

NEW UPPER BOUNDS ON THE NUMBER OF ZEROS OF ABELIAN INTEGRALS FOR THREE PERTURBED QUADRATIC REVERSIBLE SYSTEMS*

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Abstract For the quadratic reversible systems r_{12} , r_{16} and r_{18} , when they are perturbed by arbitrary polynomial of degree n , we explore the new upper bounds on the number of zeros for their Abelian integrals using the methods of Riccati equation and Picard-Fuchs equation. We consider both the highest degree and the lowest degree of the related polynomials, and more importantly, we notice their law between the terms, and take into account the range of values for variable. Some laws of related polynomials of the systems have been found, therefore, we obtain some new upper bounds, which are better than those obtained before.

Keywords Abelian integral, limit cycle, upper bound, quadratic reversible system, weakened Hilbert's 16th problem.

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1. Introduction

Consider perturbed integrable system

$$\dot{x} = \frac{H_y(x, y)}{\phi(x, y)} + \varepsilon p(x, y), \quad \dot{y} = -\frac{H_x(x, y)}{\phi(x, y)} + \varepsilon q(x, y), \quad (1.1)$$

where, ε ($0 < \varepsilon \ll 1$) is a real parameter, $H_y(x, y)/\phi(x, y)$, $H_x(x, y)/\phi(x, y)$, $p(x, y)$, $q(x, y)$ are polynomials of x and y , with $\max\{\deg(H_y(x, y)/\phi(x, y)), \deg(H_x(x, y)/\phi(x, y))\} = m$, and $\max\{\deg(p(x, y)), \deg(q(x, y))\} = n$. We suppose that when $\varepsilon = 0$, system (1.1) is an integrable system, its first integral is $H(x, y)$ with an integrating factor $\phi(x, y)$, and it has at least one center, i.e., we can define a continuous family of periodic annulus as follows:

$$\{\Gamma_h\} \subset \{(x, y) \in \mathbb{R}^2 : H(x, y) = h, h \in \Omega\},$$

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which are defined on a maximal open interval $\Omega = (h_1, h_2)$. The Abelian integral $A(h)$ (i.e. first order Melnikov function $M_1(h)$) for (1.1) is defined as

$$A(h) = \oint_{\Gamma_h} \phi(x, y) [q(x, y) dx - p(x, y) dy], \quad h \in \Omega. \quad (1.2)$$

How many limit cycles in system (1.1) can be bifurcated from periodic annulus $\{\Gamma_h\}$? If $A(h) \not\equiv 0$, the number of limit cycles of system (1.1) corresponds to the number of zeros of $A(h)$. If $A(h) \equiv 0$, which implies $M_1(h) \equiv 0$, then it is necessary to study higher-order Melnikov functions, suppose $M_2(h) \equiv 0, \dots, M_{t-1}(h) \equiv 0$, and $M_t(h) \not\equiv 0$, then the number of limit cycles of system (1.1) is determined by the number of zeros of $M_t(h)$, potentially yielding more limit cycles. However, analyzing this scenario is particularly challenging, therefore, studying the number of zeros of the Abelian integral $A(h)$ is fundamental and crucial.

If $\phi(x, y)$ is a constant, when $\varepsilon = 0$, (1.1) is a Hamiltonian system, and $H(x, y)$ is a polynomial of x and y with $\deg(H(x, y)) = m + 1$. Studying the least upper bound $Z(m, n)$ of the number of zeros for $A(h)$ is a valuable question, it is also a difficult question, where $Z(m, n)$ depends only on m, n , and has nothing to do with the specific forms of $H(x, y)$, $p(x, y)$, and $q(x, y)$. This question is called the weakened Hilbert's 16th problem [1]. For this problem, researchers have obtained many important results for some special systems [2, 4, 20, 23, 25, 26, 29–32], more details can be found in [3, 9, 21] and references therein.

If $\phi(x, y)$ is not a constant, when $\varepsilon = 0$, (1.1) is an integrable non-Hamiltonian system, for this case, the research work of zeros of $A(h)$ becomes much more difficult. So, researchers study this problem by starting from the simplest case, which is the case of $m = 2$. For the simplest case, people conjecture that the upper bounds $Z(2, n)$ are positively correlated with n and the order of the functions of the periodic orbits. Although researchers have studied some special systems and obtained some good results [5, 24–28, 33], unfortunately, this conjecture has not been resolved. We can use the Riccati equation method and the Picard-Fuchs equation method to study the upper bound of the number of zeros. Horozov and Iliev were the first to use this research method, and obtained a very good upper bound $Z(3, n) \leq 5n + 15$ [17]. References [5, 6, 18, 22, 23, 26, 28, 31–34] also used this method. Although this is a global difficult problem, many researchers have been studying it recently, some of whom are exploring the lower bounds on the number of zeros of Abelian integrals [2, 7, 8, 24, 25, 29, 30], while others are discussing the upper bounds [4, 19, 20]. Due to the difficulty of this problem, researchers began to study its simpler case—the periodic orbits are elliptic curves. Such centers—whose generic complexified periodic orbits are elliptic curves—are referred to as centers of genus one [6]. Gautier et al showed that the quadratic reversible systems of genus one can be classified into 22 cases, namely r1–r22 [6]. When they are perturbed by arbitrary polynomial of degree n , for the upper bounds of the number of zeros in their Abelian integrals, Zhao et al studied three special cases of system r1 [34]; r2 is a Hamiltonian system; Li et al studied some special cases of systems r3–r6 [22]; r7 was studied in [13]; r8, r15, and r17 were studied in [11]; r9, r13, r17, and r19 were studied in [16]; r10 was studied in [12]; r11, r16, r18, and r20 were studied in [15]; r12 and r21 were studied in [14]; r22 was studied in [10]. All of these upper bounds are positively correlated with n , some of which depend linearly on n , while the others are superior to linear correlation. In this paper, we study new upper bounds of systems r12, r16 and r18.

The specific forms of systems r18, r12, and r16 are as follows:

$$(r18) \quad \dot{x} = -xy, \quad \dot{y} = \frac{1}{4}y^2 + \frac{1}{28}x - \frac{1}{28},$$

$$H(x, y) = x^{\frac{1}{2}} \left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^7}x - \frac{1}{2^7} \right) = h, \quad h \in \left(-\frac{1}{3 \cdot 2^6}, 0 \right), \quad (1.3)$$

with an integrating factor $\phi(x, y) = x^{-1/2}$, and all periodic orbits are cubic algebraic curves.

$$(r12) \quad \begin{aligned} \dot{x} &= -xy, \quad \dot{y} = -\frac{1}{6}y^2 + \frac{1}{3^2 \cdot 2^6}x - \frac{1}{3^2 \cdot 2^6}, \\ H(x, y) &= x^{-\frac{1}{3}} \left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^7}x + \frac{1}{3 \cdot 2^6} \right) = h, \quad h \in \left(\frac{1}{2^7}, +\infty \right), \end{aligned} \quad (1.4)$$

with an integrating factor $\phi(x, y) = x^{-4/3}$, and all periodic orbits are cubic algebraic curves.

$$(r16) \quad \begin{aligned} \dot{x} &= -xy, \quad \dot{y} = -\frac{1}{8}y^2 + \frac{1}{2^{10}}x - \frac{1}{2^{10}}, \\ H(x, y) &= x^{-\frac{1}{4}} \left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^8}x + \frac{1}{2^8} \right) = h, \quad h \in \left(\frac{1}{3 \cdot 2^6}, +\infty \right), \end{aligned} \quad (1.5)$$

with an integrating factor $\phi(x, y) = x^{-5/4}$, and all periodic orbits are quartic algebraic curves.

They are integrable quadratic non-Hamiltonian systems. They have a center $(1, 0)$, an invariant curve $x = 0$, a family of periodic orbits $\{\Gamma_h\}$. All periodic orbits rotate counterclockwise, and the phase diagram of periodic orbits are shown in Figures 1–3.

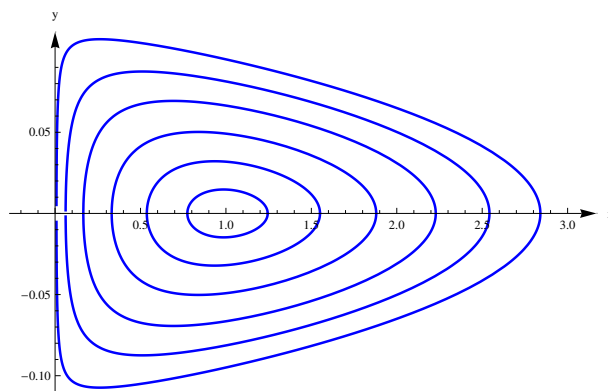


Figure 1. The periodic orbits of system $r18$.

