

A HALF-DISCRETE MULTIDIMENSIONAL HILBERT-TYPE INEQUALITY WITH THE GENERAL NONHOMOGENEOUS KERNEL AND A LOWER LIMIT FUNCTION

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Abstract By means of the weight functions, the idea of introduced parameters and Hardy's integral inequality, a new extended half-discrete multidimensional Hilbert-type inequality with the general nonhomogeneous kernel as well as one lower limit function is obtained. The equivalent statements of the best value in the new inequality related to several parameters are considered. The equivalent forms, the operator expressions and some particular cases are deduced.

Keywords Weight function, lower limit function, parameter, Hardy's integral inequality, half-discrete multidimensional Hilbert-type inequality, nonhomogeneous kernel.

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1. Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [11], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (1.1)$$

If $f(x), g(y) \geq 0$, $0 < \int_0^{\infty} f^p(x) dx < \infty$ and $0 < \int_0^{\infty} g^q(y) dy < \infty$, then we have the integral analogues of (1.1) named in Hardy-Hilbert's integral inequality with the same best value as follows (cf. [11], Theorem 316):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (1.2)$$

In 2006, by using the Euler-Maclaurin summation formula, Krnić et al. [21] gave the following extension of (1.1):

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \\ & < B(\lambda_1, \lambda_2) \left(\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right)^{\frac{1}{q}}, \end{aligned} \quad (1.3)$$

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where, $\lambda_1, \lambda_2 \in (0, 2], \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, the constant factor $B(\lambda_1, \lambda_2)$ is still the best value, and

$$B(u, v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt, u, v > 0 \quad (1.4)$$

is the Beta function.

In 2016-2017, Hong et al. [14, 16] provided some equivalent statements of the extension of (1.1) (resp. (1.2)) with several parameters related to the best value. Some related results were considered by [9, 13, 15]. In 2019, by means of (1.3) and Abel's partial summation formula, Adiyasuren et al. [1] obtained an extended application of (1.3) involving two partial sums. In 2020, Mo et al. [27] gave an extension of (1.2) involving two upper limit functions. Inequalities (1.1)-(1.2) with their extensions played an important role in analysis and its applications [2, 4, 12, 19, 22, 29, 33-36, 38]. In 2023, Hong et al. [17] obtained a more accurate half-discrete multidimensional Hardy-Hilbert's inequality involving one derivative function of m -order. In 2026, Huang et.al and Peng et al. [20, 28] gave a few extended multidimensional Hardy-Hilbert's inequality involving one partial sum. Some other results were provided by [3, 8, 10, 18, 25, 30, 32, 37]. The above extended inequalities are all with the particular homogeneous kernels.

In the following, we'll consider to build an extended half-discrete multidimensional Hilbert-type inequality with the general nonhomogeneous kernel. The construction of robust weight functions and integral inequalities in this paper provides essential analytic tools that are highly relevant to the qualitative analysis of advanced dynamic systems; indeed, establishing sharp inequality bounds is deeply connected to investigating the existence, uniqueness, and oscillatory or asymptotic characteristics of solutions in nonlinear neutral, delay, and fractional differential equations [5-7, 26, 31].

In this article, by means of the weight functions, the idea of introduced parameters and the well known Hardy's integral inequality, a new extended half-discrete multidimensional Hilbert-type inequality with the general nonhomogeneous kernel and a multiple lower limit function is obtained. The equivalent statements of the best possible constant factor in the new inequality related to several parameters are provided. The equivalent forms, the operator expressions and some particular cases are deduced.

2. Some formulas and lemmas

In what follows, we suppose that

(C₁). $n \in \mathbf{N} := \{1, 2, \dots\}$, $m \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $p > 1$ ($q > 1$), $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in \mathbf{R}_+ = (0, \infty)$, $\beta \in (0, 1]$, $\sigma, \sigma_1 \in \mathbf{R}$, $\hat{\sigma} = \frac{\sigma_1}{p} + \frac{\sigma}{q}$, $f(x)$ is a nonnegative continuous in $\mathbf{R}_+$ except on finite number of points satisfying

$$0 < \int_0^\infty x^{p(1+m-\hat{\sigma})-1} f^p(x) dx < \infty,$$

and for $k = (k_1, \dots, k_n) \in \mathbf{N}^n$, $b_k \geq 0$, $\|k\|_\beta := (\sum_{i=1}^n k_i^\beta)^{\frac{1}{\beta}}$, such that

$$0 < \sum_k \|k\|_\beta^{q(n-\alpha\hat{\sigma})-n} b_k^q < \infty.$$

If $M > 0$, $\psi(u)$ ($u > 0$) is a nonnegative measurable function, then we have the following

transfer formula (cf. [36], Lemma 9.1.5):

$$\begin{aligned} & \int \cdots \int_{\{y \in \mathbf{R}_+^n; 0 < \sum_{i=1}^n (\frac{y_i}{M})^\beta \leq 1\}} \psi \left(\sum_{i=1}^n (\frac{y_i}{M})^\beta \right) dy_1 \cdots dy_n \\ &= \frac{M^n \Gamma^n(\frac{1}{\beta})}{\beta^n \Gamma(\frac{n}{\beta})} \int_0^1 \psi(u) u^{\frac{n}{\beta}-1} du. \end{aligned} \quad (2.1)$$

