

BLOW-UP PHENOMENON OF SOLUTION TO THE SEMILINEAR VISCOUS MOORE-GIBSON-THOMPSON EQUATION WITH MEMORY TERMS*

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Abstract This article is devoted to investigating the initial boundary value problem of semilinear viscous Moore-Gibson-Thompson equation with general initial values and nonlinear memory terms $N_{\gamma,q}(u)$, $N_{\gamma,p}(u_t)$, $N_{\gamma,p,q}(u, u_t)$ on exterior domain, respectively. Blow-up dynamic of solution is deduced by taking advantage of the test function method related to an appropriate harmonic function. It is worth observing that the semilinear viscous Moore-Gibson-Thompson equation with memory terms can be considered as an approximation of viscous Moore-Gibson-Thompson equation with nonlinear terms when $\gamma \rightarrow 1^-$. Up to author's best knowledge, the results in Theorems 1.1-1.3 are new.

Keywords Viscous Moore-Gibson-Thompson equation, general initial values, nonlinear memory terms, exterior domain, blow-up.

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1. Introduction

This paper is concerned with the initial boundary value problem of semilinear viscous Moore-Gibson-Thompson equation (MGT) with memory terms

$$\begin{cases} \tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau)\Delta u_t = N(u, u_t), & x \in \Omega^c, t > 0, \\ (u, u_t, u_{tt})(x, 0) = (u_0, u_1, u_2)(x), & x \in \Omega^c, \\ u|_{\partial\Omega^c} = 0, & t > 0, \end{cases} \quad (1.1)$$

where the nonlinear term $N(u, u_t)$ is presented in the forms of

$$\begin{aligned} N(u, u_t) &= N_{\gamma,q}(u) = c_\gamma \int_0^t (t-s)^{-\gamma} |u(x, s)|^q ds, \\ N(u, u_t) &= N_{\gamma,p}(u_t) = c_\gamma \int_0^t (t-s)^{-\gamma} |u_t(x, s)|^p ds \end{aligned}$$

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and

$$\begin{aligned} N(u, u_t) &= N_{\gamma, p, q}(u, u_t) \\ &= c_\gamma \int_0^t (t-s)^{-\gamma} |u_t(x, s)|^p ds + c_\gamma \int_0^t (t-s)^{-\gamma} |u(x, s)|^q ds \end{aligned}$$

with $c_\gamma = \frac{1}{\Gamma(1-\gamma)}$, $\gamma \in (0, 1)$. $\Gamma(\cdot)$ stands for the second kind of Euler integral, namely, $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ for $s > 0$. $\tau > 0$ represents the thermal relaxation from the Cattaneo-Maxwell law of heat conduction. $\delta > 0$ denotes the diffusivity of sound, which involves of viscous coefficient in the Navier-Stokes equation. The exponents of nonlinear terms satisfy $1 < p, q < \infty$. We set $\Omega = B_1(0) = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. $\Omega^c = \mathbb{R}^n \setminus B_1(0)$ admits smooth compact boundary $\partial\Omega^c$. We suppose $B_R(0) = \{x \in \mathbb{R}^n \mid |x| \leq R\}$, where $R > 2$. Initial values u_0, u_1, u_2 are non-negative functions.

We first show some background of the third order hyperbolic MGT equation

$$\tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau) \Delta u_t = 0, \quad (1.2)$$

where $u(x, t)$ is acoustic velocity potential due to application of irrotational flow in the modeling. The parameters τ and δ stand for the thermal relaxation and diffusivity of sound, respectively. It is well-known that the MGT equation is actually linearized model for wave propagation in viscous thermally relaxing fluids. Physically, MGT equation is one of the most important models in field of acoustic waves. The origin of mathematical model of MGT equation lies in the Kuznetsov equation, which is fundamental model in nonlinear acoustics. Fourier law of heat conduction is employed in the model of Kuznetsov equation. To eliminate the paradox of infinite signal speed in wave propagation, Jordan [29] replaces Fourier law of heat conduction with the Cattaneo-Maxwell law of heat conduction in acoustic wave modeling, which leads to the mathematical model of MGT equation. According to value of parameter δ , the MGT equation can be divided into three cases: Viscous case $\delta > 0$, inviscid case $\delta = 0$ and chaotic case $\delta < 0$. MGT model is represented by the third order (in time) strictly hyperbolic partial differential equation

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0.$$

The parameter $b = \beta c^2$ characterizes diffusivity of sound with thermal relaxation parameter $\tau \in (0, \beta]$. It is worth noting that the medium property associated with dispersive, dissipative and nonlinear effects have influence on behavior of solution. Nonlinear term in problem (1.1) involves memory term self-interaction and can be considered as Riemann-Liouville integral operator

$$J_{a|t}^{1-\gamma} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{-\gamma} g(s) ds.$$

It can accurately describe the physical phenomena depending on past states. Information remembered by the system includes past history and gradually decays. Moreover, more recent information has greater impact on the system.

Many scholars are committed to studying the Cauchy problem of semilinear MGT equation

$$\tau u_{ttt} + u_{tt} - \Delta u - (\delta + \tau) \Delta u_t = f(u, u_t),$$

where $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$. In case of $\delta > 0$, exploiting the test function technique, Chen et al. [3] establish non-existence of global solution to the Cauchy problem of semilinear

MGT equation with general initial values and nonlinearity of power type $|u|^p$ in the dissipative case. Shi et al. [53] discuss the Cauchy problem of semilinear viscous MGT equation with nonlinearity of derivative type $|u_t|^p$. Blow-up of solution is deduced by utilizing the test function method $\psi(x, t) = \varphi(R^{-1}K^{-1}x)\eta(R^{-1}t)$. In case of $\delta = 0$, Chen et al. [4, 5] present upper bound lifespan estimate of solution to the Cauchy problem of semilinear MGT equation in the sub-critical and critical cases by employing test function technique and iteration argument, where nonlinear terms are $|u|^p$, $|u_t|^p$. Making use of test function method, Ming et al. [48] derive formation of singularity for solution to the semilinear MGT equation with small initial values and nonlinear terms $|u|^p$, $|u_t|^p$, $|u_t|^p + |u|^q$. Blow-up dynamic of solution to the initial boundary value problem of semilinear MGT equation with combined nonlinearities $|u_t|^p + |u|^q$ on exterior domain is considered in [47]. The proof is based on the test function technique. One can refer to the references [42, 43, 54, 61] for more details.