For three quadratic reversible systems $r18$, $r12$, and $r16$ of genus one, their periodic orbits are elliptical curves, when they are perturbed by arbitrary polynomial of degree n , we study the upper bounds on the number of zeros for their Abelian integrals again. In previous research, people mainly considered the highest degree of the polynomial function, if it is n , then it is considered that the polynomial function has at most n zeros. However, we consider both the highest degree and the lowest degree of the polynomial function, in this way, a good upper bound of the number of zeros for the polynomial function can be obtained. Most importantly, we consider the law between terms of the polynomial function, and the range of values for its variable, in this way, a very good upper bound can be obtained. Some laws of polynomial functions of the systems have been found, therefore, we obtain some new upper bounds and organize them into Theorem 1.1 below, which are better than those obtained before. The comparison between the new and old upper bounds is shown in Table 1.

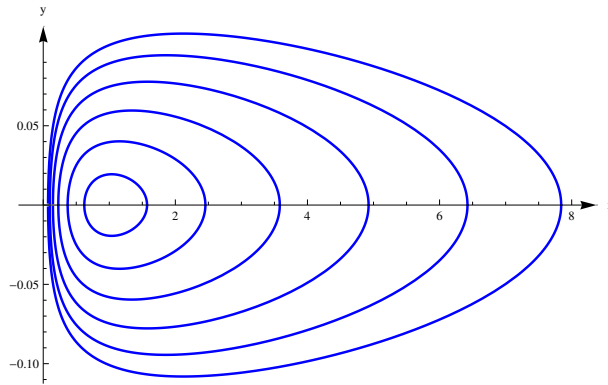


Figure 2. The periodic orbits of system $r12$.

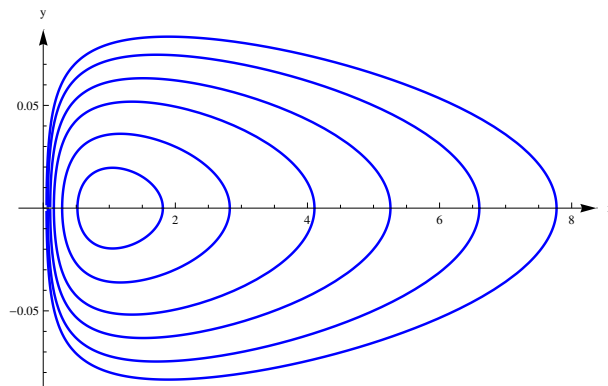


Figure 3. The periodic orbits of system $r16$.

Theorem 1.1. *If $p(x, y)$ and $q(x, y)$ are arbitrary polynomials of degree n , then the new upper bounds on the number of zeros of Abelian integrals for the above systems are positively correlated with n . More concretely, for $r18$, the new upper bounds are $4[n/3] + 2[(n-1)/3] + [(n-2)/3] + 4$ when $n \geq 2$, 3 when $n = 1$, 0 when $n = 0$; for $r12$, the new upper bounds are $6[(n-2)/2] + [(n-3)/2] + 4$ when $n \geq 4$, 8 when $1 \leq n \leq 3$, 0 when $n = 0$; for $r16$, the new upper bounds are $10[(n-2)/3] + 4[n/3] + [(n-1)/3] + 2$ when $n \geq 5$, 16 when $1 \leq n \leq 4$, 0 when $n = 0$.*

The rest part of this paper is structured as follows: In Section 2, we state Proposition 2.1 using the new statement method. In Section 3, for ease of use later, we list some Riccati equations and Picard-Fuchs equations of the systems. In Section 4, we list the relationship between $A'(h)$ and $A(h)$ (or $J'(h)$ and $J(h)$). After, we use the Riccati equation method to prove Theorem 1.1. In Section 5, we give a short conclusion.

2. Simple expressions of Abelian integrals

In this section, we derive simple expressions for the Abelian integrals $A(h)$ and their related functions $J(h)$, and thereby obtain Proposition 2.1.

We assume that $p(x, y) = \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j$, $q(x, y) = \sum_{0 \leq i+j \leq n} b_{i,j} x^i y^j$, by (1.2), $A(h)$ in

Table 1. The comparison between new and old upper bounds.

Systems	New Upper Bounds	Old Upper Bounds	Order of Periodic Orbits
$r18$	$4[n/3] + 2[(n-1)/3]$ $+[(n-2)/3] + 4(n \geq 2),$ $3(n=1).$	$8[n/3] + 4[(n-1)/3]$ $+2[(n-2)/3] + 20(n \geq 2),$ $18(n=1).$	3
$r12$	$6[(n-2)/2] + [(n-3)/2]$ $+4(n \geq 4),$ $8(1 \leq n \leq 3).$	$18[(n-2)/2] + 3[(n-3)/2]$ $+21(n \geq 4),$ $33(1 \leq n \leq 3).$	3
$r16$	$10[(n-2)/3] + 4[n/3]$ $+[(n-1)/3] + 2(n \geq 5),$ $16(1 \leq n \leq 4).$	$40[(n-2)/3] + 16[n/3]$ $+4[(n-1)/3] + 28(n \geq 5),$ $80(1 \leq n \leq 4).$	4

Note: The old upper bounds of $r18$ and $r16$ are from Reference [15], while the old upper bounds of $r18$ are from Reference [14].

Theorem 1.1 have the following forms

$$A(h) = \oint_{\Gamma_h} x^{-\rho} \left(\sum_{0 \leq i+j \leq n} b_{i,j} x^i y^j dx - \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j dy \right), \quad (2.1)$$

where $x^{-\rho}$ is an integrating factor.

For the convenience of statement, we define integral functions $I_{s,\nu}(h)$ as follows:

$$I_{s,\nu}(h) = \oint_{\Gamma_h} x^{s-\rho} y^{\nu} dx,$$

where $\nu = 0, 1, 2, \dots, n, n+1$; for $r18$, $s = -1, -1/2, 0, \dots, n-1, n-1/2, n$; for $r12$, $s = -1, -2/3, -1/3, \dots, n-2/3, n-1/3, n$; for $r16$, $s = -1, -3/4, -1/2, \dots, n-1/2, n-1/4, n$; and $0 \leq s + \nu \leq n$. When $\nu = 1$, we write $I_{s,1}(h)$ as $I_s(h)$.

For systems $r18$, $r12$ and $r16$, when $\nu = 1$, we obtain their integral functions as $I_s(h) = \oint_{\Gamma_h} x^{s-1/2} y dx$ ($s = -1, -1/2, 0, \dots, n-1, n-1/2, n$), $I_s(h) = \oint_{\Gamma_h} x^{s-4/3} y dx$ ($s = -1, -2/3, -1/3, \dots, n-2/3, n-1/3, n$) and $I_s(h) = \oint_{\Gamma_h} x^{s-5/4} y dx$ ($s = -1, -3/4, -1/2, \dots, n-1/2, n-1/4, n$), respectively.