(i) In view of $\|y\|_\beta = M[\sum_{i=1}^n (\frac{y_i}{M})^\beta]^{\frac{1}{\beta}}$, by (2.1), we have

$$\begin{aligned} & \int_{\mathbf{R}_+^n} \Psi(\|y\|_\beta) dy \\ &= \lim_{M \rightarrow \infty} \int \cdots \int_{\{y \in \mathbf{R}_+^n; 0 < \sum_{i=1}^n (\frac{y_i}{M})^\beta \leq 1\}} \Psi(M[\sum_{i=1}^n (\frac{y_i}{M})^\beta]^{\frac{1}{\beta}}) dy_1 \cdots dy_n \\ &= \lim_{M \rightarrow \infty} \frac{M^n \Gamma^n(\frac{1}{\beta})}{\beta^n \Gamma(\frac{n}{\beta})} \int_0^1 \Psi(Mu^{\frac{1}{\beta}}) u^{\frac{n}{\beta}-1} du \\ &\stackrel{v=Mu^{\frac{1}{\beta}}}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1} \Gamma(\frac{n}{\beta})} \int_0^\infty \Psi(v) v^{n-1} dv. \end{aligned} \quad (2.2)$$

(ii) If $\Psi(Mu^{\frac{1}{\beta}}) = 0, u < (\frac{b}{M})^\beta$ ($b > 0$), i.e. $\|y\|_\beta < b$, then by (2.2), we have

$$\begin{aligned} \int_{\{y \in \mathbf{R}_+^n; \|y\|_\beta \geq b\}} \Psi(\|y\|_\beta) dy &= \lim_{M \rightarrow \infty} \frac{M^n \Gamma^n(\frac{1}{\beta})}{\beta^n \Gamma(\frac{n}{\beta})} \int_{(\frac{b}{M})^\beta}^1 \Psi(Mu^{\frac{1}{\beta}}) u^{\frac{n}{\beta}-1} du \\ &\stackrel{v=Mu^{\frac{1}{\beta}}}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1} \Gamma(\frac{n}{\beta})} \int_b^\infty \Psi(v) v^{n-1} dv. \end{aligned} \quad (2.3)$$

Lemma 2.1. *With regards to the assumption (C_1) , $c \in \mathbf{R}_+$, we have the following inequalities:*

$$\frac{\Gamma^n(\frac{1}{\beta})}{\alpha c \beta^{n-1} \Gamma(\frac{n}{\beta})} \prod_{i=1}^{n-1} \frac{i}{\alpha c + i} < \sum_k \|k\|_\beta^{-\alpha c - n} < \frac{2^{\alpha c} \Gamma^n(\frac{1}{\beta})}{\alpha c \beta^{n-1} \Gamma(\frac{n}{\beta})}, \quad (2.4)$$

where, we indicate that $\prod_{i=1}^0 \frac{i}{\alpha c + i} := 1$.

Proof. Since for $\alpha, c > 0, \beta \in (0, 1], (-1)^{i-1} \frac{\partial^i}{\partial y_j^i} \|y\|_\beta^{-\alpha c - n} < 0$ ($i = 1, 2; j = 1, \dots, n$), by Hermite-Hadamard's inequality (cf. [23]) and (2.3), for $y_i \geq \frac{1}{2}$ ($i = 1, \dots, n$), $\|y\|_\beta \geq \frac{1}{2} n^{\frac{1}{\beta}} \geq \frac{1}{2}$, it follows that

$$\begin{aligned} \sum_k \|k\|_\beta^{-\alpha c - n} &< \int_{\{y \in \mathbf{R}_+^n; y_i \geq \frac{1}{2}\}} \|y\|_\beta^{-\alpha c - n} dy \\ &\leq \int_{\{y \in \mathbf{R}_+^n; \|y\|_\beta \geq \frac{1}{2}\}} \|y\|_\beta^{-\alpha c - n} dy \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1} \Gamma(\frac{n}{\beta})} \int_{\frac{1}{2}}^\infty v^{-\alpha c - n} v^{n-1} dv \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \int_{\frac{1}{2}}^{\infty} v^{-\alpha c-1} dv \\
&= \frac{2^{\alpha c}\Gamma^n(\frac{1}{\beta})}{\alpha c\beta^{n-1}\Gamma(\frac{n}{\beta})}.
\end{aligned}$$

By the decreasing property of series and (2.2), we find

$$\begin{aligned}
\sum_k \|k\|_{\beta}^{-\alpha c-n} &> \int_{\{y \in \mathbf{R}_+^n; y_i \geq 1\}} \|y\|_{\beta}^{-\alpha c-n} dy \\
&= \int_{\mathbf{R}_+^n} \|y+1\|_{\beta}^{-\alpha c-n} dy \\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \int_0^{\infty} \frac{v^{n-1} dv}{(v+1)^{\alpha c+n}} \\
&= \frac{\Gamma^n(\frac{1}{\beta})B(\alpha c, n)}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \frac{\Gamma(\alpha c)\Gamma(n)}{\Gamma(\alpha c+n)} \\
&= \frac{1}{\alpha c} \prod_{i=1}^{n-1} \frac{i}{\alpha c+i}.
\end{aligned}$$

Hence, we have (2.4).

