \right) w_k - \sum_{\ell \in \mathcal{N}_R(k)} K_{k\ell} w_\ell. \quad (4.24)$$

The entries of the matrix $A_R \in \mathbb{R}^{N^2 \times N^2}$ are thus given by

$$(A_R)_{k\ell} = \begin{cases} \sum_{\substack{m=1 \\ m \neq k}}^{N^2} K_{km} + B_k, & k = \ell, \\ -K_{k\ell}, & k \neq \ell. \end{cases} \quad (4.25)$$

Since the kernel depends only on the distance between two nodes,

$$K_{k\ell} = K_{\ell k}, \quad (4.26)$$

and consequently

$$A_R^\top = A_R. \quad (4.27)$$

For every vector $\mathbf{v} \in \mathbb{R}^{N^2}$, the quadratic form associated with A_R satisfies

$$\mathbf{v}^\top A_R \mathbf{v} = \frac{1}{2} \sum_{k=1}^{N^2} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N^2} K_{k\ell} (v_k - v_\ell)^2 + \sum_{k=1}^{N^2} B_k v_k^2. \quad (4.28)$$

Because $K_{k\ell} \geq 0$ and $B_k \geq 0$, it follows that

$$\mathbf{v}^\top A_R \mathbf{v} \geq 0. \quad (4.29)$$

If the exterior interaction connects the interior grid to the zero-valued exterior region sufficiently strongly, then

$$\mathbf{v}^\top A_R \mathbf{v} > 0 \quad \text{for every } \mathbf{v} \neq \mathbf{0}, \quad (4.30)$$

and A_R is symmetric positive definite. This property justifies the use of the conjugate-gradient method.

If the exterior contribution is omitted by setting $B_k = 0$, then the discrete operator represents a regional finite-horizon operator. In that case, constant vectors satisfy

$$A_R \mathbf{1} = \mathbf{0},$$

and the matrix may be singular. Therefore, the exterior contribution or another suitable boundary treatment is required when PCG is used to solve the Dirichlet problem.

4.2.3. Discrete nonlinear and global nonlocal terms

At the grid point ξ_k , the nonlinear auxiliary function is approximated by

$$H_k(\mathbf{w}) = 1 + \gamma e \left(-\frac{\|\xi_k - \xi_c\|^2}{2\sigma^2} \right) + \eta w_k, \quad (4.31)$$

where

$$\xi_c = (0.5, 0.5)$$

is the center of the Gaussian excitation. Define

$$G_k = \gamma e \left(-\frac{\|\xi_k - \xi_c\|^2}{2\sigma^2} \right). \quad (4.32)$$

Then

$$H_k(\mathbf{w}) = 1 + G_k + \eta w_k. \quad (4.33)$$

The discrete vector associated with the auxiliary function is

$$\mathbf{H}(\mathbf{w}) = (H_1(\mathbf{w}), \dots, H_{N^2}(\mathbf{w}))^\top. \quad (4.34)$$

The global integral appearing in the continuous problem is approximated by the nodal quadrature formula

$$\mathcal{I}_h(\mathbf{w}) = \Delta V \sum_{k=1}^{N^2} H_k(\mathbf{w}). \quad (4.35)$$

Using (4.33), this quantity can be expanded as

$$\mathcal{I}_h(\mathbf{w}) = \Delta V \left[N^2 + \sum_{k=1}^{N^2} G_k + \eta \sum_{k=1}^{N^2} w_k \right]. \quad (4.36)$$

Equation (4.36) makes the global coupling explicit. A change in one component w_k alters $\mathcal{I}_h(\mathbf{w})$ and therefore changes every component of the discrete right-hand side.

Provided that

$$H_k(\mathbf{w}) > 0, \quad k = 1, \dots, N^2, \quad (4.37)$$

one has

$$\mathcal{I}_h(\mathbf{w}) > 0. \quad (4.38)$$

The fractional powers in the nonlinear source are consequently well-defined.

The discrete right-hand side is defined componentwise by

$$F_k(\mathbf{w}) = \frac{[H_k(\mathbf{w})]^\alpha}{[\mathcal{I}_h(\mathbf{w})]^\beta}, \quad k = 1, \dots, N^2. \quad (4.39)$$

Introducing the global scaling factor

$$\Lambda_h(\mathbf{w}) = [\mathcal{I}_h(\mathbf{w})]^{-\beta}, \quad (4.40)$$

the right-hand side can be written as

$$F_k(\mathbf{w}) = \Lambda_h(\mathbf{w}) [H_k(\mathbf{w})]^\alpha. \quad (4.41)$$

In vector notation,

$$\mathbf{F}(\mathbf{w}) = \Lambda_h(\mathbf{w}) \mathbf{H}(\mathbf{w})^{[\alpha]}, \quad (4.42)$$

where

$$\mathbf{H}(\mathbf{w})^{[\alpha]} = (H_1(\mathbf{w})^\alpha, \dots, H_{N^2}(\mathbf{w})^\alpha)^\top. \quad (4.43)$$

The nonlinear discrete problem is therefore

$$A_R \mathbf{w} = \mathbf{F}(\mathbf{w}) = \frac{\mathbf{H}(\mathbf{w})^{[\alpha]}}{[\mathcal{I}_h(\mathbf{w})]^\beta}. \quad (4.44)$$

If A_R is invertible, this problem is equivalent to the fixed-point equation

$$\mathbf{w} = \mathcal{T}_h(\mathbf{w}), \quad \mathcal{T}_h(\mathbf{w}) = A_R^{-1} \mathbf{F}(\mathbf{w}). \quad (4.45)$$

The dependence of the right-hand side on $\mathbf{w}$ can be examined through its partial derivatives. Since

$$\frac{\partial H_k}{\partial w_j} = \eta \delta_{kj}, \quad (4.46)$$

where δ_{kj} is the Kronecker delta, one obtains

$$\frac{\partial \mathcal{I}_h}{\partial w_j} = \eta \Delta V. \quad (4.47)$$

Consequently,

$$\begin{aligned} \frac{\partial F_k}{\partial w_j} &= \alpha \eta \delta_{kj} \frac{H_k(\mathbf{w})^{\alpha-1}}{\mathcal{I}_h(\mathbf{w})^\beta} \\ &\quad - \beta \eta \Delta V \frac{H_k(\mathbf{w})^\alpha}{\mathcal{I}_h(\mathbf{w})^{\beta+1}}. \end{aligned} \quad (4.48)$$

The first term in (4.48) is local, whereas the second term is present for every pair (k, j) . This second contribution is the discrete manifestation of the global nonlocal coupling.

4.2.4. Relaxed Picard linearization

The nonlinear system (4.44) is solved using a relaxed Picard method. Beginning with an initial approximation

$$\mathbf{w}^{(0)} = \mathbf{0}, \quad (4.49)$$

the nonlinear terms are evaluated using the current iterate $\mathbf{w}^{(n)}$. Thus,

$$H_k^{(n)} = 1 + G_k + \eta w_k^{(n)}, \quad (4.50)$$

and

$$\mathcal{I}_h^{(n)} = \Delta V \sum_{k=1}^{N^2} H_k^{(n)}. \quad (4.51)$$

The corresponding right-hand side is

$$f_k^{(n)} = \frac{\left(H_k^{(n)}\right)^\alpha}{\left(\mathcal{I}_h^{(n)}\right)^\beta}. \quad (4.52)$$

A provisional approximation $\tilde{\mathbf{w}}^{(n+1)}$ is obtained by solving

$$A_R \tilde{\mathbf{w}}^{(n+1)} = \mathbf{F}(\mathbf{w}^{(n)}). \quad (4.53)$$

Equivalently,

$$\tilde{\mathbf{w}}^{(n+1)} = \mathcal{T}_h(\mathbf{w}^{(n)}). \quad (4.54)$$

The relaxed update is then defined by

$$\mathbf{w}^{(n+1)} = (1 - \omega) \mathbf{w}^{(n)} + \omega \tilde{\mathbf{w}}^{(n+1)}, \quad (4.55)$$

where

$$0 < \omega \leq 1.$$

For the present computation,

$$\omega = 0.9. \quad (4.56)$$

Combining (4.54) and (4.55) gives

$$\mathbf{w}^{(n+1)} = \mathcal{T}_{h,\omega}(\mathbf{w}^{(n)}), \quad (4.57)$$

where

$$\mathcal{T}_{h,\omega}(\mathbf{v}) = (1 - \omega)\mathbf{v} + \omega\mathcal{T}_h(\mathbf{v}). \quad (4.58)$$

The choice $\omega < 1$ damps abrupt changes between consecutive Picard iterates and can improve stability when the nonlinear feedback is relatively strong.

