

THE BIFURCATION PROBLEMS OF HETEROCLINIC LOOPS FOR A CLASS OF HIGH-DIMENSIONAL SYSTEMS*

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Abstract In this paper, we study the bifurcation problems of heteroclinic loops with two hyperbolic critical points for high-dimensional systems under the condition that the product of hyperbolic ratios is greater than 1. Using exponential dichotomy, the local coordinate systems and Poincaré maps are constructed in the small tubular neighborhood of heteroclinic loop. Under twisted and non-twisted conditions, we obtain existence and number of 1-homoclinic loop, 1-periodic orbit, two 1-periodic orbits and double 1-periodic orbit. In addition, the corresponding bifurcation surfaces and bifurcation diagrams are also given.

Keywords Heteroclinic loop, high-dimensional system, bifurcation, Poincaré map, periodic orbit.

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1. Introduction and hypotheses

The bifurcation problem near heteroclinic loops in dynamical systems is an important problem in global bifurcation theory. If a system has heteroclinic orbits connecting hyperbolic equilibrium points and forming a closed loop, by perturbation, rich dynamical phenomena often happen in the neighborhood of the closed loop, such as periodic orbits, homoclinic orbits, and more complex behaviors. In high-dimensional cases, the bifurcation phenomena become even more complex.

It is worth noting that limit cycle bifurcations occur near homoclinic loops — particularly in relation to Hilbert's 16th problem for planar systems. For instance, for a single homoclinic loop with a nilpotent singular point, An and Han [1] derived an expansion for the Melnikov functions with some computing formulas for its first coefficients and established a bifurcation theorem for estimating the number of limit cycles. For homoclinic loops with a hyperbolic saddle, Tian and Han [19] developed a unified computational method to find limit cycles, which has wide applications. Han et al. [6] further extended the theory to double homoclinic loops and found an important relationship between the three Melnikov functions by which a new way to find the maximum number of limit cycles was provided. These results advance the study of Hilbert's 16th problem significantly.

Early research mainly focused on homoclinic and heteroclinic loops with simple geometric structures. In [2], Chow and Hale systematically outlined the basic methods of bifurcation the-

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ory, providing a framework for subsequent studies. Wiggins [21, 22] further explored analytical methods for global bifurcations and chaos. As the theory developed, researchers began to focus on the high-dimensional and degenerate cases. The works of Han and Zhu [7], Luo [17], and Zhang [25] systematically summarized the foundations and developments of vector field bifurcation theory, particularly the methods for dealing with the problem of homoclinic and heteroclinic orbits for high-dimensional systems.

On the stability and bifurcation of heteroclinic loops, Zhu and other researchers conducted a series of foundational studies. Zhu [26] introduced exponential trichotomy to study heteroclinic bifurcations, while Zhu and Xia [29] systematically discussed basic bifurcation cases of heteroclinic loops. On this basis, Jin and Zhu [12] studied the bifurcations of rough heteroclinic loops with two saddle points, while Tian and Zhu [20] studied the case of non-twisted heteroclinic loops. For more complex structures, such as the stability and bifurcations of double homoclinic loops and heteroclinic loops accompanied by transcritical bifurcations [3, 5, 24], relevant research has also been gradually deepened.

In recent years, the effect of the twist condition on bifurcations has attracted attention. In [13], Jin studied the bifurcations of twisted double homoclinic loops with resonant condition, while [9, 10] further studied bifurcations of twisted heteroclinic loops and fine heteroclinic loops for high-dimensional systems. Lu [15] analyzed the codimension-2 bifurcations of twisted double homoclinic loops. These studies show that the presence or absence of twisting significantly influences the bifurcation outcomes.

In terms of methods, in addition to the classical Melnikov vector method [28] and exponential dichotomy, researchers have developed methods such as changing stability [14] and symmetry analysis of center manifolds [4]. Progress has also been made in the study of special structures, such as homoclinic orbit connecting nonleading eigendirections [23], loops tangent to invariant straight lines [18], and heteroclinic loops involving saddle-focus points [8]. Furthermore, some papers have conducted related studies on aspects such as high-dimensional degenerate homoclinic bifurcations [11] and the uniqueness of limit loops bifurcating from singular closed orbits [16].

Based on the above research, this paper focuses on the bifurcation problem of heteroclinic loop connecting two hyperbolic critical points for high-dimensional systems under the condition that the product of the hyperbolic ratios is greater than 1. Using exponential trichotomy, we establish local coordinates and Poincaré maps in a small tubular neighborhood of the heteroclinic loop. Under twisted and non-twisted conditions, we obtained the existence and number of 1-homoclinic loop, 1-periodic orbit, two 1-periodic orbits and double 1-periodic orbit. This paper not only presents the corresponding bifurcation surfaces and diagrams, but also contributes to and refines the study of high-dimensional heteroclinic loop bifurcations. This work integrates and draws on research results ranging from the classics [2, 7, 21] to the latest studies [4, 6, 8, 10, 14, 18], combining the theoretical framework with recent advances, with the aim of deepening the understanding of the complex bifurcation behavior of high-dimensional heteroclinic loops.

Consider the following C^r system

$$\dot{z} = f(z), \quad (1.1)$$

where $r \geq 4$, $z \in \mathbf{R}^{m+n}$. At first, we assume that the following hypotheses are valid.

(H1). (Hyperbolicity) $z = p_i$, $i = 1, 2$ are the hyperbolic critical points of (1.1), $f(p_i) = 0$. The stable manifold $W_{p_i}^s$ and the unstable manifold $W_{p_i}^u$ of p_i are m -dimensional and n -dimensional, respectively. Moreover, $-\rho_i^1$ and λ_i^1 are the simple real eigenvalues of $D_z f(p_i)$ such that any other eigenvalue σ of $D_z f(p_i)$ satisfies either $\operatorname{Re} \sigma < -\rho_i^0 < -\rho_i^1 < 0$ or $0 < \lambda_i^1 < \lambda_i^0 < \operatorname{Re} \sigma$, where ρ_i^0 and λ_i^0 are some positive constants.

(H2). (Non-degeneration) System (1.1) has a heteroclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{z = r_i(t) : t \in \mathbf{R}\}$, $r_i(+\infty) = r_{i+1}(-\infty) = p_{i+1}$, $r_3(t) = r_1(t)$, $p_3 = p_1$. $\dim(T_{r_i(t)}W_{p_i}^u \cap T_{r_i(t)}W_{p_{i+1}}^s) = 1$, $T_{r_i(t)}W_{p_i}^u$ is the tangent space of $W_{p_i}^u$ at $r_i(t)$, $T_{r_i(t)}W_{p_{i+1}}^s$ is the tangent space of $W_{p_{i+1}}^s$ at $r_i(t)$.

(H3). (Strong inclination) $\lim_{x \rightarrow +\infty} (T_{r_i(t)}W_{p_i}^u + T_{r_i(t)}W_{p_{i+1}}^s) = T_{p_{i+1}}W_{p_{i+1}}^{uu} \oplus T_{p_{i+1}}W_{p_{i+1}}^s$, $\lim_{x \rightarrow -\infty} (T_{r_i(t)}W_{p_i}^u + T_{r_i(t)}W_{p_{i+1}}^s) = T_{p_i}W_{p_i}^u \oplus T_{p_i}W_{p_i}^{ss}$, where, $W_{p_i}^{uu}$ and $W_{p_i}^{ss}$ are the strong unstable manifolds and the strong stable manifolds of p_i , respectively. $T_{p_i}W_{p_i}^{uu}$ is the tangent space of $W_{p_i}^{uu}$ at p_i , which is the generalized eigenspace corresponding to all the eigenvalues with larger real parts than λ_i^0 . $T_{p_i}W_{p_i}^{ss}$ is the tangent space of $W_{p_i}^{ss}$ at p_i , which is the generalized eigenspace corresponding to all the eigenvalues with smaller real parts than $-\rho_i^0$. $T_{p_i}W_{p_i}^u$ is the tangent space of $W_{p_i}^u$ at p_i , $T_{p_i}W_{p_i}^s$ is the tangent space of $W_{p_i}^s$ at p_i .

Denote $e_i^\pm = \lim_{t \rightarrow \mp\infty} \dot{r}_i(t)/|\dot{r}_i(t)|$, where $e_i^+ \in T_{p_i}W_{p_i}^u$ and $e_i^- \in T_{p_{i+1}}W_{p_{i+1}}^s$ are unit eigenvectors corresponding to λ_i^1 and $-\rho_{i+1}^1$, respectively. Furthermore, $\text{span}(T_{p_i}W_{p_i}^{uu}, e_i^+) = T_{p_i}W_{p_i}^u$, $\text{span}(T_{p_i}W_{p_i}^{ss}, e_i^-) = T_{p_i}W_{p_i}^s$, **(H3)** is equivalent to

$$\begin{aligned} \text{span}(T_{r_i(t)}W_{p_i}^u, T_{r_i(t)}W_{p_{i+1}}^s, e_{i+1}^+) &= \mathbf{R}^{m+n}, \quad t \gg 1, \\ \text{span}(T_{r_i(t)}W_{p_i}^u, T_{r_i(t)}W_{p_{i+1}}^s, e_{i+1}^-) &= \mathbf{R}^{m+n}, \quad t \ll -1. \end{aligned}$$

(H4). $\beta_1 = \rho_1^1/\lambda_1^1 > 1$, $\beta_2 = \rho_2^1/\lambda_2^1 < 1$, $\beta_1\beta_2 > 1$.