For system $r18$, the specific forms of main integral functions are as follows:

$$I_0(h) = \oint_{\Gamma_h} x^{-\frac{1}{2}} y dx, \quad I_{\frac{1}{2}}(h) = \oint_{\Gamma_h} y dx, \quad I_1(h) = \oint_{\Gamma_h} x^{\frac{1}{2}} y dx.$$

For system $r12$, the specific forms of main integral functions are as follows:

$$I_0(h) = \oint_{\Gamma_h} x^{-\frac{4}{3}} y dx, \quad I_{\frac{1}{3}}(h) = \oint_{\Gamma_h} x^{-1} y dx, \quad I_{\frac{2}{3}}(h) = \oint_{\Gamma_h} x^{-\frac{2}{3}} y dx.$$

For system $r16$, the specific forms of main integral functions are as follows:

$$I_0(h) = \oint_{\Gamma_h} x^{-\frac{5}{4}} y dx, \quad I_{\frac{1}{4}}(h) = \oint_{\Gamma_h} x^{-1} y dx, \quad I_{\frac{1}{2}}(h) = \oint_{\Gamma_h} x^{-\frac{3}{4}} y dx, \quad I_{\frac{3}{4}}(h) = \oint_{\Gamma_h} x^{-\frac{1}{2}} y dx.$$

Because

$$\oint_{\Gamma_h} x^{i-\rho} y^j dy = \frac{1}{j+1} \oint_{\Gamma_h} x^{i-\rho} dy^{j+1} = -\frac{i-\rho}{j+1} \oint_{\Gamma_h} x^{i-\rho-1} y^{j+1} dx = -\frac{i-\rho}{j+1} I_{i-1,j+1}(h).$$

(2.1) can be written as

$$A(h) = \sum_{0 \leq i+j \leq n} b_{i,j} I_{i,j}(h) + \sum_{0 \leq i+j \leq n} a_{i,j} \frac{i-\rho}{j+1} I_{i-1,j+1}(h) = \sum_{0 \leq \mu+\nu \leq n, -1 \leq \mu \leq n, 0 \leq \nu \leq n+1} c_{\mu,\nu} I_{\mu,\nu}(h), \quad (2.2)$$

where $c_{\mu,\nu} = b_{i,j} + (i+1-\rho)a_{i+1,j-1}/j$ ($0 \leq \mu = i \leq n-1, 1 \leq \nu = j \leq n$), $c_{-1,\nu} = -\rho a_{0,j}/(j+1)$ ($\nu = j+1, 0 \leq j \leq n$), $c_{\mu,0} = b_{i,0}$ ($0 \leq \mu = i \leq n$).

For convenience and simplicity, we introduce the following Definition 2.1.

Definition 2.1. If $B(h)$ is a polynomial consisting only of some $2k$ -degree ($k = 0, 1, 2, \dots$) monomials of h , then $B(h)$ is called a $2k$ -form polynomial of h ; if $B(h)$ is a polynomial consisting only of some $(3k+2)$ -degree ($k = 0, 1, 2, \dots$) monomials of h , then $B(h)$ is called a $(3k+2)$ -form polynomial of h ; if $B(h)$ is a polynomial consisting only of some $(4k+3)$ -degree ($k = 0, 1, 2, \dots$) monomials of h , then $B(h)$ is called a $(4k+3)$ -form polynomial of h ; etc.

The following Proposition 2.1 gives simple expressions for Abelian integrals $A(h)$.

Proposition 2.1. (i) For r18, Abelian integrals $A(h)$ can be expressed as

$$A(h) = \frac{1}{h} J(h), \quad J(h) = \alpha(h) I_0(h) + \beta(h) I_{\frac{1}{2}}(h) + \gamma(h) I_1(h), \quad (2.3)$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ are $(2k+1)$ -, $2k$ -, and $(2k+1)$ -form polynomials, respectively. When $n \geq 2$, $1 \leq \deg(\alpha(h)) \leq 2[(n-1)/3]+1$, $0 \leq \deg(\beta(h)) \leq 2[n/3]$, $1 \leq \deg(\gamma(h)) \leq 2[(n-2)/3]+1$; when $n = 1$, $\deg(\alpha(h)) = 1$, $\deg(\beta(h)) = 0$, $\gamma(h) = 0$; when $n = 0$, $\alpha(h) = 0$, $\deg(\beta(h)) = 0$, $\gamma(h) = 0$.

(ii) For r12, Abelian integrals $A(h)$ can be expressed as

$$A(h) = \alpha(h) I_0(h) + \beta(h) I_{\frac{1}{3}}(h) + \gamma(h) I_{\frac{2}{3}}(h), \quad (n \geq 1), \quad (2.4)$$

$$A(h) = \delta(h) I_{-1}(h), \quad (n = 0), \quad (2.5)$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$, $\delta(h)$ are $3k$ -, $(3k+1)$ -, $(3k+2)$ -, and $3k$ -form polynomials, respectively. When $n \geq 4$, $0 \leq \deg(\alpha(h)) \leq 3[(n-2)/2]$, $1 \leq \deg(\beta(h)) \leq 3[(n-2)/2]+1$, $2 \leq \deg(\gamma(h)) \leq 3[(n-3)/2]+2$; when $1 \leq n \leq 3$, $0 \leq \deg(\alpha(h)) \leq 3$, $\deg(\beta(h)) = 1$, $\deg(\gamma(h)) = 2$; when $n = 0$, $\deg(\delta(h)) = 0$.

(iii) For r16, Abelian integrals $A(h)$ can be expressed as

$$A(h) = \alpha(h) I_0(h) + \beta(h) I_{\frac{1}{4}}(h) + \gamma(h) I_{\frac{1}{2}}(h) + \delta(h) I_{\frac{3}{4}}(h), \quad (n \geq 1), \quad (2.6)$$

$$A(h) = \zeta(h) I_{-1}(h), \quad (n = 0), \quad (2.7)$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$, $\delta(h)$, $\zeta(h)$ are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, $(4k+3)$ -, and $4k$ -form polynomials, respectively. When $n \geq 5$, $0 \leq \deg(\alpha(h)) \leq 4[(n-2)/3]$, $1 \leq \deg(\beta(h)) \leq 4[(n-2)/3]+1$, $2 \leq \deg(\gamma(h)) \leq 4[(n-3)/3]+2$, $3 \leq \deg(\delta(h)) \leq 4[(n-4)/3]+3$; when $1 \leq n \leq 4$, $0 \leq \deg(\alpha(h)) \leq 4$, $\deg(\beta(h)) = 1$, $\deg(\gamma(h)) = 2$, $\deg(\delta(h)) = 3$; when $n = 0$, $\deg(\zeta(h)) = 0$.

Proof. Here, we will only introduce the derivation process of the conclusions of system $r16$. For $r18$ and $r12$, by utilizing the results from Reference [15] or [14], respectively, and performing some careful calculations, we can obtain their conclusions.

For $r16$, $\rho = 5/4$, from the results and equality (20) in Reference [15], we get

$$A(h) = \sum_{k=0}^n e_k I_{k-1}(h), \quad (2.8)$$

where $e_k \in \mathbb{R}$ ($k = 0, 1, \dots, n$). And

$$(4m+5)I_{m+1}(h) = 3 \cdot 2^7(8m+1)hI_{m+\frac{1}{4}}(h) - 3(4m-1)I_m(h). \quad (2.9)$$

A) If $m \geq 1$, then we rewrite (2.9) as

$$I_m(h) = \frac{3 \cdot 2^7(8m-7)}{4m+1}hI_{m-\frac{3}{4}}(h) - \frac{3(4m-5)}{4m+1}I_{m-1}(h),$$

which indicates that $I_m(h)$ can be expressed in terms of $hI_{m-3/4}(h)$ and $I_{m-1}(h)$. Then, step by step, using this formula, $I_m(h)$ can be written as a linear combination of $I_0(h)$, $I_{1/4}(h)$, $I_{1/2}(h)$ and $I_{3/4}(h)$ with polynomial coefficients about the h :

$$I_m(h) = \alpha_{m,1}(h)I_0(h) + \beta_{m,1}(h)I_{\frac{1}{4}}(h) + \gamma_{m,1}(h)I_{\frac{1}{2}}(h) + \delta_{m,1}(h)I_{\frac{3}{4}}(h),$$

where $\alpha_{m,1}(h)$, $\beta_{m,1}(h)$, $\gamma_{m,1}(h)$, $\delta_{m,1}(h)$ are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, and $(4k+3)$ -form polynomials, respectively. When $m \geq 3$, $0 \leq \deg(\alpha_{m,1}(h)) \leq 4[m-1/3]$, $1 \leq \deg(\beta_{m,1}(h)) \leq 4[m-1/3]+1$, $2 \leq \deg(\gamma_{m,1}(h)) \leq 4[m-2/3]+2$, $3 \leq \deg(\delta_{m,1}(h)) \leq 4[m-3/3]+3$; when $m = 2$, $\deg(\alpha_{m,1}(h)) = 0$, $\deg(\beta_{m,1}(h)) = 1$, $\deg(\gamma_{m,1}(h)) = 2$, $\delta_{m,1}(h) = 0$; when $m = 1$, $\deg(\alpha_{m,1}(h)) = 0$, $\deg(\beta_{m,1}(h)) = 1$, $\gamma_{m,1}(h) = 0$, $\delta_{m,1}(h) = 0$.