Subtracting $\mathbf{w}^{(n)}$ from (4.55) yields

$$\mathbf{w}^{(n+1)} - \mathbf{w}^{(n)} = \omega \left(\tilde{\mathbf{w}}^{(n+1)} - \mathbf{w}^{(n)} \right). \quad (4.59)$$

Thus, the relaxation parameter scales the unrelaxed fixed-point correction directly.

4.2.5. Preconditioned conjugate-gradient solution

At every Picard step, the matrix A_R remains unchanged and only the right-hand side varies. This makes it possible to assemble and precondition the matrix once and reuse it throughout the nonlinear iteration.

Let M be a symmetric positive-definite preconditioner approximating A_R . The preconditioned linear system is formally

$$M^{-1}A_R\tilde{\mathbf{w}}^{(n+1)} = M^{-1}\mathbf{F}(\mathbf{w}^{(n)}). \quad (4.60)$$

A diagonal preconditioner is given by

$$M_J = \text{diag}(A_R), \quad (4.61)$$

whereas an incomplete Cholesky factorization has the form

$$A_R \approx LL^T. \quad (4.62)$$

In the latter case, one may take

$$M = LL^T.$$

If $\tilde{\mathbf{w}}_q^{(n+1)}$ denotes the q th PCG approximation, the corresponding linear residual is

$$\mathbf{r}_q^{(n)} = \mathbf{F}(\mathbf{w}^{(n)}) - A_R\tilde{\mathbf{w}}_q^{(n+1)}. \quad (4.63)$$

The PCG iteration is terminated when

$$\frac{\|\mathbf{r}_q^{(n)}\|_2}{\max\{1, \|\mathbf{F}(\mathbf{w}^{(n)})\|_2\}} \leq \varepsilon_{\text{PCG}}. \quad (4.64)$$

In the numerical implementation, we use

$$\varepsilon_{\text{PCG}} = 10^{-10}. \quad (4.65)$$

This inner tolerance is chosen smaller than the nonlinear tolerances so that the error produced by the linear solver does not dominate the Picard iteration.

4.2.6. Nonlinear convergence indicators

Two complementary quantities are monitored to assess convergence. The relative change between consecutive iterates is

$$E_{\text{inc}}^{(n+1)} = \frac{\|\mathbf{w}^{(n+1)} - \mathbf{w}^{(n)}\|_2}{\max\{1, \|\mathbf{w}^{(n+1)}\|_2\}}. \quad (4.66)$$

This quantity measures the stabilization of the sequence $\{\mathbf{w}^{(n)}\}$, but it does not by itself establish that the nonlinear algebraic equation is satisfied.

The nonlinear residual vector is defined by

$$\mathbf{R}_h(\mathbf{w}) = A_R \mathbf{w} - \mathbf{F}(\mathbf{w}). \quad (4.67)$$

The corresponding relative residual is

$$E_{\text{res}}^{(n+1)} = \frac{\|\mathbf{R}_h(\mathbf{w}^{(n+1)})\|_2}{\max\{1, \|\mathbf{F}(\mathbf{w}^{(n+1)})\|_2\}}. \quad (4.68)$$

Equivalently,

$$E_{\text{res}}^{(n+1)} = \frac{\|A_R \mathbf{w}^{(n+1)} - \mathbf{F}(\mathbf{w}^{(n+1)})\|_2}{\max\{1, \|\mathbf{F}(\mathbf{w}^{(n+1)})\|_2\}}. \quad (4.69)$$

The iteration is terminated only when both conditions

$$E_{\text{inc}}^{(n+1)} \leq \varepsilon_{\text{Picard}} \quad \text{and} \quad E_{\text{res}}^{(n+1)} \leq \varepsilon_{\text{res}} \quad (4.70)$$

are satisfied. The values used in the reference computation are

$$\varepsilon_{\text{Picard}} = 10^{-4}, \quad \varepsilon_{\text{res}} = 10^{-6}. \quad (4.71)$$

The complete nonlinear update can be summarized without introducing a separate stepwise list by the recursion

$$H_k^{(n)} = 1 + G_k + \eta w_k^{(n)}, \quad (4.72)$$

$$\mathcal{I}_h^{(n)} = \Delta V \sum_{k=1}^{N^2} H_k^{(n)}, \quad (4.73)$$

$$f_k^{(n)} = \frac{\left(H_k^{(n)}\right)^\alpha}{\left(\mathcal{I}_h^{(n)}\right)^\beta}, \quad (4.74)$$

$$A_R \tilde{\mathbf{w}}^{(n+1)} = \mathbf{f}^{(n)}, \quad (4.75)$$

$$\mathbf{w}^{(n+1)} = (1 - \omega) \mathbf{w}^{(n)} + \omega \tilde{\mathbf{w}}^{(n+1)}. \quad (4.76)$$

After evaluating $E_{\text{inc}}^{(n+1)}$ and $E_{\text{res}}^{(n+1)}$, the recursion (4.72)–(4.76) is repeated until (4.70) is fulfilled or the maximum number of Picard iterations is reached.

Table 1. Reference parameters used in the two-dimensional simulation.

Parameter	Value	Description
Ω	$(0, 1)^2$	Computational domain
s	0.45	Fractional order
α	0.8	Local nonlinear exponent
β	0.5	Nonlocal normalization exponent
γ	2	Gaussian amplitude
σ	0.15	Gaussian width
η	0.3	Linear feedback coefficient
z_c	(0.5, 0.5)	Center of the localized source
N	50	Grid points in each direction
R	0.25	Interaction radius
ω	0.9	Picard relaxation parameter
$\varepsilon_{\text{Picard}}$	10^{-4}	Picard increment tolerance

4.3. Computational configuration and reproducibility

All computations are performed in MATLAB. The principal program is named

`nonlocal_fractional_2D_simulation.m`.

The reference parameter configuration is reported in Table 1.

For reproducibility, the implementation should record the following quantities at each non-linear iteration:

$$n, \quad E_{\text{inc}}^{(n)}, \quad E_{\text{res}}^{(n)}, \quad \mathcal{I}_h(\mathbf{w}^{(n)}),$$

together with the number of PCG iterations and the corresponding linear residual.

4.4. Numerical results and quantitative assessment

Spatial structure of the computed solution

Figures 1 and 2 present complementary visualizations of the converged numerical solution. The surface plot describes its amplitude and spatial localization, whereas the contour and heat-map representations provide a clearer view of its level sets and symmetry. The localized Gaussian excitation is also displayed to facilitate comparison between the spatial distribution of the source and the resulting solution.

The numerical solution is expected to attain its maximum near

$$z_c = (0.5, 0.5),$$

where the Gaussian excitation is concentrated. Figures 1(a) and 2(a) can be used to compare the position of the numerical maximum with the source center displayed in Figure 2(b).

Away from the source center, the solution amplitude is expected to decrease toward the boundary. Moreover, because the domain, the source location, and the interaction kernel are

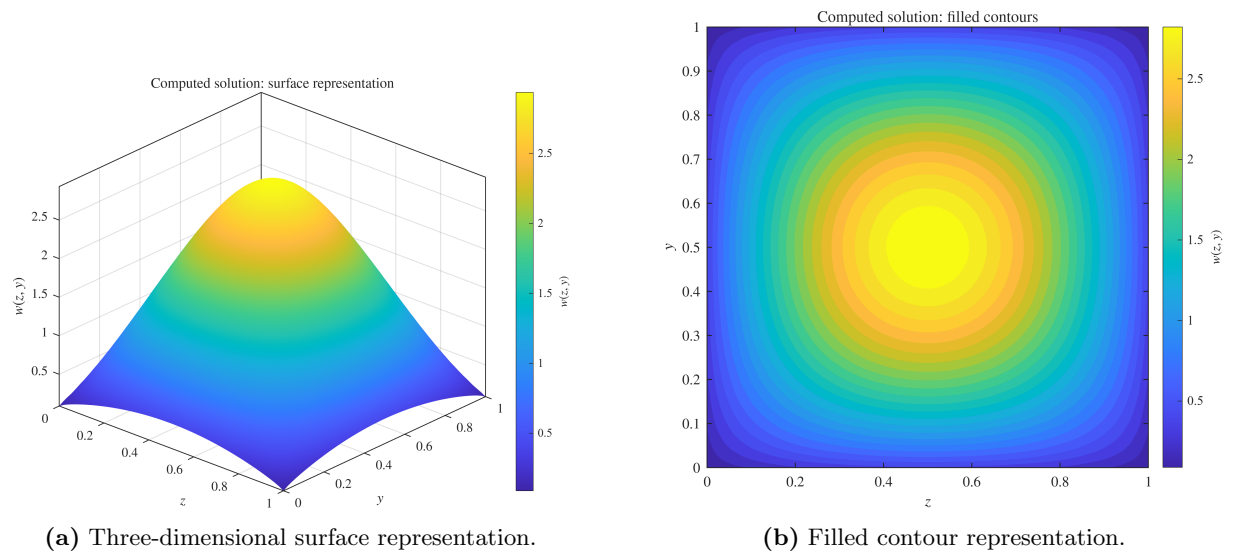


Figure 1. Principal representations of the computed solution: (a) Three-dimensional surface plot of $w(x, y)$; (b) filled contour plot showing its level sets.