Now, we study the following C^r perturbed system of (1.1)

$$\dot{z} = f(z) + g(z, \mu), \quad (1.2)$$

where $\mu \in \mathbf{R}^l$, $l \geq 2$, $0 \leq |\mu| \ll 1$, $g(z, 0) = 0$. For simplicity, we suppose that $g(p_i, \mu) = 0$, $i = 1, 2$.

2. Local coordinate systems and bifurcation equations

Denote $\mathcal{F}(z, \mu) = f(z) + g(z, \mu)$. Then, in a sufficiently small neighborhood U_i of $z = p_i$, system (1.2) has the following form

$$\dot{z} = \left(\frac{D\mathcal{F}(z, \mu)}{Dz} \Big|_{z=p_i} \right) z + o(z), \quad (2.1)$$

where, $\frac{D\mathcal{F}(z, \mu)}{Dz} \Big|_{z=p_i}$ is the coefficient matrix of the linear approximation system, $o(z)$ is the high order term of z . According to the standard theory of Jordan, there is a nonsingular linear transformation such that (2.1) has the following form

$$\dot{z} = \begin{pmatrix} \lambda_i^1(\mu) & 0 & 0 & 0 \\ 0 & -\rho_i^1(\mu) & 0 & 0 \\ 0 & 0 & B_i^1(\mu) & 0 \\ 0 & 0 & 0 & -B_i^2(\mu) \end{pmatrix} z + o(z),$$

where $B_i^1(\mu)$ is an $(n-1)$ -order matrix, $-B_i^2(\mu)$ is an $(m-1)$ -order matrix, and the real parts of the eigenvalues of $B_i^1(\mu)$ are larger than $\lambda_i^1(\mu)$, while the real parts of the eigenvalues of $-B_i^2(\mu)$

are smaller than $-\rho_i^1(\mu)$, that is $\operatorname{Re}\sigma(B_i^1(\mu)) > \lambda_i^1(\mu)$, $\operatorname{Re}\sigma(-B_i^2(\mu)) < -\rho_i^1(\mu)$. $\lambda_i^1(0) = \lambda_i^1$, $\rho_i^1(0) = \rho_i^1$, $\operatorname{Re}\sigma(B_i^1(0)) > \lambda_i^0$, $\operatorname{Re}\sigma(-B_i^2(0)) < -\rho_i^0$, $z = (x, y, u^*, v^*)^*$, $x, y \in \mathbf{R}^1$, $u \in \mathbf{R}^{n-1}$, $v \in \mathbf{R}^{m-1}$, $*$ means transposition. That is to say, the system has the following form in U_i :

$$\begin{cases} \dot{x} = [\lambda_i^1(\mu) + \cdots] x + [O(u) + O(y) + O(v)][O(u) + O(y) + O(v)], \\ \dot{y} = [-\rho_i^1(\mu) + \cdots] y + [O(v) + O(x) + O(u)][O(v) + O(x) + O(u)], \\ \dot{u} = [B_i^1(\mu) + \cdots] u + [O(x) + O(y) + O(v)][O(x) + O(y) + O(v)], \\ \dot{v} = [-B_i^2(\mu) + \cdots] v + [O(y) + O(x) + O(u)][O(y) + O(x) + O(u)]. \end{cases} \quad (2.2)$$

Moreover, we can introduce a C^r transformation to straighten $W_{p_i}^u$ and $W_{p_i}^s$, such that system (2.2) has the following form in U_i :

$$\begin{cases} \dot{x} = [\lambda_i^1(\mu) + \cdots] x + O(u)[O(u) + O(y) + O(v)], \\ \dot{y} = [-\rho_i^1(\mu) + \cdots] y + O(v)[O(v) + O(x) + O(u)], \\ \dot{u} = [B_i^1(\mu) + \cdots] u + O(x)[O(x) + O(y) + O(v)], \\ \dot{v} = [-B_i^2(\mu) + \cdots] v + O(y)[O(y) + O(x) + O(u)], \end{cases} \quad (2.3)$$

for $|\mu|$ sufficiently small. Thus, in U_i , we have

$$\begin{aligned} W_{p_i}^u &= \{z : y = 0, v = 0\}, & W_{p_i}^s &= \{z : x = 0, u = 0\}, \\ W_{p_i}^{uu} &= \{z : x = x(u), y = 0, v = 0\}, & W_{p_i}^{ss} &= \{z : x = 0, u = 0, y = y(v)\}, \\ \Gamma \cap W_{p_i}^u &= \{z : u = u(x), y = 0, v = 0\}, & \Gamma \cap W_{p_i}^s &= \{z : x = 0, u = 0, v = v(y)\}, \end{aligned}$$

where $u(0) = \dot{u}(0) = 0$, $v(0) = \dot{v}(0) = 0$, $x(0) = \dot{x}(0) = 0$, $y(0) = \dot{y}(0) = 0$.

Denote $r_i(t) = (r_i^x(t), r_i^y(t), (r_i^u(t))^*, (r_i^v(t))^*)^*$. Assume $r_i(-T_i^0) = (\delta, 0, \delta_{u_i}^*, 0^*)^*$, $r_i(T_i^1) = (0, \delta, 0^*, \delta_{v_i}^*)^*$, where $T_i^0 > 0$, $T_i^1 > 0$ are large enough and $\delta > 0$ is small enough such that $\{(x, y, u^*, v^*)^* : |x|, |y|, |u|, |v| < 2\delta\} \subset U_i$. Obviously, $|\delta_{u_i}| = o(\delta)$, $|\delta_{v_i}| = o(\delta)$.

Consider the linear variational system

$$\dot{z} = Df(r_i(t))z, \quad (2.4)$$

and its adjoint system

$$\dot{\phi} = -(Df(r_i(t)))^* \phi. \quad (2.5)$$

By [26, 28], we see (2.4) and (2.5) have exponential dichotomies in both $\mathbf{R}^-$ and $\mathbf{R}^+$. Similar to those of [20, 27], system (2.4) has a fundamental solution matrix $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ satisfying

$$\begin{aligned} z_i^1(t) &\in (T_{r_i(t)} W_{p_i}^u)^c \cap (T_{r_i(t)} W_{p_{i+1}}^s)^c, \\ z_i^2(t) &= -\dot{r}_i(t)/|\dot{r}_i^y(T_i^1)| \in T_{r_i(t)} W_{p_i}^u \cap T_{r_i(t)} W_{p_{i+1}}^s, \\ z_i^3(t) &= (z_i^{3,1}(t), \dots, z_i^{3,n-1}(t)) \in T_{r_i(t)} W_{p_i}^u \cap (T_{r_i(t)} W_{p_{i+1}}^s)^c, \\ z_i^4(t) &= (z_i^{4,1}(t), \dots, z_i^{4,m-1}(t)) \in (T_{r_i(t)} W_{p_i}^u)^c \cap T_{r_i(t)} W_{p_{i+1}}^s, \end{aligned}$$

$$Z_i(-T_i^0) = \begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41} \\ w_i^{12} & 0 & 0 & w_i^{42} \\ w_i^{13} & w_i^{23} & I & w_i^{43} \\ \tilde{w}_i^{14} & 0 & 0 & w_i^{44} \end{pmatrix}, \quad Z_i(T_i^1) = \begin{pmatrix} 1 & 0 & w_i^{31} & 0 \\ 0 & 1 & w_i^{32} & 0 \\ 0 & 0 & w_i^{33} & 0 \\ w_i^{14} & w_i^{24} & w_i^{34} & I \end{pmatrix},$$

where $W_3^{ss} = W_1^{ss}$, $w_i^{21} < 0$, $w_i^{12} \neq 0$, $\det w_i^{44} \neq 0$, $\det w_i^{33} \neq 0$. Moreover, for δ small enough, $\|w_i^{1j}(w_i^{12})^{-1}\| \ll 1$ for $j \neq 2$, $\|w_i^{2j}(w_i^{21})^{-1}\| \ll 1$ for $j = 3, 4$, $\|w_i^{3j}(w_i^{33})^{-1}\| \ll 1$ for $j \neq 3$, $\|w_i^{4j}(w_i^{44})^{-1}\| \ll 1$ for $j \neq 4$. It is not difficult to prove that we can choose an appropriate w_i^{14} such that $\tilde{w}_i^{14} = 0$.