Let us recall some known results for the Cauchy problem of classical wave equation with memory term

$$\begin{cases} u_{tt} - \Delta u = N(u, u_t), & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = \varepsilon u_0(x), \quad u_t(x, 0) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where $N(u, u_t) = N_{\gamma,p}(u)$, $N_{\gamma,p}(u_t)$. Blow-up result and lifespan estimate of solutions to problem (1.3) with memory term $N(u, u_t) = N_{\gamma,p}(u)$ in the sub-critical and critical cases are established in [6], where iteration method in the sub-critical case and slicing technique in the critical case are utilized. Ouyang et al. [49] discuss blow-up result of solution to problem (1.3) with memory term $N(u, u_t) = N_{\gamma,p}(u_t)$. Upper bound lifespan estimate of solution is obtained by employing the iteration approach and ODE blow-up argument. We note that problem (1.3) with memory term $N(u, u_t) = N_{\gamma,p}(u)$ possesses the critical exponent $p_S(n, \gamma)$, which is the biggest solution to the quadratic equation

$$(n-1)p^2 - (n+3-2\gamma)p - 2 = 0.$$

Additionally, $p_S(n, \gamma)$ satisfies $\lim_{\gamma \rightarrow 1^-} p_S(n, \gamma) = p_S(n)$. It is worth to observe that $p_S(n)$ represents the Strauss exponent of classical equation

$$u_{tt} - \Delta u = |u|^p.$$

More specifically, if $1 < p \leq p_S(n)$, there is no global (in time) solution to problem (1.3) with power nonlinearity $|u|^p$. If $p > p_S(n)$, problem (1.3) asserts a global solution (see detailed illustration in [32, 40, 59, 64, 65]). Blow-up result of solution to problem (1.3) with derivative nonlinearity $|u_t|^p$ is investigated in [17, 23, 63]. Problem (1.3) with combined nonlinearities $|u_t|^p + |u|^q$ is considered (see [20, 24, 26]). Ikeda et al. [26] illustrate upper bound lifespan estimates of solutions to the Cauchy problem of classical wave equation with different type nonlinear terms $|u|^p$, $|u_t|^p$, $|u_t|^p + |u|^q$ and corresponding weakly coupled system, where the test function method is performed.

Many researchers are interested in the investigation of blow-up for solution to the semilinear wave equation with damping term

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = f(u, u_t), & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.4)$$

where $g(u_t)$ is the damping term, nonlinear term $f(u, u_t) = |u|^p$, $|u_t|^p$, $|u_t|^p + |u|^q$. Problem (1.4) with $g(u_t) = u_t$ and $f(u, u_t) = |u|^p$ on exterior domain is discussed in [15]. Non-existence of

global solution is established by applying the test function method. Dao et al. [9,11] present blow-up result of solution to problem (1.4) with structural damping $g(u_t) = (-\Delta)^\delta u_t$ and nonlinear terms $f(u, u_t) = |u|^p, |u_t|^p$. Utilizing the test function approach $\psi(x, t) = \phi_0(x)\varphi_T^\ell(x)\eta_T^k(t)$, Chen et al. [2] derive blow-up of solution to problem (1.4) with $g(u_t) = \Delta u_t$ on exterior domain in the sub-critical and critical cases, where nonlinear terms are $f(u, u_t) = |u_t|^p, |u_t|^p + |u|^q$. We refer the readers to [21, 27, 28, 33, 34, 36–39, 41, 45, 46, 50, 55–57, 62] for more details in this direction.

The Cauchy problem of semilinear wave equation with memory term

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = N_{\gamma,p}(u), & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^n \end{cases} \quad (1.5)$$

attracts more attention (see [1, 7, 8, 10, 13, 35, 58]). Fino [13] shows non-existence of global solution to problem (1.5) with $g(u_t) = u_t$ by taking advantage of the test function technique. Blow-up result of solution to problem (1.5) with scaling invariant damping $g(u_t) = \frac{\mu}{1+t}u_t$ ($\mu > 0$) is demonstrated in [1]. The proof is based on the Kato lemma in the case $1 < p < p_S(n + \mu, \gamma)$ and iteration method in the case $p = p_S(n + 2, \gamma)$. Taking into account the test function approach, Dao et al. [10] investigate blow-up phenomenon of solution to problem (1.5) with $g(u_t) = \mu(-\Delta)^{\frac{\sigma}{2}}u_t$ ($\mu > 0, 0 < \sigma < 2$). Problem (1.5) with damping term $g(u_t) = a(x)b(t)u_t$ is discussed in [8], where $a(x)b(t) = a_0(1 + |x|^2)^{-\frac{\alpha}{2}}(1 + t)^{-\beta}$ ($a_0 > 0, \alpha, \beta \geq 0, \alpha + \beta < 1$). Non-existence of global solution to the problem is obtained by making use of the test function method. We refer the readers to [12, 16, 18, 19, 22, 25, 30, 44, 52, 60] for more details.

Enlightened by the works in [2, 3, 10, 14, 53], our main target of this paper is to acquire non-existence of global solution to problem (1.1) with nonlinear memory terms $N_{\gamma,q}(u), N_{\gamma,p}(u_t), N_{\gamma,p,q}(u, u_t)$, respectively. Dao et al. [10] derive blow-up dynamic of solution to the semilinear wave equation with structural damping and nonlinear memory term of power type $\int_0^t (t-s)^{-\gamma}|u|^p ds$ ($\gamma \in (0, 1)$), where the test function method is performed $\psi(x, t) = \phi_R(x)D_{tT}^\alpha(w(t))$. Exploiting the test function technique $\psi(x, t) = \phi_R(x)\eta_R(t)$, Chen et al. [3] show non-existence of global solution to the semilinear MGT equation with power type nonlinear term $|u|^p$ in dissipative case (namely, problem (1.1) with $N(u, u_t) = |u|^p$). Blow-up of solution to problem (1.1) with derivative type nonlinear term $|u_t|^p$ is discussed in [53]. The proof is based on the test function approach. From our observation, the research of blow-up dynamic for solution to problem (1.1) is still open. To fill this gap, we study problem (1.1) with nonlinear terms $N_{\gamma,q}(u), N_{\gamma,p}(u_t), N_{\gamma,p,q}(u, u_t)$ on exterior domain. It is worth noticing that blow-up result of solution to the semilinear wave equation with strong damping and power nonlinearity $|u|^p$ on exterior domain is demonstrated by utilizing the test function technique $\psi(x, t) = \phi_0(x)\varphi_T^\ell(x)\eta_T^k(t)$ (see [14]). Chen et al. [2] establish local (in time) existence of solution to the semilinear wave equation with strong damping and combined nonlinearities $|u_t|^p + |u|^q$ by applying the Banach fixed point theorem. Non-existence of global solution to the problem with derivative nonlinearity $|u_t|^p$, combined nonlinearities $|u_t|^p + |u|^q$ is verified by making use of the test function method, respectively. Based on the similarity of structure in equations, we expect that blow-up of solution to semilinear viscous MGT equation is the same as nonlinear wave equation in [2, 14] when $\gamma \rightarrow 1^-$. Thus, we extend the problems studied in [3, 53] to problem (1.1) with $N(u, u_t) = N_{\gamma,q}(u), N_{\gamma,p}(u_t)$ by applying test function technique, which is different from the test functions in [3, 53] (see Theorems 1.1-1.2). Moreover, we establish blow-up of solution to problem (1.1) with $N(u, u_t) = N_{\gamma,p,q}(u, u_t)$ (see Theorem 1.3). To the best of our knowledge, the results in Theorems 1.1-1.3 are new.

Throughout this paper, we write $f \lesssim g$ when there exists a positive constant C such that $f \leq Cg$.

The main results in this paper are stated as follows.