B) If $m = 0$, then $I_m(h)$ can also be a linear combination of $I_0(h)$, $I_{1/4}(h)$, $I_{1/2}(h)$ and $I_{3/4}(h)$ as $I_0(h) = I_0(h)$.

C) If $m < 0$, then we rewrite (2.9) as

$$I_m(h) = \frac{2^7(8m+1)}{4m-1}hI_{m+\frac{1}{4}}(h) - \frac{4m+5}{3(4m-1)}I_{m+1}(h),$$

which indicates that $I_m(h)$ can be expressed in terms of $hI_{m+1/4}(h)$ and $I_{m+1}(h)$. Then, step by step, using this formula, $I_{-1}(h)$ can be expressed in terms of $I_0(h)$, $I_{1/4}(h)$, $I_{1/2}(h)$ and $I_{3/4}(h)$:

$$I_{-1}(h) = \alpha_{m,2}(h)I_0(h) + \beta_{m,2}(h)I_{\frac{1}{4}}(h) + \gamma_{m,2}(h)I_{\frac{1}{2}}(h) + \delta_{m,2}(h)I_{\frac{3}{4}}(h),$$

where $\alpha_{m,2}(h)$, $\beta_{m,2}(h)$, $\gamma_{m,2}(h)$, $\delta_{m,2}(h)$ are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, and $(4k+3)$ -form polynomials, respectively, and $0 \leq \deg(\alpha_{m,2}(h)) \leq 4$, $\deg(\beta_{m,2}(h)) = 1$, $\deg(\gamma_{m,2}(h)) = 2$, $\deg(\delta_{m,2}(h)) = 3$.

As a consequence, all $I_m(h)$ can be expressed in terms of $I_0(h)$, $I_{1/4}(h)$, $I_{1/2}(h)$ and $I_{3/4}(h)$. Substituting these formulas into (2.8), we have

$$A(h) = \alpha(h)I_0(h) + \beta(h)I_{\frac{1}{4}}(h) + \gamma(h)I_{\frac{1}{2}}(h) + \delta(h)I_{\frac{3}{4}}(h),$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$, $\delta(h)$ are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, and $(4k+3)$ -form polynomials, respectively. When $n \geq 5$, $0 \leq \deg(\alpha(h)) \leq 4[(n-2)/3]$, $1 \leq \deg(\beta(h)) \leq 4[(n-2)/3]+1$,

$2 \leq \deg(\gamma(h)) \leq 4[(n-3)/3] + 2$, $3 \leq \deg(\delta(h)) \leq 4[(n-4)/3] + 3$; when $1 \leq n \leq 4$, $0 \leq \deg(\alpha(h)) \leq 4$, $\deg(\beta(h)) = 1$, $\deg(\gamma(h)) = 2$, $\deg(\delta(h)) = 3$.

When $n = 0$, from (2.8), we can get

$$A(h) = e_0 I_{-1}(h) = \zeta(h) I_{-1}(h),$$

where $\deg(\zeta(h)) = 0$. □

3. Riccati equation of the correlation function and Picard-Fuchs equation of the main integral functions

In this section, we seek Picard-Fuchs equation of the main integral functions of $A(h)$, and Riccati equation of the correlation functions $V(h)$ of $A(h)$, obtaining six Picard-Fuchs equations and three Riccati equations.

The main integral functions of $A(h)$ and their derivatives satisfy the following Lemmas 3.1–3.3 and Corollary 3.1.

Lemma 3.1. (i) [15] For $r18$, functions $I_s(h)$ for $s = 0, 1/2, 1$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I_0(h) \\ I_{\frac{1}{2}}(h) \\ I_1(h) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}h \frac{1}{2^7} & 0 \\ 0 & h \frac{1}{3 \cdot 2^6} \\ \frac{3}{8}h \frac{3}{2^9} & \frac{3}{4}h \end{pmatrix} \begin{pmatrix} I'_0(h) \\ I'_{\frac{1}{2}}(h) \\ I'_1(h) \end{pmatrix}. \quad (3.1)$$

(ii) [14] For $r12$, functions $I_s(h)$ for $s = 0, 1/3, 2/3$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I_0(h) \\ I_{\frac{1}{3}}(h) \\ I_{\frac{2}{3}}(h) \end{pmatrix} = \begin{pmatrix} h & 0 & -\frac{1}{2^7} \\ -\frac{1}{3 \cdot 2^5} & \frac{4}{3}h & 0 \\ 0 & -\frac{1}{5 \cdot 2^5} & \frac{4}{5}h \end{pmatrix} \begin{pmatrix} I'_0(h) \\ I'_{\frac{1}{3}}(h) \\ I'_{\frac{2}{3}}(h) \end{pmatrix}. \quad (3.2)$$

(iii) [15] For $r16$, functions $I_s(h)$ for $s = 0, 1/4, 1/2, 3/4$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I_0(h) \\ I_{\frac{1}{4}}(h) \\ I_{\frac{1}{2}}(h) \\ I_{\frac{3}{4}}(h) \end{pmatrix} = \begin{pmatrix} h & 0 & 0 & -\frac{1}{3 \cdot 2^6} \\ -\frac{1}{2^7} & \frac{3}{2}h & 0 & 0 \\ 0 & -\frac{1}{3 \cdot 2^6} & h & 0 \\ 0 & 0 & -\frac{1}{2^8} & \frac{3}{4}h \end{pmatrix} \begin{pmatrix} I'_0(h) \\ I'_{\frac{1}{4}}(h) \\ I'_{\frac{1}{2}}(h) \\ I'_{\frac{3}{4}}(h) \end{pmatrix}. \quad (3.3)$$

Proof. For $r18$ and $r16$, the proof can be found in [15]; for $r12$, the proof can be found in [14], so we omit their proof. □

Lemma 3.2. (i) [15] For r18, functions $I'_s(h)$ for $s = 0, 1/2, 1$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I''_0(h) \\ I''_{\frac{1}{2}}(h) \\ I''_1(h) \end{pmatrix} = \frac{1}{C(h)} \begin{pmatrix} -h^2 & \frac{1}{3^2 \cdot 2^{12}} \\ \frac{1}{3 \cdot 2^6}h & -\frac{1}{3 \cdot 2^6}h \\ -h^2 & h^2 \end{pmatrix} \begin{pmatrix} I'_0(h) \\ I'_1(h) \end{pmatrix}, \quad (3.4)$$

where $C(h) = 3h(h^2 - 1/(3^2 \cdot 2^{12})) = 3h(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))$.

(ii) [14] For r12, functions $I'_s(h)$ for $s = 0, 1/3, 2/3$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I''_0(h) \\ I''_{\frac{1}{3}}(h) \\ I''_{\frac{2}{3}}(h) \end{pmatrix} = \frac{1}{C(h)} \begin{pmatrix} -\frac{1}{2^{14}} & \frac{1}{2^7}h \\ -h^2 & \frac{1}{2^{14}} \\ -\frac{1}{2^7}h & h^2 \end{pmatrix} \begin{pmatrix} I'_{\frac{1}{3}}(h) \\ I'_{\frac{2}{3}}(h) \end{pmatrix}, \quad (3.5)$$

where $C(h) = 4(h^3 - 1/2^{21}) = 4(h - 1/2^7)(h^2 + 1/2^7h + 1/2^{14})$.

(iii) [15] For r16, functions $I'_s(h)$ for $s = 0, 1/4, 1/2, 3/4$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} I''_0(h) \\ I''_{\frac{1}{4}}(h) \\ I''_{\frac{1}{2}}(h) \\ I''_{\frac{3}{4}}(h) \end{pmatrix} = \frac{1}{C(h)} \begin{pmatrix} -\frac{1}{3^3 \cdot 2^{18}} & \frac{1}{3 \cdot 2^6}h^2 \\ -h^3 & \frac{1}{3^2 \cdot 2^{12}}h \\ -\frac{1}{3 \cdot 2^6}h^2 & \frac{1}{3^3 \cdot 2^{18}} \\ -\frac{1}{3^2 \cdot 2^{12}}h & h^3 \end{pmatrix} \begin{pmatrix} I'_{\frac{1}{4}}(h) \\ I'_{\frac{3}{4}}(h) \end{pmatrix}, \quad (3.6)$$

where $C(h) = 3(h^4 - 1/(3^4 \cdot 2^{24})) = 3(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))(h^2 + 1/(3^2 \cdot 2^{12}))$.

