

A LINEARIZED GALERKIN FINITE ELEMENT METHOD FOR NON-LINEAR KIRCHHOFF-TYPE PROBLEM WITH BOUNDARY DISSIPATION

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Abstract This work is concerned with a non-linear Kirchhoff-type wave equation with boundary dissipation, motivated by models of elastic vibrations and wave propagation in dissipative media. A linearized Galerkin finite element method to approximate solutions of the problem is developed and analysed. The scheme is constructed in both semi-discrete and fully discrete settings, with rigorous stability and convergence analyses provided. Using inverse inequalities and compactness arguments, unconditional error estimates without restrictions on the mesh grid are established. Numerical experiments are presented to illustrate the theoretical findings, confirming the efficiency and accuracy of the proposed method in capturing the complex dynamics of Kirchhoff-type equations with strong nonlinearity and boundary damping.

Keywords Nonlinear Kirchhoff-type equation, boundary dissipation, finite element method, fully discrete schemes, error estimates.

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1. Introduction

Kirchhoff-type equations have become a central focus in recent mathematical research due to their role in modeling nonlinear wave phenomena such as vibrations in elastic materials and the propagation of waves. Their analysis is essential for understanding stability and energy dynamics in complex physical systems. In recent years, Kirchhoff-type equations have attracted significant research interest, driven by advances in mathematical theory and computational methods. One of the key ingredients in this phenomenal growth is attributed to the better mathematical understanding of problems. Some interesting features on such problems and more motivation are described in [3, 5, 14, 17] and in the references therein.

This paper is concerned with the non-linear Kirchhoff-type equation with boundary dissipa-

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tion:

$$\begin{cases} u_{tt} - \phi(\|\nabla u\|^2)\Delta u - a\Delta u_t + \alpha|u_t|^{\beta-2}u_t = f(x, t) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \Gamma_1 \times (0, T), \\ \phi(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + a\frac{\partial u_t}{\partial \nu} = g(u_t) & \text{on } \Gamma_0 \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $a, \alpha > 0$ and $\beta > 2$ are constants. Here, Ω is a bounded domain of $\mathbb{R}^d (d \geq 1)$ with smooth boundary $\Gamma := \partial\Omega$ such that $\bar{\Gamma} = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures, $0 < T < \infty$. One defines $\frac{\partial}{\partial \nu}$ as the outward normal derivative on $\partial\Omega$, ν is unit outer normal vector, and u_0, u_1 and f are given smooth functions.

Equations (1.1) originates from the classical Kirchhoff–Carrier model, which was developed to describe small oscillations of an elastic stretched string. Many works have been carried out to study the properties and solutions of these equations. The theoretical analysis has been done extensively on wave equations of Kirchhoff type related to (1.1). The existence and nonexistence of solutions for a wave equation of Kirchhoff type

$$u_{tt} - \phi(\|\nabla u\|^2)\Delta u + h(u_t) = f(u) \quad \text{in } \Omega \times (0, T) \quad (1.2)$$

have been discussed by many authors for different type of the function h . For $h(u_t) = \beta u_t$, with $\beta > 0$, the global existence and blow-up results of (1.2) can be found in [19, 25]. For $h(u_t) = -\Delta u_t$, the global existence and blow-up results are in [18, 20] and the references cited therein. In [26, 27, 29], the existence and uniqueness of solution of the following problem are derived.

$$\begin{cases} u_{tt} - \phi(\|\nabla u\|^2)\Delta u - a\Delta u_t = b|u|^{\beta-2}u & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0, \quad u_t(x, 0) = u_1, & \text{in } \Omega. \end{cases} \quad (1.3)$$

In [19], Kosuke studied the global existence, the asymptotic stability and the blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff-type with strong dissipation. Lin and Li in [12] studied the problem that follows:

$$\begin{cases} u_{tt} - \phi(\|\nabla u\|^2)\Delta u - a\Delta u_t = b|u|^{\beta-2}u & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \phi(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + a\frac{\partial u_t}{\partial \nu} = g(u_t) & \text{on } \Gamma_0, \\ u(x, 0) = u_0, \quad u_t(x, 0) = u_1, & \text{in } \Omega, \end{cases} \quad (1.4)$$

under certain conditions on ϕ and g , with $\partial\bar{\Omega} = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$. They used the potential well method to prove the existence and uniqueness of a global solution by means of Galerkin method and the uniform decay rate of the energy by an integral inequality under suitable conditions on the initial data. In [28], the authors investigated the well-posedness, complete regularity, and longtime dynamics for the Kirchhoff-type wave equations with degenerate energy fractional damping on a bounded domain. Lee and Kang in [11] studied the general decay result of the viscoelastic wave equation of Kirchhoff-type with Balakrishnan-Taylor damping, weakly nonlinear time-dependent damping, and a logarithmic source term under acoustic boundary conditions. Their analysis is conducted under minimal conditions on the relaxation function g in $L^\infty(0, \infty)$. Liu et al. [13] analyzed the nonexistence of nontrivial global weak solutions

for a class of strongly damped wave equations, involving the p -Laplacian term and combined nonlinearities. They established rigorous conditions on the initial data and some exponents that ensure that solutions cannot exist globally in time. In the numerical simulations, Finite element approximation has emerged as a powerful computational tool for solving such complex non-linear equations. There is a vast literature on finite element methods for nonlinear elliptic and parabolic problems, for instance, [4, 6, 9, 16, 17]. The integration of finite element techniques with the analysis of Kirchhoff-type equations enables researchers to address challenges associated with non-linear behavior and dissipative effects, paving the way for advancements in theory and applications. Gupta et al. [7] presented a mixed finite element framework utilizing a saddle-point formulation to address parabolic biharmonic integro-differential problems of the Kirchhoff type with simply supported boundary conditions. In [2], the dual boundary integral formulation of the two-dimensional Helmholtz equation with complex wave number is derived. To address the singular and hypersingular integrals, the addition theorem is used to expand the four kernel functions, originally expressed with complex variables in the dual formulation, into purely real-variable functions in a series form. Gudi in [6] proved the well-posedness of the discrete scheme for the stationary case of (1.3) by the use of Brouwer fixed-point theorem and established optimal error estimates in H^1 -norm. Dond and Pani in [5] studied a priori and a posteriori estimates of conforming and expanded mixed finite element methods for a Kirchhoff equation of elliptic-type (that is problem (1.3) with $b|u|^{\beta-2}u = 0$). They used the Helmholtz decomposition of L^2 -vector valued function and the discrete Helmholtz decomposition of the Raviart-Thomas finite elements to achieve optimal a priori as well as a posteriori error bounds. Shi and Wu [22] applied the EQ_1^{rot} elements on anisotropic quadrilateral meshes to the two-dimensional problem (1.3) with $\phi(s) = 1 + s$, $s \in \mathbb{R}^+$. They derived optimal error estimates without time dependent constraints, in the broken H^1 norm for the semi-discrete and a linearized fully discrete schemes respectively. In Parvizi et al. [21], the Raviart-Thomas mixed finite element method was applied to approximate the solution of the coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term. They derived the L^2 -error estimate for the semi discrete scheme.

While earlier studies have established finite element frameworks for Kirchhoff-type problems, our approach introduces several key distinctions. In [6], Gudi analyzed stationary Kirchhoff-type equations using Brouwer's fixed-point theorem, focusing on well-posedness and optimal error estimates in the H^1 -norm. Shi and Wu [22] extended this to anisotropic meshes with EQ_1^{rot} elements, deriving optimal error bounds but under mesh-dependent constraints. Parvizi et al. [21] employed Raviart–Thomas mixed finite elements to handle nonlinear boundary damping with memory terms, emphasizing L^2 -error estimates. In contrast, the present work develops a linearized Galerkin finite element scheme for the wave equation with boundary dissipation, and establishes unconditional stability and error estimates without mesh restrictions.

The novelty lies in extending finite element analysis to nonlinear Kirchhoff-type wave equations with boundary damping, while ensuring unconditional error control, a feature not addressed in the cited works.

The main purpose of this research article is to present and establish the unconditionally error estimates of a linearized second-order finite element scheme for the wave equation of Kirchhoff type (1.1). The stability and error estimates for the proposed methods are derived with the help of the inverse inequality, without any restrictions on the mesh grid. The primary challenges stem from the strong nonlinearity of the weak form.

Most existing finite element approaches for Kirchhoff-type problems emphasize well-posedness

and classical error estimates, but they often lack a discussion of energy stability and structure-preserving discretizations. For example, Shi and Wu [22] achieved optimal error estimates on anisotropic meshes, yet their linearized schemes do not explicitly enforce stability properties inherent to the continuous model. Similarly, Parvizi et al. [21] addressed nonlinear boundary damping with mixed finite elements, but the method does not preserve the underlying energy structure of the Kirchhoff system.

Our proposed linearized Galerkin finite element method is designed to be unconditionally stable and to capture the dissipative energy dynamics without mesh restrictions. This positions our work as a step toward bridging the gap between classical convergence theory and modern energy-stable discretization frameworks.

This paper is organized as follows. In Section 2, preliminaries, hypotheses and important results that will be used in subsequent sections are given. Section 3 presents the finite element approximation of the spatial discrete scheme of (1.1). In Section 4, the error analysis of the fully discrete scheme is done. Section 5 is devoted to numerical results which illustrates our theoretical analysis.

2. Preliminaries and variational formulation

In this section, we introduce the notation, assumptions, and functional framework required for the analysis. We use standard Sobolev space notation throughout (see for instance [1]). We then present the variational formulation of equation (1.1) and discuss regularity properties of its solutions.

The inner product for the Hilbert space $H^k(\Omega)$, $k \geq 1$ will be denoted $(\cdot, \cdot)_k$ and its associated norm $\|\cdot\|_k$. For $k = 0$, $H^k(\Omega)$ coincides with $L^2(\Omega)$ and we use $(\cdot, \cdot)$ and $\|\cdot\|$ respectively to denote the corresponding inner product and norm. Again, for $u : [0, T] \rightarrow H^k(\Omega)$ we define the following norms:

$$\begin{aligned}\|u\|_{L^2(0,T;H^k(\Omega))} &= \left\{ \int_0^T \|u(s)\|_k^2 ds \right\}^{\frac{1}{2}}, \\ \|u\|_{L^\infty(0,T;L^2(\Omega))} &= \sup_{0 \leq t \leq T} \|u(t)\|, \\ \|u\|_{L^\infty(0,T;H_0^1(\Omega))} &= \sup_{0 \leq t \leq T} \|\nabla u(t)\|.\end{aligned}$$

We make the following assumptions:

(A1) The function ϕ is continuously differentiable on $\mathbb{R}^+$ satisfying:

$$\phi(s) \geq C_\phi > 0, \quad s\phi(s) \geq \int_0^s \phi(\theta) d\theta,$$

for a certain positive constant C_ϕ . Moreover, its first derivative is bounded, ensuring Lipschitz continuity.

(A2) The boundary damping function g is continuously differentiable, non-increasing, and has a bounded derivative. In particular, there exists $C > 0$ such that

$$|g(s)| \leq C|s| \quad \text{and} \quad sg(s) \leq 0, \quad \forall s \in \mathbb{R},$$

which guarantees dissipative behavior.

(A3) The exponent β is restricted according to the spatial dimension d . Specifically, we have

$$2 < \beta \leq \frac{2d-2}{d-2}, \quad (d > 2) \quad \text{or} \quad 2 < \beta < \infty, \quad (d = 1, 2).$$

These bounds ensure the embedding properties needed for the analysis.

Let us recall Young's inequality [1], which provides a fundamental estimate for non-negative parameters and will be used repeatedly in the analysis. That is for $a, b, \varepsilon \geq 0$, we have:

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon q}, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (2.1)$$

The following lemmas are necessary for the proof of our main results.

Lemma 2.1. [15] *For all $p \in (0, \infty)$ and $\tau \geq 0$, there exist a constant $C = C(p, d)$ such that $\forall \xi, \eta \in \mathbb{R}^d$, with $d \geq 1$, we have:*

$$\begin{aligned} \left| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right| &\leq C|\xi - \eta|^{1-\tau} (|\xi| + |\eta|)^{p-2+\tau}, \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) &\geq C|\xi - \eta|^{2+\tau} (|\xi| + |\eta|)^{p-2+\tau}. \end{aligned} \quad (2.2)$$

Lemma 2.2. [23] *Assume that $u \in C([0, T]; \mathbb{R})$, $T \in (0, \infty)$, satisfies the integral inequality*

$$u(t) \leq u_0 + \int_0^t a(s)u(s)ds + \int_0^t b(s)ds \quad \text{on} \quad (0, T),$$

for some $a, b \in L^1(0, T)$. Then u satisfies the pointwise estimate

$$u(t) \leq e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}ds, \quad \forall t \in [0, T],$$

where $A(t) = \int_0^t a(s)ds$.

Lemma 2.3. [24] *An everywhere differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ is lipschitz continuous (with Lipschitz constant $C_g = \sup |g'(x)|$) if and only if it has bounded first derivative.*

Lemma 2.4. [12] *Assume that the boundary of Ω is $\mathcal{C}^1$. Then*

$$\|u\|_{L^p(\partial\Omega)} \leq C_p \|u\|_{W^{1,p}(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega), \quad p \geq 2.$$

If $3 \leq p \leq \frac{2d}{d-2}$ ($d > 2$) or $2 \leq p < \infty$ ($d = 1, 2$) holds, then

$$\|u\|_{L^p(\partial\Omega)} \leq C_p \|\nabla u\|, \quad \text{for all } u \in V,$$

where the constant C_p depends only on Ω and p and $V := H_{\Gamma_1}^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$.

The weak formulation of problem (1.1) is: Find $u : [0, T] \rightarrow V$ such that for all test function $w \in V$,

$$\left\{ (u_{tt}, w) + \phi(\|\nabla u\|^2)(\nabla u, \nabla w) + a(\nabla u_t, \nabla w) + \alpha(|u_t|^{\beta-2}u_t, w) = (g(u_t), w)_{\Gamma_0} + (f, w), \right. \quad (2.3)$$

with initial conditions $u(0) = u_0$, $u_t(0) = u_1$. This formulation provides the foundation for both the existence theory and the finite element discretization. The result that follows gives the local existence of a solution to problem (1.1) and is proven by the Feado-Galerkin method.

Theorem 2.1. *Let $(u_0, u_1) \in H^2(\Omega) \times V$. If (A1)-(A3) hold and in addition, ϕ is monotone, then there exist a unique solution u to (2.3) satisfying for $0 < T < \infty$,*

$$u \in L^\infty(0, T; V), u_t \in L^\infty(0, T; V) \cap L^\beta(0, T, L^\beta(\Omega)), u_{tt} \in L^2(0, T; V'). \quad (2.4)$$

Proof. The proof of the existence of the solution is done using the Galerkin approximation method. This will be presented in the following steps.