Thus, we may select $z_i^1(t)$, $z_i^2(t)$, $z_i^3(t)$, $z_i^4(t)$ as a local coordinate system along Γ_i . Let $\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t)) = (Z_i^{-1}(t))^*$ be the fundamental solution matrix of (2.5). It is well known that $\phi_i^1(t)$ is bounded and tends to zero exponentially as $t \rightarrow \pm\infty$. [15, 24, 26].

Denote $\Delta_i = w_i^{12}/|w_i^{12}|$, we say that Γ_i is non-twisted as $\Delta_i = 1$, and twisted as $\Delta_i = -1$.

Let $N_i = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$, $n_i^3 = (n_i^{3,1}, \dots, n_i^{3,n-1})^*$, $n_i^4 = (n_i^{4,1}, \dots, n_i^{4,m-1})^*$, $h_i(t) = r_i(t) + Z_i(t)N_i$. We define $S_i^0 = \{z = h_i(-T_i^0) : |x|, |y|, |u|, |v| < 2\delta\}$, $S_i^1 = \{z = h_i(T_i^1) : |x|, |y|, |u|, |v| < 2\delta\}$ be the cross sections of Γ_i at $t = -T_i^0$ and $t = T_i^1$, respectively.

Now, we consider the map $F_i^0 : q_{i-1}^1 \in S_{i-1}^1 \mapsto q_i^0 \in S_i^0$, and the map $F_i^1 : q_i^0 \in S_i^0 \mapsto q_i^1 \in S_i^1$, where $S_0^1 = S_2^1$, $q_0^1 = q_2^1$. Denote $q_i^0 = (x_i^0, y_i^0, (u_i^0)^*, (v_i^0)^*)^* = r_i(-T_i^0) + Z_i(-T_i^0)N_i$, $N_i^0 = (n_i^{0,1}, 0, (n_i^{0,3})^*, (n_i^{0,4})^*)^*$, $q_i^1 = (x_i^1, y_i^1, (u_i^1)^*, (v_i^1)^*)^* = r_i(T_i^1) + Z_i(T_i^1)N_i^1$, $N_i^1 = (n_i^{1,1}, 0, (n_i^{1,3})^*, (n_i^{1,4})^*)^*$.

By the expressions of $Z_i(-T_i^0)$ and $Z_i(T_i^1)$, $i = 1, 2$, we get $y_i^1 \approx \delta$, $x_i^0 \approx \delta$, and

$$\begin{cases} n_i^{1,1} = x_i^1 - w_i^{31}(w_i^{33})^{-1}u_i^1, \\ n_i^{1,3} = (w_i^{33})^{-1}u_i^1, \\ n_i^{1,4} = -w_i^{14}x_i^1 + (w_i^{14}w_i^{31} + w_i^{24}w_i^{32} - w_i^{34})(w_i^{33})^{-1}u_i^1 + v_i^1 - \delta_{v_i}, \end{cases} \quad (2.6)$$

$$\begin{cases} n_i^{0,1} = (w_i^{12})^{-1}(y_i^0 - w_i^{42}(w_i^{44})^{-1}v_i^0), \\ n_i^{0,3} = u_i^0 - \delta_{u_i} + b_i(w_i^{12})^{-1}y_i^0 + [a_i^3 - w_i^{23}(w_i^{21})^{-1}a_i^1](w_i^{44})^{-1}v_i^0, \\ n_i^{0,4} = (w_i^{44})^{-1}v_i^0, \end{cases} \quad (2.7)$$

where, $b_i = w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}$, $a_i^j = w_i^{1j}(w_i^{12})^{-1}w_i^{42} - w_i^{4j}$, $j = 1, 3$.

Suppose that $z = h_i(t)$ is the solution of (1.2) in some small tube neighborhood of Γ_i , and substituting it into (1.2) and using $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$, we have

$$Z_i(t)(\dot{n}_i^1, 0, (\dot{n}_i^3)^*, (\dot{n}_i^4)^*)^* = g_\mu(r_i(t), 0)\mu + h.o.t.$$

Multiplying both sides by $(\Phi_i(t))^*$, and using $(\Phi_i(t))^*Z_i(t) = I$, we have

$$(\dot{n}_i^1, 0, (\dot{n}_i^3)^*, (\dot{n}_i^4)^*)^* = (\Phi_i(t))^*g_\mu(r_i(t), 0)\mu + h.o.t.$$

Thus, we obtain the maps F_i^1 defined by

$$n_i^{1,j} = n_i^{0,j} + M_i^j\mu + h.o.t., \quad j = 1, 3, 4, \quad (2.8)$$

where, $M_i^j = \int_{-\infty}^{+\infty} \phi_i^{j*}(t) g_\mu(r_i(t), 0) dt$, $i = 1, 2$, $j = 1, 3, 4$ are called Melnikov vectors.

By **(H4)** and the continuity of $\beta_i(\mu) = \rho_i^1(\mu)/\lambda_i^1(\mu)$, $i = 1, 2$, we have $\beta_1(\mu) > 1$, $\beta_2(\mu) < 1$ and $\beta_1(\mu)\beta_2(\mu) > 1$ for $|\mu| \ll 1$. We may as well denote $\beta_i(\mu) := \beta_i$, $i = 1, 2$. Let τ_i be the flying time from q_{i-1}^1 to q_i^0 , $s_1 = e^{-\lambda_1^1(\mu)\tau_1}$, $s_2 = e^{-\rho_2^1(\mu)\tau_2}$, then, using the linearization of (1.2) at p_i , we can easily get F_1^0 defined by

$$x_1^0 \approx s_1 \delta, \quad y_1^0 \approx s_1^{\beta_1} \delta, \quad u_1^0 \approx s_1^{B_1^1(\mu)/\lambda_1^1(\mu)} u_1^0, \quad v_1^0 \approx s_1^{B_1^2(\mu)/\lambda_1^1(\mu)} v_1^0, \quad (2.9)$$

and F_2^0 defined by

$$x_1^1 \approx s_2^{\frac{1}{\beta_2}} \delta, \quad y_2^0 \approx s_2 \delta, \quad u_1^1 \approx s_2^{B_2^1(\mu)/\rho_2^1(\mu)} u_2^0, \quad v_2^0 \approx s_2^{B_2^2(\mu)/\rho_2^1(\mu)} v_1^1, \quad (2.10)$$

if we neglect the higher order terms, (s_i, u_i^0, v_{i-1}^1) , $i = 1, 2$ are called Silnikov coordinates.

Thus, we have defined the Poincaré maps $F_1 = F_1^1 \circ F_1^0$: $S_2^1 \mapsto S_1^1$ as

$$\begin{cases} n_1^{1,1} = (w_1^{12})^{-1} \delta s_1^{\beta_1} + M_1^1 \mu + h.o.t., \\ n_1^{1,3} = u_1^0 - \delta_{u_1} + b_1 (w_1^{12})^{-1} \delta s_1^{\beta_1} + M_1^3 \mu + h.o.t., \\ n_1^{1,4} = (w_1^{44})^{-1} s_1^{B_1^2(\mu)/\lambda_1^1(\mu)} v_1^0 + M_1^4 \mu + h.o.t., \end{cases} \quad (2.11)$$

and $F_2 = F_2^1 \circ F_2^0$: $S_1^1 \mapsto S_2^1$ as

$$\begin{cases} n_2^{1,1} = (w_2^{12})^{-1} \delta s_2 + M_2^1 \mu + h.o.t., \\ n_2^{1,3} = u_2^0 - \delta_{u_2} + b_2 (w_2^{12})^{-1} \delta s_2 + M_2^3 \mu + h.o.t., \\ n_2^{1,4} = (w_2^{44})^{-1} s_2^{B_2^2(\mu)/\rho_2^1(\mu)} v_1^1 + M_2^4 \mu + h.o.t. \end{cases} \quad (2.12)$$

Let $G_i(q_{i-1}^1) = (G_i^1, G_i^3, G_i^4) = F_i(q_{i-1}^1) - q_i^1$, $q_0^1 = q_2^1$. Owing to (2.6), (2.9)~(2.12), we get the successor functions G_i as follows:

$$\begin{cases} G_1^1 = \delta[(w_1^{12})^{-1} s_1^{\beta_1} - s_2^{\frac{1}{\beta_2}}] + M_1^1 \mu + h.o.t., \\ G_1^3 = u_1^0 - \delta_{u_1} + b_1 (w_1^{12})^{-1} \delta s_1^{\beta_1} - (w_1^{33})^{-1} s_2^{B_2^1(\mu)/\rho_2^1(\mu)} u_2^0 + M_1^3 \mu + h.o.t., \\ G_1^4 = -v_1^1 + \delta_{v_1} + w_1^{14} \delta s_2^{\frac{1}{\beta_2}} + (w_1^{44})^{-1} s_1^{B_1^2(\mu)/\lambda_1^1(\mu)} v_1^0 + M_1^4 \mu + h.o.t., \\ G_2^1 = \delta[(w_2^{12})^{-1} s_2 - s_1] + M_2^1 \mu + h.o.t., \\ G_2^3 = u_2^0 - \delta_{u_2} + b_2 (w_2^{12})^{-1} \delta s_2 - (w_2^{33})^{-1} s_1^{B_1^1(\mu)/\lambda_1^1(\mu)} u_1^0 + M_2^3 \mu + h.o.t., \\ G_2^4 = -v_1^1 + \delta_{v_2} + w_2^{14} \delta s_1 + (w_2^{44})^{-1} s_2^{B_2^2(\mu)/\rho_2^1(\mu)} v_1^1 + M_2^4 \mu + h.o.t. \end{cases} \quad (2.13)$$

We call the following equation the bifurcation equation.