The lemma is proved. $\square$

Lemma 2.2. *With regards to the assumption (C_1) , if $h(t)t^{\sigma_1-\frac{n}{\alpha}}$ is nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$, with*

$$K(\eta) := \int_0^{\infty} h(t)t^{\eta-1} dt \in \mathbf{R}_+ \quad (\eta = \sigma, \sigma_1),$$

and we define the following weight functions:

$$\omega(\sigma, k) := \|k\|_{\beta}^{\alpha\sigma} \int_0^{\infty} h(x\|k\|_{\beta}^{\alpha}) x^{\sigma-1} dx \quad (k \in \mathbf{N}^n), \quad (2.5)$$

$$\varpi(\sigma_1, x) := x^{\sigma_1} \sum_k h(x\|k\|_{\beta}^{\alpha}) \|k\|_{\beta}^{\alpha\sigma_1-n} \quad (x \in \mathbf{R}_+), \quad (2.6)$$

then the following expression and inequality hold:

$$\omega(\sigma, k) = K(\sigma) \quad (k \in \mathbf{N}^n), \quad (2.7)$$

$$\varpi(\sigma_1, x) < \frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \quad (x \in \mathbf{R}_+). \quad (2.8)$$

Proof. In (2.5), for fixed k , setting $t = x\|k\|_{\beta}^{\alpha}$, we find

$$\omega(\sigma, k) = \|k\|_{\beta}^{\alpha\sigma} \int_0^{\infty} h(t) \left(\frac{t}{\|k\|_{\beta}^{\alpha}} \right)^{\sigma-1} \frac{1}{\|k\|_{\beta}^{\alpha}} dt = \int_0^{\infty} h(t)t^{\sigma-1} du = K(\sigma).$$

Then we have (2.7).

In (2.6), for $\varphi(v) = h(xv^\alpha)v^{\alpha\sigma_1-n}$, by the decreasing property of series and (2.2), we have

$$\begin{aligned}\varpi(\sigma_1, x) &= x^{\frac{n}{\alpha}} \sum_k h(x||k||_\beta^\alpha)(x||k||_\beta^\alpha)^{\sigma_1-\frac{n}{\alpha}} \\ &< x^{\frac{n}{\alpha}} \int_{\mathbf{R}_+^n} h(x||y||_\beta^\alpha)(x||y||_\beta^\alpha)^{\sigma_1-\frac{n}{\alpha}} \\ &= x^{\frac{n}{\alpha}} \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \int_0^\infty h(xv^\alpha)(xv^\alpha)^{\sigma_1-\frac{n}{\alpha}} v^{n-1} dv \\ &\stackrel{t=xv^\alpha}{=} \frac{\Gamma^n(\frac{1}{\beta}) \int_0^\infty h(t)t^{\sigma_1-1} dt}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \\ &= \frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})},\end{aligned}$$

and then we have (2.8).

This proves the lemma. $\square$

Lemma 2.3. *With regards to the assumption (C_1) , if $\sigma_1 - \sigma < p(1 - \sigma)$, setting $\widehat{F}_0(x) := f(x)$ ($x \in \mathbf{R}_+$), for $m \in \mathbf{N}, i = 1, \dots, m$, we define multiple lower limit functions as follows:*

$$\widehat{F}_i(x) := \int_x^\infty \left(\int_{t_i}^\infty \cdots \int_{t_2}^\infty f(t_1) dt_1 \cdots dt_{i-1} \right) dt_i \quad (x \in \mathbf{R}_+),$$

then for $m \in \mathbf{N}_0$, we have the following inequality:

$$\left[\int_0^\infty x^{p(1-\widehat{\sigma})-1} \widehat{F}_m^p(x) dx \right]^{\frac{1}{p}} \leq \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left[\int_0^\infty x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (2.9)$$

where, we denote $\prod_{i=1}^0 (i - a)^{-1} = 1$ ($a < 1$).

Proof. For $m = 0$, $\widehat{F}_0(x) = f(x)$, (2.9) keeps the form of an equality; for $m \in \mathbf{N}, i = 1, \dots, m$, by the following Hardy's integral inequality (cf. [11], Theorem 330):

$$\int_0^\infty x^{-r} \left(\int_x^\infty h(t) dt \right)^p dx \leq \left(\frac{p}{1-r} \right)^p \int_0^\infty x^{p-r} h^p(x) dx \quad (2.10)$$

($h(x) (\geq 0) \in L(\mathbf{R}_+)$, $r < 1$), for $\widehat{\sigma} = \frac{\sigma_1 - \sigma}{p} + \sigma < 1$ ($\sigma_1 - \sigma < p(1 - \sigma)$), setting $r = p(i - m - 1 + \widehat{\sigma}) + 1$ (< 1), and $h(x) = \widehat{F}_{i-1}(x)$, $\int_x^\infty h(t) dt = \widehat{F}_i(x)$, it follows that

$$\left[\int_0^\infty x^{p(1+m-i-\widehat{\sigma})-1} \widehat{F}_i^p(x) dx \right]^{\frac{1}{p}} \leq \frac{1}{m+1-i-\widehat{\sigma}} \left[\int_0^\infty x^{p(2+m-i-\widehat{\sigma})-1} \widehat{F}_{i-1}^p(x) dx \right]^{\frac{1}{p}}.$$