Step 1. Uniform estimates & space topology.

Let $(w_j)_{j \geq 1}$ be a basis of $V = H_{\Gamma_1}^1(\Omega)$ that is orthonormal in $L^2(\Omega)$ and spans the finite dimensional subspaces

$$V_m = \text{span} \{w_1, \dots, w_m\}.$$

For every m , we seek approximate solutions of the form

$$u_m(t, x) = \sum_{j=1}^m c_j^m(t) w_j(x),$$

such that

$$\begin{cases} (u_m'', w_i) + \phi(\|\nabla u_m(t)\|^2) (\nabla u_m(t), \nabla w_i) + a(\nabla u_m'(t), \nabla w_i) \\ + \alpha \left(\left| u_m'(t) \right|^{\beta-2} u_m'(t), w_i \right) - (g(u_m'), w_i)_{\Gamma_0} = (f(t), w_i), \\ u_m(0) = \sum_{j=1}^m (u_m(0), w_j) w_j \rightarrow u_0 \quad \text{in } V, \\ u_m'(0) = \sum_{j=1}^m (u_m'(0), w_j) w_j \rightarrow u_1 \quad \text{in } V. \end{cases} \quad (2.5)$$

To obtain the first a priori estimate, the i -th equation (2.5) is multiplied by $C^{m'}(t)$ and the sum from $i = 1$ to m is taken to obtain

$$\begin{aligned} & \left(u_m'', \sum_{i=1}^m C^{m'}(t) w_i \right) + \phi(\|\nabla u_m(t)\|^2) \left(\nabla u_m(t), \nabla \sum_{i=1}^m C^{m'}(t) w_i \right) \\ & + a \left(\nabla u_m'(t), \nabla \sum_{i=1}^m C^{m'}(t) w_i \right) + \alpha \left(\left| u_m'(t) \right|^{\beta-2} u_m'(t), \sum_{i=1}^m C^{m'}(t) w_i \right) \\ & - \left(g(u_m'), \sum_{i=1}^m C^{m'}(t) w_i \right)_{\Gamma_0} \\ & = \left(f(t), \sum_{i=1}^m C^{m'}(t) w_i \right). \end{aligned}$$

That is,

$$\begin{aligned} & (u_m'', u_m') + \phi(\|\nabla u_m(t)\|^2) (\nabla u_m(t), \nabla u_m') + a(\nabla u_m'(t), \nabla u_m') \\ & + \alpha \left(\left| u_m'(t) \right|^{\beta-2} u_m'(t), u_m' \right) - (g(u_m'), u_m')_{\Gamma_0} \\ & = (f(t), u_m'). \end{aligned}$$

So,

$$\frac{1}{2} \frac{d}{dt} \left[\|u'_m\|^2 + \tilde{\phi}(\|\nabla u_m\|^2) \right] + a \|\nabla u'_m\|^2 + \alpha \|u'_m\|_{L^\beta}^\beta - (g(u'_m), u'_m)_{\Gamma_0} = (f(t), u'_m),$$

where $\tilde{\phi}(\|\nabla u_m\|^2) = \int_0^{\|\nabla u_m\|^2} \phi(s) ds$.

Setting

$$E_m(t) = \frac{1}{2} \left[\|u'_m\|^2 + \tilde{\phi}(\|\nabla u_m\|^2) \right],$$

one gets Young inequalities

$$\frac{d}{dt} E_m(t) + a \|\nabla u'_m\|^2 + \alpha \|u'_m\|_{L^\beta}^\beta \leq C \|f\| \|\nabla u'_m\| \leq \frac{a}{2} \|\nabla u'_m\|^2 + C \|f\|^2.$$

Integrating over $[0, t]$ for all $t \in [0, T]$, it yields

$$E_m(t) + \frac{a}{2} \int_0^T \|\nabla u'_m(s)\|^2 ds + \alpha \int_0^T \|u'_m(s)\|_{L^\beta}^\beta ds \leq E_m(0) + C \int_0^T \|f(s)\|^2 ds,$$

where

$$\begin{aligned} E_m(0) &= \frac{1}{2} \left[\|u'_m(0)\|^2 + \int_0^{\|\nabla u_0\|^2} \phi(s) ds \right] \\ &\leq \frac{1}{2} [\|u_1\|^2 + \|\nabla u_0\|^2 \phi(\|\nabla u_0\|^2)], \text{ by hypothesis } (A_1). \end{aligned}$$

Therefore u_m is bounded in $L^\infty(0, T, V)$ and u'_m is bounded in $L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, V) \cap L^\beta(0, T, L^\beta(\Omega))$.

To show in which space u''_m is bounded, re-write the Galerkin equation as follows:

$$u''_m = -\phi(\|\nabla u_m(t)\|^2) \nabla u_m(t) - a \nabla u'_m(t) - \alpha |u'_m|^{\beta-2} u'_m(t) + g(u'_m) + f_m(t), \quad (2.6)$$

that is $u''_m \in V_m \subset V'$. Indeed, one has

- $\phi(\|\nabla u_m(t)\|^2) \nabla u_m(t)$ is the projection of the operator $u \mapsto \phi(\|\nabla u\|^2) \nabla u$. From the first a priori estimate above, ∇u_m is bounded in $L^\infty(0, T, V)$. Due to the growth of ϕ , the map $v \mapsto (\phi(\|\nabla u_m\|^2) \nabla u_m, \nabla v)$ and its norm is bounded uniformly in $L^2(0, T, V')$.
- $a \nabla u'_m(t)$ is the projection of $u \mapsto a \nabla u_t$. Similarly, $\nabla u'_m$ is bounded in $L^\infty(0, T, V)$. So, $a \nabla u'_m(t)$ is bounded in $L^2(0, T, V')$.
- $\alpha |u'_m|^{\beta-2} u'_m(t)$ is the projection of $u \mapsto \alpha |u_t|^{\beta-2} u_t$. For similar reasons, the map $v \mapsto (\alpha |u'_m|^{\beta-2} u'_m(t), v)$ is in $L^{\beta'}(0, T, V')$. Since $\beta > 2$, by embedding, the boundedness is in $L^2(0, T, V')$.
- f_m is the L^2 projection of f onto V_m .
- $g(u'_m)$ denotes the projection of the boundary force.

Thus, $u_m'' \in L^2(0, T, V')$.

Step 2. Extraction of weak/weak- limits stars.

By Banach-Alaoglu theorems, there exists a subsequence (still denoted u_m) and limit functions u, χ_1, χ_2 such that:

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u && \text{weakly-* in } L^\infty(0, T; V), \\ u_m' &\overset{*}{\rightharpoonup} u' && \text{weakly-* in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; V), \\ u_m'' &\rightharpoonup u'' && \text{weakly in } L^2(0, T; V'), \\ |u_m'|^{\beta-2} u_m' &\rightharpoonup \chi_1 && \text{weakly in } L^{\beta'}(0, T; L^{\beta'}(\Omega)) \quad \left(\frac{1}{\beta} + \frac{1}{\beta'} = 1 \right), \\ g(u_m')|_{\Gamma_0} &\rightharpoonup \chi_2 && \text{weakly in } L^2(0, T; L^2(\Gamma_0)). \end{aligned}$$

Step 3. Compactness arguments (Aubin-Lions-Simon theorem).

To handle the non-linear terms $\phi(\|\nabla u_m\|^2)$, $|u_m'|^{\beta-2} u_m'$, and $g(u_m')$, weak convergence is insufficient; strong convergence in suitable Lebesgue spaces is required.

A. Convergence of the scalar energy $\|\nabla u_m(t)\|^2$ Applying the Aubin-Lions Compactness Lemma with the embeddings:

$$V \hookrightarrow L^2(\Omega) \hookrightarrow V'.$$

Since u_m is bounded in $L^2(0, T; V)$ and u_m' is bounded in $L^2(0, T; L^2(\Omega))$, we obtain:

$$u_m \longrightarrow u \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ and } C([0, T]; L^2(\Omega)).$$

Furthermore, because $u_m \in L^\infty(0, T; V)$ and $u_m' \in L^2(0, T; V)$, the energy term satisfies $t \mapsto \|\nabla u_m(t)\|_{L^2}^2 \in C([0, T])$, and by Lebesgue's Dominated Convergence Theorem:

$$\|\nabla u_m(t)\|^2 \longrightarrow \|\nabla u(t)\|^2 \quad \text{a.e. } t \in (0, T).$$

Since ϕ is continuous and Lipschitz:

$$\phi(\|\nabla u_m(t)\|^2) \longrightarrow \phi(\|\nabla u(t)\|^2) \quad \text{a.e. } t \in (0, T).$$

B. Strong convergence of derivatives for boundary/damping terms applying Aubin-Lions to u_m' :

$$u_m' \text{ bounded in } L^2(0, T; V), \quad u_m'' \text{ bounded in } L^2(0, T; V').$$

Because $V \hookrightarrow L^2(\Omega)$ continuously and compactly:

$$u_m' \longrightarrow u' \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ and a.e. in } \Omega \times (0, T).$$

By the trace theorem $V = H_{\Gamma_1}^1(\Omega) \hookrightarrow H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$:

$$u_m'|_{\Gamma_0} \longrightarrow u'|_{\Gamma_0} \quad \text{strongly in } L^2(0, T; L^2(\Gamma_0)) \text{ and a.e. on } \Gamma_0 \times (0, T).$$

Combining $\phi(\|\nabla u_m\|^2) \rightarrow \phi(\|\nabla u\|^2)$ almost everywhere with $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^2(0, T; L^2(\Omega))$, we get:

$$\phi(\|\nabla u_m\|^2) \nabla u_m \rightharpoonup \phi(\|\nabla u\|^2) \nabla u \quad \text{weakly in } L^2(0, T; L^2(\Omega)).$$

Since g is continuous and $u'_m|_{\Gamma_0} \rightarrow u'|_{\Gamma_0}$ almost everywhere on $\Gamma_0 \times (0, T)$, it follows that:

$$g(u'_m) \longrightarrow g(u') \quad \text{a.e. on } \Gamma_0 \times (0, T).$$

Combined with the uniform bound in $L^2(0, T; L^2(\Gamma_0))$, vital application of Lions' Lemma yields $\chi_2 = g(u')$.

Since $u'_m \rightarrow u'$ almost everywhere in $\Omega \times (0, T)$:

$$|u'_m|^{\beta-2} u'_m \longrightarrow |u'|^{\beta-2} u' \quad \text{a.e. in } \Omega \times (0, T).$$

Hence $\chi_1 = |u'|^{\beta-2} u'$.

Step 4. Passing to the limit in the variational equality.

Multiply the discrete equation by a test function $v(t) = \theta(t)w_j$ where $\theta \in D(0, T)$ and w_j is a basis vector:

$$\begin{aligned} & \int_0^T (u''_m, w_j) \theta(t) dt + \int_0^T \phi(\|\nabla u_m\|^2) (\nabla u_m, \nabla w_j) \theta(t) dt + a \int_0^T (\nabla u'_m, \nabla w_j) \theta(t) dt \\ & + \alpha \int_0^T (|u'_m|^{\beta-2} u'_m, w_j) \theta(t) dt - \int_0^T (g(u'_m), w_j)_{\Gamma_0} \theta(t) dt \\ & = \int_0^T (f, w_j) \theta(t) dt. \end{aligned}$$

Taking $m \rightarrow \infty$ and applying the weak and strong convergences identified in Section 4:

$$\begin{aligned} & \int_0^T \langle u'', w_j \rangle \theta(t) dt + \int_0^T \phi(\|\nabla u\|^2) (\nabla u, \nabla w_j) \theta(t) dt + a \int_0^T (\nabla u', \nabla w_j) \theta(t) dt \\ & + \alpha \int_0^T (|u'|^{\beta-2} u', w_j) \theta(t) dt - \int_0^T (g(u'), w_j)_{\Gamma_0} \theta(t) dt \\ & = \int_0^T (f, w_j) \theta(t) dt. \end{aligned}$$

Since the set of linear combinations of $\{w_j \theta(t)\}$ is dense in $L^2(0, T; V)$, this identity holds for all test functions $v \in L^2(0, T; V)$, proving the existence of the weak solution.

Since $u_m \rightarrow u$ strongly in $C([0, T]; L^2(\Omega))$ and $u'_m \rightarrow u'$ weakly in $C([0, T]; V')$, evaluating at $t = 0$ confirms:

$$u(0) = \lim_{m \rightarrow \infty} u_m(0) = u_0, \quad u'(0) = \lim_{m \rightarrow \infty} u'_m(0) = u_1 \quad \text{in } L^2(\Omega).$$

For the uniqueness, let u, v be two weak solutions corresponding to the same initial data

$$u(0) = v(0) = u_0, \quad u_t(0) = v_t(0) = u_1.$$

Set $\psi := u - v$. Subtracting the weak formulations for u and v , and choosing $w = \psi_t$ as test function, we obtain

$$\begin{aligned} & (\psi_{tt}, \psi_t) + ((\phi(\nabla u) \nabla u - \phi(\nabla v) \nabla v), \nabla \psi_t) + a \|\nabla \psi_t\|^2 \\ & + \alpha \left((|u_t|^{\beta-2} u_t - |v_t|^{\beta-2} v_t), \psi_t \right) + ((g(u_t) - g(v_t)), \psi_t) = 0. \end{aligned}$$

Evaluating each term,

$$(\psi_{tt}, \psi_t) = \frac{1}{2} \frac{d}{dt} (\|\psi_t\|^2).$$

By the relation (2.2), one has

$$\alpha \int_{\Omega} (|u_t|^{\beta-2} u_t - |v_t|^{\beta-2} v_t)(u_t - v_t) dx \geq \alpha \int_{\Omega} |u_t - v_t|^2 (|u_t| + |v_t|)^{\beta-2} dx \geq 0.$$

Using the local Lipschitz continuity of $S(\nabla u) = \phi(|\nabla u|^2) \nabla u$ and since $\phi(\cdot)$ remains bounded, there exists $L > 0$ such that

$$|S(\nabla u) - S(\nabla v)| \leq L |\nabla u - \nabla v|.$$

Hence,

$$\left| \int_{\Omega} (S(\nabla u) - S(\nabla v)) \cdot \nabla \psi_t dx \right| \leq L \|\nabla \psi\| \|\nabla \psi_t\| \leq \frac{a}{2} \|\nabla \psi_t\|^2 + \frac{L^2}{2a} \|\nabla \psi\|^2.$$