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0. \quad (2.14)$$

Thus, there is a one-to-one correspondence between the 1-heteroclinic orbit, 1-homoclinic orbit, 1-periodic orbit of (1.2) and the solution $Q = (s_1, s_2, u_1^0, u_2^0, v_1^1, v_2^1)$ of (2.14) with $s_1 \geq 0$, $s_2 \geq 0$.

It is easy to see that the equation $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ always has a solution $u_i^0 = u_i^0(s_1, s_2, \mu)$, $v_i^1 = v_i^1(s_1, s_2, \mu)$ $i = 1, 2$ for $\delta, |\mu|, s_1, s_2$ sufficiently small. Substituting it into $(G_1^1, G_2^1) = 0$, we get

$$\begin{cases} (w_1^{12})^{-1} s_1^{\beta_1} - s_2^{\frac{1}{\beta_2}} + \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ (w_2^{12})^{-1} s_2 - s_1 + \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases} \quad (2.15)$$

Thus, we only need to consider the solutions $s_1 \geq 0, s_2 \geq 0$ of (2.15).

3. The preservations of heteroclinic orbits and bifurcations of homoclinic loops

Denote $R_1^2 = \{\mu : M_1^1 \mu > 0, \Delta_2 M_2^1 \mu < 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : \Delta_1 M_1^1 \mu < 0, M_2^1 \mu > 0, |\mu| \ll 1\}$. Then, we have the following theorem.

Theorem 3.1. *Suppose that the hypotheses (H1)~(H4) are valid, and $\text{rank}(M_1^1, M_2^1) = 2$, $|\mu| \ll 1$. Then*

- (i) *There exists a $(l-1)$ -dimensional surface L_i with normal vector M_i^1 at $\mu = 0$, such that (1.2) has a heteroclinic orbit joining p_1 and p_2 near Γ_i if and only if $\mu \in L_i$, $i = 1, 2$. Moreover, (1.2) has a heteroclinic loop near Γ if and only if $\mu \in L_{12} = L_1 \cap L_2$ which is a $(l-2)$ -dimensional surface with normal plane $\text{span}\{M_1^1, M_2^1\}$ at $\mu = 0$.*
- (ii) *There exists a $(l-1)$ -dimensional surface $L_1^2 \subset R_1^2$ which is tangent to L_1 at $\mu = 0$ such that (1.2) has a unique homoclinic loop Γ_1^2 connecting p_1 for $\mu \in L_1^2$. Meanwhile, there also exists a $(l-1)$ -dimensional surface $L_2^1 \subset R_2^1$ which is tangent to L_1 at $\mu = 0$ such that (1.2) has a unique homoclinic loop Γ_2^1 connecting p_2 for $\mu \in L_2^1$.*

Proof. If (2.15) has solution $s_1 = 0, s_2 = 0$, then we get

$$M_i^1 \mu + h.o.t. = 0, i = 1, 2. \quad (3.1)$$

Denote by L_i the $(l-1)$ -dimensional surface defined by (3.1). Then the conclusion (i) holds. If (2.15) has a solution $s_1 = 0, s_2 > 0$, then we get

$$\begin{aligned} s_2 &= -\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t., \\ (-\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{1}{\beta_2}} &= \delta^{-1} M_1^1 \mu + h.o.t. \end{aligned} \quad (3.2)$$

If (2.15) has a solution $s_1 > 0, s_2 = 0$, then we get

$$\begin{aligned} s_1 &= \delta^{-1} M_2^1 \mu + h.o.t., \\ (\delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1} &= -\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. \end{aligned} \quad (3.3)$$

Let L_1^2 be the $(l-1)$ -dimensional surface defined by (3.2), and L_2^1 be the $(l-1)$ -dimensional surface defined by (3.3), then, the conclusion (ii) holds.

Moreover, for the case $\Delta_1 = \Delta_2 = -1$, if $-w_1^{12}(-w_2^{12})^{\frac{1}{\beta_2}} > 1$, then, by (3.2), (3.3) and $\beta_1 > \frac{1}{\beta_2} > 1$, we get

$$\delta^{-1} M_1^1 \mu|_{L_1} = 0$$

$$\begin{aligned}
&< \delta^{-1} M_1^1 \mu|_{L_2^1} \\
&= (-w_1^{12})^{-1} (\delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1} + h.o.t. \\
&< (-w_2^{12})^{\frac{1}{\beta_2}} (\delta^{-1} M_2^1 \mu + h.o.t.)^{\frac{1}{\beta_2}} + h.o.t. \\
&= \delta^{-1} M_1^1 \mu|_{L_1^2}
\end{aligned}$$

for $\mu \in \{M_1^1 \mu > 0, M_2^1 \mu > 0, |\mu| \ll 1\}$, this is to say, L_2^1 is located in the open region bounded by L_1 and L_1^2 . $\square$

4. Twisted bifurcations and nontwisted bifurcations

If (2.15) has solutions $s_1 \geq 0, s_2 \geq 0$, then, we have

$$(w_2^{12}(s_1 - \delta^{-1} M_2^1 \mu + h.o.t.))^{\frac{1}{\beta_2}} = (w_1^{12})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^1 \mu + h.o.t., \quad (4.1)$$

$$((w_2^{12})^{-1} s_2 + \delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1} = w_1^{12} (s_2^{\frac{1}{\beta_2}} - \delta^{-1} M_1^1 \mu) + h.o.t. \quad (4.2)$$

Denote $\mathcal{L}_1(s_1)$ and $\mathcal{R}_1(s_1)$ as the left and right hand of (4.1), $\mathcal{L}_2(s_2)$ and $\mathcal{R}_2(s_2)$ as the left and right hand of (4.2), respectively.

4.1. Twisted bifurcations

(A1). $\Delta_1 = \Delta_2 = -1$.

In this case, $R_1^2 = R_2^1 = \{\mu : M_1^1 \mu > 0, M_2^1 \mu > 0, |\mu| \ll 1\}$, Denote $R := R_1^2 = R_2^1$, $R_0 = \{\mu : M_1^1 \mu < 0, |\mu| \ll 1\} \cup \{\mu : M_2^1 \mu < 0, |\mu| \ll 1\}$.

Theorem 4.1. *Suppose that the hypotheses (H1)~(H4) and (A1) hold. Then system (1.2) has not any 1-periodic and 1-homoclinic loop near Γ as $\mu \in R_0$.*

Proof. Because of $w_1^{12} < 0, w_2^{12} < 0$, so, it is easy to see that, (2.15) has not any nonnegative solution satisfying $s_1 \geq 0, s_2 \geq 0$ if $M_1^1 \mu < 0$ or $M_2^1 \mu < 0$. $\square$

So, we only need to consider the bifurcations in R . If (2.15) has solutions $s_1 \geq 0, s_2 \geq 0$ for $\mu \in R$, then, we have

$$\begin{aligned}
s_1 &\leq \delta^{-1} M_2^1 \mu + h.o.t., & s_2 &\leq -\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t., \\
s_1^{\beta_1} &\leq -\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t., & s_2^{\frac{1}{\beta_2}} &\leq \delta^{-1} M_1^1 \mu + h.o.t.
\end{aligned} \quad (4.3)$$

Lemma 4.1. *Suppose that the hypotheses (H1)~(H4) and (A1) hold. Then the curve $h = \mathcal{L}_2(s_2)$ is tangent to $h = \mathcal{R}_2(s_2)$ at some point $s_2 \in (0, -\delta^{-1} w_2^{12} M_2^1 \mu)$, if and only if $\mu \in R$, $|M_1^1 \mu| \ll |M_2^1 \mu|$ and $\mu \in \tilde{L}_2^1$, where $\tilde{L}_2^1$ is defined by*

$$\begin{aligned}
\delta^{-1} M_1^1 \mu &= \left((w_2^{12})^{-1} \left(-\frac{\beta_1 \beta_2 M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{\frac{1}{\beta_2} - 1} + \delta^{-1} M_2^1 \mu + h.o.t. \right)^{\beta_1 - 1} \\
&\quad \times \left(-\delta^{-1} (w_1^{12})^{-1} M_2^1 \mu \right)
\end{aligned} \quad (4.4)$$

in the open region bounded by L_1 and L_2^1 . Moreover, $\tilde{L}_2^1$ is tangent to L_1 at $\mu = 0$.