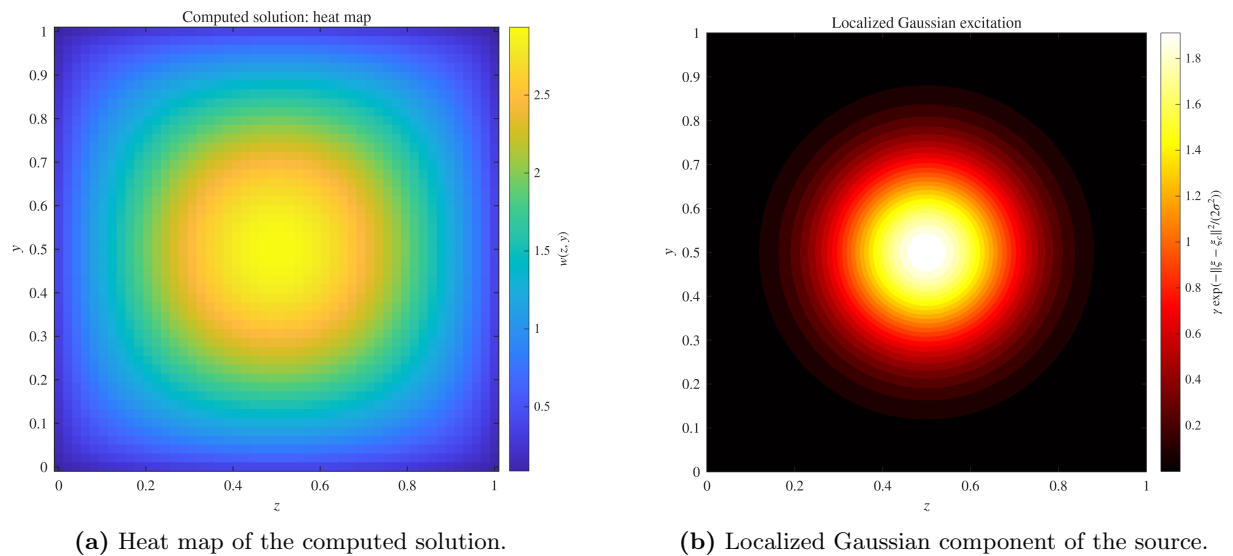


Figure 2. Comparison between the solution and the spatial excitation: (a) Heat map of $w(x, y)$; (b) localized Gaussian excitation centered at $z_c = (0.5, 0.5)$.

symmetric with respect to the lines

$$x = 0.5 \quad \text{and} \quad y = 0.5,$$

the computed solution should exhibit approximately symmetric level sets, as shown in Figures 1(b) and 2(a). These qualitative observations are examined quantitatively through the cross-sectional and symmetry indicators introduced below.

Cross-sectional and radial profiles

To examine the spatial decay more precisely, we consider the two central cross-sections

$$P_x(x) = w(x, 0.5), \quad P_y(y) = w(0.5, y). \quad (4.77)$$

The agreement between P_x and P_y provides a numerical measure of the reflection symmetry of the computed solution.

To describe the decay with respect to the distance from the source center, we also introduce a discrete radial average. Let

$$\mathcal{S}_m = \left\{ z_k : r_m - \frac{\Delta r}{2} \leq \|z_k - z_c\| < r_m + \frac{\Delta r}{2} \right\} \quad (4.78)$$

denote the set of grid points belonging to the radial shell centered at r_m . The corresponding radial profile is defined by

$$P_r(r_m) = \frac{1}{\#\mathcal{S}_m} \sum_{z_k \in \mathcal{S}_m} w_k, \quad (4.79)$$

where $\#\mathcal{S}_m$ denotes the number of grid points contained in $\mathcal{S}_m$.

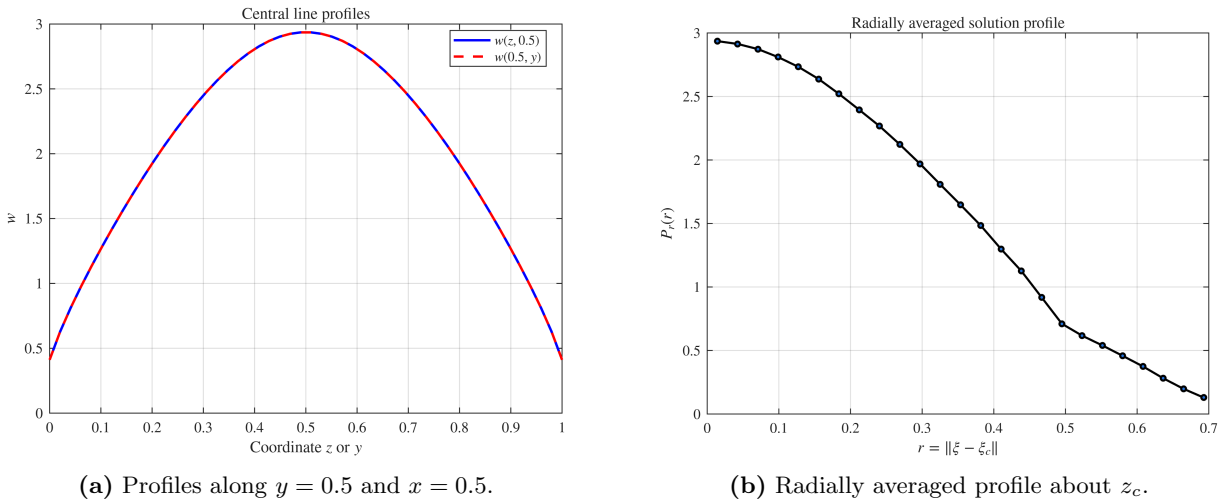


Figure 3. Cross-sectional and radial behavior of the converged solution: (a) Central horizontal and vertical profiles; (b) radially averaged profile with respect to $z_c = (0.5, 0.5)$.

Figure 3(a) compares the two central profiles, whereas Figure 3(b) describes the average variation of the solution with respect to the distance from the source center.

The relative midline discrepancy is defined by

$$E_{\text{sym}} = \frac{\|P_x - P_y\|_2}{\max\{1, \|P_x\|_2\}}. \quad (4.80)$$

A small value of E_{sym} indicates that the numerical discretization approximately preserves the reflection symmetry of the domain and the localized source.

The radial decay may also be examined through the discrete differences

$$D_m = P_r(r_{m+1}) - P_r(r_m). \quad (4.81)$$

Within the numerical tolerance, a nonincreasing radial profile is characterized by

$$D_m \leq \varepsilon_{\text{mon}} \quad \text{for all admissible } m, \quad (4.82)$$

where $\varepsilon_{\text{mon}} > 0$ accounts for discretization and radial binning errors.

Convergence of the nonlinear iteration

Figure 4 presents the convergence history of the relaxed Picard–PCG method. The first panel reports the relative iterate increment and nonlinear residual, while the second panel shows the number of PCG iterations required to solve each linearized system.

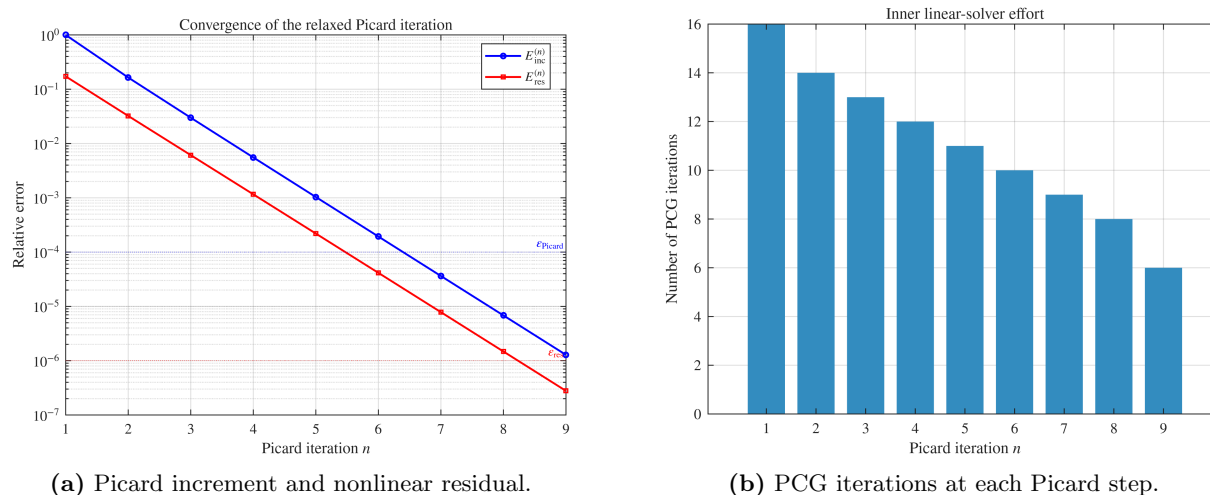


Figure 4. Convergence diagnostics for the iterative solution procedure: (a) Relative Picard increment and nonlinear residual; (b) number of PCG iterations required at each Picard step.