Since g is Lipschitz, and by Lemma 2.4, it can be estimated as follows

$$\int_{\Gamma_0} (g(u_t) - g(v_t)) \psi_t ds \leq \int_{\Gamma_0} L |\psi_t|^2 ds < L \|\psi_t\|_{\Gamma_0}^2 < C \|\nabla \psi_t\|^2.$$

Combining the above estimates yields

$$\frac{1}{2} \frac{d}{dt} (\|\psi_t\|^2) + \frac{a}{2} \|\nabla \psi_t\|^2 \leq \frac{L^2}{2a} \|\nabla \psi\|^2 + C \|\nabla \psi_t\|^2. \quad (2.7)$$

Note that

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \psi\|^2) = \int_{\Omega} \nabla \psi \cdot \nabla \psi_t dx \leq \frac{\varepsilon}{2} \|\nabla \psi_t\|^2 + \frac{1}{2\varepsilon} \|\nabla \psi\|^2.$$

Let

$$E(t) := \frac{1}{2} \|\psi_t(t)\|^2 + \frac{\gamma}{2} \|\nabla \psi(t)\|^2.$$

Choosing ε and γ appropriately, we obtain,

$$E'(t) \leq C E(t),$$

for some $C > 0$. Since $\psi(0) = 0$ and $\psi_t(0) = 0$, we have $E(0) = 0$. By Gronwall's lemma,

$$E(t) \leq E(0) e^{Ct} = 0 \quad \text{for all } t \in [0, T].$$

Therefore $E(t) \equiv 0$, which implies $\psi_t \equiv 0$ and $\nabla \psi \equiv 0$, hence $\psi \equiv 0$. Thus, the weak solution is unique. $\square$

3. Semi-discrete approximation and error estimate

Let us now introduce the spatial discretization of problem (1.1) using finite element methods. Let $\mathcal{T}_h$ denote a regular triangulation of the domain with mesh size h . For each element $T \in \mathcal{T}_h$,

we consider polynomial spaces of degree at most r . The discrete space $V_h \subset V$ is then defined as the set of continuous piecewise polynomials vanishing on the boundary Γ_1 . We define

$$V_h = \{v_h \in \mathcal{C}^0(\bar{\Omega}) : v_h|_T \in \mathbb{P}_r(T), \forall T \in \mathcal{T}_h, v_h|_{\Gamma_1} = 0\},$$

with $V_h \subset V$. We define the Ritz operator $R_h : H_{\Gamma_1}^1(\Omega) \rightarrow V_h$ by

$$(\nabla(u - R_h u), \nabla w) = 0, \quad \forall w \in H_{\Gamma_1}^1(\Omega). \quad (3.1)$$

The following properties from [23] states that for $u \in H^{r+1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$, we have

$$\begin{aligned} \|u - \Pi_h u\| + h\|\nabla(u - \Pi_h u)\| &\leq Ch^{r+1}\|u\|_{r+1}, \\ \|u - R_h u\| + h\|\nabla(u - R_h u)\| &\leq Ch^{r+1}\|u\|_{r+1}, \\ \|R_h u\|_1 + \|\Pi_h u\|_1 &\leq C\|u\|_1, \end{aligned} \quad (3.2)$$

where C is a positive constant that does not depend on h . The operator Π_h denotes the interpolation operator.

The semi-discrete formulation seeks $u_h : [0, T] \rightarrow V_h$ such that

$$\begin{cases} (u_{htt}, w_h) + \phi(\|\nabla u_h\|^2)(\nabla u_h, \nabla w_h) + a(\nabla u_{ht}, \nabla w_h) + \alpha(|u_{ht}|^{\beta-2} u_{ht}, w_h) \\ \quad - (g(u_{ht}), w_h)_{\Gamma_0} \\ = (f, w_h), \quad \forall w_h \in V_h, t \in [0, T], \\ u_h(x, 0) = \Pi_h u_0(x), \quad u_{ht}(x, 0) = \Pi_h u_1(x) \quad \text{on } \Omega. \end{cases} \quad (3.3)$$

The system (3.3) is a second order non-linear ODE with ϕ, g and f continuous. So, it has a unique solution on $[0, T]$. Considering suitable regularity conditions on the exact solution, we have the following results on the stability and convergence of the semi-discrete scheme.

Theorem 3.1. *If the initial data $u_0, u_1 \in H_0^1(\Omega)$ and*

$$\int_0^{\|\nabla \Pi_h u_0\|^2} \phi(s) ds \leq C,$$

then the semi-discret solution u_h of (3.3) remains stable. Specifically, it satisfies

$$\|u_{ht}\|_{L^\infty(0,T;L^2(\Omega))} + \|u_{ht}\|_{L^2(0,T;H_0^1(\Omega))} \leq C(\|\Pi_h u_1\|^2 + \int_0^T \|f(t)\|^2 dt). \quad (3.4)$$

Proof. Replacing w_h in (3.3) by u_{ht} and use the fact that $-(g(u_{ht}), u_{ht})_{\Gamma_0} > 0$, we get

$$\frac{1}{2} \frac{d}{dt} [\|u_{ht}\|^2 + \tilde{\phi}(\|\nabla u_h\|^2)] + a\|\nabla u_{ht}\|^2 + \alpha(|u_{ht}|^{\beta-2} u_{ht}, u_{ht}) \leq C\|f\|^2 + \frac{a}{2}\|u_{ht}\|^2,$$

where $\tilde{\phi}(\|\nabla u\|^2) = \int_0^{\|\nabla u\|^2} \phi(s) ds$.

Considering the fact that $\alpha(|u_{ht}|^{\beta-2} u_{ht}, u_{ht}) > 0$ and integrating from 0 to t , we have

$$\|u_{ht}\|^2 + \tilde{\phi}(\|\nabla u_h\|^2) + a \int_0^t \|\nabla u_{ht}(s)\|^2 ds \leq \|\Pi_h u_1\|^2 + \int_0^{\|\nabla \Pi_h u_0\|^2} \phi(s) ds + C \int_0^t \|f(s)\|^2 ds.$$

By applying the Gronwall's lemma with $t \in [0, T]$, we have the desired result. $\square$

The following theorem gives the error estimate of the semi-discrete scheme.

Theorem 3.2. *Let u denote the exact solution of (2.3) and u_h the semi-discrete solution of (3.3). Then there exists a constant C independent of h such that:*

$$\begin{aligned} & \| (u - u_h)_t \|_{L^\infty(0,T;L^2(\Omega))} + \| u - u_h \|_{L^\infty(0,T;H_0^1(\Omega))} + \| (u - u_h)_t \|_{L^2(0,T;H_0^1(\Omega))} \\ & \leq Ch^r \left[\int_0^T \| u_{tt} \|_{r+1}^2 + \| u \|_{r+1}^2 + \| u_t \|_{r+1}^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (3.5)$$

Proof. We set $e = u - u_h = (u - R_h u) + (R_h u - u_h) = \eta + \xi$.

From Lemma 3.2, we have

$$\| \eta \| \leq Ch^{r+1} \| u \|_{r+1}. \quad (3.6)$$

To estimate ξ , we subtract (3.3) from (2.3) with $w = w_h$, that is,

$$\begin{aligned} & (\xi_{tt}, w_h) + \phi(\| \nabla u \|^2) (\nabla \xi, \nabla w_h) + a(\nabla \xi_t, \nabla w_h) \\ & = -(\eta_{tt}, w_h) - [\phi(\| \nabla u \|^2) - \phi(\| \nabla u_h \|^2)] (\nabla u_h, \nabla w_h) \\ & \quad - \alpha \left(|u_t|^{\beta-2} u_t - |u_{ht}|^{\beta-2} u_{ht}, w_h \right) + (g(u_t) - g(u_{ht}), w_h)_{\Gamma_0}. \end{aligned}$$

Taking $w_h = \xi_t$ in the above relation leads to

$$\begin{aligned} & (\xi_{tt}, \xi_t) + \phi(\| \nabla u \|^2) (\nabla \xi, \nabla \xi_t) + a(\nabla \xi_t, \nabla \xi_t) \\ & = -(\eta_{tt}, \xi_t) - [\phi(\| \nabla u \|^2) - \phi(\| \nabla u_h \|^2)] (\nabla u_h, \nabla \xi_t) \\ & \quad - \alpha \left(|u_t|^{\beta-2} u_t - |u_{ht}|^{\beta-2} u_{ht}, \xi_t \right) + (g(u_t) - g(u_{ht}), \xi_t)_{\Gamma_0}. \end{aligned}$$

That is

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\| \xi_t \|^2 + \phi(\| \nabla u \|^2) \| \nabla \xi \|^2] + a \| \nabla \xi_t \|^2 \\ & = -(\eta_{tt}, \xi_t) - [\phi(\| \nabla u \|^2) - \phi(\| \nabla u_h \|^2)] (\nabla u_h, \nabla \xi_t) \\ & \quad - \alpha \left(|u_t|^{\beta-2} u_t - |u_{ht}|^{\beta-2} u_{ht}, \xi_t \right) + (g(u_t) - g(u_{ht}), \xi_t)_{\Gamma_0} \\ & \quad + \phi_t(\| \nabla u \|^2) (\nabla u, \nabla u_t) \| \nabla \xi \|^2 \\ & := \sum_{i=1}^5 A_i. \end{aligned} \quad (3.7)$$

The terms A_i , $i = 1, \dots, 3$ are bounded as follows using Young's inequality:

$$A_1 \leq C \| \eta_{tt} \| \| \nabla \xi_t \| \leq \varepsilon_1 \| \eta_{tt} \|^2 + \frac{\varepsilon_1}{8} \| \nabla \xi_t \|^2, \quad (3.8)$$

$$\begin{aligned} A_2 & \leq \gamma \left| \| \nabla u \|^2 - \| \nabla u_h \|^2 \right| \sup_{0 \leq s \leq T} \| \nabla u_h(s) \| \| \nabla \xi_t \| \\ & \leq C \left(\gamma, \| u \|_{L^\infty(0,T;V)}, \| u_h \|_{L^\infty(0,T;V)} \right) \| \nabla(u - u_h) \| \| \nabla \xi_t \| \\ & \leq C_1 (\| \nabla \eta \| \| \nabla \xi_t \| + \| \nabla \xi \| \| \nabla \xi_t \|) \quad (\text{where } C_1 = C(\gamma, \| u \|_{L^\infty(0,T;V)}, \| u_h \|_{L^\infty(0,T;V)})) \\ & \leq \varepsilon_2 \| \nabla \eta \|^2 + \frac{\varepsilon_2'}{8} \| \nabla \xi_t \|^2 + \varepsilon_3 \| \nabla \xi \|^2, \end{aligned} \quad (3.9)$$

$$A_3 \leq C \alpha \left(|u_t - u_{ht}| (|u_t| + |u_{ht}|)^{\beta-2}, |\xi_t| \right)$$

$$\begin{aligned}
&\leq C(\alpha, \|u_t\|_{L^\infty(0,T;L^2(\Omega))}, \|u_{ht}\|_{L^\infty(0,T;L^2(\Omega))}) \|u_t - u_{ht}\| \|\xi_t\| \\
&\leq C_2 (\|\eta\| \|\xi_t\| + \|\xi_t\|^2) \quad (\text{where } C_2 = C(\alpha, \|u_t\|_{L^\infty(0,T;L^2(\Omega))}, \|u_{ht}\|_{L^\infty(0,T;L^2(\Omega))})) \\
&\leq C'_2 \|\eta\| \|\nabla \xi_t\| + C_2 \|\xi_t\|^2 \\
&\leq \varepsilon_4 \|\eta\|^2 + \frac{\varepsilon_4}{8} \|\nabla \xi_t\|^2 + C_2 \|\xi_t\|^2.
\end{aligned} \tag{3.10}$$

To bound A_4 , we use Lemmas 2.3 and 2.4 to have

$$\begin{aligned}
A_4 &\leq C_g \|u_t - u_{ht}\|_{\Gamma_0} \|\xi_t\|_{\Gamma_0} \\
&\leq C_g C_p^2 \|\nabla(u - u_h)_t\| \|\nabla \xi_t\| \\
&\leq \varepsilon_5 \|\nabla \eta_t\|^2 + \frac{\varepsilon_5}{8} \|\nabla \xi_t\|^2 + C_3 \|\nabla \xi_t\|^2 \quad (\text{where } C_3 = C_g C_p^2).
\end{aligned} \tag{3.11}$$

Since ϕ has bounded first derivative, we have

$$A_5 \leq C (\|u\|_{L^\infty(0,T;V)}, \|u_t\|_{L^\infty(0,T;V)}) \|\nabla \xi\|^2 \equiv C_4 \|\nabla \xi\|^2. \tag{3.12}$$

Substituting equations (3.8) to (3.12) in (3.7), yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} [\|\xi_t\|^2 + \phi(\|\nabla u\|^2) \|\nabla \xi\|^2] + a \|\nabla \xi_t\|^2 \\
&\leq C_5 [\|\eta_{tt}\|^2 + \|\nabla \eta\|^2 + \|\eta_t\|^2 + \|\nabla \eta_t\|^2] \\
&\quad + \frac{\varepsilon_1 + \varepsilon'_2 + \varepsilon_4 + \varepsilon_5}{8} \|\nabla \xi_t\|^2 + C_6 [\|\xi_t\|^2 + \|\nabla \xi\|^2 + \|\nabla \xi_t\|^2],
\end{aligned}$$

where $C_5 = \max(\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5)$ and $C_6 = \max(\varepsilon_3, C_2, C_3, C_4)$.