Theorem 1.1. *Let*

$$\begin{cases} 1 < q \leq \frac{1+\theta}{\gamma}, & n=1, \\ 1 < q < \frac{3}{\gamma}, & n=2, \\ 1 < q \leq 1 + \frac{3-\gamma}{n+\gamma-2}, & n \geq 3, \end{cases}$$

where θ is the positive root of $2\theta^2 - \gamma\theta - 2 = 0$. Suppose

$$u_0 \in L^1(\Omega^c), u_1\phi_0 \in L^1(\Omega^c), u_2\phi_0 \in L^1(\Omega^c).$$

It holds that

$$\int_{\Omega^c} (\tau u_2(x) + u_1(x))\phi_0(x)dx > 0. \quad (1.6)$$

Then, there is no global weak solution to problem (1.1) with $N(u, u_t) = N_{\gamma,q}(u)$.

Theorem 1.2. *Let*

$$\begin{cases} 1 < p \leq \frac{2\theta+1}{2\theta+\gamma-1}, & n=1, \\ 1 < p < \frac{3}{1+\gamma}, & n=2, \\ 1 < p \leq 1 + \frac{2-\gamma}{n+\gamma-1}, & n \geq 3, \end{cases}$$

where θ is the positive root of $2\theta^2 - (1+\gamma)\theta - 2 = 0$. Assume

$$(u_0, u_1, u_2)(x) \in (H_0^1(\Omega^c) \cap L^1(\Omega^c)) \times H^1(\Omega^c) \times H^1(\Omega^c),$$

$u_1\phi_0 \in L^1(\Omega^c)$ and $u_2\phi_0 \in L^1(\Omega^c)$. It holds that

$$\int_{\Omega^c} (\tau u_2(x) + u_1(x))\phi_0(x)dx > 0.$$

Then, there is no global weak solution to problem (1.1) with $N(u, u_t) = N_{\gamma,p}(u_t)$.

Theorem 1.3. *Let p, q satisfy*

$$\begin{cases} 1 < p \text{ and } 1 < q \leq \frac{1+\theta_1}{\gamma}, \\ 1 < p \leq \frac{1+\theta}{1+\gamma} \text{ and } 1 < q \leq \frac{2\theta+1}{2\theta+\gamma-2}, & \theta < \theta_1, \\ 1 < p \leq \frac{2\theta+1}{2\theta+\gamma-1} \text{ and } 1 < q \leq \frac{1+\theta}{\gamma}, & \theta_1 < \theta < \theta_2, \\ 1 < p \leq \frac{1+\theta_2}{1+\gamma} \text{ and } q > 1, \end{cases}$$

where θ_1 is the positive root of $2\theta^2 - \gamma\theta - 2 = 0$, θ_2 is the positive root of

$$2\theta^2 - (1+\gamma)\theta - 2 = 0$$

in the case $n = 1$. Suppose that

$$\begin{cases} 1 < p < \frac{3}{1+\gamma} \text{ or } 1 < q < \frac{3}{\gamma}, & n = 2, \\ 1 < p \leq 1 + \frac{2-\gamma}{n+\gamma-1} \text{ or } 1 < q \leq 1 + \frac{3-\gamma}{n+\gamma-2}, & n \geq 3. \end{cases}$$

Assume

$$(u_0, u_1, u_2)(x) \in (H_0^1(\Omega^c) \cap L^1(\Omega^c)) \times H^1(\Omega^c) \times H^1(\Omega^c),$$

$u_1\phi_0 \in L^1(\Omega^c)$ and $u_2\phi_0 \in L^1(\Omega^c)$. It holds that

$$\int_{\Omega^c} (\tau u_2(x) + u_1(x))\phi_0(x)dx > 0.$$

Then, there is no global weak solution to problem (1.1) with

$$N(u, u_t) = N_{\gamma,p,q}(u, u_t).$$

Remark 1.1. We recognize that non-existence of global solution to problem (1.1) with power nonlinearity $|u|^p$ and derivative nonlinearity $|u_t|^p$ in whole space $\mathbb{R}^n$ ($n \geq 1$) is discussed in [3, 53]. But there is no related result about blow-up phenomenon of solution to problem (1.1) with memory terms $N_{\gamma,q}(u)$, $N_{\gamma,p}(u_t)$ on exterior domain, respectively. As a result, we supplement blow-up dynamic of solution to problem (1.1) (see Theorems 1.1-1.2). Moreover, we present blow-up of solution to problem (1.1) with $N_{\gamma,p,q}(u, u_t)$ (see Theorem 1.3).

Remark 1.2. It is worth noticing that blow-up results in Theorems 1.1-1.3 are same as the results in [2, 14] when $\gamma \rightarrow 1^-$.

2. Proof of Theorem 1.1

2.1. Several definitions and related lemmas

Definition 2.1. [51] Set $f(t) \in L^1(0, T)$. The Riemann-Liouville left and right sided frictional integrals of the order $\alpha \in (0, 1)$ are defined by

$$\begin{aligned} I_{0|t}^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-(1-\alpha)} f(\tau) d\tau, \quad t > 0, \\ I_{t|T}^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{-(1-\alpha)} f(\tau) d\tau, \quad t < T. \end{aligned} \quad (2.1)$$

Definition 2.2. [51] Let $f(t) \in AC[0, T]$. The Riemann-Liouville left and right sided fractional derivatives of the order $\alpha \in (0, 1)$ are defined by

$$\begin{aligned} D_{0|t}^\alpha f(t) &= \frac{d}{dt} I_{0|t}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad t > 0, \\ D_{t|T}^\alpha f(t) &= -\frac{d}{dt} I_{t|T}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{-\alpha} f(\tau) d\tau, \quad t < T. \end{aligned} \quad (2.2)$$

Definition 2.3. Let $(u_0, u_1, u_2) \in L^1(\Omega^c) \times L^1(\Omega^c) \times L^1(\Omega^c)$. $u \in L^q_{loc}([0, \infty) \times \Omega^c)$ when $N(u, u_t) = N_{\gamma, q}(u)$. $u_t \in L^p_{loc}([0, \infty) \times \Omega^c)$ when $N(u, u_t) = N_{\gamma, p}(u_t)$. $u \in L^q_{loc}([0, \infty) \times \Omega^c)$ and $u_t \in L^p_{loc}([0, \infty) \times \Omega^c)$ when $N(u, u_t) = N_{\gamma, p, q}(u, u_t)$. It holds that

$$\begin{aligned} & \int_0^\infty \int_{\Omega^c} (-\tau u_{tt} \phi_t - u_t \phi_t - u \Delta \phi + (\delta + \tau) u \Delta \phi_t)(x, s) dx ds \\ & - \tau \int_{\Omega^c} u_2(x) \phi(x, 0) dx - \int_{\Omega^c} u_1(x) \phi(x, 0) dx + (\delta + \tau) \int_{\Omega^c} \Delta u_0(x) \phi(x, 0) dx \\ & = \int_0^\infty \int_{\Omega^c} N(u, u_t) \phi(x, s) dx ds, \end{aligned} \quad (2.3)$$

where $\phi \in C_0^\infty([0, \infty) \times \Omega^c)$. Then, u is called a global (in time) weak solution to problem (1.1).