Proof. $h = \mathcal{L}_2(s_2)$ is tangent to $h = \mathcal{R}_2(s_2)$ at point s_2 if and only if $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ and $\dot{\mathcal{L}}_2(s_2) = \dot{\mathcal{R}}_2(s_2)$, that is (4.2) and

$$\beta_1(w_2^{12})^{-1} \left((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t. \right)^{\beta_1-1} = \frac{1}{\beta_2}w_1^{12}s_2^{\frac{1}{\beta_2}-1} + h.o.t. \quad (4.5)$$

By dividing the two sides of the two formulas (4.2) and (4.5), and some simple calculations, it is not difficult to get that (4.2) and (4.5) have a unique small positive solution

$$s_2 = \left(-\frac{\beta_1\beta_2M_1^1\mu}{w_2^{12}M_2^1\mu} \right)^{\frac{1}{\frac{1}{\beta_2}-1}} + h.o.t.,$$

for $|M_1^1\mu| \ll |M_2^1\mu|$. Substituting it into (4.5), we get that (4.4) holds.

Due to $\beta_1 > 1$, $\beta_2 < 1$, $\beta_1\beta_2 > 1$, we have $\frac{\beta_2}{1-\beta_2} > 0$, $\frac{(\beta_1-1)\beta_2}{1-\beta_2} > 1$. Therefore, $\tilde{L}_2^1$ is tangent to L_1 at $\mu = 0$.

By (3.3) and (4.4), we have

$$\begin{aligned} \delta^{-1}M_1^1\mu|_{\tilde{L}_2^1} &= \left((w_2^{12})^{-1} \left(-\frac{\beta_1\beta_2M_1^1\mu}{w_2^{12}M_2^1\mu} \right)^{\frac{1}{\frac{1}{\beta_2}-1}} + \delta^{-1}M_2^1\mu + h.o.t. \right)^{\beta_1-1} \\ &\quad \times \left(-\delta^{-1}(w_1^{12})^{-1}M_2^1\mu \right) \\ &< (\delta^{-1}M_2^1\mu + h.o.t.)^{\beta_1-1} \times \left(-\delta^{-1}(w_1^{12})^{-1}M_2^1\mu \right) \\ &= (-w_1^{12})^{-1}(\delta^{-1}M_2^1\mu + h.o.t.)^{\beta_1} \\ &= \delta^{-1}M_1^1\mu|_{L_2^1}. \end{aligned}$$

So, $\tilde{L}_2^1$ located in the open region bounded by L_1 and L_2^1 . $\square$

Thus, we define the following four open sub-regions of R : R_1 is the region whose boundaries are L_1 and $\tilde{L}_2^1$, R_2 is the region whose boundaries are $\tilde{L}_2^1$ and L_2^1 , R_3 is the region whose boundaries are L_2^1 and L_1^2 , R_4 is the region whose boundaries are L_1^2 and L_2 , all of R_i , $i = 1, 2, 3, 4$ have nonempty intersection with R .

Lemma 4.2. Suppose that the hypotheses (H1)~(H4) and (A1) hold, and $-w_1^{12}(-w_2^{12})^{\frac{1}{\beta_2}} > 1$. Then

- (i) (4.2) has not any positive solution $s_2 > 0$ for $\mu \in R_1$.
- (ii) (4.2) has a double positive solution $s_{20} > 0$ for $\mu \in \tilde{L}_2^1$.
- (iii) (4.2) has two single positive solutions $s_{22} > 0$, $s_{23} > 0$ for $\mu \in R_2$.
- (iv) For $\mu \in L_2^1$, (4.2) has a unique positive solution $\hat{s}_2 > 0$ in addition to the zero solution $s_2 = 0$. Moreover, the positive solution is persistent for μ in the neighborhood of L_2^1 .
- (v) (4.2) has a unique positive solution $s_{21} > 0$ for $\mu \in R_3$.

Proof.

$$\begin{aligned} \mathcal{L}_2(s_2) &= ((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t.)^{\beta_1}, \quad \mathcal{R}_2(s_2) = w_1^{12}(s_2^{\frac{1}{\beta_2}} - \delta^{-1}M_1^1\mu) + h.o.t., \\ \dot{\mathcal{L}}_2(s_2) &= \beta_1(w_2^{12})^{-1} \left((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t. \right)^{\beta_1-1} \begin{cases} < 0, & 0 < s_2 < -\delta^{-1}w_2^{12}M_2^1\mu, \\ = 0, & s_2 = -\delta^{-1}w_2^{12}M_2^1\mu, \end{cases} \end{aligned}$$

$$\dot{\mathcal{R}}_2(s_2) = \frac{1}{\beta_2} w_1^{12} s_2^{\frac{1}{\beta_2}-1} + h.o.t. \begin{cases} < 0, & 0 < s_2 < -\delta^{-1} w_2^{12} M_2^1 \mu, \\ = 0, & s_2 = 0, \end{cases}$$

$$\ddot{\mathcal{L}}_2(s_2) = \beta_1(\beta_1 - 1)(w_2^{12})^{-2} ((w_2^{12})^{-1} s_2 + \delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1-2} \begin{cases} > 0, & 0 < s_2 < -\delta^{-1} w_2^{12} M_2^1 \mu, \\ = 0, & s_2 = -\delta^{-1} w_2^{12} M_2^1 \mu, \end{cases}$$

$$\ddot{\mathcal{R}}_2(s_2) = \frac{1}{\beta_2} \left(\frac{1}{\beta_2} - 1 \right) w_1^{12} s_2^{\frac{1}{\beta_2}-2} + h.o.t. \begin{cases} < 0, & 0 < s_2 < -\delta^{-1} w_2^{12} M_2^1 \mu, \\ = 0, & s_2 = 0. \end{cases}$$

For $\mu \in L_2^1$, by (3.3), we know that $\mathcal{L}_2(0) = \mathcal{R}_2(0)$. Therefore, there exists $\bar{s}_2 > 0$, such that for all $0 < s_2 \leq \bar{s}_2$, $\mathcal{L}_2(s_2) < \mathcal{R}_2(s_2)$.

If $-w_1^{12}(-w_2^{12})^{\frac{1}{\beta_2}} > 1$, that is, $(-w_2^{12})^{\frac{1}{\beta_2}} > -(w_1^{12})^{-1}$, then, by $\beta_1 > \frac{1}{\beta_2}$, we have

$$(-\delta^{-1} w_2^{12} M_2^1 \mu)^{\frac{1}{\beta_2}} > -(w_1^{12})^{-1} (\delta^{-1} M_2^1 \mu)^{\beta_1}.$$

Due to

$$-(w_1^{12})^{-1} (\delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1} = \delta^{-1} M_1^1 \mu + h.o.t.$$

So

$$(-\delta^{-1} w_2^{12} M_2^1 \mu)^{\frac{1}{\beta_2}} > \delta^{-1} M_1^1 \mu.$$

Let $\bar{s}_2 = -\delta^{-1} w_2^{12} M_2^1 \mu$, then $\mathcal{L}_2(\bar{s}_2) = 0$, $\mathcal{R}_2(\bar{s}_2) = w_1^{12} \left((-\delta^{-1} w_2^{12} M_2^1 \mu)^{\frac{1}{\beta_2}} - \delta^{-1} M_1^1 \mu \right) + h.o.t. < 0$. So, $\mathcal{L}_2(\bar{s}_2) > \mathcal{R}_2(\bar{s}_2)$.

Let $\bar{\bar{s}}_2 = (\delta^{-1} M_1^1 \mu)^{\beta_2}$, then $\mathcal{R}_2(\bar{\bar{s}}_2) = 0$. Notice that $\mathcal{R}_2(\bar{s}_2) < 0$ and the monotonicity of $\mathcal{R}_2(s_2)$, we get $\bar{s}_2 > \bar{\bar{s}}_2$. Again by the monotonicity of $\mathcal{L}_2(s_2)$ and $\mathcal{L}_2(\bar{s}_2) = 0$, we have $\mathcal{L}_2(\bar{\bar{s}}_2) > 0$. So $\mathcal{L}_2(\bar{\bar{s}}_2) > \mathcal{R}_2(\bar{\bar{s}}_2)$.

Therefore, there is $\hat{s}_2 \in (\bar{s}_2, \bar{\bar{s}}_2) \subset (\bar{s}_2, \bar{\bar{s}}_2)$, such that $\mathcal{L}_2(\hat{s}_2) = \mathcal{R}_2(\hat{s}_2)$.

Moreover, by the monotonicity and convexity of $\mathcal{L}_2(s_2)$ and $\mathcal{R}_2(s_2)$, we can easily get that $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has a unique solution in $(\bar{s}_2, \bar{\bar{s}}_2)$.