Taking $\varepsilon_1 = \varepsilon'_2 = \varepsilon_4 = \varepsilon_5 = a$ and integrating from 0 to t , we have

$$\begin{aligned}
&\|\xi_t\|^2 + C_\phi \|\nabla \xi\|^2 + a \int_0^t \|\nabla \xi_t(s)\|^2 ds \\
&\leq \|\xi_t(0)\|^2 + \phi(\|\nabla u_0\|^2) \|\nabla \xi(0)\|^2 \\
&\quad + C_5 \int_0^t [\|\eta_{tt}(s)\|^2 + \|\nabla \eta(s)\|^2 + \|\eta_t(s)\|^2 + \|\nabla \eta_t(s)\|^2] ds \\
&\quad + C_6 \int_0^t [\|\xi_t(s)\|^2 + \|\nabla \xi(s)\|^2 + \|\nabla \xi_t(s)\|^2] ds.
\end{aligned}$$

Applying Gronwall's Lemma 2.2 over $[0, T]$ we have

$$\begin{aligned}
&\|\xi_t\|^2 + \|\nabla \xi\|^2 + \int_0^T \|\nabla \xi_t(s)\|^2 ds \\
&\leq C \left(\|\xi_t(0)\|^2 + \phi(\|\nabla u_0\|^2) \|\nabla \xi(0)\|^2 + \int_0^T [\|\eta_{tt}(s)\|^2 + \|\nabla \eta(s)\|^2 + \|\eta_t(s)\|^2 + \|\nabla \eta_t(s)\|^2] ds \right).
\end{aligned}$$

From (3.2), we have

$$\begin{aligned}
&\|\xi_t\|^2 + \|\nabla \xi\|^2 + \int_0^T \|\nabla \xi_t(s)\|^2 ds \\
&\leq C h^{2r} \left(\int_0^T [\|u_{tt}(s)\|_{r+1}^2 + \|u(s)\|_{r+1}^2 + \|u_t(s)\|_{r+1}^2] ds \right).
\end{aligned}$$

Thus,

$$\begin{aligned} & \|\xi_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\xi\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\xi_t\|_{L^2(0,T;H_0^1(\Omega))} \\ & \leq Ch^r \left(\int_0^T [\|u_{tt}(s)\|_{r+1}^2 + \|u(s)\|_{r+1}^2 + \|u_t(s)\|_{r+1}^2] ds \right)^{\frac{1}{2}}. \end{aligned}$$

□

4. The fully discrete scheme and error analysis

We now consider a full discretization in time. Let $\{t_n\}_{n=0}^N$ be a uniform partition of the interval $[0, T]$ with step size τ . For any sequence of functions, $(u^n)_{n=0}^N$, we define:

$$\begin{aligned} \bar{\partial}_{tt}u^n &= \frac{1}{\tau^2} (u^{n+1} - 2u^n + u^{n-1}); \quad \bar{\partial}_t u^n = \frac{1}{2\tau} (u^{n+1} - u^{n-1}); \quad \tilde{\partial}_t u^n = \frac{1}{\tau} (u^n - u^{n-1}), \\ \bar{u}^n &= \frac{1}{2} (u^n + u^{n-1}); \quad u^{n,\frac{1}{4}} = \frac{1}{4} (u^{n+1} + 2u^n + u^{n-1}), \\ \hat{u}^n &= \frac{1}{4} (3u^n + 2u^{n-1} - u^{n-2}); \quad \tilde{u}^n = \frac{u^n + u^{n-2}}{2}, \quad n = 2, \dots, N, \\ \bar{\partial}_{tt}u^n &= \frac{1}{\tau} (\tilde{\partial}_t u^{n+1} - \tilde{\partial}_t u^n); \quad \bar{\partial}_t u^n = \frac{1}{2} (\tilde{\partial}_t u^{n+1} + \tilde{\partial}_t u^n) = \frac{1}{\tau} (\bar{u}^{n+1} - \bar{u}^n); \\ u^{n,\frac{1}{4}} &= \frac{1}{2} (\bar{u}^{n+1} + \bar{u}^n). \end{aligned}$$

Let U_h^n be an approximation of $u_h(t_n)$. The fully discrete scheme is constructed as a linearized second-order method, defined in three steps to approximate the solution at successive time levels.

- Step 1. Find $U_h^{2,0} \in V_h$ such that:

$$\begin{aligned} & \left(\frac{U_h^{2,0} - 2U_h^1 + U_h^0}{\tau^2}, w_h \right) + \phi(\|\nabla \bar{U}_h^1\|^2) \left(\nabla \frac{U_h^{2,0} + 2U_h^1 + U_h^0}{4}, \nabla w_h \right) + a \left(\nabla \frac{U_h^{2,0} - U_h^0}{2\tau}, \nabla w_h \right) \\ & + \alpha \left(\left| \frac{U_h^1 - U_h^0}{2\tau/2} \right|^{\beta-2} \frac{U_h^{2,0} - U_h^0}{2\tau}, w_h \right) - \left(g \left(\frac{U_h^1 - U_h^0}{2\tau/2} \right), w_h \right)_{\Gamma_0} \\ & = \left(f^{2,\frac{1}{4}}, w_h \right), \quad \forall w_h \in V_h. \end{aligned} \tag{4.1}$$

- Step 2. Find $U_h^2 \in V_h$ such that:

$$\begin{aligned} & (\bar{\partial}_{tt}U_h^1, w_h) + \phi \left(\left\| \nabla \frac{U_h^{2,0} + U_h^0}{2} \right\|^2 \right) \left(\nabla U_h^{1,\frac{1}{4}}, \nabla w_h \right) + a (\nabla \bar{\partial}_t U_h^1, \nabla w_h) \\ & + \alpha \left(\left| \frac{U_h^{2,0} - U_h^0}{2\tau} \right|^{\beta-2} \bar{\partial}_t U_h^1, w_h \right) - \left(g \left(\frac{U_h^{2,0} - U_h^0}{2\tau} \right), w_h \right)_{\Gamma_0} \\ & = \left(f^{2,\frac{1}{4}}, w_h \right), \quad \forall w_h \in V_h. \end{aligned} \tag{4.2}$$

- Step 3. Find $U_h^n \in V_h$ for $n \geq 2$ such that:

$$(\bar{\partial}_{tt}U_h^n, w_h) + \phi(\|\nabla \hat{U}_h^n\|^2) \left(\nabla U_h^{n,\frac{1}{4}}, \nabla w_h \right) + a (\nabla \bar{\partial}_t U_h^n, \nabla w_h)$$

$$+ \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t U_h^n, w_h \right) - (g(\bar{\partial}_t U_h^{n-1}), w_h)_{\Gamma_0} = (f^{n, \frac{1}{4}}, w_h), \quad \forall w_h \in V_h, \quad (4.3)$$

where

$$\begin{cases} U_h^0 = \Pi_h u_0(x), \\ U_h^1 = \Pi_h \left[u_0(x) + u_1(x)\tau + \frac{1}{2}u_{tt}(x, 0)\tau^2 \right] \end{cases}$$

with

$$u_{tt}(x, 0) = a\Delta u_1(x) + \phi(\|\nabla u_0(x)\|^2)\Delta u_0(x) - \alpha|u_1(x)|^{\beta-2}u_1(x) + f(x, 0).$$

For our subsequent analysis, the exact solution satisfies

$$\begin{cases} (\bar{\partial}_{tt}u^n, w) + \phi(\|\nabla \hat{u}^n\|^2) (\nabla u^{n, \frac{1}{4}}, \nabla w) + a(\nabla \bar{\partial}_t u^n, \nabla w) + \alpha(|\hat{u}^n|^{\beta-2}u^{n, \frac{1}{4}}, w) \\ = (g(\bar{\partial}_t u^n), w)_{\Gamma_0} + (f^{n, \frac{1}{4}}, w) + (R_1^n, w) + (\vec{R}_2^n, w) + (\vec{R}_3^n, w) + (R_4^n, w) + (R_5^n, w)_{\Gamma_0}, \\ \forall w \in V, t \in [0, T], \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{on } \Omega, \end{cases} \quad (4.4)$$

where,

$$\begin{aligned} R_1^n &= \bar{\partial}_{tt}u^n - u_{tt}^{n, \frac{1}{4}}, \\ \vec{R}_2^n &= \phi(\|\nabla \hat{u}^n\|^2)\nabla u^{n, \frac{1}{4}} - (\phi(\|\nabla u\|^2)\nabla u)^{n, \frac{1}{4}}, \\ \vec{R}_3^n &= a\nabla \bar{\partial}_t u^n - a(\nabla u_t)^{n, \frac{1}{4}}, \\ R_4^n &= \alpha|\bar{\partial}_t u^{n-1}|^{\beta-2}\bar{\partial}_t u^n - \alpha(|u_t|^{\beta-2}u_t)^{n, \frac{1}{4}}, \\ R_5^n &= g\left(u_t^{n, \frac{1}{4}}\right) - g(\bar{\partial}_t u^{n-1}). \end{aligned}$$

By Taylor's expansion (see for instance in [8]), one has

$$\sum_{n=1}^N \tau \left(\|R_1^n\|^2 + \|\vec{R}_2^n\|^2 + \|\vec{R}_3^n\|^2 + \|R_4^n\|^2 + \|R_5^n\|_{\Gamma_0}^2 \right) \leq C\tau^4. \quad (4.5)$$

4.1. Stability analysis

The following results gives the stability of the fully discrete scheme.

Lemma 4.1. *If the initial data $u_0, u_1 \in H_{\Gamma_1}^1(\Omega)$ then the discrete solution satisfies a uniform stability bound. That is, there exists a generic constant C independent on h and τ such that,*

$$\|\tilde{\partial}_t U_h^{2,0}\|^2 + \|\nabla \bar{U}_h^{2,0}\|^2 + \tau \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 \leq C(\|u_0\|_1^2 + \|u_1\|_1^2 + \tau \|u_1\|_1^2 + \tau \|f^{2, \frac{1}{4}}\|^2). \quad (4.6)$$

Proof. Through out this proof, we use the following notations

$$\begin{aligned} \bar{\partial}_{tt}U_h^{2,0} &= \frac{U_h^{2,0} - 2U_h^1 + U_h^0}{\tau^2} = \frac{1}{\tau} [\tilde{\partial}_t U_h^{2,0} - \tilde{\partial}_t U_h^1], \\ \tilde{\partial}_t U_h^{2,0} &= \frac{U_h^{2,0} - U_h^1}{\tau}, \end{aligned}$$

$$\begin{aligned}\bar{\partial}_t U_h^{2,0} &= \frac{U_h^{2,0} - U_h^0}{2\tau} = \frac{1}{2} [\tilde{\partial}_t U_h^{2,0} + \tilde{\partial}_t U_h^1] = \frac{1}{\tau} [\bar{U}_h^{2,0} - \bar{U}_h^1], \\ U_h^{0, \frac{1}{4}} &= \frac{U_h^{2,0} + 2U_h^1 + U_h^0}{4} = \frac{1}{2} [\bar{U}_h^{2,0} + \bar{U}_h^1], \\ \bar{U}_h^{2,0} &= \frac{U_h^{2,0} + U_h^1}{2}.\end{aligned}$$

Replacing w_h by $\bar{\partial}_t U_h^{2,0}$ in (4.1), we get

$$\begin{aligned}& \left(\frac{U_h^{2,0} - 2U_h^1 + U_h^0}{\tau^2}, \bar{\partial}_t U_h^{2,0} \right) + \phi(\|\nabla \bar{U}_h^1\|^2) \left(\nabla \frac{U_h^{2,0} + 2U_h^1 + U_h^0}{4}, \nabla \bar{\partial}_t U_h^{2,0} \right) \\& + a \left(\nabla \frac{U_h^{2,0} - U_h^0}{2\tau}, \nabla \bar{\partial}_t U_h^{2,0} \right) + \alpha \left(\left| \frac{U_h^1 - U_h^0}{2\tau/2} \right|^{\beta-2} \frac{U_h^{2,0} - U_h^0}{2\tau}, \bar{\partial}_t U_h^{2,0} \right) \\& - \left(g \left(\frac{U_h^1 - U_h^0}{2\tau/2} \right), \bar{\partial}_t U_h^{2,0} \right)_{\Gamma_0} \\& = \left(f^{2, \frac{1}{4}}, \bar{\partial}_t U_h^{2,0} \right), \\& \Rightarrow \frac{1}{2\tau} [\|\tilde{\partial}_t U_h^{2,0}\|^2 - \|\tilde{\partial}_t U_h^1\|^2] + \frac{1}{2\tau} \phi(\|\nabla \bar{U}_h^1\|^2) [\|\nabla \bar{U}_h^{2,0}\|^2 - \|\nabla \bar{U}_h^1\|^2] + a \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 \\& + \alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t U_h^{2,0}, \bar{\partial}_t U_h^{2,0} \right) - \left(g(\tilde{\partial}_t U_h^1), \bar{\partial}_t U_h^{2,0} \right)_{\Gamma_0} \\& = \left(f^{2, \frac{1}{4}}, \bar{\partial}_t U_h^{2,0} \right).\end{aligned}$$

We see that, $\alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t U_h^{2,0}, \bar{\partial}_t U_h^{2,0} \right) > 0$. By Young's inequality, Lemma (2.4) and hypothesis (A1),

$$\begin{aligned}& \frac{1}{2\tau} [\|\tilde{\partial}_t U_h^{2,0}\|^2 - \|\tilde{\partial}_t U_h^1\|^2] + \frac{C_\phi}{2\tau} [\|\nabla \bar{U}_h^{2,0}\|^2 - \|\nabla \bar{U}_h^1\|^2] + a \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 \\& \leq C_g \|\tilde{\partial}_t U_h^1\|_{\Gamma_0} \|\bar{\partial}_t U_h^{2,0}\|_{\Gamma_0} + \|f^{2, \frac{1}{4}}\| \|\bar{\partial}_t U_h^{2,0}\| \\& \leq C \|\nabla \tilde{\partial}_t U_h^1\| \|\nabla \bar{\partial}_t U_h^{2,0}\| + C \|f^{2, \frac{1}{4}}\| \|\nabla \bar{\partial}_t U_h^{2,0}\| \\& \leq C \|\nabla \tilde{\partial}_t U_h^1\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 + C \|f^{2, \frac{1}{4}}\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^{2,0}\|^2.\end{aligned}$$

That is

$$\|\tilde{\partial}_t U_h^{2,0}\|^2 + C_\phi \|\nabla \bar{U}_h^{2,0}\|^2 + a\tau \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 \leq \|\tilde{\partial}_t U_h^1\|^2 + \|\nabla \bar{U}_h^1\|^2 + C\tau (\|\nabla \tilde{\partial}_t U_h^1\|^2 + \|f^{2, \frac{1}{4}}\|^2).$$

By the definition of U_h^0 and U_h^1 , one has for example $\|\nabla \bar{U}_h^1\| \leq C(\|\nabla u_0\| + \tau \|\nabla u_1\|)$ and $\|\nabla \tilde{\partial}_t U_h^1\| \leq C \|\nabla u_1\|$. Therefore, one gets (4.6). $\square$

Theorem 4.1. *Under the same assumptions, the fully discrete solution is stable for all time steps. The discrete energy norm is bounded uniformly, ensuring that the scheme does not exhibit numerical blow-up. That is, there exists a generic constant C independent on h and τ such that,*

$$\|\tilde{\partial}_t U_h^2\|^2 + \|\nabla \bar{U}_h^2\|^2 + \tau \|\nabla \bar{\partial}_t U_h^1\|^2 \leq C(\|u_0\|_1^2 + \|u_1\|_1^2 + \tau \|u_1\|_1^2 + \tau \|f^{2, \frac{1}{4}}\|^2). \quad (4.7)$$