Proof. For r18 and r16, the proof can be found in [15]; for r12, the proof can be found in [14], so we omit their proof. $\square$

Lemma 3.3. (i) For r18, $I_s(-1/(3 \cdot 2^6)) = 0$ ($s = 0, 1/2, 1$); when $h \in (-1/(3 \cdot 2^6), 0)$, $I_s(h) < 0$, $I'_s(h) > 0$ ($s = 0, 1/2, 1$).

(ii) For r12, $I_s(1/2^7) = 0$ ($s = 0, 1/3, 2/3$); when $h \in (1/2^7, +\infty)$, $I_s(h) < 0$, $I'_s(h) > 0$ ($s = -1, 0, 1/3, 2/3$).

(iii) For r16, $I_s(1/(3 \cdot 2^6)) = 0$ ($s = 0, 1/4, 1/2, 3/4$); when $h \in (1/(3 \cdot 2^6), +\infty)$, $I_s(h) < 0$, $I'_s(h) > 0$ ($s = -1, 0, 1/4, 1/2, 3/4$).

For r18, $I_s(h) = \oint_{\Gamma_h} x^{s-1/2} y dx$; for r12, $I_s(h) = \oint_{\Gamma_h} x^{s-4/3} y dx$; for r16, $I_s(h) = \oint_{\Gamma_h} x^{s-5/4} y dx$. And $I'_s(h) = \oint_{\Gamma_h} x^{s-1} y dx$. The proof only contains some simple calculation, we omit it.

Corollary 3.1. (i) [15] For r18, assume that $V(h) := I'_0(h)/I'_1(h)$, then $V(h)$ satisfies the following Riccati equation

$$C(h)V'(h) = h^2V^2(h) - 2h^2V(h) + \frac{1}{3^2 \cdot 2^{12}}, \quad (3.7)$$

where $C(h) = 3h(h^2 - 1/(3^2 \cdot 2^{12})) = 3h(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))$.

(ii) [14] For r12, assume that $V(h) := I'_{1/3}(h)/I'_{2/3}(h)$, then $V(h)$ satisfies the following Riccati equation

$$C(h)V'(h) = \frac{1}{2^7}hV^2(h) - 2h^2V(h) + \frac{1}{2^{14}}, \quad (3.8)$$

where $C(h) = 4(h^3 - 1/2^{21}) = 4(h - 1/2^7)(h^2 + h/2^7 + 1/2^{14})$.

(iii) [15] For r16, assume that $V(h) := I'_{1/4}(h)/I'_{3/4}(h)$, then $V(h)$ satisfies the following Riccati equation

$$C(h)V'(h) = \frac{1}{3^2 \cdot 2^{12}}hV^2(h) - 2h^3V(h) + \frac{1}{3^2 \cdot 2^{12}}h, \quad (3.9)$$

where $C(h) = 3(h^4 - 1/(3^4 \cdot 2^{24})) = 3(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))(h^2 + 1/(3^2 \cdot 2^{12}))$.

Proof. For r18 and r16, the proof can be found in [15]; for r12, the proof can be found in [14], so we omit their proof. $\square$

4. New upper bounds on the number of zeros of Abelian integrals

In this section, we seek relationship between the $A(h)$ and $A'(h)$, and relationship between the correlation functions $J(h)$ and $J'(h)$ of $A(h)$, obtaining four first order linear ordinary differential equations and three Riccati equations. Then, we prove Theorem 1.1 by using Lemmas 4.2 and 4.3.

(i) For r18, if $n \geq 1$, by (2.3) and (3.1), we get

$$J(h) = \alpha_1(h)I'_0(h) + \beta_1(h)I'_{\frac{1}{2}}(h) + \gamma_1(h)I'_1(h), \quad (4.1)$$

where $\alpha_1(h) = 3h\alpha(h)/2 + 3h\gamma(h)/8$, $\beta_1(h) = \alpha(h)/2^7 + h\beta(h) + 3\gamma(h)/2^9$, $\gamma_1(h) = \beta(h)/(3 \cdot 2^6) + 3h\gamma(h)/4$. Thus, $\alpha_1(h)$, $\beta_1(h)$, $\gamma_1(h)$ are $2k$ -, $(2k+1)$ -, and $2k$ -form polynomials, respectively. When $n \geq 2$, $2 \leq \deg(\alpha_1(h)) \leq 2[(n-1)/3] + 2$, $1 \leq \deg(\beta_1(h)) \leq 2[n/3] + 1$, and $0 \leq \deg(\gamma_1(h)) \leq 2[(n+1)/3]$; when $n = 1$, $\deg(\alpha_1(h)) = 2$, $\deg(\beta_1(h)) = 1$, and $\deg(\gamma_1(h)) = 0$.

Differentiating both sides of (2.3) with respect to h and using (3.1), we obtain

$$J'(h) = \alpha_2(h)I'_0(h) + \beta_2(h)I'_{\frac{1}{2}}(h) + \gamma_2(h)I'_1(h), \quad (4.2)$$

where $\alpha_2(h) = 3h\alpha'(h)/2 + \alpha(h) + 3h\gamma'(h)/8$, $\beta_2(h) = \alpha'(h)/2^7 + \beta(h) + h\beta'(h) + 3\gamma'(h)/2^9$, $\gamma_2(h) = \gamma(h) + \beta'(h)/(3 \cdot 2^6) + 3h\gamma'(h)/4$. So, $\alpha_2(h)$, $\beta_2(h)$, $\gamma_2(h)$ are $(2k+1)$ -, $2k$ -, and $(2k+1)$ -form polynomials, respectively. When $n \geq 2$, $1 \leq \deg(\alpha_2(h)) \leq 2[(n-1)/3] + 1$, $0 \leq \deg(\beta_2(h)) \leq 2[n/3]$, $1 \leq \deg(\gamma_2(h)) \leq 2[(n-2)/3] + 1$; when $n = 1$, $\deg(\alpha_2(h)) = 1$, $\deg(\beta_2(h)) = 0$, $\gamma_2(h) = 0$.

By (4.1) and (4.2), we have

$$\begin{aligned} \beta_1(h)J'(h) &= \beta_2(h)J(h) + T(h), \\ T(h) &= K(h)I'_0(h) + L(h)I'_1(h), \end{aligned} \quad (4.3)$$

where $K(h) = \beta_1(h)\alpha_2(h) - \beta_2(h)\alpha_1(h)$, $L(h) = \beta_1(h)\gamma_2(h) - \beta_2(h)\gamma_1(h)$. Thus $K(h)$, $L(h)$ are $2k$ -form polynomials. When $n \geq 2$, $2 \leq \deg(K(h)) \leq 2[n/3] + 2[(n-1)/3] + 2$, $0 \leq \deg(L(h)) \leq 2[(n+1)/3] + 2[n/3]$; when $n = 1$, $\deg(K(h)) = 2$, $\deg(L(h)) = 0$.

(ii) For r12, if $n \geq 1$, by (2.4) and (3.2), we obtain

$$A(h) = \alpha_1(h)I'_0(h) + \beta_1(h)I'_{\frac{1}{3}}(h) + \gamma_1(h)I'_{\frac{2}{3}}(h), \quad (4.4)$$

where $\alpha_1(h) = h\alpha(h) - \beta(h)/(3 \cdot 2^5)$, $\beta_1(h) = 4h\beta(h)/3 - \gamma(h)/(5 \cdot 2^5)$, $\gamma_1(h) = 4h\gamma(h)/5 - \alpha(h)/2^7$. Thus, $\alpha_1(h)$, $\beta_1(h)$, $\gamma_1(h)$ are $(3k+1)$ -, $(3k+2)$ -, and $3k$ -form polynomials, respectively. When $n \geq 4$, $1 \leq \deg(\alpha_1(h)) \leq 3[(n-2)/2] + 1$, $2 \leq \deg(\beta_1(h)) \leq 3[(n-2)/2] + 2$, $0 \leq \deg(\gamma_1(h)) \leq 3[(n-1)/2]$; when $1 \leq n \leq 3$, $1 \leq \deg(\alpha_1(h)) \leq 4$, $\deg(\beta_1(h)) = 2$, $0 \leq \deg(\gamma_1(h)) \leq 3$.