Lemma 2.1 (Lemmas 2.1-2.3, [2]). Suppose that $\phi_0(x) \in C^2(\Omega^c) \cap C(\overline{\Omega^c})$ satisfies

$$\begin{cases} \Delta \phi_0(x) = 0, & x \in \Omega^c, \\ \phi_0(x) = 0, & x \in \partial \Omega^c, \end{cases}$$

$\phi_0(x) \rightarrow 1$ as $|x| \rightarrow \infty$ and $0 < \phi_0(x) < 1$ for all $x \in \Omega^c$ when $n \geq 3$. $\phi_0(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $0 < \phi_0(x) \leq C_1 \ln(|x|)$ for all $x \in \Omega^c$ when $n = 2$. $\phi_0(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $C_2 x \leq \phi_0(x) \leq C_3 x$ for all $x > 0$ when $n = 1$. Here, C_1, C_2 and C_3 are positive constants. In addition, it holds that $|\nabla \phi_0(x)| \leq C|x|^{1-n}$, where C is a positive constant.

Lemma 2.2 (Lemmas 2.1, 2.3 [11]). Set $s \in (0, 1]$, $R > 2$ and $p > 1$. $\varphi(x) \in C^2(\mathbb{R}^n)$ satisfy

$$\varphi(x) = \begin{cases} 1, & |x| \leq 1, \\ \langle x \rangle^{-n-2s}, & |x| \geq 1, \end{cases}$$

where $\langle x \rangle = (1 + (|x| - 1)^4)^{\frac{1}{4}}$ for $x \in \mathbb{R}^n$. Moreover, it holds that

$$\int_{\mathbb{R}^n} (\varphi_R(x))^{-\frac{1}{p-1}} |(-\Delta)^s \varphi_R(x)|^{\frac{p}{p-1}} dx \lesssim R^{-\frac{2sp}{p-1} + n},$$

where $\varphi_R(x) = \varphi(R^{-1}x)$.

Lemma 2.3 ([51]). Set $T > 0$ and $\alpha \in (0, 1)$. Assume

$$\begin{aligned} I_{0|t}^\alpha(L^q(0, T)) &= \{f(t) = I_{0|t}^\alpha h(t) \mid h(t) \in L^q(0, T)\}, \\ I_{t|T}^\alpha(L^p(0, T)) &= \{f(t) = I_{t|T}^\alpha h(t) \mid h(t) \in L^p(0, T)\}. \end{aligned}$$

It holds that

$$\int_0^T f(t) D_{0|t}^\alpha g(t) dt = \int_0^T g(t) D_{t|T}^\alpha f(t) dt, \quad (2.4)$$

where $f(t) \in I_{t|T}^\alpha(L^p(0, T))$ and $g(t) \in I_{0|t}^\alpha(L^q(0, T))$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$ with $1 < p, q < \infty$.

Lemma 2.4 ([31]). Let $\alpha \in (0, 1)$ and $t \in (0, T)$ with $T > 0$. Then

$$D_{0|t}^\alpha I_{0|t}^\alpha f(t) = f(t)$$

for $f(t) \in L^r(0, T)$ with $1 \leq r \leq \infty$. Moreover, it holds that

$$(-1)^m D^m D_{t|T}^\alpha f(t) = D_{t|T}^{m+\alpha} f(t)$$

for $f(t) \in AC^{m+1}[0, T]$.

Lemma 2.5 (Lemmas 2.1-2.2 [10]). *Suppose that $w(t) = (1 - \frac{t}{T})^\beta$ with $T > 0$, $\beta \gg 1$. It holds that*

$$D_{t|T}^{m+\alpha} w(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m-\alpha)} T^{-(m+\alpha)} (1 - \frac{t}{T})^{\beta-\alpha-m},$$

where $m \geq 0$, $0 < \alpha < 1$. It holds that

$$\int_0^T (w(t))^{-\frac{1}{p-1}} |D_{t|T}^{m+\alpha} w(t)|^{\frac{p}{p-1}} dt = CT^{1-(m+\alpha)\frac{p}{p-1}}$$

for some positive constant C .

2.2. Proof of Theorem 1.1 for $n \geq 3$

Proof. We denote $\tilde{\psi} = \phi_0(x)\varphi_R(x)w(t)$ and the test function

$$\psi(x, t) = D_{t|T}^\alpha(\tilde{\psi}(x, t)) = \phi_0(x)\varphi_R(x)D_{t|T}^\alpha(w(t))$$

with $\alpha = 1 - \gamma$.

Replacing $\phi(x, s)$ in (2.3) with $N(u, u_t) = N_{\gamma, q}(u)$ by $\psi(x, s)$ and applying (2.1), (2.2), Lemmas 2.3-2.4, we achieve

$$\begin{aligned} & J_R + T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-1-\alpha} \tau \int_{\Omega^c} (u_1(x) - T^{-1} u_0(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-\alpha} (\delta + \tau) \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx - T^{-1-\alpha} \int_{\Omega^c} u_0(x) \phi_0(x) \varphi_R(x) dx \\ & = \int_0^T \int_{\Omega^c} u(x, s) (-\tau \psi_{ttt} + \psi_{tt} + (\delta + \tau) \Delta \psi_t - \Delta \psi)(x, s) dx ds \\ & = \int_0^T \int_{\Omega^c} u(x, s) (\tau \phi_0(x) \varphi_R(x) D_{t|T}^{3+\alpha}(w(s)) + \phi_0(x) \varphi_R(x) D_{t|T}^{2+\alpha}(w(s)) \\ & \quad - (\delta + \tau) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^{1+\alpha}(w(s)) - \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(w(s))) dx ds \\ & = J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{2.5}$$

where $J_R = \int_0^T \int_{\Omega^c} |u(x, s)|^q \tilde{\psi}(x, s) dx ds$.

Making use of the change of variables $\tilde{t} = \frac{t}{T}$, $\tilde{x} = \frac{x}{R}$, Lemmas 2.1, 2.5 and (2.5) yields

$$\begin{aligned} |J_1| & \leq \tau \int_0^T \int_{\Omega^c} |u(x, s)| \phi_0(x) \varphi_R(x) |D_{t|T}^{3+\alpha}(w(s))| dx ds \\ & \leq C J_R^{\frac{1}{q}} \left(\int_0^T \int_{\Omega^c} \phi_0(x) \varphi_R(x) (w(s))^{-\frac{q'}{q}} |D_{t|T}^{3+\alpha}(w(s))|^{q'} dx ds \right)^{\frac{1}{q'}} \\ & \leq \frac{1}{8} J_R + CT^{-(3+\alpha)q'+1} R^n, \end{aligned} \tag{2.6}$$

$$\begin{aligned} |J_2| & \leq \int_0^T \int_{\Omega^c} |u(x, s)| \phi_0(x) \varphi_R(x) |D_{t|T}^{2+\alpha}(w(s))| dx ds \\ & \leq \frac{1}{8} J_R + CT^{-(2+\alpha)q'+1} R^n. \end{aligned} \tag{2.7}$$