And for all $n \geq 2$,

$$\|\tilde{\partial}_t U_h^n\|^2 + \|\nabla \bar{U}_h^n\|^2 + \tau \sum_{k=1}^{n-1} \|\nabla \bar{\partial}_t U_h^k\|^2 \leq C \left[\|\tilde{\partial}_t U_h^1\|^2 + \|\nabla \bar{U}_h^1\|^2 + \tau \sum_{k=1}^{n-1} \|f^{k, \frac{1}{4}}\|^2 \right]. \quad (4.8)$$

Proof. To prove (4.7), take $w_h = \bar{\partial}_t U_h^1$ in (4.2), and we have

$$\begin{aligned} & \left\{ (\bar{\partial}_{tt} U_h^1, \bar{\partial}_t U_h^1) + \phi \left(\left\| \nabla \frac{U_h^{2,0} + U_h^0}{2} \right\|^2 \right) \left(\nabla U_h^{1, \frac{1}{4}}, \nabla \bar{\partial}_t U_h^1 \right) + a (\nabla \bar{\partial}_t U_h^1, \nabla \bar{\partial}_t U_h^1) \right. \\ & \left. + \alpha \left(\left| \frac{U_h^{2,0} - U_h^0}{2\tau} \right|^{\beta-2} \bar{\partial}_t U_h^1, \bar{\partial}_t U_h^1 \right) - \left(g \left(\frac{U_h^{2,0} - U_h^0}{2\tau} \right), \bar{\partial}_t U_h^1 \right)_{\Gamma_0} \right\} = (f^{2, \frac{1}{4}}, \bar{\partial}_t U_h^1), \\ & \Rightarrow \frac{1}{2\tau} [\|\tilde{\partial}_t U_h^2\|^2 - \|\tilde{\partial}_t U_h^1\|^2] + \frac{1}{2\tau} \phi \left(\left\| \nabla \frac{U_h^{2,0} + U_h^0}{2} \right\|^2 \right) [\|\nabla \bar{U}_h^2\|^2 - \|\nabla \bar{U}_h^1\|^2] + a \|\nabla \bar{\partial}_t U_h^1\|^2 \\ & + \alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t U_h^1, \bar{\partial}_t U_h^1 \right) - \left(g(\bar{\partial}_t U_h^{2,0}), \bar{\partial}_t U_h^1 \right)_{\Gamma_0} = (f^{2, \frac{1}{4}}, \bar{\partial}_t U_h^1). \end{aligned}$$

We drop the term $\alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t U_h^1, \bar{\partial}_t U_h^1 \right) > 0$. By Young's inequality, Lemma (2.4) and hypothesis (A1), we have

$$\begin{aligned} & \frac{1}{2\tau} [\|\tilde{\partial}_t U_h^2\|^2 - \|\tilde{\partial}_t U_h^1\|^2] + \frac{C_\phi}{2\tau} [\|\nabla \bar{U}_h^2\|^2 - \|\nabla \bar{U}_h^1\|^2] + a \|\nabla \bar{\partial}_t U_h^1\|^2 \\ & \leq C_g \|\bar{\partial}_t U_h^{2,0}\|_{\Gamma_0} \|\bar{\partial}_t U_h^1\|_{\Gamma_0} + \|f^{2, \frac{1}{4}}\| \|\bar{\partial}_t U_h^1\| \\ & \leq C \|\nabla \bar{\partial}_t U_h^{2,0}\| \|\nabla \bar{\partial}_t U_h^1\| + C \|f^{2, \frac{1}{4}}\| \|\nabla \bar{\partial}_t U_h^1\| \\ & \leq C \|\nabla \bar{\partial}_t U_h^{2,0}\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^1\|^2 + C \|f^{2, \frac{1}{4}}\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^1\|^2, \end{aligned}$$

that is

$$\|\tilde{\partial}_t U_h^2\|^2 + C_\phi \|\nabla \bar{U}_h^2\|^2 + a\tau \|\nabla \bar{\partial}_t U_h^1\|^2 \leq \|\tilde{\partial}_t U_h^1\|^2 + \|\nabla \bar{U}_h^1\|^2 + C\tau [\|\nabla \bar{\partial}_t U_h^{2,0}\|^2 + \|f^{2, \frac{1}{4}}\|^2].$$

Then using (4.6), we conclude (4.7).

For (4.8), replace $w_h = \bar{\partial}_t U_h^n$ in step (4.3), we have

$$\begin{aligned} & \frac{1}{2\tau} [\|\tilde{\partial}_t U_h^{n+1}\|^2 - \|\tilde{\partial}_t U_h^n\|^2] + \frac{1}{2\tau} \phi \left(\|\nabla \bar{U}_h^{n+1}\|^2 \right) [\|\nabla \bar{U}_h^{n+1}\|^2 - \|\nabla \bar{U}_h^n\|^2] + a \|\nabla \bar{\partial}_t U_h^n\|^2 \\ & + \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t U_h^n, \bar{\partial}_t U_h^n \right) - \left(g(\bar{\partial}_t U_h^{n-1}), \bar{\partial}_t U_h^n \right)_{\Gamma_0} \\ & = (f^{n, \frac{1}{4}}, \bar{\partial}_t U_h^n). \end{aligned}$$

The term $\alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t U_h^n, \bar{\partial}_t U_h^n \right) > 0$ is dropped. With the use of Young's inequality, Lemma 2.4 and hypothesis (A1), one has

$$\frac{1}{2\tau} [\|\tilde{\partial}_t U_h^{n+1}\|^2 - \|\tilde{\partial}_t U_h^n\|^2] + \frac{C_\phi}{2\tau} [\|\nabla \bar{U}_h^{n+1}\|^2 - \|\nabla \bar{U}_h^n\|^2] + a \|\nabla \bar{\partial}_t U_h^n\|^2$$

$$\begin{aligned}
&\leq C_g \|\bar{\partial}_t U_h^{n-1}\|_{\Gamma_0} \|\bar{\partial}_t U_h^n\|_{\Gamma_0} + \|f^{n,\frac{1}{4}}\| \|\bar{\partial}_t U_h^n\| \\
&\leq C \|\nabla \bar{\partial}_t U_h^{n-1}\| \|\nabla \bar{\partial}_t U_h^n\| + C \|f^{n,\frac{1}{4}}\| \|\nabla \bar{\partial}_t U_h^n\| \\
&\leq C \|\nabla \bar{\partial}_t U_h^{n-1}\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^n\|^2 + C \|f^{n,\frac{1}{4}}\|^2 + \frac{a}{4} \|\nabla \bar{\partial}_t U_h^n\|^2.
\end{aligned}$$

Multiplying through by 2τ

$$\begin{aligned}
&\|\tilde{\partial}_t U_h^{n+1}\|^2 - \|\tilde{\partial}_t U_h^n\|^2 + C_\phi [\|\nabla \bar{U}_h^{n+1}\|^2 - \|\nabla \bar{U}_h^n\|^2] + a\tau \|\nabla \bar{\partial}_t U_h^n\|^2 \\
&\leq C\tau \left[\|\nabla \bar{\partial}_t U_h^{n-1}\|^2 + \|f^{n,\frac{1}{4}}\|^2 \right].
\end{aligned} \tag{4.9}$$

Summing up this inequality from $k = 1$ to $n - 1 \leq N$ and using the discrete Gronwall lemma, we obtain

$$\|\tilde{\partial}_t U_h^n\|^2 + \|\nabla \bar{U}_h^n\|^2 + \tau \sum_{k=1}^{n-1} \|\nabla \bar{\partial}_t U_h^k\|^2 \leq C \left[\|\tilde{\partial}_t U_h^1\|^2 + \|\nabla \bar{U}_h^1\|^2 + \sum_{k=1}^{n-1} \tau \|f^{k,\frac{1}{4}}\|^2 \right].$$

□

4.2. Error analysis

To establish convergence, we decompose the error into interpolation and discretization components.

$$\begin{aligned}
\xi^{2,0} &= R_h u^2 - U_h^{2,0}, \\
\eta^n &= u^n - R_h u^n, \quad \xi^n = R_h u^n - U_h^n, \quad \xi^{n,1/4} = R_h u^{n,1/4} - U_h^{n,1/4}.
\end{aligned}$$

Using Taylor expansions and discrete Gronwall-type arguments, we derive bounds for each term. This leads to the following main result.

Lemma 4.2. *Let u^2 be the exact solution of (4.4) with $n = 2$ and $U_h^{2,0}$ the solution of (4.1). Assume that $u, u_t, u_{tt} \in L^\infty(0, T; H^{r+1}(\Omega))$, $u_{ttt} \in L^\infty(0, T; L^2(\Omega))$. Then there exists a generic constant C independent of τ and h such that*

$$\|\xi^{2,0}\|^2 + \tau^2 \|\nabla \xi^{2,0}\|^2 + \tau \|\nabla \xi^{2,0}\|^2 \leq C (h^{2r} + \tau^4).$$

Proof. Subtracting equation (4.1) from (4.4) for $n = 1$, we have

$$\begin{aligned}
&(\bar{\partial}_{tt} \xi^{2,0}, w_h) + \phi(\|\nabla \bar{U}_h^1\|^2) \left(\nabla \xi^{0,\frac{1}{4}}, \nabla w_h \right) + a(\nabla \bar{\partial}_t \xi^{2,0}, \nabla w_h) + \alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t \xi^{2,0}, w_h \right) \\
&= -(\bar{\partial}_{tt} \eta^1, w_h) - [\phi(\|\nabla \bar{u}^1\|^2) - \phi(\|\nabla \bar{U}_h^1\|^2)] \left(\nabla u^{1,\frac{1}{4}}, \nabla w_h \right) \\
&\quad - \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t u^1 - \left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t R_h u^1, w_h \right) + \left(g(\tilde{\partial}_t u^1) - g(\tilde{\partial}_t U_h^1), w_h \right)_{\Gamma_0} \\
&\quad + (R_1^1, w_h) + (\bar{R}_2^1, w_h) + (\bar{R}_3^1, w_h) + (R_4^1, w_h) + (R_5^1, w_h)_{\Gamma_0}.
\end{aligned}$$

Taking $w_h = \bar{\partial}_t \xi^{2,0}$, one obtains

$$(\bar{\partial}_{tt} \xi^{2,0}, \bar{\partial}_t \xi^{2,0}) + \phi(\|\nabla \bar{U}_h^1\|^2) \left(\nabla \xi^{0,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^{2,0} \right) + a(\nabla \bar{\partial}_t \xi^{2,0}, \nabla \bar{\partial}_t \xi^{2,0})$$

$$\begin{aligned}
& + \alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t \xi^{2,0}, \bar{\partial}_t \xi^{2,0} \right) \\
& = - (\bar{\partial}_{tt} \eta^1, \bar{\partial}_t \xi^{2,0}) - [\phi(\|\nabla \bar{u}^1\|^2) - \phi(\|\nabla \bar{U}_h^1\|^2)] \left(\nabla u^{1,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^{2,0} \right) \\
& \quad - \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t u^1 - \left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) + \left(g(\tilde{\partial}_t u^1) - g(\tilde{\partial}_t U_h^1), \bar{\partial}_t \xi^{2,0} \right)_{\Gamma_0} \\
& \quad + (R_1^1, \bar{\partial}_t \xi^{2,0}) + (\bar{R}_2^1, \bar{\partial}_t \xi^{2,0}) + (\bar{R}_3^1, \bar{\partial}_t \xi^{2,0}) + (R_4^1, \bar{\partial}_t \xi^{2,0}) + (R_5^1, \bar{\partial}_t \xi^{2,0})_{\Gamma_0} \\
& = \sum_{i=1}^9 D_i.
\end{aligned}$$

That is,

$$\begin{aligned}
& \frac{1}{2\tau} \left[\|\tilde{\partial}_t \xi^{2,0}\|^2 - \|\tilde{\partial}_t \xi^1\|^2 \right] + \frac{1}{2\tau} \phi(\|\nabla \bar{U}_h^1\|^2) \left[\|\nabla \bar{\xi}^{2,0}\|^2 - \|\nabla \bar{\xi}^1\|^2 \right] \\
& + a \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 + \alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t \xi^{2,0}, \bar{\partial}_t \xi^{2,0} \right) = \sum_{i=1}^9 D_i.
\end{aligned} \tag{4.10}$$

By Cauchy Schwarz and Young's inequalities, the terms of the right side of (4.10), are given by

$$D_1 \leq \|\bar{\partial}_{tt} \eta^1\| \|\bar{\partial}_t \xi^{2,0}\| \leq C \|\bar{\partial}_{tt} \eta^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \leq C \|\bar{\partial}_{tt} \eta^1\|^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2, \tag{4.11}$$

$$\begin{aligned}
D_2 & = [\phi(\|\nabla \bar{u}^1\|^2) - \phi(\|\nabla \bar{U}_h^1\|^2)] \left(\nabla u^{1,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^{2,0} \right) \\
& \leq L_\phi \left| \|\nabla \bar{u}^1\|^2 - \|\nabla \bar{U}_h^1\|^2 \right| \|\nabla u^{1,\frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
& \leq L_\phi (\|\nabla \bar{u}^1\| + \|\nabla \bar{U}_h^1\|) \|\nabla \bar{u}^1 - \nabla \bar{U}_h^1\| \|\nabla u^{1,\frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
& \leq C \left(\|\nabla \bar{u}^1\|, \|\nabla \bar{U}_h^1\|, \|\nabla u^{1,\frac{1}{4}}\| \right) \|\nabla \bar{u}^1 - \nabla \bar{U}_h^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
& \leq C \left(\|\nabla \bar{u}^1\|, \|\nabla \bar{U}_h^1\|, \|\nabla u^{1,\frac{1}{4}}\| \right) \|\nabla \bar{u}^1 - \nabla \bar{U}_h^1\|^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
D_3 & = \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t u^1 - \left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) + \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) \\
& \quad - \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) \\
& = \alpha \left(\left| \tilde{\partial}_t u^1 \right|^{\beta-2} \bar{\partial}_t \eta^1, \bar{\partial}_t \xi^{2,0} \right) + \alpha \left(\left[\left| \tilde{\partial}_t u^1 \right|^{\beta-2} - \left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \right] \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) \\
& \leq C \alpha \|\tilde{\partial}_t u^1\|_\infty \|\bar{\partial}_t \eta^1\| \|\bar{\partial}_t \xi^{2,0}\| + C \alpha \left(|\tilde{\partial}_t u^1 - \tilde{\partial}_t U_h^1| \left[|\tilde{\partial}_t u^1| + |\tilde{\partial}_t U_h^1| \right]^{\beta-3} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^{2,0} \right) \\
& \leq C \alpha \|\tilde{\partial}_t u^1\|_\infty \|\bar{\partial}_t \eta^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
& \quad + C \left(\alpha, \|\bar{\partial}_t R_h u^1\|_\infty, \|\tilde{\partial}_t u^1\|_\infty, \|\tilde{\partial}_t U_h^1\|_\infty \right) \|\tilde{\partial}_t u^1 - \tilde{\partial}_t U_h^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
& \leq C \left(\alpha, \|\bar{\partial}_t R_h u^1\|_\infty, \|\tilde{\partial}_t u^1\|_\infty, \|\tilde{\partial}_t U_h^1\|_\infty \right) \left[\|\bar{\partial}_t \eta^1\|^2 + \|\tilde{\partial}_t (u^1 - U_h^1)\|^2 \right] + \frac{2a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
D_4 &\leq L_g \|\tilde{\partial}_t(u^1 - U_h^1)\|_{\Gamma_0} \|\bar{\partial}_t \xi^{2,0}\|_{\Gamma_0} \\
&\leq C \|\tilde{\nabla} \partial_t(u^1 - U_h^1)\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
&\leq C \|\tilde{\nabla} \partial_t(u^1 - U_h^1)\|^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
D_5 + D_8 &= (R_1^1, \bar{\partial}_t \xi^{2,0}) + (R_4^1, \bar{\partial}_t \xi^{2,0}) \\
&\leq C [\|R_1^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| + \|R_4^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\|] \\
&\leq C [\|R_1^1\|^2 + \|R_4^1\|^2] + \frac{2a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
D_6 + D_7 &= (\bar{R}_2^1, \nabla \bar{\partial}_t \xi^{2,0}) + (\bar{R}_3^1, \nabla \bar{\partial}_t \xi^{2,0}) \\
&\leq \|\bar{R}_2^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| + \|\bar{R}_3^1\| \|\nabla \bar{\partial}_t \xi^{2,0}\| \\
&\leq C [\|\bar{R}_2^1\|^2 + \|\bar{R}_3^1\|^2] + \frac{2a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2,
\end{aligned} \tag{4.16}$$