Differentiating both sides of (2.4) with respect to h and using (3.2), we obtain

$$A'(h) = \alpha_2(h)I'_0(h) + \beta_2(h)I'_{\frac{1}{3}}(h) + \gamma_2(h)I'_{\frac{2}{3}}(h), \quad (4.5)$$

where $\alpha_2(h) = \alpha(h) + h\alpha'(h) - \beta'(h)/(3 \cdot 2^5)$, $\beta_2(h) = \beta(h) + 4h\beta'(h)/3 - \gamma'(h)/(5 \cdot 2^5)$, $\gamma_2(h) = \gamma(h) + 4h\gamma'(h)/5 - \alpha'(h)/2^7$. Thus, $\alpha_2(h)$, $\beta_2(h)$, $\gamma_2(h)$ are $3k$ -, $(3k+1)$ -, and $(3k+2)$ -form polynomials, respectively. When $n \geq 4$, $0 \leq \deg(\alpha_2(h)) \leq 3[(n-2)/2]$, $1 \leq \deg(\beta_2(h)) \leq 3[(n-2)/2] + 1$, $2 \leq \deg(\gamma_2(h)) \leq 3[(n-3)/2] + 2$; when $1 \leq n \leq 3$, $0 \leq \deg(\alpha_2(h)) \leq 3$, $\deg(\beta_2(h)) = 1$, $\deg(\gamma_2(h)) = 2$.

By (4.4) and (4.5), we get

$$\begin{aligned} \alpha_1(h)A'(h) &= \alpha_2(h)A(h) + T(h), \\ T(h) &= K(h)I'_{\frac{1}{3}}(h) + L(h)I'_{\frac{2}{3}}(h), \end{aligned} \quad (4.6)$$

where $K(h) = \alpha_1(h)\beta_2(h) - \alpha_2(h)\beta_1(h)$, $L(h) = \alpha_1(h)\gamma_2(h) - \alpha_2(h)\gamma_1(h)$. Thus, $K(h)$, $L(h)$ are $(3k+2)$ -, and $3k$ -form polynomials, respectively. When $n \geq 4$, $2 \leq \deg(K(h)) \leq 6[(n-2)/2] + 2$, $0 \leq \deg(L(h)) \leq 3[(n-1)/2] + 3[(n-2)/2]$; when $1 \leq n \leq 3$, $2 \leq \deg(K(h)) \leq 5$, $0 \leq \deg(L(h)) \leq 6$.

(iii) For $r16$, if $n \geq 1$, by (2.6) and (3.3), we get

$$A(h) = \alpha_1(h)I'_0(h) + \beta_1(h)I'_{\frac{1}{4}}(h) + \gamma_1(h)I'_{\frac{1}{2}}(h) + \delta_1(h)I'_{\frac{3}{4}}(h), \quad (4.7)$$

where $\alpha_1(h) = h\alpha(h) - \beta(h)/2^7$, $\beta_1(h) = 3h\beta(h)/2 - \gamma(h)/(3 \cdot 2^6)$, $\gamma_1(h) = h\gamma(h) - \delta(h)/2^8$, $\delta_1(h) = 3h\delta(h)/4 - \alpha(h)/(3 \cdot 2^6)$. Thus $\alpha_1(h)$, $\beta_1(h)$, $\gamma_1(h)$, $\delta_1(h)$ are $(4k+1)$ -, $(4k+2)$ -, $(4k+3)$ -, and $4k$ -form polynomials, respectively. When $n \geq 5$, $1 \leq \deg(\alpha_1(h)) \leq 4[(n-2)/3] + 1$, $2 \leq \deg(\beta_1(h)) \leq 4[(n-2)/3] + 2$, $3 \leq \deg(\gamma_1(h)) \leq 4[(n-3)/3] + 3$, and $0 \leq \deg(\delta_1(h)) \leq 4[(n-1)/3]$; when $1 \leq n \leq 4$, $1 \leq \deg(\alpha_1(h)) \leq 5$, $\deg(\beta_1(h)) = 2$, $\deg(\gamma_1(h)) = 3$, and $0 \leq \deg(\delta_1(h)) \leq 4$.

Differentiating both sides of (2.6) with respect to h and using (3.3), we obtain

$$A'(h) = \alpha_2(h)I'_0(h) + \beta_2(h)I'_{\frac{1}{4}}(h) + \gamma_2(h)I'_{\frac{1}{2}}(h) + \delta_2(h)I'_{\frac{3}{4}}(h), \quad (4.8)$$

where $\alpha_2(h) = h\alpha'(h) - \beta'(h)/2^7 + \alpha(h)$, $\beta_2(h) = 3h\beta'(h)/2 - \gamma'(h)/(3 \cdot 2^6) + \beta(h)$, $\gamma_2(h) = h\gamma'(h) - \delta'(h)/2^8 + \gamma(h)$, $\delta_2(h) = 3h\delta'(h)/4 - \alpha'(h)/(3 \cdot 2^6) + \delta(h)$. So $\alpha_2(h)$, $\beta_2(h)$, $\gamma_2(h)$, $\delta_2(h)$ are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, and $(4k+3)$ -form polynomials, respectively. When $n \geq 5$, $0 \leq \deg(\alpha_2(h)) \leq 4[(n-2)/3]$, $1 \leq \deg(\beta_2(h)) \leq 4[(n-2)/3] + 1$, $2 \leq \deg(\gamma_2(h)) \leq 4[(n-3)/3] + 2$, $3 \leq \deg(\delta_2(h)) \leq 4[(n-4)/3] + 3$; when $1 \leq n \leq 4$, $0 \leq \deg(\alpha_2(h)) \leq 4$, $\deg(\beta_2(h)) = 1$, $\deg(\gamma_2(h)) = 2$, $\deg(\delta_2(h)) = 3$.

By (4.7) and (4.8), we have

$$\alpha_1(h)A'(h) = \alpha_2(h)A(h) + T_1(h), \quad (4.9)$$

$$T_1(h) = \beta_3(h)I'_{\frac{1}{4}}(h) + \gamma_3(h)I'_{\frac{1}{2}}(h) + \delta_3(h)I'_{\frac{3}{4}}(h), \quad (4.10)$$

where $\beta_3(h) = \alpha_1(h)\beta_2(h) - \alpha_2(h)\beta_1(h)$, $\gamma_3(h) = \alpha_1(h)\gamma_2(h) - \alpha_2(h)\gamma_1(h)$, $\delta_3(h) = \alpha_1(h)\delta_2(h) - \alpha_2(h)\delta_1(h)$. Thus $\beta_3(h)$, $\gamma_3(h)$, $\delta_3(h)$ are $(4k+2)$ -, $(4k+3)$ -, and $4k$ -form polynomials, respectively. When $n \geq 5$, $2 \leq \deg(\beta_3(h)) \leq 8[(n-2)/3] + 2$, $3 \leq \deg(\gamma_3(h)) \leq 4[(n-2)/3] + 4[(n-$

$3)/3] + 3$, $4 \leq \deg(\delta_3(h)) \leq 4[(n-2)/3] + 4[(n-4)/3] + 4$; when $1 \leq n \leq 4$, $2 \leq \deg(\beta_3(h)) \leq 6$, $3 \leq \deg(\gamma_3(h)) \leq 7$, $0 \leq \deg(\delta_3(h)) \leq 8$.

Differentiating both sides of (4.10) with respect to h and using (3.6), we obtain

$$C(h)T_1'(h) = \beta_4(h)I_{\frac{1}{4}}'(h) + C(h)\gamma_3'(h)I_{\frac{1}{2}}'(h) + \delta_4(h)I_{\frac{3}{4}}'(h), \quad (4.11)$$

where $\beta_4(h) = C(h)\beta_3'(h) - h^3\beta_3(h) - h^2\gamma_3(h)/(3 \cdot 2^6) - h\delta_3(h)/(3^2 \cdot 2^{12})$, $\delta_4(h) = C(h)\delta_3'(h) + h\beta_3(h)/(3^2 \cdot 2^{12}) + \gamma_3(h)/(3^3 \cdot 2^{18}) + h^3\delta_3(h)$. So $\beta_4(h)$, $\delta_4(h)$ are $(4k+1)$ -, and $(4k+3)$ -form polynomials, respectively. When $n \geq 5$, $1 \leq \deg(\beta_4(h)) \leq 8[(n-2)/3] + 5$, $3 \leq \deg(\delta_4(h)) \leq 4[(n-1)/3] + 4[(n-2)/3] + 3$; when $1 \leq n \leq 4$, $1 \leq \deg(\beta_4(h)) \leq 9$, $3 \leq \deg(\delta_4(h)) \leq 11$.