The relative iterate increment is defined by

$$E_{\text{inc}}^{(n+1)} = \frac{\|\mathbf{w}^{(n+1)} - \mathbf{w}^{(n)}\|_2}{\max\{1, \|\mathbf{w}^{(n+1)}\|_2\}}, \quad (4.83)$$

whereas the relative nonlinear residual is

$$E_{\text{res}}^{(n+1)} = \frac{\|A_R \mathbf{w}^{(n+1)} - \mathbf{F}(\mathbf{w}^{(n+1)})\|_2}{\max\{1, \|\mathbf{F}(\mathbf{w}^{(n+1)})\|_2\}}. \quad (4.84)$$

The numerical iteration is considered converged when

$$E_{\text{inc}}^{(n+1)} \leq \varepsilon_{\text{Picard}} \quad \text{and} \quad E_{\text{res}}^{(n+1)} \leq \varepsilon_{\text{res}}. \quad (4.85)$$

The simultaneous use of both conditions is important because a small change between two consecutive iterates does not necessarily imply that the nonlinear discrete system has been solved accurately.

As shown in Figure 4(a), the convergence behavior should be assessed by comparing the final values of $E_{\text{inc}}^{(n)}$ and $E_{\text{res}}^{(n)}$ with their prescribed tolerances. Figure 4(b) describes the computational effort of the inner PCG solver during the nonlinear iteration.

The principal convergence statistics are summarized in Table 2. The computation includes the discrete exterior Dirichlet contribution B_k and uses the Jacobi preconditioner $M_J = \text{diag}(A_R)$. At every nonlinear step, PCG is initialized with the current Picard iterate. The nonlinear iteration terminated when both $E_{\text{inc}}^{(n)} \leq 10^{-4}$ and $E_{\text{res}}^{(n)} \leq 10^{-6}$ were satisfied. The reported CPU time includes matrix assembly and the nonlinear solution procedure.

Table 2. Convergence statistics for the reference simulation.

Quantity	Computed value
Number of Picard iterations	9
Final relative increment $E_{\text{inc}}^{(n)}$	1.2806×10^{-6}
Final nonlinear residual $E_{\text{res}}^{(n)}$	2.7798×10^{-7}
Total number of PCG iterations	240
Final nonlocal integral $\mathcal{I}_h(\mathbf{w})$	1.76249344
Maximum solution value $\max_k w_k$	2.93568070
Minimum solution value $\min_k w_k$	$8.87544826 \times 10^{-2}$
Location of the discrete maximum	$\{24/49, 25/49\}^2$
Relative symmetry error E_{sym}	4.96×10^{-16}
CPU time in seconds	0.75

Mesh-refinement assessment

To investigate the dependence of the numerical solution on the spatial resolution, the problem is solved on the sequence of grids

$$N_m \in \{25, 50, 75, 100\}, \quad h_m = \frac{1}{N_m - 1}. \quad (4.86)$$

Let $\mathbf{w}_{N_m}$ denote the solution computed on the $N_m \times N_m$ grid.

Because these grids are not all related by a constant factor of two, the coarse-grid solution is interpolated onto the next grid in the sequence. Let

$$\mathcal{P}_{N_m \rightarrow N_{m+1}}$$

denote the corresponding interpolation operator. The relative difference between two successive numerical solutions is defined by

$$E_{N_m \rightarrow N_{m+1}} = \frac{\|\mathcal{P}_{N_m \rightarrow N_{m+1}} \mathbf{w}_{N_m} - \mathbf{w}_{N_{m+1}}\|_2}{\max \{1, \|\mathbf{w}_{N_{m+1}}\|_2\}}. \quad (4.87)$$

The use of $\mathcal{P}_{N_m \rightarrow N_{m+1}}$ is necessary because the two solutions are represented on different grids and therefore cannot be subtracted directly. A decreasing sequence of relative differences, together with stabilization of the maximum solution value and the nonlocal integral, provides numerical evidence of mesh convergence.

The mesh-refinement results are reported in Table 3. For each pair of consecutive grids, the coarser-grid solution is interpolated bilinearly onto the finer grid before the relative difference $E_{N_m \rightarrow N_{m+1}}$ is evaluated. The final row has no relative difference because no finer reference grid is included in the present sequence.

Table 3. Mesh-refinement results for the numerical solution.

N_m	h_m	$\max \mathbf{w}_{N_m}$	$\mathcal{I}_{h_m}(\mathbf{w}_{N_m})$	Picard iterations	$E_{N_m \rightarrow N_{m+1}}$
25	1/24	3.24342590	1.88062008	9	1.3835×10^{-1}
50	1/49	2.93568070	1.76249344	9	3.3830×10^{-2}
75	1/74	2.87832621	1.73088266	9	1.5269×10^{-2}
100	1/99	2.85402272	1.71622276	9	—

To assess the stabilization of a scalar numerical quantity, such as the maximum solution amplitude, we additionally define

$$E_{\max}^{(m)} = \frac{|\max \mathbf{w}_{N_{m+1}} - \max \mathbf{w}_{N_m}|}{\max \{1, |\max \mathbf{w}_{N_{m+1}}|\}}. \quad (4.88)$$

Similarly, the relative variation of the global nonlocal integral is

$$E_{\mathcal{I}}^{(m)} = \frac{|\mathcal{I}_{h_{m+1}}(\mathbf{w}_{N_{m+1}}) - \mathcal{I}_{h_m}(\mathbf{w}_{N_m})|}{\max \{1, |\mathcal{I}_{h_{m+1}}(\mathbf{w}_{N_{m+1}})|\}}. \quad (4.89)$$

For the successive transitions $25 \rightarrow 50$, $50 \rightarrow 75$, and $75 \rightarrow 100$, the computed scalar variations are

$$\begin{aligned} E_{\max}^{(m)} &= (1.0483 \times 10^{-1}, 1.9926 \times 10^{-2}, 8.5155 \times 10^{-3}), \\ E_{\mathcal{I}}^{(m)} &= (6.7022 \times 10^{-2}, 1.8263 \times 10^{-2}, 8.5420 \times 10^{-3}). \end{aligned} \quad (4.90)$$

Thus, the three refinement indicators $E_{N_m \rightarrow N_{m+1}}$, $E_{\max}^{(m)}$, and $E_{\mathcal{I}}^{(m)}$ decrease consistently as the mesh is refined. In particular, the successive solution differences decrease from 1.3835×10^{-1} to 1.5269×10^{-2} , while the maximum solution amplitude and the nonlocal integral progressively stabilize. These results provide quantitative evidence of mesh convergence for the implemented finite-horizon discretization.

Sensitivity to the fractional order and interaction radius

The influence of the fractional order and the finite interaction radius is examined by varying one parameter at a time while keeping all remaining parameters fixed at the reference values reported in Table 1. The selected parameter sets are

$$s \in \{0.25, 0.35, 0.45, 0.55, 0.65\}, \quad R \in \{0.15, 0.20, 0.25, 0.30, 0.40\}. \quad (4.91)$$

For each value of s , the interaction radius is fixed at

$$R = 0.25,$$

whereas, for each value of R , the fractional order is fixed at

$$s = 0.45.$$

This one-parameter-at-a-time procedure isolates the numerical influence of each parameter and avoids combining two different effects in the same comparison.

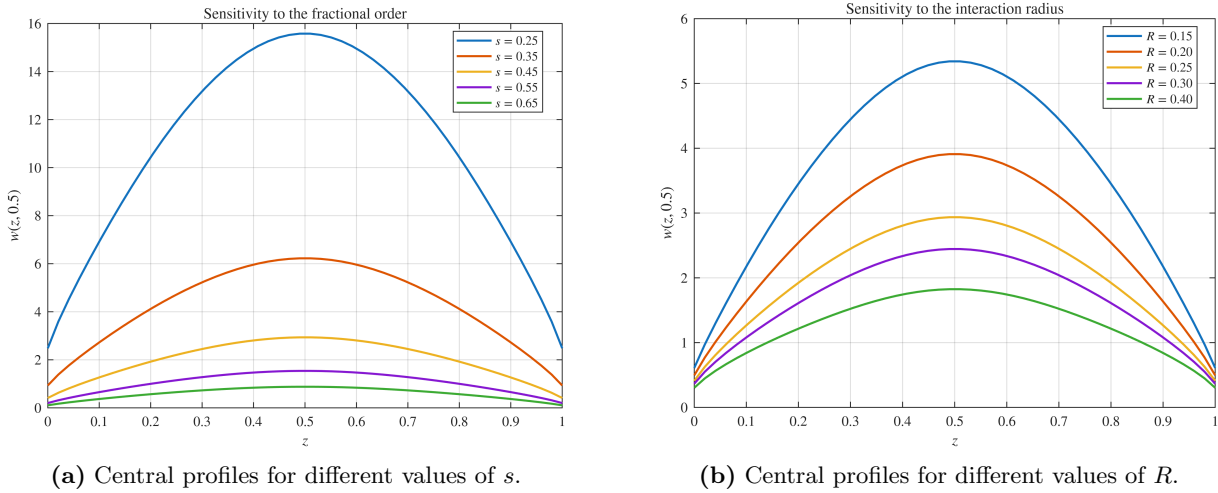


Figure 5. Sensitivity of the numerical solution to the nonlocal parameters: **(a)** Influence of the fractional order s for $R = 0.25$; **(b)** influence of the interaction radius R for $s = 0.45$.