Substitution of $i = 1, \dots, m$ in the above inequality, by simplifications, we obtain

$$\left[\int_0^\infty x^{p(1-\widehat{\sigma})-1} \widehat{F}_m^p(x) dx \right]^{\frac{1}{p}} \leq \prod_{i=1}^m \frac{1}{m+1-i-\widehat{\sigma}} \left[\int_0^\infty x^{p(1+m-\widehat{\sigma})-1} \widehat{F}_0^p(x) dx \right]^{\frac{1}{p}}$$

$$= \prod_{i=1}^m \frac{1}{i - \hat{\sigma}} \left[\int_0^\infty x^{p(1+m-\hat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}},$$

namely, (2.9) holds.

This proves the lemma. $\square$

Lemma 2.4. *With regards to the assumptions of (C_1) and Lemma 2.2, for $m \in \mathbf{N}_0$, we have the following inequality:*

$$\begin{aligned} I &:= \sum_k \int_0^\infty h(x \|k\|_\beta^\alpha) \widehat{F}_m(x) b_k dx \\ &< \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \left[\int_0^\infty x^{p(1-\hat{\sigma})-1} \widehat{F}_m^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\hat{\sigma})-n} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (2.11)$$

Proof. By Hölder's inequality (cf. [23]), (2.5) and (2.6), we have

$$\begin{aligned} I &= \sum_k \int_0^\infty h(x \|k\|_\beta^\alpha) \left[\frac{x^{(1-\sigma)/q} \widehat{F}_m(x)}{\|k\|_\beta^{(n-\alpha\sigma_1)/p}} \right] \left[\frac{\|k\|_\beta^{(n-\alpha\sigma_1)/p} b_k}{x^{(1-\sigma)/q}} \right] dx \\ &\leq \left\{ \int_0^\infty \sum_k h(x \|k\|_\beta^\alpha) \frac{x^{(1-\sigma)(p-1)}}{\|k\|_\beta^{n-\alpha\sigma_1}} \widehat{F}_m^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_k \int_0^\infty h(x \|k\|_\beta^\alpha) \frac{\|k\|_\beta^{(n-\alpha\sigma_1)(q-1)}}{x^{1-\sigma}} b_k^q dx \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \left[x^{\sigma_1} \sum_k h(x \|k\|_\beta^\alpha) \|k\|_\beta^{\alpha\sigma_1-n} dy \right] x^{p(1-\hat{\sigma})-1} \widehat{F}_m^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_k \left[\|k\|_\beta^{\alpha\sigma_1} \int_0^\infty h(x \|k\|_\beta^\alpha) x^{\sigma-1} dx \right] \|k\|_\beta^{q(n-\alpha\hat{\sigma})-n} b_k^q \right\}^{\frac{1}{q}} \\ &= \left[\int_0^\infty \varpi(\sigma_1, x) x^{p(1-\hat{\sigma})-1} \widehat{F}_m^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \omega(\sigma, k) \|k\|_\beta^{q(n-\alpha\hat{\sigma})-n} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (2.12)$$

Since by (2.9) and $H := \int_0^\infty x^{p(1+m-\hat{\sigma})-1} f^p(x) dx < \infty$, it follows that $L := \int_0^\infty x^{p(1-\hat{\sigma})-1} \widehat{F}_m^p(x) dx < \infty$. If $L = 0$, then $\widehat{F}_m(x) = 0$ a.e. in $\mathbf{R}_+$, which means that $f(x) = 0$ a.e. in $\mathbf{R}_+$, and $H = 0$. It contradicts the fact that $H > 0$. Hence, we have $0 < L < \infty$. In view of (2.12), (2.7) and (2.8), we have (2.11).

This proves the lemma. $\square$

3. Main results

By (2.11) and (2.9), we have

Theorem 3.1. *With regards to the assumption (C_1) , if $h(t)t^{\sigma_1-\frac{n}{\alpha}}$ is nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$, with*

$$K(\eta) := \int_0^\infty h(t)t^{\eta-1} dt \in \mathbf{R}_+ \quad (\eta = \sigma, \sigma_1),$$

then for $\sigma_1 - \sigma < p(1 - \sigma)$, we have the following extended half-discrete multidimensional Hilbert-type inequality with the nonhomogeneous kernel and one multiple lower limit function:

$$\begin{aligned} I &= \sum_k \int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) b_k dx \\ &< \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \\ &\quad \times \left[\int_0^\infty x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\widehat{\sigma})-n} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (3.1)$$

In particular, for $\sigma_1 = \sigma$ (< 1), in view of the assumption (C_1) , we have

$$0 < \int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx < \infty, 0 < \sum_k \|k\|_\beta^{q(n-\alpha\sigma)-n} b_k^q < \infty,$$

and the following inequality

$$\begin{aligned} &\sum_k \int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) b_k dx \\ &< \prod_{i=1}^m \frac{1}{i - \sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma) \\ &\quad \times \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\sigma)-n} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (3.2)$$

Theorem 3.2. With regards to the assumption of Theorem 3.1, if $\sigma_1 = \sigma$ (< 1), there exists constants $b \in (\sigma, 1)$ and $M_b > 0$, such that

$$h(t) \leq M_b t^{-b} \quad (t \in (0, \infty)), \quad (3.3)$$

then the constant factor

$$\prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (3.1) is the best possible.