$$D_9 = (R_5^1, \bar{\partial}_t \xi^{2,0})_{\Gamma_0} \leq \|R_5^1\|_{\Gamma_0} \|\bar{\partial}_t \xi^{2,0}\|_{\Gamma_0} \leq C \|R_5^1\|_{\Gamma_0}^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2. \tag{4.17}$$

Taking (4.11)–(4.17) into (4.10), one has

$$\begin{aligned}
&\frac{1}{2\tau} [\|\tilde{\partial}_t \xi^{2,0}\|^2 - \|\tilde{\partial}_t \xi^1\|^2] + \frac{C_\phi}{2\tau} [\|\nabla \bar{\xi}^{2,0}\|^2 - \|\nabla \bar{\xi}^1\|^2] + a \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \\
&+ \alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t \xi^{2,0}, \bar{\partial}_t \xi^{2,0} \right) \\
&\leq \frac{a}{2} \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 + C \|\bar{\partial}_{tt} \eta^1\|^2 + C \|\bar{\partial}_t \eta^1\|^2 \\
&+ C \|\nabla(\bar{u}^1 - \bar{U}_h^1)\|^2 + C \|\tilde{\partial}_t(u^1 - U_h^1)\|^2 + C \|\nabla \tilde{\partial}_t(u^1 - U_h^1)\|^2 \\
&+ C [\|R_1^1\|^2 + \|\bar{R}_2^1\|^2 + \|\bar{R}_3^1\|^2 + \|R_4^1\|^2 + \|R_5^1\|_{\Gamma_0}^2].
\end{aligned} \tag{4.18}$$

Taking into consideration the fact that $\alpha \left(\left| \tilde{\partial}_t U_h^1 \right|^{\beta-2} \bar{\partial}_t \xi^{2,0}, \bar{\partial}_t \xi^{2,0} \right) > 0$, $C_\phi > 0$ and multiplying through by 2τ , one gets

$$\begin{aligned}
&\|\tilde{\partial}_t \xi^{2,0}\|^2 - \|\tilde{\partial}_t \xi^1\|^2 + \|\nabla \bar{\xi}^{2,0}\|^2 - \|\nabla \bar{\xi}^1\|^2 + a\tau \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \\
&\leq C\tau [\|\bar{\partial}_{tt} \eta^1\|^2 + \|\bar{\partial}_t \eta^1\|^2 + \|\nabla \bar{\eta}^1\|^2 + \|\tilde{\partial}_t \eta^1\|^2 + \|\nabla \tilde{\partial}_t \eta^1\|^2] \\
&+ C\tau [\|\nabla \bar{\xi}^1\|^2 + \|\tilde{\partial}_t \xi^1\|^2 + \|\nabla \tilde{\partial}_t \xi^1\|^2] \\
&+ C\tau [\|R_1^1\|^2 + \|\bar{R}_2^1\|^2 + \|\bar{R}_3^1\|^2 + \|R_4^1\|^2 + \|R_5^1\|_{\Gamma_0}^2].
\end{aligned} \tag{4.19}$$

By (4.5) and Lemma 3.2, we have

$$\begin{aligned}
&\|\tilde{\partial}_t \xi^{2,0}\|^2 + \|\nabla \bar{\xi}^{2,0}\|^2 + \tau \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \\
&\leq C\tau h^{2r} [\|u_{tt}^1\|_{r+1}^2 + \|u_t^1\|_{r+1}^2 + \|u^1\|_{r+1}^2] \\
&+ \|\nabla \bar{\xi}^1\|^2 + \|\tilde{\partial}_t \xi^1\|^2 + C\tau [\|\nabla \bar{\xi}^1\|^2 + \|\tilde{\partial}_t \xi^1\|^2 + \|\nabla \tilde{\partial}_t \xi^1\|^2] \\
&+ C\tau^2 [\|u_{ttt}^1\|^2 + \|u_{ttt}^1\|_1^2 + \|u_{tt}^1\|_1^2].
\end{aligned} \tag{4.20}$$

Since $\xi^0 \equiv 0$ and thus $\nabla \xi^0 \equiv 0$, it follows that

$$\|\bar{\partial}_t \xi^{2,0}\|^2 = \frac{\|\xi^{2,0}\|^2}{4\tau^2}, \quad \|\bar{\xi}^1\|^2 = \frac{\|\xi^1\|^2}{4}, \quad \text{and} \quad \|\tilde{\partial}_t \xi^1\|^2 = \frac{\|\xi^1\|^2}{\tau^2}.$$

Moreover, we have

$$\frac{\|\xi^{2,0}\|^2}{4} \leq \|\bar{\xi}^{2,0}\|^2.$$

The relation (4.20) becomes

$$\begin{aligned} \frac{\|\xi^{2,0}\|^2}{\tau^2} + \frac{\|\nabla \xi^{2,0}\|^2}{4} + \tau \frac{\|\nabla \xi^{2,0}\|^2}{4\tau^2} &\leq C\tau h^{2r} [\|u_{tt}^1\|_{r+1}^2 + \|u_t^1\|_{r+1}^2 + \|u^1\|_{r+1}^2] \\ &\quad + \frac{\|\nabla \xi^1\|^2}{4} + \frac{\|\xi^1\|^2}{\tau^2} + C\tau^2 [\|u_{tttt}^1\|^2 + \|u_{ttt}^1\|_1^2 + \|u_{tt}^1\|_1^2] \\ &\quad + C\tau \left[\frac{\|\nabla \xi^1\|^2}{4} + \frac{\|\xi^1\|^2}{\tau^2} + \frac{\|\nabla \xi^1\|^2}{\tau^2} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \|\xi^{2,0}\|^2 + \tau^2 \|\nabla \xi^{2,0}\|^2 + \tau \|\nabla \xi^{2,0}\|^2 &\leq Ch^{2r} [\|u_{tt}^1\|_{r+1}^2 + \|u_t^1\|_{r+1}^2 + \|u^1\|_{r+1}^2] \\ &\quad + C\tau^4 [\|u_{tttt}^1\|^2 + \|u_{ttt}^1\|_1^2 + \|u_{tt}^1\|_1^2] + C\|\xi^1\|^2 \\ &\quad + C\tau [\tau^2 \|\nabla \xi^1\|^2 + \|\xi^1\|^2 + \|\nabla \xi^1\|^2]. \end{aligned}$$

□

We state now the main result of this section.

Theorem 4.2. *Let u^n be the exact solution and U_h^n the fully discret solution. Assume that $u, u_t, u_{tt} \in L^\infty(0, T; H^{r+1}(\Omega))$, $u_{tttt} \in L^\infty(0, T; L^2(\Omega))$. Then there exists a generic constant C independent of τ and h such that*

$$\|\tilde{\partial}_t \xi^2\|^2 + \|\nabla \bar{\xi}^2\|^2 + \tau \|\nabla \bar{\partial}_t \xi^2\|^2 \leq C(h^{2r} + \tau^4). \quad (4.21)$$

And for all

$$\|\tilde{\partial}_t \xi^n\|^2 + \|\nabla \bar{\xi}^n\|^2 + \sum_{k=2}^{n-2} \tau \|\nabla \bar{\partial}_t \xi^k\|^2 \leq C(h^{2r} + \tau^4). \quad (4.22)$$

Proof. Subtracting equation (4.4) with $n = 1$ from (4.2), one gets

$$\begin{aligned} &(\bar{\partial}_{tt} \xi^1, w_h) + \phi \left(\|\nabla \bar{U}_h^{2,0}\|^2 \right) \left(\nabla \xi^{1, \frac{1}{4}}, \nabla w_h \right) + a \left(\nabla \bar{\partial}_t \xi^1, \nabla w_h \right) + \alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t \xi^1, w_h \right) \\ &= -(\bar{\partial}_{tt} \eta^1, w_h) - \left[\phi \left(\|\nabla \bar{u}^{2,0}\|^2 \right) - \phi \left(\|\nabla \bar{U}_h^{2,0}\|^2 \right) \right] \left(\nabla u^{1, \frac{1}{4}}, \nabla w_h \right) \\ &\quad - \alpha \left(\left| \bar{\partial}_t u^{2,0} \right|^{\beta-2} \bar{\partial}_t u^1 - \left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t R_h u^1, w_h \right) + \left(g \left(\bar{\partial}_t u^{2,0} \right) - g \left(\bar{\partial}_t U_h^{2,0} \right), w_h \right)_{\Gamma_0} \\ &\quad + (R_1^1, w_h) + \left(\vec{R}_2^1, \nabla w_h \right) + \left(\vec{R}_3^1, \nabla w_h \right) + (R_4^1, w_h) + (R_5^1, w_h)_{\Gamma_0}. \end{aligned}$$

Taking $w_h = \bar{\partial}_t \xi^1$ in the above relation, one obtains

$$\begin{aligned}
& (\bar{\partial}_{tt} \xi^1, \bar{\partial}_t \xi^1) + \phi \left(\|\nabla \bar{U}_h^{2,0}\|^2 \right) \left(\nabla \xi^{1,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^1 \right) + a \left(\nabla \bar{\partial}_t \xi^1, \nabla \bar{\partial}_t \xi^1 \right) + \alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t \xi^1, \bar{\partial}_t \xi^1 \right) \\
& = - (\bar{\partial}_{tt} \eta^1, \bar{\partial}_t \xi^1) - \left[\phi \left(\|\nabla \bar{u}^{2,0}\|^2 \right) - \phi \left(\|\nabla \bar{U}_h^{2,0}\|^2 \right) \right] \left(\nabla u^{1,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^1 \right) \\
& \quad - \alpha \left(\left| \bar{\partial}_t u^{2,0} \right|^{\beta-2} \bar{\partial}_t u^1 - \left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^1 \right) + \left(g \left(\bar{\partial}_t u^{2,0} \right) - g \left(\bar{\partial}_t U_h^{2,0} \right), \bar{\partial}_t \xi^1 \right)_{\Gamma_0} \\
& \quad + (R_1^1, \bar{\partial}_t \xi^1) + \left(\bar{R}_2^1, \nabla \bar{\partial}_t \xi^1 \right) + \left(\bar{R}_3^1, \nabla \bar{\partial}_t \xi^1 \right) + (R_4^1, \bar{\partial}_t \xi^1) + (R_5^1, \bar{\partial}_t \xi^1)_{\Gamma_0}.
\end{aligned}$$

That is,

$$\begin{aligned}
& \frac{1}{2\tau} (\|\tilde{\partial}_t \xi^2\|^2 - \|\tilde{\partial}_t \xi^1\|^2) + \frac{1}{2\tau} \phi(\|\nabla \bar{U}_h^{2,0}\|^2) (\|\nabla \bar{\xi}^2\|^2 - \|\nabla \bar{\xi}^1\|^2) + a \|\nabla \bar{\partial}_t \xi^1\|^2 \\
& \quad + \alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t \xi^1, \bar{\partial}_t \xi^1 \right) \\
& = \sum_{i=1}^9 T_i.
\end{aligned} \tag{4.23}$$