Figure 5(a) compares the central profiles

$$P_x^{(s)}(x) = w_s(x, 0.5)$$

obtained for different fractional orders. The corresponding discrete interaction weights are

$$K_{k\ell}^{(s)} = C_{2,s} \frac{h^2}{\|z_k - z_\ell\|^{2+2s}} \mathbb{I}_{\{0 < \|z_k - z_\ell\| \leq R\}}, \quad (4.92)$$

where $\mathbb{I}$ denotes the indicator function. Changing s modifies both the exponent of the interaction kernel and the normalization constant $C_{2,s}$. Consequently, the effect of s should be evaluated from the computed profiles and quantitative statistics, rather than inferred from the denominator in (4.92) alone.

To quantify the difference between the solution corresponding to a given fractional order and the reference solution computed with $s_{\text{ref}} = 0.45$, one may define

$$E_s = \frac{\|\mathbf{w}_s - \mathbf{w}_{s_{\text{ref}}}\|_2}{\max\{1, \|\mathbf{w}_{s_{\text{ref}}}\|_2\}}. \quad (4.93)$$

The quantity E_s measures the global relative variation of the discrete solution induced by a change in the fractional order.

Figure 5(b) displays the central profiles

$$P_x^{(R)}(x) = \mathbf{w}_R(x, 0.5)$$

obtained using different interaction radii. For a grid point z_k , define the discrete interaction neighborhood

$$\mathcal{N}_R(k) = \{\ell \neq k : 0 < \|z_k - z_\ell\| \leq R\}. \quad (4.94)$$

If $R_1 < R_2$, then

$$\mathcal{N}_{R_1}(k) \subseteq \mathcal{N}_{R_2}(k). \quad (4.95)$$

Thus, increasing R enlarges the nonlocal stencil and increases the number of interactions contributing to

$$(A_R \mathbf{w})_k = \sum_{\ell \in \mathcal{N}_R(k)} K_{k\ell}(w_k - w_\ell) + B_k(R)w_k. \quad (4.96)$$

The term $B_k(R)$ represents the finite-horizon exterior contribution associated with the homogeneous nonlocal Dirichlet condition.

The relative variation with respect to the reference radius $R_{\text{ref}} = 0.25$ may be measured by

$$E_R = \frac{\|\mathbf{w}_R - \mathbf{w}_{R_{\text{ref}}}\|_2}{\max\{1, \|\mathbf{w}_{R_{\text{ref}}}\|_2\}}. \quad (4.97)$$

In addition to modifying the numerical solution, increasing R generally raises the computational cost because the number of nonzero matrix entries increases with the cardinality of $\mathcal{N}_R(k)$.

The quantitative results of the fractional-order study are reported in Table 4. In every computation, the interaction radius is fixed at $R = 0.25$, while the solution corresponding to $s_{\text{ref}} = 0.45$ is used as the reference in the definition of E_s .

Table 4. Sensitivity of the numerical solution to the fractional order s with $R = 0.25$.

s	$\max \mathbf{w}_s$	$\mathcal{I}_h(\mathbf{w}_s)$	Picard iterations	E_{sym}	E_s
0.25	15.57656466	3.69950046	12	3.72×10^{-16}	4.39411668
0.35	6.22460165	2.26359341	10	4.63×10^{-16}	1.13684066
0.45	2.93568070	1.76249344	9	4.54×10^{-16}	0
0.55	1.53901007	1.55144365	8	3.32×10^{-16}	$4.79240209 \times 10^{-1}$
0.65	$8.79216065 \times 10^{-1}$	1.45254132	7	4.64×10^{-16}	$7.04234663 \times 10^{-1}$

Table 4 shows that increasing the fractional order substantially decreases the solution amplitude. In particular, the maximum value decreases from 15.57656466 at $s = 0.25$ to 0.87921607 at $s = 0.65$. The global nonlocal integral decreases simultaneously from 3.69950046 to 1.45254132.

The relative differences E_s confirm that the lower fractional orders produce the largest departure from the reference solution. The symmetry errors remain below 5×10^{-16} in every case, showing that the discrete solutions preserve the reflection symmetry of the square domain and centered source to machine precision.

The interaction-radius results are summarized in Table 5. The fractional order is fixed at $s = 0.45$, and the solution corresponding to $R_{\text{ref}} = 0.25$ is used as the reference in the definition of E_R . The reported CPU times include both sparse-matrix assembly and the nonlinear solution procedure and are the median values obtained from five runs performed under identical computational conditions.

Table 5. Sensitivity of the numerical solution to the interaction radius R with $s = 0.45$.

R	$\max \mathbf{w}_R$	$\mathcal{I}_h(\mathbf{w}_R)$	Picard iterations	CPU time (s)	E_R
0.15	5.34120737	2.09415498	10	0.101	$7.87802149 \times 10^{-1}$
0.20	3.90943118	1.89709381	9	0.152	$3.19235233 \times 10^{-1}$
0.25	2.93568070	1.76249344	9	0.229	0
0.30	2.44472166	1.69427226	8	0.279	$1.61359193 \times 10^{-1}$
0.40	1.82451305	1.60739300	8	0.467	$3.66095036 \times 10^{-1}$

The numerical results demonstrate that increasing the interaction radius reduces the maximum solution amplitude and the global nonlocal integral. Indeed, the maximum decreases from 5.34120737 at $R = 0.15$ to 1.82451305 at $R = 0.40$. At the same time, the computational cost increases because the number of interacting grid-point pairs grows with R . Accordingly, the median CPU time increases from 0.101 seconds for $R = 0.15$ to 0.467 seconds for $R = 0.40$. The values of E_R quantify the global change relative to the reference radius and confirm that both shorter- and longer-range interactions produce appreciable variations in the computed solution.

Interpretation of the numerical observations

The numerical observations should be interpreted through a combined examination of the graphical results and the quantitative indicators. The surface, contour, and heat-map representations in Figures 1 and 2 describe the spatial localization of the computed solution. Its numerical nonnegativity is verified by checking

$$\min_{1 \leq k \leq N^2} w_k \geq -\varepsilon_{\text{pos}}, \quad (4.98)$$

where ε_{pos} is a small tolerance consistent with the accuracy of the nonlinear and linear solvers.

The relationship between the source location and the numerical maximum is examined through

$$E_{\text{loc}} = \|z_{\text{max}} - z_c\|_2, \quad z_{\text{max}} = \arg \max_{z_k} w_k. \quad (4.99)$$

A small value of E_{loc} indicates that the discrete maximum is located close to the center of the Gaussian excitation.

The horizontal and vertical profiles shown in Figure 3(a) are compared using the relative symmetry error E_{sym} . The radial profile in Figure 3(b) is used to determine whether the solution

decreases, on average, as the distance from the source center increases. These quantitative tests provide stronger evidence of symmetry and decay than a visual inspection of the plots alone.

The convergence history in Figure 4 and the statistics in Table 2 determine whether the Picard iteration and the inner PCG solver have satisfied their prescribed tolerances. The numerical solution should be accepted only when both the relative iterate increment and the nonlinear residual satisfy

$$E_{\text{inc}}^{(n)} \leq \varepsilon_{\text{Picard}} \quad \text{and} \quad E_{\text{res}}^{(n)} \leq \varepsilon_{\text{res}}.$$

The mesh-refinement data in Table 3 indicate whether the principal numerical quantities stabilize as the number of grid points increases. Evidence of mesh convergence is provided by decreasing values of $E_{N_m \rightarrow N_{m+1}}$ and by stabilization of $\max \mathbf{w}_{N_m}$ and $\mathcal{I}_{h_m}(\mathbf{w}_{N_m})$.