Proof. We need to prove that the constant factor in (3.2) is the best possible.

There exists $\delta_0 \in (0, 1 - \sigma)$, such that $b \in (\sigma + \delta_0, 1)$. For any $0 < \varepsilon < p\delta_0$ ($< p(1 - \sigma)$), we set

$$\begin{aligned} \widetilde{f}(x) &:= \begin{cases} \prod_{i=1}^m (i - \sigma - \frac{\varepsilon}{p}) x^{\sigma - m + \frac{\varepsilon}{p} - 1}, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \quad (m \in \mathbf{N}_0), \\ \widetilde{b}_k &:= \|k\|_\beta^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - n} \quad (k \in \mathbf{N}^n). \end{aligned}$$

Then we find $\tilde{F}_0(x) = \tilde{f}(x)$, and for $m \in \mathbf{N}$,

$$\begin{aligned}\tilde{F}_m(x) &= \int_x^\infty \left(\int_{t_m}^\infty \cdots \int_{t_2}^\infty \tilde{f}(t_1) dt_1 \cdots dt_{m-1} \right) dt_m \\ &= \begin{cases} \int_x^1 \left(\int_{t_m}^1 \cdots \int_{t_2}^1 t_1^{\sigma-m+\frac{\varepsilon}{p}-1} dt_1 \cdots dt_{m-1} \right) dt_m, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \\ &= \begin{cases} x^{\sigma+\frac{\varepsilon}{p}-1} - P_{m-1}(x), & 0 < x \leq 1, \\ 0, & x > 1, \end{cases}\end{aligned}$$

where, $P_{m-1}(x)$ is indicated as a nonnegative polynomial of $(m-1)$ -degree satisfying

$$P_{m-1}^{(i)}(1) = (x^{\sigma+\frac{\varepsilon}{p}-1})^{(i)}|_{x=1} \quad (i = 0, \dots, m-1).$$

We denote $P_{-1}(x) := 0$, the above expression satisfies for $m \in \mathbf{N}_0$.

Since $P_{m-1}(x) \rightarrow \text{Constant}$ ($x \rightarrow 0^+$), there exists a positive constant C , such that

$$P_{m-1}(x) \leq C \quad (x \in (0, 1]). \quad (3.4)$$

If there exists a positive constant M , with

$$M \leq \prod_{i=1}^m \frac{1}{i-\sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma) \quad (3.5)$$

such that (3.2) is valid when we replace the constant factor by M , then in particular, by (2.4) (for $c = \varepsilon$), we have

$$\begin{aligned}\tilde{I} &:= \sum_k \int_0^\infty h(x \|k\|_\beta^\alpha) \tilde{F}_m(x) \tilde{b}_k dx \\ &< M \left[\int_0^\infty x^{p(1+m-\sigma)-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\sigma)-n} \tilde{b}_k^q \right]^{\frac{1}{q}} \\ &= M \prod_{i=1}^m \left(i - \sigma - \frac{\varepsilon}{p} \right) \left(\int_0^1 x^{-1+\varepsilon} dx \right)^{\frac{1}{p}} \left(\sum_k \|k\|_\beta^{-\alpha\varepsilon-n} \right)^{\frac{1}{q}} \\ &< \frac{M}{\varepsilon} \prod_{i=1}^m \left(i - \sigma - \frac{\varepsilon}{p} \right) \left(\frac{2^{\alpha\varepsilon} \Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}.\end{aligned}$$

By (2.3) and Fubini theorem (cf. [24]), we have

$$\tilde{I} = \sum_k \|k\|_\beta^{\alpha(\sigma-\frac{\varepsilon}{q})-n} \int_0^1 h(x \|k\|_\beta^\alpha) (x^{\sigma+\frac{\varepsilon}{p}-1} - P_{m-1}(x)) dx = A - (A_1 + A_2),$$

where, we define

$$A := \sum_k \|k\|_\beta^{-\alpha\varepsilon-n} \left[\|k\|_\beta^{\alpha(\sigma+\frac{\varepsilon}{p})} \int_0^\infty h(x \|k\|_\beta^\alpha) x^{\sigma+\frac{\varepsilon}{p}-1} dx \right],$$

$$A_1 := \sum_k \|k\|_\beta^{\alpha(\sigma - \frac{\varepsilon}{q}) - n} \int_1^\infty h(x \|k\|_\beta^\alpha) x^{\sigma + \frac{\varepsilon}{p} - 1} dx,$$

$$A_2 := \sum_k \|k\|_\beta^{\alpha(\sigma - \frac{\varepsilon}{q}) - n} \int_0^1 h(x \|k\|_\beta^\alpha) P_{m-1}(x) dx.$$