It is deduced from Lemmas 2.1, 2.2, 2.5 and (2.5) that

$$\begin{aligned}
 |J_3| &\leq (\delta + \tau) \int_0^T \int_{\Omega^c} |u(x, s)| \Delta(\phi_0(x) \varphi_R(x)) |D_{t|T}^{1+\alpha}(w(s))| dx ds \\
 &\leq C \int_0^T \int_{\Omega^c} |u(x, s)| |\nabla \phi_0(x)| |\nabla \varphi_R(x)| |D_{t|T}^{1+\alpha}(w(s))| dx ds \\
 &\quad + C \int_0^T \int_{\Omega^c} |u(x, s)| \phi_0(x) |\Delta \varphi_R(x)| |D_{t|T}^{1+\alpha}(w(s))| dx ds \\
 &\leq \frac{1}{8} \tilde{J}_R + CT^{-(\alpha+1)q'+1} R^{-2q'+n}, \tag{2.8}
 \end{aligned}$$

$$\begin{aligned}
 |J_4| &\leq \int_0^T \int_{\Omega^c} |u(x, s)| \Delta(\phi_0(x) \varphi_R(x)) |D_{t|T}^\alpha(w(s))| dx ds \\
 &\leq \frac{1}{8} \tilde{J}_R + CT^{-\alpha q'+1} R^{-2q'+n}, \tag{2.9}
 \end{aligned}$$

where $\tilde{J}_R = \int_0^T \int_{\Omega^c \cap \{|x| \geq R\}} |u(x, s)|^q \tilde{\psi}(x, s) dx ds$.

Taking into account (2.5)-(2.9) and choosing $R = T$, we obtain

$$\begin{aligned}
 &T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\
 &- T^{-1-\alpha} \tau \int_{\Omega^c} (u_1(x) - T^{-1} u_0(x)) \phi_0(x) \varphi_R(x) dx \\
 &- T^{-\alpha} (\delta + \tau) \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\
 &- T^{-1-\alpha} \int_{\Omega^c} u_0(x) \phi_0(x) \varphi_R(x) dx \\
 &\leq CT^{-(3+\alpha)q'+1} R^n + CT^{-(2+\alpha)q'+1} R^n \\
 &\quad + CT^{-(\alpha+1)q'+1} R^{-2q'+n} + CT^{-\alpha q'+1} R^{-2q'+n} \\
 &\leq CT^{-(2+\alpha)q'+n+1}. \tag{2.10}
 \end{aligned}$$

It is worth pointing out that $q \in (1, 1 + \frac{3-\gamma}{n+\gamma-2}]$ is equivalent to

$$-(2 + \alpha)q' + n + 1 \leq 0.$$

Consequently, our considerations are divided into two cases.

In the sub-critical case $-(2 + \alpha)q' + n + 1 < 0$, sending $R, T \rightarrow \infty$ in (2.10) leads to

$$\begin{aligned}
 &\lim_{R \rightarrow \infty} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\
 &\leq \lim_{R \rightarrow \infty} (\delta + \tau) \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx.
 \end{aligned}$$

Applying the condition $u_0, u_1 \phi_0, u_2 \phi_0 \in L^1(\Omega^c)$ and $\Delta(\phi_0(x) \varphi_R(x)) \leq \frac{C}{T^2}$, we arrive at a contradiction to (1.6).

In the critical case $-(2 + \alpha)q' + n + 1 = 0$, employing the Holder inequality, (2.5) and taking $R = TK^{-1}$ give rise to

$$|J_1| \leq C J_R^{\frac{1}{q}} T^{-1} K^{-\frac{n}{q'}}, \quad |J_2| \leq C J_R^{\frac{1}{q}} K^{-\frac{n}{q'}}, \tag{2.11}$$

$$|J_3| \leq C J_R^{\frac{1}{q}} T^{-1} K^{-\left(\frac{n}{q}-2\right)}, \quad |J_4| \leq C J_R^{\frac{1}{q}} K^{-\left(\frac{n}{q}-2\right)}. \quad (2.12)$$

Combining (2.5), (2.11) and (2.12), we conclude the desired result by taking K big enough. $\square$

2.3. Proof of Theorem 1.1 for $n = 2$

Proof. Similar to the derivation in (2.6)-(2.9), applying Lemma 2.1, we derive

$$|J_1| \leq \frac{1}{8} J_R + C T^{-(3+\alpha)q'+1} R^2 \ln R, \quad (2.13)$$

$$|J_2| \leq \frac{1}{8} J_R + C T^{-(2+\alpha)q'+1} R^2 \ln R, \quad (2.14)$$

$$|J_3| \leq \frac{1}{8} \tilde{J}_R + C T^{-(1+\alpha)q'+1} R^{-2q'+2} \ln R + C T^{-(1+\alpha)q'+1} R^{-2q'+2}, \quad (2.15)$$

$$|J_4| \leq \frac{1}{8} \tilde{J}_R + C T^{-\alpha q'+1} R^{-2q'+2} \ln R + C T^{-\alpha q'+1} R^{-2q'+2}. \quad (2.16)$$

An application of (2.5), (2.13)-(2.16) and taking $R = T$ show

$$\begin{aligned} & T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-1-\alpha} \tau \int_{\Omega^c} (u_1(x) - T^{-1} u_0(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-\alpha} (\delta + \tau) \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\ & - T^{-1-\alpha} \int_{\Omega^c} u_0(x) \phi_0(x) \varphi_R(x) dx \\ & \leq C T^{-(2+\alpha)q'+3} \ln T \\ & \leq C T^{-\frac{(2+\alpha)q'}{2} + \frac{3}{2}}, \end{aligned} \quad (2.17)$$

where we have utilized the fact $\ln T \leq C T^{\frac{(2+\alpha)q'}{2} - \frac{3}{2}}$.

Similar to the proof in Sub-section 2.2, we acquire that the solution blows up in finite time when $q \in (1, \frac{3}{\gamma})$. $\square$

2.4. Proof of Theorem 1.1 for $n = 1$

Proof. Similar to the derivation in (2.6)-(2.9), exploiting Lemma 2.1, (2.5) and taking $R = T^\theta$ yield

$$|J_1| \leq \frac{1}{8} J_R + C T^{-(3+\alpha)q'+1} R^2 \leq \frac{1}{8} J_R + C T^{-(3+\alpha)q'+1+2\theta}, \quad (2.18)$$

$$|J_2| \leq \frac{1}{8} J_R + C T^{-(2+\alpha)q'+1} R^2 \leq \frac{1}{8} J_R + C T^{-(2+\alpha)q'+1+2\theta}, \quad (2.19)$$

$$\begin{aligned} |J_3| & \leq \frac{1}{8} \tilde{J}_R + C T^{-(1+\alpha)q'+1} R^{-q'+1} + C T^{-(1+\alpha)q'+1} R^{-2q'+2} \\ & \leq \frac{1}{8} \tilde{J}_R + C T^{-(1+\alpha+\theta)q'+1+\theta}, \end{aligned} \quad (2.20)$$

$$|J_4| \leq \frac{1}{8} \tilde{J}_R + C T^{-\alpha q'+1} R^{-q'+1} + C T^{-\alpha q'} R^{-2q'+2}$$

$$\leq \frac{1}{8} \tilde{J}_R + CT^{-(\alpha+\theta)q'+1+\theta}. \quad (2.21)$$