Thus, (iv) holds. Moreover, by (3.3) and some simple calculations, we have that conclusions (i), (ii), (iii), (v) hold, these are

- (i) For $\mu \in R_1$, we have $\mathcal{L}_2(0) > \mathcal{R}_2(0)$, and (4.2) has not any positive solution satisfying $s_2 > 0$.
- (ii) For $\mu \in \tilde{L}_2^1$, we have $\mathcal{L}_2(0) > \mathcal{R}_2(0)$, and (4.2) has a double positive solution $s_{20} > 0$.
- (iii) For $\mu \in R_2$, we have $\mathcal{L}_2(0) > \mathcal{R}_2(0)$, and (4.2) has two single positive solutions $s_{22} > 0$, $s_{23} > 0$.
- (iv) For $\mu \in L_2^1$, we have $\mathcal{L}_2(0) = \mathcal{R}_2(0)$, and (4.2) has a unique positive solution $\hat{s}_2 > 0$ in addition to the zero solution $s_2 = 0$.
- (v) For $\mu \in R_3$, we have $\mathcal{L}_2(0) < \mathcal{R}_2(0)$, and (4.2) has a unique positive solution $s_{21} > 0$.

□

Lemma 4.3. Suppose that the hypotheses **(H1)~(H4)** and **(A1)** hold. Then

- (i) In addition to the zero solution $s_1 = 0$, (4.1) have not any positive solution $s_1 > 0$ for $\mu \in L_1^2$.

- (ii) (4.1) has a unique positive solution $s_{11} > 0$ for $\mu \in R_4$.
 (iii) (4.1) has not any positive solution $s_1 > 0$ for $\mu \in R_3$.
 (iv) The curve $h = \mathcal{L}_1(s_1)$ can not be tangent to $h = \mathcal{R}_1(s_1)$ at any point s_1 satisfying $0 \leq s_1 \ll 1$.

Proof.

$$\begin{aligned}\mathcal{L}_1(s_1) &= (w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.))^{\frac{1}{\beta_2}}, \quad \mathcal{R}_1(s_1) = (w_1^{12})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^1\mu + h.o.t., \\ \dot{\mathcal{L}}_1(s_1) &= \frac{1}{\beta_2}w_2^{12}(w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.))^{\frac{1}{\beta_2}-1} \begin{cases} < 0, & 0 \leq s_1 < \delta^{-1}M_2^1\mu, \\ = 0, & s_1 = \delta^{-1}M_2^1\mu, \end{cases} \\ \dot{\mathcal{R}}_1(s_1) &= \beta_1(w_1^{12})^{-1}s_1^{\beta_1-1} \begin{cases} = 0, & s_1 = 0, \\ < 0, & 0 < s_1 \leq \delta^{-1}M_2^1\mu, \end{cases} \\ \ddot{\mathcal{L}}_1(s_1) &= \frac{1}{\beta_2}(\frac{1}{\beta_2} - 1)(w_2^{12})^2(w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.))^{\frac{1}{\beta_2}-2} \begin{cases} > 0, & 0 \leq s_1 < \delta^{-1}M_2^1\mu, \\ = 0, & s_1 = \delta^{-1}M_2^1\mu, \end{cases} \\ \ddot{\mathcal{R}}_1(s_1) &= \beta_1(\beta_1 - 1)(w_1^{12})^{-1}s_1^{\beta_1-2} \begin{cases} = 0, & s_1 = 0, \\ < 0, & 0 < s_1 \leq \delta^{-1}M_2^1\mu. \end{cases}\end{aligned}$$

If $-w_1^{12}(-w_2^{12})^{\frac{1}{\beta_2}} > 1$, that is, $(-w_2^{12})^{\frac{1}{\beta_2}} > -(w_1^{12})^{-1}$, then, by $\beta_1 > \frac{1}{\beta_2}$, we have

$$(-\delta^{-1}w_2^{12}M_2^1\mu)^{\frac{1}{\beta_2}} > -(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{\beta_1}.$$

(a) For $\mu \in L_1^2$, $\mathcal{L}_1(0) = \mathcal{R}_1(0)$, that is

$$(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{\beta_2}} = \delta^{-1}M_1^1\mu + h.o.t.$$

So

$$-(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{\beta_1} < \delta^{-1}M_1^1\mu.$$

Let $\bar{s}_1 = \delta^{-1}M_2^1\mu$, then, $\mathcal{L}_1(\bar{s}_1) = 0$,

$$\mathcal{R}_1(\bar{s}_1) = (w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{\beta_1} + \delta^{-1}M_1^1\mu + h.o.t. > 0.$$

Let $\bar{\bar{s}}_1 = (-\delta^{-1}w_1^{12}M_1^1\mu)^{\frac{1}{\beta_1}}$, then, $\mathcal{R}_1(\bar{\bar{s}}_1) = 0$. Notice that $\mathcal{R}_1(\bar{s}_1) > 0$ and the monotonicity of $\mathcal{R}_1(s_1)$, we get $\bar{\bar{s}}_1 > \bar{s}_1$.

Therefore, by the monotonicity and convexity of $\mathcal{L}_1(s_1)$ and $\mathcal{R}_1(s_1)$, we can easily get, there is not any $\hat{s}_1 \in (0, \bar{s}_1) \subset (0, \bar{\bar{s}}_1)$, such that $\mathcal{L}_1(\hat{s}_1) = \mathcal{R}_1(\hat{s}_1)$. So, (i) holds.

(b) By (3.2) and some simple calculations, we have

$$\begin{cases} \mathcal{L}_1(0) < \mathcal{R}_1(0), & \mu \in R_4, \\ \mathcal{L}_1(0) = \mathcal{R}_1(0), & \mu \in L_1^2, \\ \mathcal{L}_1(0) > \mathcal{R}_1(0), & \mu \in R_3. \end{cases}$$

Therefore, we get the conclusions (ii), (iii), (iv). $\square$

Theorem 4.2. Suppose that the hypotheses **(H1)~(H4)** and **(A1)** hold. Then in the open region R , there are three $(l-1)$ -dimensional surfaces L_1^2 , $\tilde{L}_2^1$ and L_2^1 , all tangent to L_1 at $\mu = 0$, and the four open regions: R_1 , R_2 , R_3 , R_4 , such that the following conclusions are true (see Figure 1).

- (i) System (1.2) has no any 1-periodic orbit and 1-homoclinic loop near Γ as $\mu \in R_1 \cup R_0 \cup R_4$.
- (ii) System (1.2) has a unique double 1-periodic orbit near Γ as $\mu \in \tilde{L}_2^1$.
- (iii) System (1.2) has exactly two simple 1-periodic orbits near Γ as $\mu \in R_2$.
- (iv) System (1.2) has exactly one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$.
- (v) System (1.2) has exactly one simple 1-periodic orbit near Γ as $\mu \in R_3$.
- (vi) System (1.2) has exactly one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$.

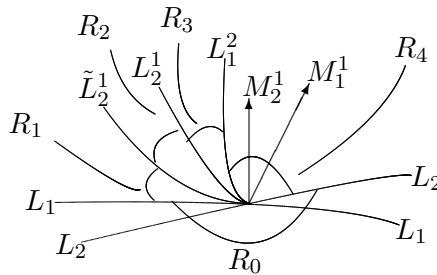


Figure 1. Bifurcation graph for $\Delta_1 = -1$, $\Delta_2 = -1$.

(A2). $\Delta_1 = 1$, $\Delta_2 = -1$.

In this case, $R_1^2 = \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu < 0, M_2^1\mu > 0, |\mu| \ll 1\}$, denote $R_0 = \{\mu : M_2^1\mu < 0, |\mu| \ll 1\}$.

Thus, we can discuss in a similar way and obtain the following theorem.

Theorem 4.3. Suppose that the hypotheses **(H1)~(H4)** and **(A2)** hold. Then, there are four open regions: $(R_1^2)_0$ is the region whose boundaries are L_2 and L_1^2 , $(R_1^1)_1$ is the region whose boundaries are L_1^2 and L_1 , $(R_2^1)_0$ is the region whose boundaries are L_2 and L_2^1 , $(R_2^1)_1$ is the region whose boundaries are L_2^1 and L_1 , such that the following conclusions are true.

- (i) System (1.2) has not any 1-periodic orbit or 1-homoclinic loop near Γ as $\mu \in R_0 \cup (R_1^2)_0 \cup (R_2^1)_0$.
- (ii) System (1.2) has exactly one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$, but no any simple 1-periodic orbit.
- (iii) System (1.2) has exactly one simple 1-periodic orbit near Γ as $\mu \in (R_1^1)_1 \cup (R_2^1)_1$.
- (iv) System (1.2) has exactly one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$, but no any simple 1-periodic orbit.

(A3). $\Delta_1 = -1$, $\Delta_2 = 1$.

In this case, $R_1^2 = \{\mu : M_1^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\}$, denote $R_0 = \{\mu : M_1^1\mu < 0, |\mu| \ll 1\}$.

We can also discuss in a similar way and obtain the following theorem.