By (2.7) and (2.4), we find

$$\begin{aligned} \varepsilon A &= \varepsilon \sum_k \|k\|_\beta^{-\alpha\varepsilon - n} \omega(\sigma + \frac{\varepsilon}{p}, k) \\ &= \varepsilon K(\sigma + \frac{\varepsilon}{p}) \sum_k \|k\|_\beta^{-\alpha\varepsilon - n} \\ &> \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} K(\sigma + \frac{\varepsilon}{p}) \prod_{i=1}^{n-1} \frac{i}{\alpha\varepsilon + i}. \end{aligned}$$

For $b > \sigma + \delta_0$, by the assumption and (2.4), we find

$$\begin{aligned} 0 &< A_1 \\ &\leq M_b \sum_k \|k\|_\beta^{\alpha(\sigma - \frac{\varepsilon}{q}) - n} \int_1^\infty (x \|k\|_\beta^\alpha)^{-b} x^{\sigma + \frac{\varepsilon}{p} - 1} dx \\ &\leq M_b \sum_k \|k\|_\beta^{-\alpha(b - \sigma) - n} \int_1^\infty x^{-b + \sigma + \delta_0 - 1} dx \\ &< \frac{M_b}{b - \sigma - \delta_0} \cdot \frac{2^{\alpha(b - \sigma)} \Gamma^n(\frac{1}{\beta})}{\alpha(b - \sigma) \beta^{n-1} \Gamma(\frac{n}{\beta})} \\ &\leq M_1 \\ &< \infty. \end{aligned}$$

In view of the assumption and (2.4), it follows that

$$\begin{aligned} 0 &\leq A_2 \\ &\leq CM_b \sum_k \|k\|_\beta^{\alpha\sigma - n} \int_0^1 (x \|k\|_\beta^\alpha)^{-b} dx \\ &= CM_b \sum_k \|k\|_\beta^{-\alpha(b - \sigma) - n} \frac{1}{1 - b} \\ &\leq \frac{CM_b}{1 - b} \cdot \frac{2^{\alpha(b - \sigma)} \Gamma^n(\frac{1}{\beta})}{\alpha(b - \sigma) \beta^{n-1} \Gamma(\frac{n}{\beta})} \\ &\leq M_2 \\ &< \infty. \end{aligned}$$

Based on the above results, we have the following inequality

$$\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma + \frac{\varepsilon}{p})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \prod_{i=1}^{n-1} \frac{i}{\alpha\varepsilon + i} - \varepsilon(A_1 + A_2) < \varepsilon \tilde{I} < M \prod_{i=1}^m (i - \sigma - \frac{\varepsilon}{p}) \left(\frac{2^{\alpha\varepsilon} \Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}.$$

For $\varepsilon \rightarrow 0^+$, in view of Fatou lemma (cf. [24]) and $\varepsilon(A_1 + A_2) \rightarrow 0$, we have

$$\frac{\Gamma^n(\frac{1}{\beta})K(\sigma)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \leq M \prod_{i=1}^m (i - \sigma) \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}},$$

namely,

$$\prod_{i=1}^m \frac{1}{i - \sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma) \leq M.$$

In view of (3.5), it follows that

$$M = \prod_{i=1}^m \frac{1}{i - \sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma)$$

is the best value in (3.2) (namely, for $\sigma_1 = \sigma$ in (3.1)).

This proves the theorem. $\square$

Theorem 3.3. *With regards to the assumption of Theorem 3.1, if $0 \leq \sigma_1 - \sigma < p(1 - \sigma)$, the same constant factor in (3.1) is the best possible, then we have $\sigma_1 = \sigma$.*

Proof. For $\sigma_1 - \sigma < p(1 - \sigma)$, we have $\hat{\sigma} = \frac{\sigma_1 - \sigma}{p} + \sigma < 1$; for $\hat{\sigma} = \frac{\sigma - \sigma_1}{q} + \sigma_1$, $\sigma - \sigma_1 \leq 0$, we observe that $h(t)t^{\hat{\sigma} - \frac{n}{\alpha}} = t^{\frac{\sigma - \sigma_1}{q}} (h(t)t^{\sigma_1 - \frac{n}{\alpha}})$ is also nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$.

By Hölder's inequality (cf. [23]), we obtain

$$\begin{aligned} 0 &< K(\hat{\sigma}) \\ &= \int_0^\infty h(u) u^{\frac{\sigma_1}{p} + \frac{\sigma}{q} - 1} du \\ &= \int_0^\infty h(u) (u^{\frac{\sigma_1 - 1}{p}}) (u^{\frac{\sigma - 1}{q}}) du \\ &\leq \left(\int_0^\infty h(u) u^{\sigma_1 - 1} du \right)^{\frac{1}{p}} \left(\int_0^\infty h(u) u^{\sigma - 1} du \right)^{\frac{1}{q}} \\ &= K^{\frac{1}{p}}(\sigma_1) K^{\frac{1}{q}}(\sigma) \\ &< \infty. \end{aligned} \tag{3.6}$$