We observe that $q < \frac{1+\theta}{\gamma} < \frac{\theta}{2(\theta-1)}$ indicates

$$CT^{-(2+\alpha)q'+1+2\theta} < CT^{-(\alpha+\theta)q'+1+\theta}.$$

As a result, using (2.5) and (2.18)-(2.21), we deduce

$$\begin{aligned} & J_R + T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-1-\alpha} \tau \int_{\Omega^c} (u_1(x) - T^{-1} u_0(x)) \phi_0(x) \varphi_R(x) dx \\ & - T^{-\alpha} (\delta + \tau) \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\ & - T^{-1-\alpha} \int_{\Omega^c} u_0(x) \phi_0(x) \varphi_R(x) dx \\ & \leq CT^{-(3+\alpha)q'+1+2\theta} + CT^{-(2+\alpha)q'+1+2\theta} \\ & \quad + CT^{-(1+\alpha+\theta)q'+1+\theta} + CT^{-(\alpha+\theta)q'+1+\theta} \\ & \leq CT^{-(\alpha+\theta)q'+1+\theta}. \end{aligned} \quad (2.22)$$

Similar to the proof in Sub-section 2.2, we achieve that the solution blows up in finite time when $q \in (1, \frac{1+\theta}{\gamma}]$. This finishes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

3.1. Proof of Theorem 1.2 for $n \geq 3$

Proof. We suppose

$$\begin{aligned} I_R &= \int_0^T \int_{\Omega^c} |u_t(x, s)|^p \tilde{\psi}(x, s) dx ds, \\ \tilde{I}_R &= \int_0^T \int_{\Omega^c \cap \{|x| \geq R\}} |u_t(x, s)|^p \tilde{\psi}(x, s) dx ds. \end{aligned}$$

Let $\Psi(t) = \int_t^\infty w(\tau) d\tau$. Then, we obtain $\Psi'(t) = -w(t)$.

Replacing $\phi(x, s)$ in (2.3) with $N(u, u_t) = N_{\gamma, p}(u_t)$ by

$$\psi = D_{t|T}^\alpha(\tilde{\psi}(x, s)) = \phi_0(x) \varphi_R(x) D_{t|T}^\alpha(w(s))$$

and exploiting (2.1), (2.2), Lemmas 2.3-2.4 give rise to

$$\begin{aligned} & I_R + T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\ & - \tau T^{-1-\alpha} \int_{\Omega^c} u_1(x) \phi_0(x) \varphi_R(x) dx + \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(0)) dx \\ & = \int_0^T \int_{\Omega^c} u_t(x, s) (\tau \psi_{tt} - \psi_t - (\delta + \tau) \Delta \psi)(x, s) dx ds \end{aligned}$$

$$\begin{aligned}
& - \int_0^T \int_{\Omega^c} u_t(x, s) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(s)) dx ds \\
& = \int_0^T \int_{\Omega^c} u_t(x, s) (\tau \phi_0(x) \varphi_R(x) D_{t|T}^{2+\alpha}(w(s)) + \phi_0(x) \varphi_R(x) D_{t|T}^{1+\alpha}(w(s)) \\
& \quad - (\delta + \tau) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(w(s))) dx ds \\
& \quad - \int_0^T \int_{\Omega^c} u_t(x, s) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(s)) dx ds \\
& = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.1}$$

Taking into account Lemmas 2.1, 2.5 and (3.1), we deduce

$$\begin{aligned}
|I_1| & \leq \tau \int_0^T \int_{\Omega^c} |u_t(x, s)| \phi_0(x) \varphi_R(x) |D_{t|T}^{2+\alpha}(w(s))| dx ds \\
& \leq C I_R^{\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \phi_0(x) \varphi_R(x) (w(s))^{-\frac{p'}{p}} |D_{t|T}^{2+\alpha}(w(s))|^{p'} dx ds \right)^{\frac{1}{p'}} \\
& \leq \frac{1}{8} I_R + C T^{-(2+\alpha)p'+1} R^n,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
|I_2| & \leq \int_0^T \int_{\Omega^c} |u_t(x, s)| \phi_0(x) \varphi_R(x) |D_{t|T}^{1+\alpha}(w(s))| dx ds \\
& \leq \frac{1}{8} I_R + C T^{-(1+\alpha)p'+1} R^n.
\end{aligned} \tag{3.3}$$

From Lemmas 2.1, 2.2, 2.5 and (3.1), we achieve

$$\begin{aligned}
|I_3| & \leq (\delta + \tau) \int_0^T \int_{\Omega^c} |u_t(x, s)| \Delta(\phi_0(x) \varphi_R(x)) |D_{t|T}^\alpha(w(s))| dx ds \\
& \leq C \int_0^T \int_{\Omega^c} |u_t(x, s)| |\nabla \phi_0(x)| |\nabla \varphi_R(x)| |D_{t|T}^\alpha(w(s))| dx ds \\
& \quad + C \int_0^T \int_{\Omega^c} |u_t(x, s)| \phi_0(x) |\Delta \varphi_R(x)| |D_{t|T}^\alpha(w(s))| dx ds \\
& \leq \frac{1}{8} \tilde{I}_R + C T^{-\alpha p'+1} R^{-2p'+n},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
|I_4| & \leq \int_0^T \int_{\Omega^c} |u_t(x, s)| \Delta(\phi_0(x) \varphi_R(x)) |D_{t|T}^\alpha(\Psi(s))| dx ds \\
& \leq C \int_0^T \int_{\Omega^c} |u_t(x, s)| |\nabla \phi_0(x)| |\nabla \varphi_R(x)| |D_{t|T}^\alpha(\Psi(s))| dx ds \\
& \quad + C \int_0^T \int_{\Omega^c} |u_t(x, s)| \phi_0(x) |\Delta \varphi_R(x)| |D_{t|T}^\alpha(\Psi(s))| dx ds \\
& \leq \frac{1}{8} \tilde{I}_R + C T^{-(\alpha-1)p'+1} R^{-2p'+n},
\end{aligned} \tag{3.5}$$

where we have employed the fact $(w(s))^{-\frac{p'}{p}} (D_{t|T}^\alpha \Psi(s))^{p'} \leq T^{(1-\alpha)p'}$.