Theorem 4.4. *Suppose that the hypotheses (H1)~(H4) and (A3) hold. Then, there are four open regions: $(R_1^2)_0$ is the region whose boundaries are L_1 and L_1^2 , $(R_1^2)_1$ is the region whose boundaries are L_1^2 and L_2 , $(R_2^1)_0$ is the region whose boundaries are L_1 and L_2^1 , $(R_2^1)_1$ is the region whose boundaries are L_2^1 and L_2 , such that the following conclusions are true.*

- (i) *System (1.2) has not any 1-periodic orbit or 1-homoclinic loop near Γ as $\mu \in R_0 \cup (R_1^2)_0 \cup (R_2^1)_0$.*
- (ii) *System (1.2) has exactly one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$, but no any simple 1-periodic orbit.*
- (iii) *System (1.2) has exactly one simple 1-periodic orbit near Γ as $\mu \in (R_1^2)_1 \cup (R_2^1)_1$.*
- (iv) *System (1.2) has exactly one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$, but no any simple 1-periodic orbit.*

4.2. Nontwisted bifurcations

(A4). $\Delta_1 = \Delta_2 = 1$.

In this case, $R_1^2 = \{\mu : M_1^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu < 0, M_2^1\mu > 0, |\mu| \ll 1\}$.

Lemma 4.4. *Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then system (1.2) has no any 1-periodic orbit near Γ for $\mu \in L_1^2$ except the 1-homoclinic loop Γ_1^2 .*

Proof. The condition $\mu \in L_1^2$ and $|\mu| \ll 1$ means that (1.2) has one 1-homoclinic loop Γ_1^2 homoclinic to p_1 .

$$\begin{aligned}\mathcal{L}_1(s_1) &= (w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.))^{\frac{1}{\beta_2}}, \\ \mathcal{R}_1(s_1) &= (w_1^{12})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^1\mu + h.o.t.,\end{aligned}$$

by (3.2), we get $\mathcal{L}_1(0) = \mathcal{R}_1(0)$, as $\mu \in L_1^2$. Furthermore,

$$\begin{aligned}\dot{\mathcal{L}}_1(s_1) &= \frac{1}{\beta_2}w_2^{12}(w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.))^{\frac{1}{\beta_2}-1}, \\ \dot{\mathcal{R}}_1(s_1) &= \beta_1(w_1^{12})^{-1}s_1^{\beta_1-1}.\end{aligned}$$

Noting that $1 < \frac{1}{\beta_2} < \beta_1$, we get $\dot{\mathcal{L}}_1(s_1) > \frac{1}{\beta_2}w_2^{12}(w_2^{12})^{\frac{1}{\beta_2}-1}s_1^{\frac{1}{\beta_2}-1} > \dot{\mathcal{R}}_1(s_1)$ for $0 \leq s_1 \ll 1$. Therefore, $\mathcal{L}_1(s_1) > \mathcal{R}_1(s_1)$ for $0 < s_1 \ll 1$. The proof is complete. $\square$

Lemma 4.5. *Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then, in addition to the 1-homoclinic loop Γ_2^1 , system (1.2) has exactly one simple 1-periodic orbit near Γ for $\mu \in L_2^1$. Moreover, the 1-periodic orbit is persistent for μ in the neighborhood of L_2^1 .*

Proof. If $\mu \in L_2^1$ and $|\mu| \ll 1$, then we know that (1.2) has one 1-homoclinic loop Γ_2^1 homoclinic to p_2 .

$$\begin{aligned}\mathcal{L}_2(s_2) &= ((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t.)^{\beta_1}, \\ \mathcal{R}_2(s_2) &= w_1^{12}(s_2^{\frac{1}{\beta_2}} - \delta^{-1}M_1^1\mu) + h.o.t.,\end{aligned}$$

by (3.3), we get $\mathcal{L}_2(0) = \mathcal{R}_2(0)$, as $\mu \in L_2^1$. Moreover,

$$\dot{\mathcal{L}}_2(s_2) = \beta_1(w_2^{12})^{-1}((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t.)^{\beta_1-1},$$

$$\dot{\mathcal{R}}_2(s_2) = \frac{1}{\beta_2} w_1^{12} s_2^{\frac{1}{\beta_2}-1} + h.o.t.,$$

so, $\dot{\mathcal{L}}_2(s_2)\big|_{s_2=0} > \dot{\mathcal{R}}_2(s_2)\big|_{s_2=0} = 0$, for $\beta_1 > 1, \beta_2 < 1$. Therefore, there is a $\tilde{s}_2$, $0 < \tilde{s}_2 \ll 1$ such that for $0 < s_2 < \tilde{s}_2$, $\mathcal{L}_2(s_2) > \mathcal{R}_2(s_2)$.

Denote $s_{24} = \delta^{-1} w_2^{12} M_2^1 \mu$. Then

$$\begin{aligned}\mathcal{L}_2(s_{24}) &= (2w_2^{12})^{-\beta_1} s_{24}^{\beta_1} + h.o.t., \\ \mathcal{R}_2(s_{24}) &= w_1^{12} \left(s_{24}^{\frac{1}{\beta_2}} - \delta^{-1} M_1^1 \mu \right) + h.o.t.\end{aligned}$$

It is not difficult to check that $\mathcal{L}_2(s_{24}) < \mathcal{R}_2(s_{24})$, since $|\mu|, s_{24} \ll 1$ and $1 < \frac{1}{\beta_2} < \beta_1$.

Thus, we get that $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has at least one solution s_2^* satisfying $0 < s_2^* < s_{24}$. That is to say, (1.2) has at least one 1-periodic orbit near Γ for $\mu \in L_2^1$ and $0 < |\mu| \ll 1$.

It is easy to see that $\mathcal{L}_2(s_2) < \mathcal{R}_2(s_2)$ as $s_{24} \leq s_2 \ll 1$. In fact,

$$\begin{aligned}\dot{\mathcal{L}}_2(s_2) &= \beta_1 (w_2^{12})^{-1} ((w_2^{12})^{-1} s_2 + \delta^{-1} M_2^1 \mu + h.o.t.)^{\beta_1-1} \\ &= \beta_1 (w_2^{12})^{-\beta_1} (s_2 + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\beta_1-1} \\ &\leq \beta_1 (w_2^{12})^{-\beta_1} 2^{\beta_1-1} s_2^{\beta_1-1} \\ &< \frac{1}{\beta_2} w_1^{12} s_2^{\frac{1}{\beta_2}-1} \\ &= \dot{\mathcal{R}}_2(s_2).\end{aligned}$$

Combining it with $\mathcal{L}_2(s_{24}) < \mathcal{R}_2(s_{24})$, we can obtain $\mathcal{L}_2(s_2) < \mathcal{R}_2(s_2)$ as $s_{24} \leq s_2 \ll 1$ immediately.

Next, we prove the uniqueness of the sufficiently small positive solution of equation $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$.

Noting that $\mathcal{L}_2(s_2) - \mathcal{R}_2(s_2) = 0$ has solutions $s_2 = 0$ and $s_2 = s_2^*$, we know that $\dot{\mathcal{L}}_2(s_2) - \dot{\mathcal{R}}_2(s_2) = 0$ surely has a solution $s_2 = s_{25}$ in $(0, s_{24})$ according to Rolle's theorem. Thus we have

$$\begin{aligned}\beta_1 (w_2^{12})^{-\beta_1} (s_{25} + s_{24})^{\beta_1-1} - \frac{1}{\beta_2} w_1^{12} (s_{25})^{\frac{1}{\beta_2}-1} &= 0, \\ \frac{s_{25}}{s_{25} + s_{24}} &= (\beta_1 \beta_2)^{\frac{1}{\beta_1-1}} \left[(w_1^{12})^{-1} (w_2^{12})^{-\beta_1} \right]^{\frac{1}{\beta_1-1}} (s_{25})^{\frac{\beta_2-1}{\beta_2(\beta_1-1)}} \ll 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2 [\mathcal{L}_2(s_{25}) - \mathcal{R}_2(s_{25})]}{ds_2^2} &= (\beta_1 - 1) \beta_1 (w_2^{12})^{-\beta_1} (s_{25} + s_{24})^{\beta_1-2} \\ &\quad - \left(\frac{1}{\beta_2} - 1 \right) \frac{1}{\beta_2} w_1^{12} (s_{25})^{\frac{1}{\beta_2}-2} + h.o.t. \\ &= \frac{\beta_1 - 1}{s_{25} + s_{24}} \left(1 - \frac{1 - \beta_2}{\beta_2 (\beta_1 - 1)} \frac{s_{25} + s_{24}}{s_{25}^{\beta_1}} \right) \dot{\mathcal{L}}_2(s_{25}) \\ &> 0.\end{aligned}$$

It turns out that $s_2 = s_2^*$ is the unique sufficiently small positive solution of the equation $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$. Due to (2.15) and (3.3), (2.15) has a sufficiently small positive solution $s_1 = s_1^*$ corresponding to $s_2 = s_2^*$. Moreover, s_2^* is a simple zero of $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ which is persistent under a small perturbation of μ .