Each term of the right side of (4.23) will be bound as follows

$$T_1 \leq \|\bar{\partial}_{tt} \eta^1\| \|\bar{\partial}_t \xi^1\| \leq C \|\bar{\partial}_{tt} \eta^1\| \|\nabla \bar{\partial}_t \xi^1\| \leq C \|\bar{\partial}_{tt} \eta^1\|^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^1\|^2, \tag{4.24}$$

$$\begin{aligned}
T_2 & = \left[\phi \left(\|\nabla \bar{u}^{2,0}\|^2 \right) - \phi \left(\|\nabla \bar{U}_h^{2,0}\|^2 \right) \right] \left(\nabla u^{1,\frac{1}{4}}, \nabla \bar{\partial}_t \xi^1 \right) \\
& \leq L_\phi \left| \|\nabla \bar{u}^{2,0}\|^2 - \|\nabla \bar{U}_h^{2,0}\|^2 \right| \|\nabla u^{1,\frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \leq L_\phi \left(\|\nabla \bar{u}^{2,0}\| + \|\nabla \bar{U}_h^{2,0}\| \right) \|\nabla \bar{u}^{2,0} - \nabla \bar{U}_h^{2,0}\| \|\nabla u^{1,\frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \leq C \left(\|\nabla \bar{u}^{2,0}\|, \|\nabla \bar{U}_h^{2,0}\|, \|\nabla u^{1,\frac{1}{4}}\| \right) \|\nabla \bar{u}^{2,0} - \nabla \bar{U}_h^{2,0}\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \leq C \left[\|\nabla \bar{\eta}^{2,0}\|^2 + \|\nabla \bar{\xi}^{2,0}\|^2 \right] + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^1\|^2,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
T_3 & = \alpha \left(\left| \bar{\partial}_t u^{2,0} \right|^{\beta-2} \bar{\partial}_t \eta^1, \bar{\partial}_t \xi^1 \right) + \alpha \left(\left[\left| \bar{\partial}_t u^{2,0} \right|^{\beta-2} - \left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \right] \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^1 \right) \\
& \leq C \alpha \|\bar{\partial}_t u^{2,0}\|_\infty \|\bar{\partial}_t \eta^1\| \|\bar{\partial}_t \xi^1\| + C \alpha \left(\left| \bar{\partial}_t u^{2,0} - \bar{\partial}_t U_h^{2,0} \right| \left[\left| \bar{\partial}_t u^{2,0} \right| + \left| \bar{\partial}_t U_h^{2,0} \right| \right]^{\beta-3} \bar{\partial}_t R_h u^1, \bar{\partial}_t \xi^1 \right) \\
& \leq C \alpha \|\bar{\partial}_t u^{2,0}\|_\infty \|\bar{\partial}_t \eta^1\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \quad + C \left(\alpha, \|\bar{\partial}_t R_h u^1\|_\infty, \|\bar{\partial}_t u^{2,0}\|_\infty, \|\bar{\partial}_t U_h^{2,0}\|_\infty \right) \|\bar{\partial}_t u^{2,0} - \bar{\partial}_t U_h^{2,0}\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \leq C \left[\|\bar{\partial}_t \eta^1\|^2 + \|\bar{\partial}_t \eta^{2,0}\|^2 + \|\bar{\partial}_t \xi^{2,0}\|^2 \right] + \frac{2a}{20} \|\nabla \bar{\partial}_t \xi^1\|^2,
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
T_4 & \leq L_g \|\bar{\partial}_t (u^{2,0} - U_h^{2,0})\|_{\Gamma_0} \|\bar{\partial}_t \xi^1\|_{\Gamma_0} \\
& \leq C \|\nabla \bar{\partial}_t (u^{2,0} - U_h^{2,0})\| \|\nabla \bar{\partial}_t \xi^1\| \\
& \leq C \left[\|\nabla \bar{\partial}_t \eta^{2,0}\|^2 + \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \right] + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^1\|^2,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
T_5 + T_6 + T_7 + T_8 + T_9 &= (R_1^1, \bar{\partial}_t \xi^1) + (\vec{R}_2^1, \nabla \bar{\partial}_t \xi^1) + (\vec{R}_3^1, \nabla \bar{\partial}_t \xi^1) + (R_4^1, \bar{\partial}_t \xi^1) \\
&\quad + (R_5^1, \bar{\partial}_t \xi^1)_{\Gamma_0} \\
&\leq C \left[\|R_1^1\|^2 + \|\vec{R}_2^1\|^2 + \|\vec{R}_3^1\|^2 + \|R_4^1\|^2 + \|R_5^1\|_{\Gamma_0}^2 \right] + \frac{5a}{20} \|\nabla \bar{\partial}_t \xi^1\|^2.
\end{aligned} \tag{4.28}$$

Taking (4.24)–(4.28) into (4.23), one has

$$\begin{aligned}
&\frac{1}{2\tau} \left[\|\tilde{\partial}_t \xi^2\|^2 - \|\tilde{\partial}_t \xi^1\|^2 \right] + \frac{C_\phi}{2\tau} \left[\|\nabla \bar{\xi}^2\|^2 - \|\nabla \bar{\xi}^1\|^2 \right] + a \|\nabla \bar{\partial}_t \xi^1\|^2 \\
&\quad + \alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t \xi^1, \bar{\partial}_t \xi^1 \right) \\
&\leq \frac{a}{2} \|\nabla \bar{\partial}_t \xi^1\|^2 + C \left[\|\bar{\partial}_{tt} \eta^1\|^2 + \|\bar{\partial}_t \eta^1\|^2 + \|\nabla \bar{\eta}^{2,0}\|^2 + \|\nabla \bar{\partial}_t \eta^{2,0}\|^2 \right] \\
&\quad + C \left[\|\nabla \bar{\xi}^{2,0}\|^2 + \|\bar{\partial}_t \xi^{2,0}\|^2 + \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \right] \\
&\quad + C \left[\|R_1^1\|^2 + \|\vec{R}_2^1\|^2 + \|\vec{R}_3^1\|^2 + \|R_4^1\|^2 + \|R_5^1\|_{\Gamma_0}^2 \right].
\end{aligned} \tag{4.29}$$

Dropping $\alpha \left(\left| \bar{\partial}_t U_h^{2,0} \right|^{\beta-2} \bar{\partial}_t \xi^1, \bar{\partial}_t \xi^1 \right) > 0$, and multiplying through by 2τ , one gets

$$\begin{aligned}
&\|\tilde{\partial}_t \xi^2\|^2 - \|\tilde{\partial}_t \xi^1\|^2 + \|\nabla \bar{\xi}^2\|^2 - \|\nabla \bar{\xi}^1\|^2 + a\tau \|\nabla \bar{\partial}_t \xi^1\|^2 \\
&\leq C\tau \left[\|\bar{\partial}_{tt} \eta^1\|^2 + \|\bar{\partial}_t \eta^1\|^2 + \|\nabla \bar{\eta}^{2,0}\|^2 + \|\nabla \bar{\partial}_t \eta^{2,0}\|^2 \right] \\
&\quad + C\tau \left[\|\nabla \bar{\xi}^{2,0}\|^2 + \|\bar{\partial}_t \xi^{2,0}\|^2 + \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \right] \\
&\quad + C\tau \left[\|R_1^1\|^2 + \|\vec{R}_2^1\|^2 + \|\vec{R}_3^1\|^2 + \|R_4^1\|^2 + \|R_5^1\|_{\Gamma_0}^2 \right].
\end{aligned} \tag{4.30}$$

By Lemma (3.2), we have

$$\begin{aligned}
&\|\tilde{\partial}_t \xi^2\|^2 + \|\nabla \bar{\xi}^2\|^2 + \tau \|\nabla \bar{\partial}_t \xi^1\|^2 \\
&\leq Ch^{2r} \tau \left[\|u_{tt}^1\|_{r+1}^2 + \|u_t^1\|_{r+1}^2 + \|\nabla \bar{u}^1\|_{r+1}^2 \right] \\
&\quad + \|\nabla \bar{\xi}^1\|^2 + \|\tilde{\partial}_t \xi^1\|^2 + C\tau \left[\|\nabla \bar{\xi}^{2,0}\|^2 + \|\bar{\partial}_t \xi^{2,0}\|^2 + \|\nabla \bar{\partial}_t \xi^{2,0}\|^2 \right] \\
&\quad + C\tau^5 \left[\|u_{ttt}^1\|^2 + \|u_{ttt}^1\|_1^2 + \|u_{tt}^1\|_1^2 \right].
\end{aligned} \tag{4.31}$$

To estimate (4.22), subtracting (4.3) from (4.4) lead to

$$\begin{aligned}
&(\bar{\partial}_{tt} \xi^n, w_h) + \phi(\|\nabla \hat{u}^n\|^2) \left(\nabla \xi^{n, \frac{1}{4}}, \nabla w_h \right) + a(\nabla \bar{\partial}_t \xi^n, \nabla w_h) + \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t \xi^n, w_h \right) \\
&= -(\bar{\partial}_{tt} \eta^n, w_h) - \left[\phi(\|\nabla \hat{u}^n\|^2) - \phi(\|\nabla \hat{U}_h^n\|^2) \right] \left(\nabla u^{n, \frac{1}{4}}, \nabla w_h \right) \\
&\quad - \alpha \left(\left| \bar{\partial}_t u^{n-1} \right|^{\beta-2} \bar{\partial}_t u^n - \left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t R_h u^n, w_h \right) + (g(\bar{\partial}_t u^{n-1}) - g(\bar{\partial}_t U_h^{n-1}), w_h)_{\Gamma_0} \\
&\quad + (R_1^n, w_h) + (\vec{R}_2^n, \nabla w_h) + (\vec{R}_3^n, \nabla w_h) + (R_4^n, w_h) + (R_5^n, w_h)_{\Gamma_0}.
\end{aligned}$$

Taking $w_h = \bar{\partial}_t \xi^n$, one obtains

$$\begin{aligned}
 & (\bar{\partial}_{tt} \xi^n, \bar{\partial}_t \xi^n) + \phi(\|\nabla \hat{u}^n\|^2) \left(\nabla \xi^{n, \frac{1}{4}}, \nabla \bar{\partial}_t \xi^n \right) + a(\nabla \bar{\partial}_t \xi^n, \nabla \bar{\partial}_t \xi^n) + \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t \xi^n, \bar{\partial}_t \xi^n \right) \\
 &= -(\bar{\partial}_{tt} \eta^n, \bar{\partial}_t \xi^n) - \left[\phi(\|\nabla \hat{u}^n\|^2) - \phi(\|\nabla \hat{U}_h^n\|^2) \right] \left(\nabla u^{n, \frac{1}{4}}, \nabla \bar{\partial}_t \xi^n \right) \\
 &\quad - \alpha \left(\left| \bar{\partial}_t u^{n-1} \right|^{\beta-2} \bar{\partial}_t u^n - \left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t R_h u^n, \bar{\partial}_t \xi^n \right) + (g(\bar{\partial}_t u^{n-1}) - g(\bar{\partial}_t U_h^{n-1}), \bar{\partial}_t \xi^n)_{\Gamma_0} \\
 &\quad + (R_1^n, \bar{\partial}_t \xi^n) + (\vec{R}_2^n, \nabla \bar{\partial}_t \xi^n) + (\vec{R}_3^n, \nabla \bar{\partial}_t \xi^n) + (R_4^n, \bar{\partial}_t \xi^n) + (R_5^n, \bar{\partial}_t \xi^n)_{\Gamma_0}.
 \end{aligned}$$

That is,

$$\begin{aligned}
 & \frac{1}{2\tau} \left[\|\tilde{\partial}_t \xi^{n+1}\|^2 - \|\tilde{\partial}_t \xi^n\|^2 \right] + \frac{1}{2\tau} \phi(\|\nabla \hat{u}^n\|^2) \left[\|\nabla \bar{\xi}^{n+1}\|^2 - \|\nabla \bar{\xi}^n\|^2 \right] \\
 & + a \|\nabla \bar{\partial}_t \xi^n\|^2 + \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t \xi^n, \bar{\partial}_t \xi^n \right) = \sum_{i=1}^9 P_i.
 \end{aligned} \tag{4.32}$$

Now, we estimate the terms of the right side of (4.32)

$$P_1 \leq \|\bar{\partial}_{tt} \eta^n\| \|\bar{\partial}_t \xi^n\| \leq C \|\bar{\partial}_{tt} \eta^n\| \|\nabla \bar{\partial}_t \xi^n\| \leq C \|\bar{\partial}_{tt} \eta^n\|^2 + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^n\|^2, \tag{4.33}$$

$$\begin{aligned}
 P_2 &= \left[\phi(\|\nabla \hat{u}^n\|^2) - \phi(\|\nabla \hat{U}_h^n\|^2) \right] \left(\nabla u^{n, \frac{1}{4}}, \nabla \bar{\partial}_t \xi^n \right) \\
 &\leq L_\phi \left| \|\nabla \hat{u}^n\|^2 - \|\nabla \hat{U}_h^n\|^2 \right| \|\nabla u^{n, \frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\leq L_\phi \left(\|\nabla \hat{u}^n\| + \|\nabla \hat{U}_h^n\| \right) \|\nabla \hat{u}^n - \nabla \hat{U}_h^n\| \|\nabla u^{n, \frac{1}{4}}\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\leq C \left(\|\nabla \hat{u}^n\|, \|\nabla \hat{U}_h^n\|, \|\nabla u^{n, \frac{1}{4}}\| \right) \|\nabla \hat{u}^n - \nabla \hat{U}_h^n\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\leq C \left[\|\nabla \hat{\eta}^n\|^2 + \|\nabla \hat{\xi}^n\|^2 \right] + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^n\|^2,
 \end{aligned} \tag{4.34}$$

$$\begin{aligned}
 P_3 &= \alpha \left(\left| \bar{\partial}_t u^{n-1} \right|^{\beta-2} \bar{\partial}_t \eta^n, \bar{\partial}_t \xi^n \right) + \alpha \left(\left[\left| \bar{\partial}_t u^{n-1} \right|^{\beta-2} - \left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \right] \bar{\partial}_t R_h u^n, \bar{\partial}_t \xi^n \right) \\
 &\leq C \alpha \|\bar{\partial}_t u^{n-1}\|_\infty \|\bar{\partial}_t \eta^n\| \|\bar{\partial}_t \xi^n\| \\
 &\quad + C \alpha \left(\left| \bar{\partial}_t u^{n-1} - \bar{\partial}_t U_h^{n-1} \right| \left[|\bar{\partial}_t u^{n-1}| + |\bar{\partial}_t U_h^{n-1}| \right]^{\beta-3} \bar{\partial}_t R_h u^n, \bar{\partial}_t \xi^n \right) \\
 &\leq C \alpha \|\bar{\partial}_t u^{n-1}\|_\infty \|\bar{\partial}_t \eta^n\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\quad + C \left(\alpha, \|\bar{\partial}_t R_h u^n\|_\infty, \|\bar{\partial}_t u^{n-1}\|_\infty, \|\bar{\partial}_t U_h^{n-1}\|_\infty \right) \|\bar{\partial}_t u^{n-1} - \bar{\partial}_t U_h^{n-1}\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\leq C \left[\|\bar{\partial}_t \eta^n\|^2 + \|\bar{\partial}_t \eta^{n-1}\|^2 + \|\bar{\partial}_t \xi^{n-1}\|^2 \right] + \frac{2a}{20} \|\nabla \bar{\partial}_t \xi^n\|^2,
 \end{aligned} \tag{4.35}$$

$$\begin{aligned}
 P_4 &\leq L_g \|\bar{\partial}_t(u^{n-1} - U_h^{n-1})\|_{\Gamma_0} \|\bar{\partial}_t \xi^n\|_{\Gamma_0} \\
 &\leq C \|\nabla \bar{\partial}_t(u^{n-1} - U_h^{n-1})\| \|\nabla \bar{\partial}_t \xi^n\| \\
 &\leq C \left[\|\nabla \bar{\partial}_t \eta^{n-1}\|^2 + \|\nabla \bar{\partial}_t \xi^{n-1}\|^2 \right] + \frac{a}{20} \|\nabla \bar{\partial}_t \xi^n\|^2,
 \end{aligned} \tag{4.36}$$

$$P_5 + P_6 + P_7 + P_8 + P_9 = (R_1^n, \bar{\partial}_t \xi^n) + (\vec{R}_2^n, \nabla \bar{\partial}_t \xi^n) + (\vec{R}_3^n, \nabla \bar{\partial}_t \xi^n) + (R_4^n, \bar{\partial}_t \xi^n)$$

$$\begin{aligned}
& + (R_5^n, \bar{\partial}_t \xi^n)_{\Gamma_0} \\
& \leq C \left[\|R_1^n\|^2 + \|\vec{R}_2^n\|^2 + \|\vec{R}_3^n\|^2 + \|R_4^n\|^2 + \|R_5^n\|_{\Gamma_0}^2 \right] + \frac{5a}{20} \|\nabla \bar{\partial}_t \xi^n\|^2.
\end{aligned} \tag{4.37}$$