Finally, Figure 5 and Tables 4 and 5 quantify the dependence of the solution on s and R . Differences in amplitude or spatial spread should be discussed using the computed values of E_s and E_R . Because the implemented operator employs a finite interaction radius, the numerical results represent a truncated nonlocal model. They should only be identified with the full fractional Laplacian when the neglected long-range interactions and the exterior contribution are incorporated consistently.

4.5. Discussion of the numerical results

The numerical experiment illustrates how the localized nonlinear source, the global integral coefficient, and the truncated fractional diffusion operator jointly determine the computed solution. The reference parameter values are reported in Table 1. The spatial structure of the solution is presented in Figures 1 and 2, while its cross-sectional and radial behavior is examined in Figure 3.

For the discrete nonlinear problem

$$A_R \mathbf{w} = \frac{\mathbf{H}(\mathbf{w})^{[\alpha]}}{[\mathcal{I}_h(\mathbf{w})]^\beta}, \quad (4.100)$$

the notation $\mathbf{H}(\mathbf{w})^{[\alpha]}$ denotes componentwise exponentiation:

$$\mathbf{H}(\mathbf{w})^{[\alpha]} = (H_1(\mathbf{w})^\alpha, \dots, H_{N^2}(\mathbf{w})^\alpha)^\top. \quad (4.101)$$

The discrete nonlocal integral is given by

$$\mathcal{I}_h(\mathbf{w}) = h^2 \sum_{k=1}^{N^2} H_k(\mathbf{w}). \quad (4.102)$$

Consequently, the scalar quantity

$$\Lambda_h(\mathbf{w}) = [\mathcal{I}_h(\mathbf{w})]^{-\beta} \quad (4.103)$$

acts simultaneously on all components of the discrete right-hand side:

$$A_R \mathbf{w} = \Lambda_h(\mathbf{w}) \mathbf{H}(\mathbf{w})^{[\alpha]}. \quad (4.104)$$

Equation (4.104) shows that the integral term introduces a global coupling. A change in the solution at a single grid point modifies $H_k(\mathbf{w})$, which changes $\mathcal{I}_h(\mathbf{w})$ and therefore rescales the

forcing throughout the computational domain. The effect of the nonlocal integral is thus not restricted to the point at which the solution is perturbed.

The surface representation in Figure 1(a) describes the amplitude and spatial localization of the numerical solution. The filled contour plot in Figure 1(b) and the heat map in Figure 2(a) provide complementary representations of its level sets. Their comparison with the Gaussian excitation in Figure 2(b) determines whether the dominant response is concentrated near the prescribed source center

$$z_c = (0.5, 0.5).$$

Let

$$w_{\min} = \min_{1 \leq k \leq N^2} w_k, \quad w_{\max} = \max_{1 \leq k \leq N^2} w_k. \quad (4.105)$$

The computed solution may be regarded as numerically nonnegative when

$$w_{\min} \geq -\varepsilon_{\text{pos}}, \quad (4.106)$$

where $\varepsilon_{\text{pos}} > 0$ is chosen consistently with the PCG and Picard tolerances. The minimum value, maximum value, and location of the discrete maximum must be reported using the computed data summarized in Table 2.

The distance between the discrete maximum and the Gaussian center may be quantified by

$$E_{\text{loc}} = \|z_{\max} - z_c\|_2, \quad z_{\max} = \arg \max_{z_k} w_k. \quad (4.107)$$

A small value of E_{loc} indicates that the dominant numerical response is localized near the center of the excitation.

The central profiles shown in Figure 3(a) provide a more precise assessment of reflection symmetry than the two-dimensional plots alone. These profiles are defined by

$$P_x(x) = w(x, 0.5), \quad P_y(y) = w(0.5, y). \quad (4.108)$$

Their relative discrepancy is measured by

$$E_{\text{sym}} = \frac{\|P_x - P_y\|_2}{\max\{1, \|P_x\|_2\}}. \quad (4.109)$$

A small value of E_{sym} provides quantitative evidence that the discretization approximately preserves the symmetry of the square domain, the centered source, and the interaction kernel. The reported symmetry claim should therefore be supported by the value of E_{sym} in Table 2, rather than by visual inspection alone.

The radial profile displayed in Figure 3(b) describes the average dependence of the solution on the distance from z_c . Its discrete variation can be examined through

$$D_m = P_r(r_{m+1}) - P_r(r_m). \quad (4.110)$$

A numerically nonincreasing radial profile satisfies

$$D_m \leq \varepsilon_{\text{mon}} \quad \text{for all admissible } m, \quad (4.111)$$

where $\varepsilon_{\text{mon}} > 0$ accounts for discretization, interpolation, and radial-binning errors. Because the computational domain is square, exact radial symmetry is not expected close to the boundary, even though the localized source and interaction kernel are radially symmetric about z_c .

The convergence behavior of the nonlinear solution procedure is presented in Figure 4. In particular, Figure 4(a) displays the relative iterate increment and nonlinear residual, whereas Figure 4(b) reports the number of PCG iterations required at each Picard step. The nonlinear iteration is accepted as converged only when

$$E_{\text{inc}}^{(n)} \leq \varepsilon_{\text{Picard}} \quad \text{and} \quad E_{\text{res}}^{(n)} \leq \varepsilon_{\text{res}}. \quad (4.112)$$

A small iterate increment alone is not sufficient because consecutive iterates may stagnate before satisfying the nonlinear discrete equation. The final increment, nonlinear residual, iteration counts, nonlocal integral, symmetry error, and computational time are summarized in Table 2.

The mesh-refinement results in Table 3 assess whether the principal numerical quantities stabilize as the spatial resolution is increased. For two successive grids N_m and N_{m+1} , the relative solution difference is

$$E_{N_m \rightarrow N_{m+1}} = \frac{\|\mathcal{P}_{N_m \rightarrow N_{m+1}} \mathbf{w}_{N_m} - \mathbf{w}_{N_{m+1}}\|_2}{\max\{1, \|\mathbf{w}_{N_{m+1}}\|_2\}}. \quad (4.113)$$

Here, $\mathcal{P}_{N_m \rightarrow N_{m+1}}$ interpolates the coarse-grid solution onto the next finer grid. This formulation is appropriate for the sequence

$$N_m \in \{25, 50, 75, 100\},$$

whose successive grids are not all related by a factor of two.

Numerical evidence of mesh convergence is provided when $E_{N_m \rightarrow N_{m+1}}$ decreases and the scalar quantities

$$\max \mathbf{w}_{N_m} \quad \text{and} \quad \mathcal{I}_{h_m}(\mathbf{w}_{N_m})$$

stabilize as N_m increases. Their relative variations may be measured by

$$E_{\text{max}}^{(m)} = \frac{|\max \mathbf{w}_{N_{m+1}} - \max \mathbf{w}_{N_m}|}{\max\{1, |\max \mathbf{w}_{N_{m+1}}|\}} \quad (4.114)$$

and

$$E_{\mathcal{I}}^{(m)} = \frac{|\mathcal{I}_{h_{m+1}}(\mathbf{w}_{N_{m+1}}) - \mathcal{I}_{h_m}(\mathbf{w}_{N_m})|}{\max\{1, |\mathcal{I}_{h_{m+1}}(\mathbf{w}_{N_{m+1}})|\}}. \quad (4.115)$$

A convergence order should not be claimed unless the mesh-refinement results in Table 3 display a sufficiently stable asymptotic trend.

The dependence of the solution on the fractional order and interaction radius is shown in Figure 5. Figure 5(a) compares the central profiles for different values of s , while Figure 5(b) compares the corresponding profiles for different values of R .

For a fixed interaction radius, the discrete interaction weights depend on s through

$$K_{k\ell}(s) = C_{2,s} \frac{h^2}{\|z_k - z_\ell\|^{2+2s}} \mathbb{I}_{\{0 < \|z_k - z_\ell\| \leq R\}}. \quad (4.116)$$

Changing s modifies both the distance-dependent power and the normalization constant $C_{2,s}$. Moreover, the resulting solution remains coupled to the global factor $\Lambda_h(\mathbf{w})$. Consequently, the influence of s on the solution amplitude and spatial spread should be determined from Figure 5(a) and Table 4.

Relative to the reference value $s_{\text{ref}} = 0.45$, the global variation is measured by

$$E_s = \frac{\|\mathbf{w}_s - \mathbf{w}_{s_{\text{ref}}}\|_2}{\max\{1, \|\mathbf{w}_{s_{\text{ref}}}\|_2\}}. \quad (4.117)$$

The computed values of E_s , the maximum amplitude, the nonlocal integral, the Picard iteration count, and the symmetry error are reported in Table 4.

The effect of the interaction radius is determined by the neighborhood

$$\mathcal{N}_R(k) = \{\ell \neq k : 0 < \|z_k - z_\ell\| \leq R\}. \quad (4.118)$$

For $R_1 < R_2$, one has

$$\mathcal{N}_{R_1}(k) \subseteq \mathcal{N}_{R_2}(k). \quad (4.119)$$

Increasing R therefore enlarges the discrete interaction neighborhood and generally increases both the number of nonzero matrix coefficients and the computational cost.