Since the constant factor

$$\prod_{i=1}^m \frac{1}{i - \hat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (3.1) is the best possible. comparing with the constant factors in (3.1) and (3.2) (by substitution of $\sigma = \hat{\sigma}$), we have

$$\prod_{i=1}^m \frac{1}{i - \hat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \leq \prod_{i=1}^m \frac{1}{i - \hat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\hat{\sigma}),$$

namely, $K(\hat{\sigma}) \geq K^{\frac{1}{p}}(\sigma_1)K^{\frac{1}{q}}(\sigma)$. Hence, (3.6) keeps the form of an equality. The necessary and sufficient condition for taking an equal sign is that there exist constants A and B , such that they are not both zero, and (cf. [23]) $Au^{\sigma_1-1} = Bu^{\sigma-1}$ a.e. in $\mathbf{R}_+$. Assuming that $A \neq 0$, we have $u^{\sigma_1-\sigma} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$. It follows that $\sigma_1 - \sigma = 0$, and then $\sigma_1 = \sigma$.

This proves the theorem. $\square$

For $n = 1$ in Theorem 3.1–Theorem 3.3 with the assumptions, if $h(t)t^{\sigma_1-\frac{1}{\alpha}}$ is nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$, then we have

Corollary 3.1. *The following inequality holds:*

$$\begin{aligned} \sum_{k=1}^{\infty} \int_0^{\infty} h(xk^{\alpha}) \widehat{F}_m(x) b_k dx &< \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{1}{\alpha} K(\sigma_1) \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \\ &\times \left[\int_0^{\infty} x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} k^{q(1-\alpha\widehat{\sigma})-1} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (3.7)$$

In particular, for $\sigma_1 = \sigma$ (< 1), we have

$$0 < \int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx < \infty, 0 < \sum_{k=1}^{\infty} k^{q(1-\alpha\sigma)-1} b_k^q < \infty,$$

and the following inequality:

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_0^{\infty} h(xk^{\alpha}) \widehat{F}_m(x) b_k dx \\ &< \prod_{i=1}^m \frac{1}{i - \sigma} \frac{K(\sigma)}{\alpha^{1/p}} \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} k^{q(1-\alpha\sigma)-1} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (3.8)$$

Corollary 3.2. *If $\sigma_1 = \sigma$ (< 1), there exists constants $b \in (\sigma, 1)$ and $M_b > 0$, such that (3.3) holds, then the constant factor*

$$\prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{1}{\alpha} K(\sigma_1) \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (3.7) is the best possible. On the other hand, if $0 \leq \sigma_1 - \sigma < p(1 - \sigma)$, the same constant factor in (3.3) is the best possible, then we have $\sigma_1 = \sigma$.

4. Equivalent forms and operator expressions

Theorem 4.1. *With regards to the assumption of Theorem 3.1, we have the following inequality equivalent to (3.1):*

$$\begin{aligned} J &:= \left\{ \sum_k \|k\|_{\beta}^{p\alpha\widehat{\sigma}-n} \left[\int_0^{\infty} h(x\|k\|_{\beta}^{\alpha}) \widehat{F}_m(x) dx \right]^p \right\}^{\frac{1}{p}} \\ &< \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \left[\int_0^{\infty} x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (4.1)$$

In particular, for $\sigma_1 = \sigma$ (< 1), we obtain the following inequality equivalent to (3.2):

$$\begin{aligned} & \left\{ \sum_k \|k\|_\beta^{p\alpha\sigma-n} \left[\int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) dx \right]^p \right\}^{\frac{1}{p}} \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma) \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (4.2)$$

Proof. Suppose that (4.1) is value. By Hölder inequality (cf. [23]), we have

$$\begin{aligned} I &= \sum_k \left(\|k\|_\beta^{\alpha\widehat{\sigma}-\frac{n}{p}} \int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) dx \right) \left(\|k\|_\beta^{\frac{n}{p}-\alpha\widehat{\sigma}} b_k \right) \\ &\leq J \left[\sum_k \|k\|_\beta^{q(n-\alpha\widehat{\sigma})-n} b_k^q \right]^{\frac{1}{q}}. \end{aligned} \quad (4.3)$$

Then by (4.1), we obtain (3.1).

In contrast, suppose that (3.1) is valid. We set

$$b_k := \|k\|_\beta^{p\alpha\widehat{\sigma}-n} \left[\int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) dx \right]^{p-1}, \quad k \in \mathbb{N}^n.$$

If $J = 0$, then (4.1) is true; if $J = \infty$, then (4.1) is not true, implying that $J < \infty$. For $0 < J < \infty$, based on (3.1), we obtain

$$\begin{aligned} & \sum_k \|k\|_\beta^{q(n-\alpha\widehat{\sigma})-n} b_k^q = J^p = I \\ & < \prod_{i=1}^m \frac{1}{i-\widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \\ & \quad \times \left[\int_0^\infty x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\widehat{\sigma})-n} b_k^q \right]^{\frac{1}{q}}, \\ & \left[\sum_k \|k\|_\beta^{q(n-\alpha\widehat{\sigma})-n} b_k^q \right]^{\frac{1}{p}} = J \\ & < \prod_{i=1}^m \frac{1}{i-\widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} K(\sigma_1) \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \left[\int_0^\infty x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (4.4)$$

Thus, (4.1) is true, which is equivalent to (3.1).