The proof is complete. $\square$

Lemma 4.6. *Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then the curve $h = \mathcal{L}_1(s_1)$ cannot be tangent to $h = \mathcal{R}_1(s_1)$ at any point s_1 satisfying $0 \leq s_1 \ll 1$.*

Proof. By direct computation, we can see that $\mathcal{L}_1(s_1) = \mathcal{R}_1(s_1)$ and $\dot{\mathcal{L}}_1(s_1) = \dot{\mathcal{R}}_1(s_1)$ are equivalent to (4.1) and

$$\frac{1}{\beta_2} w_2^{12} (w_2^{12} (s_1 - \delta^{-1} M_2^1 \mu + h.o.t.))^{1/\beta_2 - 1} = \beta_1 (w_1^{12})^{-1} s_1^{\beta_1 - 1}. \quad (4.6)$$

By dividing the two sides of equations (4.1) and (4.6), and some simple calculations, it is not difficult to show that (4.1) and (4.6) have a unique small positive solution

$$s_1 = \left(-\frac{w_1^{12} M_1^1 \mu}{\beta_1 \beta_2 M_2^1 \mu} \right)^{1/\beta_1 - 1} + h.o.t.,$$

for $|M_1^1 \mu| \ll |M_2^1 \mu|$. Substituting it into (4.6), we get

$$-\delta^{-1} M_2^1 \mu = \left(-\frac{M_1^1 \mu}{M_2^1 \mu} \right)^{\frac{\beta_2}{1-\beta_2}} (w_2^{12})^{1/\beta_2 - 1} - \left(\frac{w_1^{12}}{\beta_1 \beta_2} \right)^{1/\beta_1 - 1} \left(-\frac{M_1^1 \mu}{M_2^1 \mu} \right)^{1/\beta_1 - 1}. \quad (4.7)$$

Due to $\beta_1 > 1$, $\beta_2 < 1$, $\beta_1 \beta_2 > 1$, we have $\frac{\beta_2}{1-\beta_2} > \frac{1}{\beta_1 - 1}$. Thus, for $|M_1^1 \mu| \ll |M_2^1 \mu|$, the right-hand side of (4.7) will be negative, but $-\delta^{-1} M_2^1 \mu > 0$. This is a contradiction.

The proof is complete. $\square$

Lemma 4.7. *Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then the curve $h = \mathcal{L}_2(s_2)$ is tangent to $h = \mathcal{R}_2(s_2)$ at some point s_2 satisfying $0 \leq s_2 \ll 1$ if and only if $\mu \in R_2^1$, $|M_1^1 \mu| \ll |M_2^1 \mu|$ and*

$$\delta^{-1} M_2^1 \mu = \left(-\frac{M_1^1 \mu}{M_2^1 \mu} \right)^{1/\beta_1 - 1} (w_1^{12})^{1/\beta_1 - 1} - (w_2^{12})^{-1} \left(\frac{\beta_1 \beta_2}{w_2^{12}} \right)^{\frac{\beta_2}{1-\beta_2}} \left(-\frac{M_1^1 \mu}{M_2^1 \mu} \right)^{\frac{\beta_2}{1-\beta_2}}. \quad (4.8)$$

Proof. $h = \mathcal{L}_2(s_2)$ is tangent to $h = \mathcal{R}_2(s_2)$ at point s_2 if and only if $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ and $\dot{\mathcal{L}}_2(s_2) = \dot{\mathcal{R}}_2(s_2)$, that is, (4.2) and (4.5) hold.

By dividing the two sides of equations (4.2) and (4.5), and some simple calculations, it is not difficult to show that (4.2) and (4.5) have a unique small positive solution

$$s_2 = \left(-\frac{\beta_1 \beta_2 M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{1/\beta_2 - 1} + h.o.t.,$$

for $|M_1^1 \mu| \ll |M_2^1 \mu|$. Substituting it into (4.5), we get the lemma. $\square$

If M_1^1 and M_2^1 are linearly independent, then (4.8) has a solution which defines a $(l-1)$ dimensional surface $\bar{L}_2^1$ in the neighborhood of $\mu = 0$. There is no difficulty to see that $\bar{L}_2^1$ is tangent to L_1 at $\mu = 0$.

Owing to $\beta_1 > 1$, $\beta_2 < 1$, $\beta_1\beta_2 > 1$, we have $\frac{\beta_2}{1-\beta_2} > \frac{1}{\beta_1-1}$, and so we have

$$\begin{aligned} \delta^{-1}M_2^1\mu|_{L_2^1} &= \left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{1}{\beta_1-1}}(w_1^{12})^{\frac{1}{\beta_1-1}} \\ &> \left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{1}{\beta_1-1}}(w_1^{12})^{\frac{1}{\beta_1-1}} - (w_2^{12})^{-1}\left(\frac{\beta_1\beta_2}{w_2^{12}}\right)^{\frac{\beta_2}{1-\beta_2}}\left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}} \\ &= \delta^{-1}M_2^1\mu|_{\bar{L}_2^1} \\ &> 0 \\ &\approx \delta^{-1}M_2^1\mu|_{L_2} \end{aligned}$$

for $\mu \in R_2^1$ and $|M_1^1\mu| \ll |M_2^1\mu|$.

We get that $\bar{L}_2^1$ will be in the open region bounded by L_2 and L_2^1 . Thus, we can define the following three open regions:

$(R_2^1)_1$ is the region whose boundaries are L_2 and $\bar{L}_2^1$, $(R_2^1)_2$ is the region whose boundaries are $\bar{L}_2^1$ and L_2^1 , $(R_2^1)_0$ is the region whose boundaries are L_2^1 and L_1 .

Now, we consider the non-negative solutions of $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$. Using Lemmas 4.5 and 4.7, it is not difficult to prove that:

If $\mu \in (R_2^1)_1$, then $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has exactly one small positive solution.

If $\mu \in L_2^1$, then $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has exactly one small positive and one zero solution.

If $\mu \in (R_2^1)_2$, then $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has exactly two small positive solutions.

If $\mu \in \bar{L}_2^1$, then $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has exactly one small two-fold positive solution.

If $\mu \in (R_2^1)_0$, then $\mathcal{L}_2(s_2) = \mathcal{R}_2(s_2)$ has no any small non-negative solution.

Thus, we have shown the following theorem.

Theorem 4.5. Suppose that the hypotheses (H1)~(H4) and (A4) hold, and $\mu \in R_2^1$. Then the following conclusions are true.

- (i) System (1.2) has exactly one simple 1-periodic orbit near Γ as $\mu \in (R_2^1)_1$.
- (ii) System (1.2) has exactly one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$.
- (iii) System (1.2) has exactly two simple 1-periodic orbits near Γ as $\mu \in (R_2^1)_2$.
- (iv) System (1.2) has a unique two-fold 1-periodic orbit near Γ as $\mu \in \bar{L}_2^1$.
- (v) System (1.2) has not any 1-periodic orbit and 1-homoclinic loop near Γ as $\mu \in (R_2^1)_0$.

Similarly, we can define two open regions $(R_1^2)_0$ and $(R_1^2)_1$ in R_1^2 and obtain the following Theorem 4.6, where $(R_1^2)_0$ is the region whose boundaries are L_2 and L_1^2 , $(R_1^2)_1$ is the region whose boundaries are L_1^2 and L_1 .

Theorem 4.6. Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then System (1.2) has not any 1-periodic and 1-homoclinic loop near Γ as $\mu \in (R_1^2)_0$, has exactly one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$, and has exactly one simple 1-periodic orbit near Γ as $\mu \in (R_1^2)_1$, respectively.

Denote D_1^2 as the open region whose boundaries are L_1 and L_2 , such that

$$D_1^2 \cap \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\} \neq \emptyset,$$

D_2^1 as the open region whose boundaries are L_2 and L_1 , such that

$$D_2^1 \cap \{\mu : M_1^1 \mu < 0, M_2^1 \mu < 0, |\mu| \ll 1\} \neq \emptyset.$$

By Lemmas 4.6 and 4.7, we obtain the following theorem.

Theorem 4.7. *Suppose that the hypotheses (H1)~(H4) and (A4) hold. Then*

- (i) *System (1.2) has exactly one simple 1-periodic orbit near Γ as $\mu \in D_1^2$.*
- (ii) *System (1.2) has no any 1-periodic orbit near Γ as $\mu \in D_2^1$.*

Figure 2 is the bifurcation diagram of Theorem 4.5, Theorem 4.6 and Theorem 4.7.

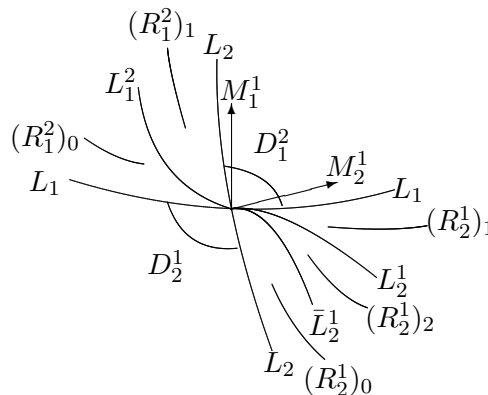


Figure 2. Bifurcation graph for $\Delta_1 = 1$, $\Delta_2 = 1$.

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