By (4.10) and (4.11), we have

$$\begin{aligned} C(h)\gamma_3(h)T_1'(h) &= C(h)\gamma_3'(h)T_1(h) + T_2(h), \\ T_2(h) &= K(h)I_{\frac{1}{4}}'(h) + L(h)I_{\frac{3}{4}}'(h), \end{aligned} \quad (4.12)$$

where $K(h) = \beta_4(h)\gamma_3(h) - C(h)\gamma_3'(h)\beta_3(h)$, $L(h) = \delta_4(h)\gamma_3(h) - C(h)\gamma_3'(h)\delta_3(h)$. Thus $K(h)$, $L(h)$ are $4k$ -, and $(4k+2)$ -form polynomials, respectively. When $n \geq 5$, $4 \leq \deg(K(h)) \leq 12[(n-2)/3] + 4[(n-3)/3] + 8$, $6 \leq \deg(L(h)) \leq 8[(n-2)/3] + 4[(n-1)/3] + 4[(n-3)/3] + 6$; when $1 \leq n \leq 4$, $4 \leq \deg(K(h)) \leq 16$, $2 \leq \deg(L(h)) \leq 18$.

Using Corollary 3.1, after simple calculations, we obtain the following Lemma 4.1.

Lemma 4.1. (i) For r18, when $n \geq 1$, assume that $W(h) := T(h)/I_1'(h)$, then $W(h)$ satisfies the following Riccati equation

$$C(h)K(h)W'(h) = h^2W^2(h) + M(h)W(h) + N(h), \quad (4.13)$$

where $N(h) = C(h)K(h)L'(h) + h^2L^2(h) - C(h)K'(h)L(h) + 2h^2K(h)L(h) + K^2(h)/(3^2 \cdot 2^{12})$, $M(h) = C(h)K'(h) - 2h^2K(h) - 2h^2L(h)$. Thus, $M(h)$ and $N(h)$ are $2k$ -form polynomials. When $n \geq 2$, $2 \leq \deg(M(h)) \leq 2[n/3] + 2[(n-1)/3] + 4$, $2 \leq \deg(N(h)) \leq 4[n/3] + 2[(n-1)/3] + 2[(n-2)/3] + 6$; when $n = 1$, $2 \leq \deg(M(h)) \leq 4$, $2 \leq \deg(N(h)) \leq 4$.

(ii) For r12, when $n \geq 1$, assume that $W(h) := T(h)/I_{2/3}'(h)$, then $W(h)$ satisfies the following Riccati equation

$$C(h)K(h)W'(h) = \frac{1}{2^7}hW^2(h) + M(h)W(h) + N(h), \quad (4.14)$$

where $N(h) = K^2(h)/2^{14} + hL^2(h)/2^7 + C(h)K(h)L'(h) - C(h)K'(h)L(h) + 2h^2K(h)L(h)$, $M(h) = C(h)K'(h) - 2h^2K(h) - hL(h)/2^6$. Thus, $M(h)$ and $N(h)$ are $(3k+1)$ -form polynomials. When $n \geq 4$, $1 \leq \deg(M(h)) \leq 6[(n-2)/2] + 4$, $1 \leq \deg(N(h)) \leq 9[(n-2)/2] + 3[(n-3)/2] + 7$; when $1 \leq n \leq 3$, $1 \leq \deg(M(h)) \leq 7$, $1 \leq \deg(N(h)) \leq 13$.

(iii) For r16, when $n \geq 1$, assume that $W(h) := T_2(h)/J_{3/4}'(h)$, then $W(h)$ satisfies the following Riccati equation

$$C(h)K(h)W'(h) = \frac{1}{3^2 \cdot 2^{12}}hW^2(h) + M(h)W(h) + N(h), \quad (4.15)$$

where $M(h) = C(h)K'(h) - 2h^3K(h) - hL(h)/(3^2 \cdot 2^{11})$, $N(h) = C(h)K(h)L'(h) + 2h^3K(h)L(h) - C(h)K'(h)L(h) + hK^2(h)/(3^2 \cdot 2^{12}) + hL^2(h)/(3^2 \cdot 2^{12})$. Thus $M(h)$ and $N(h)$ are $(4k+3)$ -, and $(4k+1)$ -form polynomials, respectively. When $n \geq 5$, $3 \leq \deg(M(h)) \leq 12[(n-2)/3] + 4[(n-3)/3] + 11$, $9 \leq \deg(N(h)) \leq 20[(n-2)/3] + 8[(n-3)/3] + 4[(n-1)/3] + 17$; when $1 \leq n \leq 4$, $3 \leq \deg(M(h)) \leq 19$, $5 \leq \deg(N(h)) \leq 37$.

For conciseness, we use $\sharp A(h)$ to denote the number of zeros of functions $A(h)$ in Ω . However, we need the following Lemmas 4.2 and 4.3.

Lemma 4.2. [22] *If the smooth functions $D(h)$, $\eta(h)$, $\theta(h)$, $\kappa(h)$, and $\lambda(h)$ satisfy*

$$\eta(h)D'(h) = \theta(h)D^2(h) + \kappa(h)D(h) + \lambda(h),$$

then

$$\sharp D(h) \leq \sharp \eta(h) + \sharp \lambda(h) + 1.$$

Lemma 4.2 is Lemma 5.3 in [22], therefore, we omit its proof.

Lemma 4.3. (i) *If $E(h)$ is a $2k$ -form polynomial, and $0 \leq \deg(E(h)) \leq 2t$ ($t = 0, 1, 2, \dots$), further $h \in (-\infty, 0)$ or $h \in (0, +\infty)$, then $\sharp E(h) \leq t$.*

(ii) *If $E(h)$ is a $4k$ -form polynomial, and $0 \leq \deg(E(h)) \leq 4t$ ($t = 0, 1, 2, \dots$), further $h \in (-\infty, 0)$ or $h \in (0, +\infty)$, then $\sharp E(h) \leq t$.*

(iii) *If $E(h)$ is a $(4k+3)$ -form polynomial, and $3 \leq \deg(E(h)) \leq 4t+3$ ($t = 0, 1, 2, \dots$), further $h \in (-\infty, 0)$ or $h \in (0, +\infty)$, then $\sharp E(h) \leq t$.*

Proof. (i) Assume that $s := h^2$, we get $E(h) = U(s)$, since $0 \leq \deg(E(h)) \leq 2t$, so, $0 \leq \deg(U(s)) \leq t$. The polynomial $U(s)$ has at most t zeros in $(-\infty, +\infty)$, i.e. $\sharp U(s) \leq t$, moreover, a definite s corresponds to the unique $h = -\sqrt{s} \in (-\infty, 0)$ or $h = \sqrt{s} \in (0, +\infty)$, so, the polynomial $E(h)$ has at most t zeros in $(-\infty, 0)$ or $(0, +\infty)$, i.e. $\sharp E(h) \leq t$.

(ii) Assume that $s := h^4$, we get $E(h) = U(s)$, since $0 \leq \deg(E(h)) \leq 4t$, so, $0 \leq \deg(U(s)) \leq t$. The polynomial $U(s)$ has at most t zeros in $(0, +\infty)$, i.e. $\sharp U(s) \leq t$, moreover, a definite s corresponds to the unique $h = -\sqrt[4]{s} \in (-\infty, 0)$ or $h = \sqrt[4]{s} \in (0, +\infty)$, so, the polynomial $E(h)$ has at most t zeros in $(-\infty, 0)$ or $(0, +\infty)$, i.e. $\sharp E(h) \leq t$.

(iii) Assume that $h^3 E_1(h) := E(h)$, since $3 \leq \deg(E(h)) \leq 4t+3$, so, $E_1(h)$ is a $4k$ -form polynomial, and $0 \leq \deg(E_1(h)) \leq 4t$, further $h \in (-\infty, 0)$ or $h \in (0, +\infty)$, so, $h \neq 0$, $\sharp E(h) = \sharp E_1(h)$. Using the conclusion in (2) above, we get $\sharp E_1(h) \leq t$, therefore, $\sharp E(h) \leq t$. $\square$

Similarly, we can get other similar results and omit them for brevity. Then, we will complete the proof of Theorem 1.1 by using Lemmas 4.2 and 4.3.