According to (3.1)-(3.5) and choosing $R = T$ lead to

$$T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx$$

$$\begin{aligned}
& -\tau T^{-1-\alpha} \int_{\Omega^c} u_1(x) \phi_0(x) \varphi_R(x) dx \\
& + T^{1-\alpha} \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\
& \leq CT^{-(2+\alpha)p'+1} R^n + CT^{-(1+\alpha)p'+1} R^n \\
& \quad + CT^{-\alpha p'+1} R^{-2p'+n} + CT^{-(\alpha-1)p'+1} R^{-2p'+n} \\
& \leq CT^{-(1+\alpha)p'+n+1},
\end{aligned} \tag{3.6}$$

where we have performed the fact

$$\int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(0)) dx \leq T^{1-\alpha} \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx.$$

It is worth to observe that $p \in (1, 1 + \frac{2-\gamma}{n+\gamma-1}]$ is equivalent to

$$-(1+\alpha)p' + n + 1 \leq 0.$$

Similar to the proof of Theorem 1.1 for $n \geq 3$, we obtain that the solution blows up in finite time when $p \in (1, 1 + \frac{2-\gamma}{n+\gamma-1}]$. $\square$

3.2. Proof of Theorem 1.2 for $n = 2$

Proof. Similar to the derivation in (3.2)-(3.5), utilizing Lemma 2.1 yields

$$|I_1| \leq \frac{1}{8} I_R + CT^{-(2+\alpha)p'+1} R^2 \ln R, \tag{3.7}$$

$$|I_2| \leq \frac{1}{8} I_R + CT^{-(1+\alpha)p'+1} R^2 \ln R, \tag{3.8}$$

$$|I_3| \leq \frac{1}{8} \tilde{I}_R + CT^{-\alpha p'+1} R^{-2p'+2} + CT^{-\alpha p'+1} R^{-2p'+2} \ln R, \tag{3.9}$$

$$|I_4| \leq \frac{1}{8} \tilde{I}_R + CT^{-(\alpha-1)p'+1} R^{-2p'+2} + CT^{-(\alpha-1)p'+1} R^{-2p'+2} \ln R. \tag{3.10}$$

Making use of (3.1) and (3.7)-(3.10) and taking $R = T$, we derive

$$\begin{aligned}
& T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\
& -\tau T^{-1-\alpha} \int_{\Omega^c} u_1(x) \phi_0(x) \varphi_R(x) dx \\
& + T^{1-\alpha} \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\
& \leq CT^{-(1+\alpha)p'+3} \ln T \\
& \leq CT^{-\frac{(1+\alpha)p'}{2} + \frac{3}{2}},
\end{aligned} \tag{3.11}$$

where we have applied the fact $\ln T \leq CT^{\frac{(1+\alpha)p'}{2} - \frac{3}{2}}$.

Similar to the proof of Theorem 1.1 for $n = 2$, we conclude that the solution blows up in finite time when $p \in (1, \frac{3}{1+\gamma})$. $\square$

3.3. Proof of Theorem 1.2 for $n = 1$

Proof. Similar to the derivation in (3.2)-(3.5), applying Lemma 2.1 and choosing $R = T^\theta$ lead to

$$|I_1| \leq \frac{1}{8}I_R + CT^{-(2+\alpha)p'+1}R^2 \leq \frac{1}{8}I_R + CT^{-(2+\alpha)p'+1+2\theta}, \quad (3.12)$$

$$|I_2| \leq \frac{1}{8}I_R + CT^{-(1+\alpha)p'+1}R^2 \leq \frac{1}{8}I_R + CT^{-(1+\alpha)p'+1+2\theta}, \quad (3.13)$$

$$\begin{aligned} |I_3| &\leq \frac{1}{8}\tilde{I}_R + CT^{-\alpha p'+1}R^{-p'+1} + CT^{-\alpha p'+1}R^{-2p'+2} \\ &\leq \frac{1}{8}\tilde{I}_R + CT^{-(\alpha+\theta)p'+1+\theta}, \end{aligned} \quad (3.14)$$

$$\begin{aligned} |I_4| &\leq \frac{1}{8}\tilde{I}_R + CT^{-(\alpha-1)p'+1}R^{-p'+1} + CT^{-(\alpha-1)p'+1}R^{-2p'+2} \\ &\leq \frac{1}{8}\tilde{I}_R + CT^{-(\alpha-1+\theta)p'+1+\theta}. \end{aligned} \quad (3.15)$$

Combining (3.1) and (3.12)-(3.14), we arrive at

$$\begin{aligned} &T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\ &- \tau T^{-1-\alpha} \int_{\Omega^c} u_1(x) \phi_0(x) \varphi_R(x) dx \\ &+ T^{1-\alpha} \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\ &\leq CT^{-(2+\alpha)p'+1+2\theta} + CT^{-(1+\alpha)p'+1+2\theta} \\ &\quad + CT^{-(\alpha+\theta)p'+1+\theta} + CT^{-(\alpha-1+\theta)p'+1+\theta} \\ &\leq CT^{-(1+\alpha)p'+1+2\theta}. \end{aligned} \quad (3.16)$$

It is worth noticing that $p \in (1, \frac{2\theta+1}{2\theta+\gamma-1}]$ is equivalent to $-(1+\alpha)p' + 1 + 2\theta \leq 0$.

Similar to the proof of Theorem 1.1 for $n = 1$, we derive that the solution blows up in finite time when $p \in (1, \frac{2\theta+1}{2\theta+\gamma-1}]$. This completes the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.3

4.1. Proof of Theorem 1.3 for $n \geq 3$

Proof. Replacing $\phi(x, s)$ in (2.3) with $N(u, u_t) = N_{\gamma, p, q}(u, u_t)$ by

$$\psi(x, t) = D_{t|T}^\alpha(\tilde{\psi}(x, t)) = \phi_0(x) \varphi_R(x) D_{t|T}^\alpha(w(t))$$

and choosing $R = T$, we obtain that $J_1 - J_4$ in (2.5) are replaced by $J_5 - J_8$, $I_1 - I_4$ in (3.1) are replaced by $I_5 - I_8$, respectively.

Similar to the estimates of $J_1 - J_4$ in (2.6)-(2.9), $I_1 - I_4$ in (3.2)-(3.5), we acquire

$$|J_5| \leq \varepsilon J_R + CT^{-(3+\alpha)q'+n+1}, \quad |J_6| \leq \varepsilon J_R + CT^{-(2+\alpha)q'+n+1}, \quad (4.1)$$

$$|J_7| \leq \varepsilon \tilde{J}_R + CT^{-(3+\alpha)q'+n+1}, \quad |J_8| \leq \varepsilon \tilde{J}_R + CT^{-(2+\alpha)q'+n+1}, \quad (4.2)$$

$$|I_5| \leq \varepsilon I_R + CT^{-(2+\alpha)p'+n+1}, \quad |I_6| \leq \varepsilon I_R + CT^{-(1+\alpha)p'+n+1}, \quad (4.3)$$

$$|I_7| \leq \varepsilon \tilde{I}_R + CT^{-(2+\alpha)p'+n+1} - T^{-\alpha}(\delta + \tau) \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx, \quad (4.4)$$

$$|I_8| \leq \varepsilon \tilde{I}_R + CT^{-(1+\alpha)p'+n+1} - \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(0)) dx. \quad (4.5)$$

Thus, we need to discuss the following 16 cases.