Relative to the reference value $R_{\text{ref}} = 0.25$, the solution variation is measured by

$$E_R = \frac{\|\mathbf{w}_R - \mathbf{w}_{R_{\text{ref}}}\|_2}{\max\{1, \|\mathbf{w}_{R_{\text{ref}}}\|_2\}}. \quad (4.120)$$

The resulting changes in amplitude, global integral, nonlinear iteration count, CPU time, and E_R are reported in Table 5. These results should be interpreted together with Figure 5(b).

Taken together, Figures 1, 2, 3, 4, and 5, together with Tables 1, 2, 3, 4, and 5, provide complementary numerical assessments of the model. The spatial plots describe localization and level-set geometry. The profile and error measurements quantify symmetry and radial decay. The iteration history evaluates nonlinear and linear solver convergence. The mesh study examines discretization stability, while the sensitivity study measures the dependence on s and R . Once all MATLAB-generated values have been inserted into the tables, these diagnostics determine which qualitative properties of the theoretical model are reproduced by the finite-dimensional approximation.

The same computational strategy can be extended to a variable-exponent model, although the resulting discrete operator is nonlinear. For a symmetric exponent

$$p_{k\ell} = \mathbf{p}(z_k, z_\ell),$$

a representative discrete interaction operator is

$$(\mathcal{A}_{R,\mathbf{p}}\mathbf{w})_k = \sum_{\ell \in \mathcal{N}_R(k)} C_{2,s,p_{k\ell}} \frac{|w_k - w_\ell|^{p_{k\ell}-2} (w_k - w_\ell)}{\|z_k - z_\ell\|^{2+sp_{k\ell}}} h^2 + B_{k,\mathbf{p}}(\mathbf{w}). \quad (4.121)$$

Because the operator in (4.121) depends nonlinearly on $\mathbf{w}$, it cannot generally be evaluated through a single PCG solve with a fixed matrix. Its numerical treatment requires an appropriate nonlinear linearization procedure, such as a damped Picard, Newton, semismooth Newton, or nonlinear Krylov method, together with suitable regularization and preconditioning.

Algorithm 1 Relaxed Picard–PCG method for the two-dimensional nonlocal problem**1. Specify the model and numerical parameters.**

Set

$$s, \alpha, \beta, \gamma, \sigma, \eta, z_c, N, R, \omega,$$

together with the tolerances

$$\varepsilon_{\text{PCG}}, \quad \varepsilon_{\text{Picard}}, \quad \varepsilon_{\text{res}},$$

and the maximum iteration counts $N_{\text{PCG}}^{\max}$ and $N_{\text{Picard}}^{\max}$.**2. Construct the spatial grid.**

Set

$$h = \frac{1}{N-1}, \quad z_{i,j} = (ih, jh), \quad i, j = 0, \dots, N-1,$$

and let

$$\Delta V = h^2.$$

Map each pair (i, j) to a global index $k = 1, \dots, N^2$.**3. Assemble the truncated fractional operator.**For every pair $k \neq \ell$, compute

$$r_{k\ell} = \|z_k - z_\ell\|.$$

Define

$$K_{k\ell} = \begin{cases} C_{2,s} \frac{\Delta V}{r_{k\ell}^{2+2s}}, & 0 < r_{k\ell} \leq R, \\ 0, & r_{k\ell} > R. \end{cases} \quad (\text{A.1})$$

If the exterior Dirichlet contribution is included, evaluate

$$B_k = C_{2,s} \int_{\substack{\mathbb{R}^2 \setminus \Omega \\ \|z_k - \zeta\| \leq R}} \frac{1}{\|z_k - \zeta\|^{2+2s}} d\zeta. \quad (\text{A.2})$$

Otherwise, set $B_k = 0$ and interpret the discretization as a regional finite-horizon operator.Assemble A_R according to

$$(A_R)_{k\ell} = \begin{cases} \sum_{\substack{m=1 \\ m \neq k}}^{N^2} K_{km} + B_k, & k = \ell, \\ -K_{k\ell}, & k \neq \ell. \end{cases} \quad (\text{A.3})$$

4. Select the initial approximation.

Set

$$\mathbf{w}^{(0)} = \mathbf{0}, \quad n = 0.$$

Algorithm 2 Relaxed Picard–PCG method (continued)

5. Perform the relaxed Picard iteration.

While $n < N_{\text{Picard}}^{\max}$, carry out the following steps:

(a) Evaluate the nonlinear auxiliary function:

$$H_k^{(n)} = 1 + \gamma e \left(-\frac{\|z_k - z_c\|^2}{2\sigma^2} \right) + \eta w_k^{(n)}. \quad (\text{A.4})$$

(b) Approximate the global integral:

$$I_h^{(n)} = \Delta V \sum_{k=1}^{N^2} H_k^{(n)}. \quad (\text{A.5})$$

(c) Form the Picard right-hand side:

$$f_k^{(n)} = \frac{(H_k^{(n)})^\alpha}{(I_h^{(n)})^\beta}, \quad k = 1, \dots, N^2. \quad (\text{A.6})$$

(d) Solve the linearized system

$$A_R \tilde{\mathbf{w}}^{(n+1)} = \mathbf{f}^{(n)} \quad (\text{A.7})$$

using PCG with tolerance ε_{PCG} and maximum iteration count $N_{\text{PCG}}^{\max}$.

(e) Apply the relaxation step:

$$\mathbf{w}^{(n+1)} = (1 - \omega) \mathbf{w}^{(n)} + \omega \tilde{\mathbf{w}}^{(n+1)}. \quad (\text{A.8})$$

(f) Re-evaluate the nonlinear function at $\mathbf{w}^{(n+1)}$ and compute

$$E_{\text{inc}}^{(n+1)} = \frac{\|\mathbf{w}^{(n+1)} - \mathbf{w}^{(n)}\|_2}{\max\{1, \|\mathbf{w}^{(n+1)}\|_2\}} \quad (\text{A.9})$$

and

$$E_{\text{res}}^{(n+1)} = \frac{\|A_R \mathbf{w}^{(n+1)} - \mathbf{F}(\mathbf{w}^{(n+1)})\|_2}{\max\{1, \|\mathbf{F}(\mathbf{w}^{(n+1)})\|_2\}}. \quad (\text{A.10})$$

(g) Record

$$n + 1, \quad E_{\text{inc}}^{(n+1)}, \quad E_{\text{res}}^{(n+1)}, \quad I_h^{(n+1)},$$

together with the PCG iteration count and linear residual.

(h) If

$$E_{\text{inc}}^{(n+1)} \leq \varepsilon_{\text{Picard}} \quad \text{and} \quad E_{\text{res}}^{(n+1)} \leq \varepsilon_{\text{res}},$$

terminate the iteration. Otherwise, set $n \leftarrow n + 1$ and continue.

Algorithm 3 Relaxed Picard–PCG method (continued)**6. Compute diagnostic quantities.**

Evaluate

$$w_{\min} = \min_k w_k, \quad w_{\max} = \max_k w_k, \quad E_{\text{sym}} = \frac{\|P_x - P_y\|_2}{\max\{1, \|P_x\|_2\}},$$

and record the location of the discrete maximum, final residuals, total PCG iterations, and CPU time.

7. Perform post-processing.

Generate the surface, contour, heat-map, source, midline-profile, radial-profile, iteration-history, and parameter-sensitivity plots shown in Figures 1–5. Export the numerical statistics required by Tables 2–5.

A. Numerical algorithm for the two-dimensional nonlocal fractional problem

This appendix summarizes the computational procedure used to solve (4.1). The algorithm distinguishes the provisional PCG solution from the relaxed Picard iterate and monitors both the iterate increment and the nonlinear residual.

Algorithmic remarks.

- The matrix A_R is symmetric when

$$K_{k\ell} = K_{\ell k}.$$

PCG additionally requires the imposed boundary treatment to make the discrete system positive definite.

- The exterior term B_k is required for a consistent finite-horizon approximation of the homogeneous exterior Dirichlet condition. If it is omitted, the implemented operator should be described as a regional truncated operator.
- The numerical effect of s must be assessed together with the normalization constant $C_{2,s}$; it should not be inferred solely from the kernel exponent.
- The statement that the method converges in fewer than a specified number of iterations should be made only after inserting the observed count from Table 2.
- For a variable exponent $p_{k\ell} = \mathbf{p}(z_k, z_\ell)$, the interaction term becomes nonlinear:

$$\frac{|w_k - w_\ell|^{p_{k\ell}-2}(w_k - w_\ell)}{\|z_k - z_\ell\|^{2+sp_{k\ell}}}.$$

Such a problem generally requires a nonlinear linearization strategy rather than direct application of PCG to a fixed matrix.