Replacing (4.33)–(4.37) into (4.32), one has

$$\begin{aligned}
& \frac{1}{2\tau} \left[\|\bar{\partial}_t \xi^{n+1}\|^2 - \|\bar{\partial}_t \xi^n\|^2 \right] + \frac{C_\phi}{2\tau} \left[\|\nabla \bar{\xi}^{n+1}\|^2 - \|\nabla \bar{\xi}^n\|^2 \right] + a \|\nabla \bar{\partial}_t \xi^n\|^2 \\
& + \alpha \left(\left| \bar{\partial}_t U_h^{n-1} \right|^{\beta-2} \bar{\partial}_t \xi^n, \bar{\partial}_t \xi^n \right) \\
& \leq C \left[\|\bar{\partial}_{tt} \eta^n\|^2 + \|\bar{\partial}_t \eta^n\|^2 + \|\nabla \hat{\eta}^n\|^2 + \|\bar{\partial}_t \eta^{n-1}\|^2 + \|\nabla \bar{\partial}_t \eta^{n-1}\|^2 \right] \\
& + \frac{a}{2} \|\nabla \bar{\partial}_t \xi^n\|^2 + C \left[\|\nabla \hat{\xi}^n\|^2 + \|\bar{\partial}_t \xi^{n-1}\|^2 + \|\nabla \bar{\partial}_t \xi^{n-1}\|^2 \right] \\
& + C \left[\|R_1^n\|^2 + \|\vec{R}_2^n\|^2 + \|\vec{R}_3^n\|^2 + \|R_4^n\|^2 + \|R_5^n\|_{\Gamma_0}^2 \right].
\end{aligned} \tag{4.38}$$

That is

$$\begin{aligned}
& \|\bar{\partial}_t \xi^{n+1}\|^2 - \|\bar{\partial}_t \xi^n\|^2 + \|\nabla \bar{\xi}^{n+1}\|^2 - \|\nabla \bar{\xi}^n\|^2 + a\tau \|\nabla \bar{\partial}_t \xi^n\|^2 \\
& \leq C\tau \left[\|\bar{\partial}_{tt} \eta^n\|^2 + \|\bar{\partial}_t \eta^n\|^2 + \|\nabla \hat{\eta}^n\|^2 + \|\bar{\partial}_t \eta^{n-1}\|^2 + \|\nabla \bar{\partial}_t \eta^{n-1}\|^2 \right] \\
& + C\tau \left[\|\nabla \hat{\xi}^n\|^2 + \|\bar{\partial}_t \xi^{n-1}\|^2 + \|\nabla \bar{\partial}_t \xi^{n-1}\|^2 \right] \\
& + C\tau \left[\|R_1^n\|^2 + \|\vec{R}_2^n\|^2 + \|\vec{R}_3^n\|^2 + \|R_4^n\|^2 + \|R_5^n\|_{\Gamma_0}^2 \right].
\end{aligned} \tag{4.39}$$

Replacing n by k in (4.39) and summing up from $k = 2$ to $n - 1$, one gets

$$\begin{aligned}
& \|\bar{\partial}_t \xi^n\|^2 - \|\bar{\partial}_t \xi^2\|^2 + \|\nabla \bar{\xi}^n\|^2 - \|\nabla \bar{\xi}^2\|^2 + \tau \sum_{k=2}^{n-1} \|\nabla \bar{\partial}_t \xi^k\|^2 \\
& \leq C\tau \sum_{k=2}^{n-1} \left[\|\bar{\partial}_{tt} \eta^k\|^2 + \|\bar{\partial}_t \eta^k\|^2 + \|\nabla \hat{\eta}^k\|^2 + \|\bar{\partial}_t \eta^{k-1}\|^2 + \|\nabla \bar{\partial}_t \eta^{k-1}\|^2 \right] \\
& + C\tau \sum_{k=2}^{n-1} \left[\|\nabla \hat{\xi}^k\|^2 + \|\bar{\partial}_t \xi^{k-1}\|^2 + \|\nabla \bar{\partial}_t \xi^{k-1}\|^2 \right] \\
& + C \sum_{k=2}^{n-1} \tau \left[\|R_1^k\|^2 + \|\vec{R}_2^k\|^2 + \|\vec{R}_3^k\|^2 + \|R_4^k\|^2 + \|R_5^k\|_{\Gamma_0}^2 \right].
\end{aligned} \tag{4.40}$$

By Lemma (3.2) and Taylor's expansion (4.5), we have

$$\begin{aligned}
& \|\bar{\partial}_t \xi^n\|^2 + \|\nabla \bar{\xi}^n\|^2 + \tau \sum_{k=2}^{n-1} \|\nabla \bar{\partial}_t \xi^k\|^2 \\
& \leq Ch^{2r} \sum_{k=2}^{n-1} \tau \left[\|u_{tt}^k\|_{r+1}^2 + \|u_t^k\|_{r+1}^2 + \|\hat{u}^k\|_{r+1}^2 + \|u_t^{k-1}\|_{r+1}^2 \right] \\
& + \|\nabla \bar{\xi}^2\|^2 + \|\bar{\partial}_t \xi^2\|^2 + C\tau \sum_{k=2}^{n-1} \left[\|\nabla \hat{\xi}^k\|^2 + \|\bar{\partial}_t \xi^{k-1}\|^2 + \|\nabla \bar{\partial}_t \xi^{k-1}\|^2 \right]
\end{aligned}$$

$$+ C\tau^4 \sum_{k=2}^{n-1} \tau \left[\|u_{tttt}^k\|^2 + \|u_{ttt}^k\|_1^2 + \|u_{tt}^k\|_1^2 \right]. \quad (4.41)$$

Using the fact that

$$\|\nabla \hat{\xi}^n\|^2 \leq C \left[\|\nabla \xi^n\|^2 + \|\nabla \xi^{n-1}\|^2 + \|\nabla \xi^{n-2}\|^2 \right],$$

and by the discrete Gronwall lemma on (4.41), one gets

$$\begin{aligned} & \|\tilde{\partial}_t \xi^n\|^2 + \|\nabla \bar{\xi}^n\|^2 + \tau \sum_{k=2}^{n-1} \|\nabla \bar{\partial}_t \xi^k\|^2 \\ & \leq Ch^{2r} \sum_{k=2}^{n-1} \tau \left[\|u_{tt}^k\|_{r+1}^2 + \|u_t^k\|_{r+1}^2 + \|\hat{u}^k\|_{r+1}^2 + \|u_t^{k-1}\|_{r+1}^2 \right] \\ & \quad + \|\nabla \bar{\xi}^2\|^2 + \|\tilde{\partial}_t \xi^2\|^2 + C\tau^4 \sum_{k=2}^{n-1} \tau \left[\|u_{tttt}^k\|^2 + \|u_{ttt}^k\|_1^2 + \|u_{tt}^k\|_1^2 \right]. \end{aligned} \quad (4.42)$$

□

5. Numerical results

To illustrate the effectiveness of the proposed finite element scheme, we present numerical experiments on representative Kirchhoff-type problems with boundary dissipation. The simulations were carried out on uniform triangulations of the domain, using polynomial spaces of degree $r = 1$. To implement the algorithm (4.2)-(4.3), we use the finite element solver FreeFem++ [10]. The problem (1.1) is solved on the square $\Omega = (0, 1)^2$ and the interval $[0, T]$ with $T = 1$. The boundary $\partial\Omega$ is divided into two part: Γ_0 and Γ_1 with $\Gamma_0 = \{(x, 1); \quad 0 \leq x \leq 1\}$, $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. We impose the homogeneous Dirichlet boundary condition on Γ_1 and the boundary dissipation given by (1.1)-c on Γ_0 .

We set $a = 1$, $\alpha = 1$, $\beta = 3.5$

$$\phi(s) = 1 + s^2; \quad g(s) = 0.$$

The functions f , u_0 and u_1 are chosen such that the analytical solution is given by

$$u_{ex}(x, y, t) = \exp(-t) \sin(\pi x) \sin\left(\frac{\pi}{2}y\right).$$

Figure 1 shows that the computed solutions align closely with the exact solutions. The approximate solution is computed with linear finite element approximation with $h = 1/15$ and $\tau = h^2$.

To confirm our error estimates, we solved the same problem using the algorithm (4.1)-(4.3) on uniform meshes.

For the convergence with respect to the mesh size h , we compute the errors for different values of $h = \frac{1}{2^{n+3}}$, with $n = 0, 1, 2, 3$ and set $\tau = h$. Figure 2-(a) shows that the slopes of $\log(\|u_{ex} - u_h^n\|)$ and $\log(\|u_{ex} - u_h^n\|_1)$ are 1 which confirm the linear order convergence with respect to the mesh size h . For the convergence with respect to the time steps, we compute the errors for different values of $\tau = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{70}$ and fix the mesh size $h = 1/128$. Figure 2-(b)

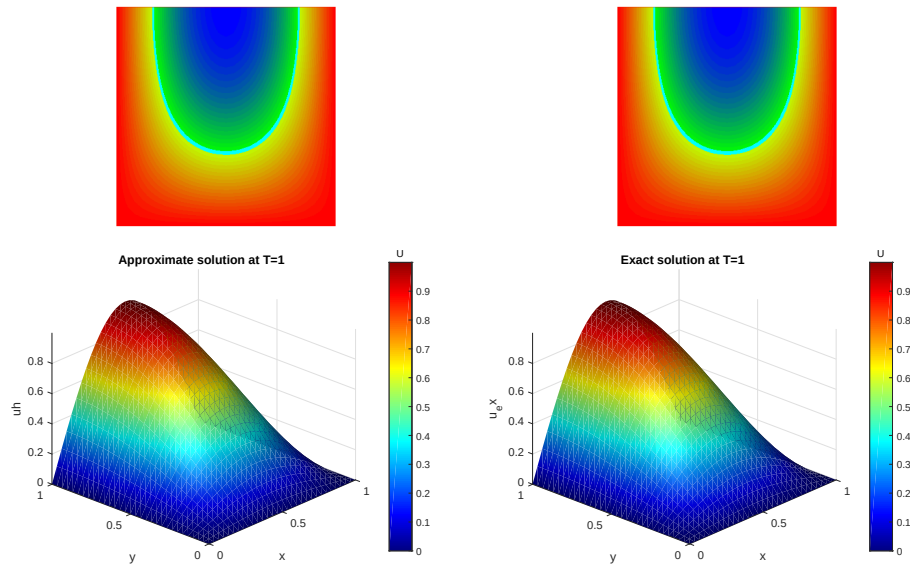


Figure 1. The graphics of approximate solution and exact solution at $T = 1$ respectively.

shows that the slopes of $\log(\|u_{ex} - u_h^n\|)$ and $\log(\|u_{ex} - u_h^n\|_1)$ are 2 which confirm the second order convergence with respect to the time step.

In our experiments, the proposed scheme achieved second-order convergence in time and optimal spatial accuracy. For mesh sizes $h = \frac{1}{2^{n+3}}$, with $n = 0, 1, 2, 3$, the L^2 -error decreased consistently, confirming the theoretical error bounds. Moreover, the discrete energy functional exhibited monotone decay, in line with the dissipative nature of the continuous problem. These results demonstrate that the linearized Galerkin scheme not only provides accurate approximations but also captures the qualitative energy dynamics of Kirchhoff-type wave equations. Unlike classical schemes that require mesh-dependent stability conditions, our method remains stable under arbitrary discretizations. However, the absence of explicit energy-preserving features suggests that future work should explore structure-preserving variants to enhance long-time reliability.

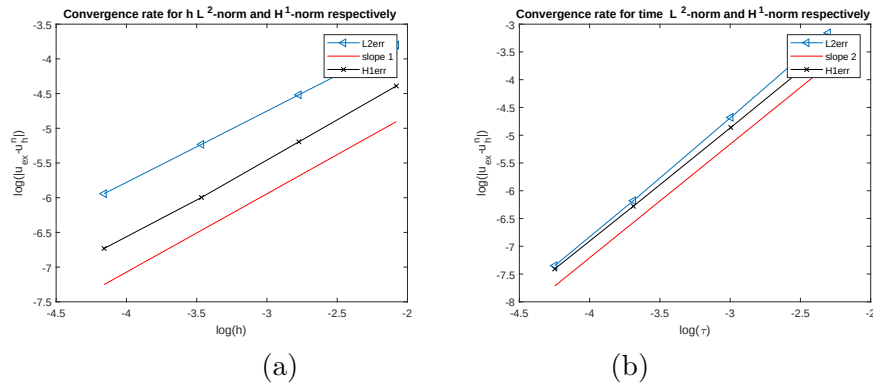


Figure 2. (a) L^2 and H^1 - errors with respect the mesh size h . (b) L^2 and H^1 - errors with respect to the time step.

6. Conclusion

In this paper, we presented a linearized second-order finite element scheme for non-linear Kirchhoff-type problems with boundary dissipation. The analysis demonstrates that the method is stable and convergent under general assumptions, with error bounds derived independently of mesh constraints. The theoretical framework combines variational formulations, Galerkin approximations, and discrete energy estimates to handle the strong nonlinearities inherent in the model. Numerical simulations validate the theoretical predictions, showing that the scheme provides reliable approximations of the solution and preserves the dissipative nature of the system. These results highlight the potential of finite element techniques as robust computational tools for tackling complex non-linear wave equations, paving the way for further applications in engineering and applied sciences. Although the proposed linearized Galerkin finite element method achieves unconditional stability and convergence without mesh restrictions, several limitations remain. First, the analysis is restricted to smooth bounded domains; extending the framework to irregular geometries or domains with complex boundaries may require additional stabilization techniques. Second, while the scheme captures dissipative energy dynamics, it does not explicitly enforce structure-preserving properties such as symplecticity or discrete energy conservation, which may be critical for very long-time simulations. Third, the linearization strategy simplifies implementation but may introduce efficiency trade-offs compared to fully nonlinear solvers, especially for strongly coupled systems. Finally, the numerical experiments, though illustrative, are limited in scope; broader testing on high-dimensional problems and real-world applications would be necessary to fully validate the robustness of the method.

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