	Est. I_5, J_5	Est. I_6, J_6	Est. I_7, J_7	Est. I_8, J_8
Case 1	J_5	J_6	J_7	J_8
Case 2	I_5	J_6	J_7	J_8
Case 3	J_5	J_6	J_7	I_8
Case 4	I_5	J_6	J_7	I_8
Case 5	J_5	J_6	I_7	J_8
Case 6	I_5	J_6	I_7	J_8
Case 7	J_5	I_6	J_7	J_8
Case 8	I_5	I_6	J_7	J_8
Case 9	J_5	I_6	I_7	I_8
Case 10	I_5	I_6	I_7	I_8
Case 11	J_5	J_6	I_7	I_8
Case 12	I_5	J_6	I_7	I_8
Case 13	J_5	I_6	J_7	I_8
Case 14	I_5	I_6	J_7	I_8
Case 15	J_5	I_6	I_7	J_8
Case 16	I_5	I_6	I_7	J_8

To verify blow-up dynamic of solution, we notice that it is sufficient to discuss case 1 and case 10 by direct calculation.

Case 1, according to (2.3) and (4.1)-(4.2) gives rise to

$$\begin{aligned}
& T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx \\
& - T^{-1-\alpha} \tau \int_{\Omega^c} (u_1(x) - T^{-1} u_0(x)) \phi_0(x) \varphi_R(x) dx \\
& - T^{-\alpha} (\delta + \tau) \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\
& - T^{-1-\alpha} \int_{\Omega^c} u_0(x) \phi_0(x) \varphi_R(x) dx \\
& \leq CT^{-(2+\alpha)q'+n+1}.
\end{aligned} \quad (4.6)$$

Consequently, we acquire blow-up of solution when $p > 1$ and $1 < q \leq 1 + \frac{3-\gamma}{n+\gamma-2}$.

Case 10, it is deduced from (2.3) and (4.3)-(4.5) that

$$T^{-\alpha} \int_{\Omega^c} (\tau u_2(x) + u_1(x)) \phi_0(x) \varphi_R(x) dx$$

$$\begin{aligned}
& -\tau T^{-1-\alpha} \int_{\Omega^c} u_1(x) \phi_0(x) \varphi_R(x) dx \\
& + T^{1-\alpha} \int_{\Omega^c \cap \{|x| \geq R\}} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx \\
& \leq CT^{-(1+\alpha)p'+n+1}.
\end{aligned} \tag{4.7}$$

Therefore, we deduce blow-up of solution when $1 < p \leq 1 + \frac{2-\gamma}{n+\gamma-1}$ and $q > 1$. $\square$

4.2. Proof of Theorem 1.3 for $n = 2$

Proof. Similar to the estimates of $J_1 - J_4$ in (2.13)-(2.16), $I_1 - I_4$ in (3.7)-(3.10), taking $R = T$, we have

$$|J_5| \leq \varepsilon J_R + CT^{-(3+\alpha)q'+3} \ln T, \quad |J_6| \leq \varepsilon J_R + CT^{-(2+\alpha)q'+3} \ln T, \tag{4.8}$$

$$|J_7| \leq \varepsilon \tilde{J}_R + CT^{-(3+\alpha)q'+3} \ln T + CT^{-(3+\alpha)q'+3}, \tag{4.9}$$

$$|J_8| \leq \varepsilon \tilde{J}_R + CT^{-(2+\alpha)q'+3} \ln T + CT^{-(2+\alpha)q'+3}, \tag{4.10}$$

$$|I_5| \leq \varepsilon I_R + CT^{-(2+\alpha)p'+3} \ln T, \quad |I_6| \leq \varepsilon I_R + CT^{-(1+\alpha)p'+3} \ln T, \tag{4.11}$$

$$\begin{aligned}
& |I_7| \leq \varepsilon \tilde{I}_R + CT^{-(2+\alpha)p'+3} \ln T + CT^{-(2+\alpha)p'+3} \\
& \quad - T^{-\alpha}(\delta + \tau) \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
& |I_8| \leq \varepsilon \tilde{I}_R + CT^{-(1+\alpha)p'+3} \ln T + CT^{-(1+\alpha)p'+3} \\
& \quad - \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(0)) dx.
\end{aligned} \tag{4.13}$$

Similar to the Sub-section 4.1, for the 16 cases, we only need to compute the following two cases to illustrate blow-up phenomenon.

In case 1, from (2.3) and (4.8)-(4.10), we arrive at blow-up of solution when $p > 1$ and $1 < q < \frac{3}{\gamma}$.

In case 10, utilizing (2.3) and (4.11)-(4.13) leads to blow-up of solution when $1 < p < \frac{3}{1+\gamma}$ and $q > 1$. $\square$

4.3. Proof of Theorem 1.3 for $n = 1$

Proof. Similar to the estimates of $J_1 - J_4$ in (2.18)-(2.21), $I_1 - I_4$ in (3.12)-(3.15) and taking $R = T^\theta$, we conclude

$$|J_5| \leq \varepsilon J_R + CT^{-(3+\alpha)q'+1+2\theta}, \quad |J_6| \leq \varepsilon J_R + CT^{-(2+\alpha)q'+1+2\theta}, \tag{4.14}$$

$$|J_7| \leq \varepsilon \tilde{J}_R + CT^{-(1+\alpha+\theta)q'+1+\theta}, \quad |J_8| \leq \varepsilon \tilde{J}_R + CT^{-(\alpha+\theta)q'+1+\theta}, \tag{4.15}$$

$$|I_5| \leq \varepsilon I_R + CT^{-(2+\alpha)p'+1+2\theta}, \quad |I_6| \leq \varepsilon I_R + CT^{-(1+\alpha)p'+1+2\theta}, \tag{4.16}$$

$$\begin{aligned}
& |I_7| \leq \varepsilon \tilde{I}_R + CT^{-(\alpha+\theta)p'+1+\theta} \\
& \quad - T^{-\alpha}(\delta + \tau) \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) dx,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
& |I_8| \leq \varepsilon \tilde{I}_R + CT^{-(\alpha-1+\theta)p'+1+\theta} \\
& \quad - \int_{\Omega^c} u_0(x) \Delta(\phi_0(x) \varphi_R(x)) D_{t|T}^\alpha(\Psi(0)) dx.
\end{aligned} \tag{4.18}$$

Similar to the Sub-section 4.1, for the 16 cases, we only need to calculate the following four cases to verify blow-up result.

In case 1, making use of (2.3) and (4.14)-(4.15), we achieve that the solution blows up in finite time when $p > 1$ and $1 < q \leq \frac{1+\theta_1}{\gamma}$.

In case 3, applying (2.3), (4.14)-(4.15) and (4.18), we deduce that the solution blows up in finite time when $1 < p \leq \frac{1+\theta}{1+\gamma}$ and $1 < q \leq \frac{2\theta+1}{2\theta+\gamma-2}$.

In case 8, utilizing (2.3), (4.15)-(4.16), we conclude that the solution blows up in finite time when $1 < p \leq \frac{2\theta+1}{2\theta+\gamma-1}$ and $1 < q \leq \frac{1+\theta}{\gamma}$.

In case 10, it follows from (2.3), (4.16)-(4.18) that the solution blows up in finite time when $1 < p \leq \frac{1+\theta_2}{1+\gamma}$ and $q > 1$. The proof of Theorem 1.3 is completed. $\square$

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