- Higher-dimensional extensions preserve the overall algorithmic structure, but their memory and computational costs increase rapidly because each grid point interacts with all points contained in its nonlocal neighborhood.

B. Conclusion and perspective

In this work, we investigated a class of nonlocal fractional $\mathbf{p}(z, y)$ -Laplacian problems involving spatially variable exponents $\alpha(z)$ and $\beta(z)$. By applying the Tychonoff and Schauder fixed-point theorems, we established two distinct existence results for weak solutions under suitable structural assumptions on the nonlinear terms $h(z, w)$ and $f(z, w)$. These results extend and unify several existing models of nonlocal fractional equations with variable growth conditions, highlighting the flexibility of fixed-point techniques in treating nonlinear and nonlocal operators of variable order. Furthermore, a two-dimensional numerical illustration for the simplified fractional case ($\mathbf{p} = 2$) demonstrated the qualitative behavior of the model, confirming the analytical predictions and emphasizing the interplay between nonlocality and variable exponents. From a broader perspective, this study opens several avenues for future research. A natural continuation would be to investigate the multiplicity and regularity of weak solutions, as well as their stability with respect to changes in the exponents and boundary data. Extending the analysis to systems of coupled nonlocal equations, or to problems with time-dependent and stochastic components, would also be of significant theoretical and practical interest. Moreover, developing efficient numerical schemes for the general fractional $\mathbf{p}(z, y)$ -Laplacian operator could provide deeper insight into the behavior of solutions and support applications in physics, biology, and material sciences where spatial heterogeneity and nonlocal diffusion coexist.

Declarations

Funding. The authors received no financial support or funding for the research, authorship, or publication of this article.

Conflict of interest. The authors declare that they have no conflict of interest regarding the publication of this paper.

Ethical approval. This article does not contain any studies involving human participants or animals performed by any of the authors.

Data availability. All data generated or analyzed during this study are included in this published article. Additional numerical data or MATLAB codes used for the simulations are available from the corresponding author, **Salah Boulaaras**, upon reasonable request.

Authors' contributions. Omar Djidel and Salah Boulaaras developed the main theoretical framework and carried out the analytical proofs. Mokhtar Boumaaza implemented the numerical simulations and assisted in the interpretation of the results. All authors contributed to writing and reviewing the manuscript and approved the final version.

Acknowledgements. The authors have no specific acknowledgements to declare.

References

- [1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., 2002, 164, 213–259.
- [2] R. P. Agarwal, E. Azroul, N. Kamali and M. Shimi, *On a class of $p(x, \cdot)$ -integro-differential Kirchhoff-type problem with singular kernel*, J. Appl. Anal. Comput., 2025, 15(1), 333–353.
- [3] D. Arcoya, T. Leonori and A. Primo, *Existence of solutions for semilinear nonlocal elliptic problems via a Bolzano theorem*, Acta Appl. Math., 2013, 127, 87–104.

- [4] E. Azroul, A. Benkirane and M. Shimi, *General fractional Sobolev space with variable exponent and applications to nonlocal problems*, Adv. Oper. Theory, 2020, 5, 1512–1540.
- [5] E. Azroul, A. Benkirane, M. Shimi and M. Srati, *On a class of fractional $p(x)$ -Kirchhoff type problems*, Appl. Anal., 2021, 100, 383–402.
- [6] E. Azroul, N. Kamali, M. A. Ragusa and M. Shimi, *Multiple solutions for anisotropic nonlocal problems with variable exponents*, Fract. Calc. Appl. Anal., 2025, 28(5), 2333–2356.
- [7] A. Bahrouni and V. D. Rădulescu, *On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent*, Discrete Contin. Dyn. Syst. Ser. S, 2018, 11(3), 379–389.
- [8] G. M. Bisci, V. D. Rădulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, Cambridge, 2016.
- [9] S. Boulaaras, *Optimal shape design for two-dimensional fractional heat equations with source terms: Long-time behavior and Γ -convergence*, J. Nonlinear Math. Phys., 2026. DOI: 10.1007/s44198-026-00458-x.
- [10] S. Boulaaras and R. Guefaifia, *Global existence and blow-up phenomena for a hyperbolic p -Laplacian equation with logarithmic nonlinearity and nonlinear boundary conditions*, J. Appl. Anal. Comput., 2026, 16(6), 2887–2912.
- [11] S. Boulaaras, R. Guefaifia, B. Cherif and T. Radwan, *Existence result for a Kirchhoff elliptic system involving p -Laplacian operator with variable parameters and additive right-hand side via sub- and supersolution methods*, AIMS Math., 2021, 6(3), 2315–2329.
- [12] F. J. S. A. Corrêa and G. M. Figueiredo, *On an elliptic equation of p -Kirchhoff type via variational methods*, Bull. Aust. Math. Soc., 2006, 74(2), 263–277.
- [13] F. J. S. A. Corrêa and G. M. Figueiredo, *On a p -Kirchhoff equation via Krasnoselskii's genus*, Appl. Math. Lett., 2009, 22, 819–822.
- [14] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1998, 15(4), 493–516.
- [15] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Heidelberg, 2011.
- [16] O. Djidel, K. Biroud, M.-S. Rezaoui, S. Boulaaras and R. Guefaifia, *A study of the nonlocal solution of p -Laplacian fractional elliptic problems via approximation methods*, Bound. Value Probl., 2024, Article 169.
- [17] X. Fan, J. Shen and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$* , J. Math. Anal. Appl., 2001, 262, 749–760.
- [18] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$* , J. Math. Anal. Appl., 2001, 263, 424–446.
- [19] R. Guefaifia and S. Boulaaras, *A qualitative study on the existence of positive solutions for local and nonlocal elliptic integro-differential quasilinear systems*, Bound. Value Probl., 2024, Article 159.

- [20] R. Guefaifia, S. Boulaaras and F. Kamache, *Existence of weak solutions for a new class of fractional boundary value impulsive systems with Riemann–Liouville derivatives*, J. Integral Equations Appl., 2021, 33(3), 301–313.
- [21] R. Guefaifia, G. C. G. dos Santos, T. Bouali, R. Jan, S. Boulaaras and A. Alharbi, *Sub-super solutions method combined with Schauder’s fixed point for existence of positive weak solutions for anisotropic non-local elliptic systems*, Mathematics, 2022, 10, Article 4479.
- [22] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 2014.
- [23] H. El-Houari, E. Arhrrabi and H. Moussa, *Study of Schrödinger–Choquard problem with $p(\cdot)$ -Laplacian operator*, J. Appl. Anal. Comput., 2026, 16(1), 479–494.
- [24] A. Iannizzotto, S. Liu, K. Perera and M. Squassina, *Existence results for fractional p -Laplacian problems via Morse theory*, Adv. Calc. Var., 2016, 9(2), 101–125.
- [25] F. Kamache, R. Guefaifia, S. Boulaaras and A. Alharbi, *Existence of weak solutions for a new class of fractional p -Laplacian boundary value systems*, Mathematics, 2020, 8, Article 475.
- [26] U. Kaufmann, J. Rossi and R. Vidal, *Fractional Sobolev spaces with variable exponents and fractional $p(x)$ -Laplacians*, Electron. J. Qual. Theory Differ. Equ., 2017, 76, 1–10.
- [27] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* , Czechoslovak Math. J., 1991, 41, 592–618.
- [28] D. Liu, *On a p -Kirchhoff equation via fountain theorem and dual fountain theorem*, Nonlinear Anal., 2010, 72, 302–308.
- [29] M. Louchaich and K. Saoudi, *On a weak solutions for a singular problems involving the fractional $p(\cdot)$ -Laplacian type operators*, Complex Var. Elliptic Equ., 2026, 71(5), 1043–1063.
- [30] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., 2012, 136(5), 521–573.
- [31] R. Pei, *On ground state of fractional p -Kirchhoff equation involving subcritical and critical exponential growth*, J. Appl. Anal. Comput., 2024, 14(5), 2653–2672.
- [32] K. Perera, M. Squassina and Y. Yang, *Bifurcation and multiplicity results for critical fractional p -Laplacian problems*, Math. Nachr., 2016, 289(2–3), 332–342.
- [33] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.
- [34] A. Yahiaoui, M.-S. Rezaoui, O. Djidel, R. Guefaifia and S. Boulaaras, *Kirchhoff-type problems with the non-local fractional $d(z, \cdot)$ -Laplacian operator*, J. Inequal. Appl., 2025, Article 43.
- [35] E. Zaouche, *Nontrivial weak solutions for nonlocal nonhomogeneous elliptic problems*, Appl. Anal., 2022, 101(4), 1261–1270.
- [36] C. Zhang and X. Zhang, *Renormalized solutions for the fractional $p(x)$ -Laplacian equation with L^1 data*, 2017. arXiv:1708.04481.

Received March 2026; Accepted September 2026; Available online October 2026.