Proof. (i) For $r18$, from (4.3), using Lemma 4.2, we have

$$\sharp J(h) \leq \sharp \beta_1(h) + \sharp T(h) + 1.$$

Using (4.13) and Lemma 4.2, we get

$$\sharp T(h) = \sharp W(h) \leq \sharp C(h) + \sharp K(h) + \sharp N(h) + 1.$$

Using (2.3) and the above two relations, therefore

$$\sharp A(h) = \sharp J(h) \leq \sharp \beta_1(h) + \sharp C(h) + \sharp K(h) + \sharp N(h) + 2,$$

where $C(h) = 3h(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))$ has no zero in $(-1/(3 \cdot 2^6), 0)$, and $\beta_1(h)$, $K(h)$, $N(h)$ are $(2k+1)$ -, $2k$ -, and $2k$ -form polynomials, respectively.

When $n \geq 2$, since $1 \leq \deg(\beta_1(h)) \leq 2[n/3] + 1$, $2 \leq \deg(K(h)) \leq 2[n/3] + 2[(n-1)/3] + 2$, and $2 \leq \deg(N(h)) \leq 4[n/3] + 2[(n-1)/3] + 2[(n-2)/3] + 6$, using Lemma 4.3, we have

$$\sharp A(h) \leq \left\lfloor \frac{n}{3} \right\rfloor + 0 + \left(\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n-1}{3} \right\rfloor \right) + \left(2 \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n-2}{3} \right\rfloor + 2 \right) + 2$$

$$= 4 \left\lfloor \frac{n}{3} \right\rfloor + 2 \left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n-2}{3} \right\rfloor + 4.$$

When $n = 1$, since $\deg(\beta_1(h)) = 1$, $\deg(K(h)) = 2$, and $2 \leq \deg(N(h)) \leq 4$, using Lemma 4.3, we have

$$\sharp A(h) \leq 0 + 0 + 0 + 1 + 2 = 3.$$

When $n = 0$, from (2.3), since $\alpha(h) = 0$, $\gamma(h) = 0$, $\deg(\beta(h)) = 0$, $I_{1/2}(h) < 0$, we have

$$\sharp A(h) = 0.$$

(ii) For $r12$, when $n \geq 1$, from (4.6), using Lemma 4.2, we get

$$\sharp A(h) \leq \sharp \alpha_1(h) + \sharp T(h) + 1.$$

Using (4.14) and Lemma 4.2, we get

$$\sharp T(h) = \sharp W(h) \leq \sharp C(h) + \sharp K(h) + \sharp N(h) + 1.$$

Using the two inequalities above, we obtain

$$\sharp A(h) \leq \sharp \alpha_1(h) + \sharp C(h) + \sharp K(h) + \sharp N(h) + 2,$$

where $C(h) = 4(h - 1/2^7)(h^2 + h/2^7 + 1/2^{14})$ has no zero in $(1/2^7, +\infty)$, and $\alpha_1(h)$, $K(h)$, $N(h)$ are $(3k+1)$ -, $(3k+2)$ -, and $(3k+1)$ -form polynomials, respectively.

When $n \geq 4$, since $1 \leq \deg(\alpha_1(h)) \leq 3[(n-2)/2] + 1$, $2 \leq \deg(K(h)) \leq 6[(n-2)/2] + 2$, and $1 \leq \deg(N(h)) \leq 9[(n-2)/2] + 3[(n-3)/2] + 7$, using Lemma 4.3, we obtain

$$\begin{aligned} \sharp A(h) &\leq \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 2 \left\lfloor \frac{n-2}{2} \right\rfloor + \left(3 \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{2} \right\rfloor + 2 \right) + 2 \\ &= 6 \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{2} \right\rfloor + 4. \end{aligned}$$

When $1 \leq n \leq 3$, since $1 \leq \deg(\alpha_1(h)) \leq 4$, $2 \leq \deg(K(h)) \leq 5$, and $1 \leq \deg(N(h)) \leq 13$, using Lemma 4.3, we obtain

$$\sharp A(h) \leq 1 + 0 + 1 + 4 + 2 = 8.$$

When $n = 0$, from (2.5), since $\deg(\delta(h)) = 0$, $I_{-1}(h) < 0$, we get

$$\sharp A(h) = 0.$$

(iii) For $r16$, when $n \geq 1$, from (4.9) and (4.12), using Lemma 4.2, we have

$$\sharp A(h) \leq \sharp \alpha_1(h) + \sharp T_1(h) + 1, \quad \sharp T_1(h) \leq \sharp C(h) + \sharp \gamma_3(h) + \sharp T_2(h) + 1.$$

Using (4.15) and Lemma 4.2, we get

$$\sharp T_2(h) = \sharp W(h) \leq \sharp C(h) + \sharp K(h) + \sharp N(h) + 1.$$

Using the three inequalities above, therefore

$$\sharp A(h) \leq \sharp \alpha_1(h) + 2\sharp C(h) + \sharp \gamma_3(h) + \sharp K(h) + \sharp N(h) + 3,$$

where $C(h) = 3(h - 1/(3 \cdot 2^6))(h + 1/(3 \cdot 2^6))(h^2 + 1/(3^2 \cdot 2^{12}))$ has no zero in $(1/(3 \cdot 2^6), +\infty)$, and $\alpha_1(h)$, $\gamma_3(h)$, $K(h)$, $N(h)$ are $(4k+1)$ -, $(4k+3)$ -, $4k$ -, and $(4k+1)$ -form polynomials, respectively.

When $n \geq 5$, since $1 \leq \deg(\alpha_1(h)) \leq 4[(n-2)/3] + 1$, $3 \leq \deg(\gamma_3(h)) \leq 4[(n-2)/3] + 4[(n-3)/3] + 3$, $4 \leq \deg(K(h)) \leq 12[(n-2)/3] + 4[(n-3)/3] + 8$, and $9 \leq \deg(N(h)) \leq 20[(n-2)/3] + 8[(n-3)/3] + 4[(n-1)/3] + 17$, using Lemma 4.3, we have

$$\begin{aligned} \sharp A(h) &\leq \left\lfloor \frac{n-2}{3} \right\rfloor + 0 + \left(\left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{n-3}{3} \right\rfloor \right) + \left(3 \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{n-3}{3} \right\rfloor + 1 \right) \\ &\quad + \left(5 \left\lfloor \frac{n-2}{3} \right\rfloor + 2 \left\lfloor \frac{n-3}{3} \right\rfloor + \left\lfloor \frac{n-1}{3} \right\rfloor + 2 \right) + 3 \\ &= 10 \left\lfloor \frac{n-2}{3} \right\rfloor + 4 \left\lfloor \frac{n-3}{3} \right\rfloor + \left\lfloor \frac{n-1}{3} \right\rfloor + 6. \end{aligned}$$

When $1 \leq n \leq 4$, since $1 \leq \deg(\alpha_1(h)) \leq 5$, $3 \leq \deg(\gamma_3(h)) \leq 7$, $4 \leq \deg(K(h)) \leq 16$, and $5 \leq \deg(N(h)) \leq 37$, using Lemma 4.3, we get

$$\sharp A(h) \leq 1 + 0 + 1 + 3 + 8 + 3 = 16.$$

When $n = 0$, from (2.7), since $\deg(\zeta(h)) = 0$, $I_{-1}(h) < 0$, we have

$$\sharp A(h) = 0.$$

□

5. Conclusion

For the quadratic reversible systems $r12$, $r16$ and $r18$ of genus one, when they are perturbed by arbitrary polynomial of degree n , we explore the new upper bounds on the number of zeros for their Abelian integrals using the methods of Riccati equation and Picard-Fuchs equation. We consider both the highest degree and the lowest degree of the related polynomials, and more importantly, we notice their law between the terms, and take into account the range of values for variable. Some laws of related polynomials of the systems have been found, therefore, we obtain some new upper bounds, which are better than those obtained before.

We could see that for $r18$, the related polynomials are $2k$ - and $(2k+1)$ -form, respectively, and the new upper bounds are about $1/2$ of the old ones; for $r12$, the related polynomials are $3k$ -, $(3k+1)$ -, and $(3k+2)$ -form, respectively, and the new upper bounds are about $1/3$ of the old ones; for $r16$, the related polynomials are $4k$ -, $(4k+1)$ -, $(4k+2)$ -, and $(4k+3)$ -form, respectively, and the new upper bounds are about $1/4$ of the old ones. Many systems have these laws, by using these laws, we can obtain some very good upper bounds.

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