The theorem is proved. $\square$

Theorem 4.2. With regards to the assumption of Theorem 3.2, if $\sigma_1 = \sigma$ (< 1), then the constant factor

$$\prod_{i=1}^m \frac{1}{i-\widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta})K(\sigma_1)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (4.1) is the best possible. On contrast, if $0 \leq \sigma_1 - \sigma < p(1 - \sigma)$, the same constant factor in (4.1) is the best possible, then we have $\sigma_1 = \sigma$.

Proof. According to Theorem 3.2, for $\sigma_1 = \sigma$, the constant factor

$$\prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (3.1) is the best possible. The same constant in (4.1) remains the best possible. Alternatively, based on (4.3) (for $\sigma_1 = \sigma$), it is a contradiction that the constant factor in (3.2) is not the best possible.

In addition, if the constant factor in (4.1) is the best possible, then using the equivalence between (4.1) and (3.1), and by considering $J^p = I$ (see (4.4)), the constant factor in (3.1) is the best possible. Based on the assumption and Theorem 3.3, we derive that $\sigma_1 = \sigma$.

The theorem is proved. $\square$

For $n = 1$ in Theorem 4.1-Theorem 4.2 with the assumptions, if $h(t)t^{\sigma_1 - \frac{1}{\alpha}}$ is nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$, then we have

Corollary 4.1. *We have the following inequality equivalent to (3.7):*

$$\left\{ \sum_{k=1}^{\infty} k^{p\alpha\widehat{\sigma}-1} \left[\int_0^{\infty} h(xk^{\alpha}) \widehat{F}_m(x) dx \right]^p \right\}^{\frac{1}{p}} < \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{1}{\alpha} K(\sigma_1) \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \left[\int_0^{\infty} x^{p(1+m-\widehat{\sigma})-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (4.5)$$

In particular, for $\sigma_1 = \sigma$ (< 1), we have the following inequality equivalent to (3.8):

$$\left\{ \sum_{k=1}^{\infty} k^{p\alpha\sigma-1} \left[\int_0^{\infty} h(xk^{\alpha}) \widehat{F}_m(x) dx \right]^p \right\}^{\frac{1}{p}} < \prod_{i=1}^m \frac{1}{i - \sigma} \frac{K(\sigma)}{\alpha^{1/p}} \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (4.6)$$

Corollary 4.2. *If $\sigma_1 = \sigma$ (< 1), there exists constants $b \in (\sigma, 1)$ and $M_b > 0$, such that (3.3) holds, then the constant factor*

$$\prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{1}{\alpha} K(\sigma_1) \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (4.5) is the best possible. On the other hand, if $0 \leq \sigma_1 - \sigma < p(1 - \sigma)$, the same constant factor in (4.5) is the best possible, then we have $\sigma_1 = \sigma$.

Setting functions $\phi_i(x) := x^{p(1+i-\widehat{\sigma})-1}$ ($i = 0, m$), $\psi(k) := \|k\|_{\beta}^{q(n-\alpha\widehat{\sigma})-n}$, wherefrom,

$$\psi^{1-p}(k) = \|k\|_{\beta}^{p\alpha\widehat{\sigma}-n} \quad (x \in \mathbf{R}_+, k \in \mathbf{N}^n),$$

we define the real normed linear spaces as follows:

$$L_{p,\phi_i}(\mathbf{R}_+) := \left\{ f = f(x); \|f\|_{p,\phi_i} := \left(\int_0^{\infty} \phi_i(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_{k_1, \dots, k_n}\}; \|b\|_{q,\psi} := \left(\sum_k \psi(k) |b_k|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_{k_1, \dots, k_n}\}; \|c\|_{q,\psi} := \left(\sum_k \psi^{1-p}(k) |c_k|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$\widehat{L}(\mathbf{R}_+) := \{\widehat{F}_m = \widehat{F}_m(x) \in L_{p,\phi_0}(\mathbf{R}_+); f(x) (\geq 0) \text{ is continuous unless at finite points in } \mathbf{R}_+, f \in L_{p,\phi_m}(\mathbf{R}_+)\}.$$

For any $\widehat{F}_m \in \widehat{L}(\mathbf{R}_+)$, setting $c_k := \int_0^\infty h(x) \|k\|_\beta^\alpha \widehat{F}_m(x) dx, k \in \mathbf{N}^n$, (4.1) is rewritten as follows:

$$\|c\|_{p,\psi^{1-p}} \leq \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \|f\|_{p,\phi_m} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 4.1. Define an extended half-discrete multidimensional Hilbert-type operator $T : \widehat{L}(\mathbf{R}_+) \rightarrow l_{p,\psi^{1-p}}$ as follows: For any $\widehat{F}_m \in \widehat{L}(\mathbf{R}_+)$, there exists a unique representation $c = T\widehat{F}_m \in l_{p,\psi^{1-p}}$, such that for any $k \in \mathbf{N}^n$, $(T\widehat{F}_m)(k) = c_k$. Define the formal inner product of $T\widehat{F}_m$ and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(T\widehat{F}_m, b) := \sum_k b_k \int_0^\infty K_\lambda(x, \|k\|_\beta^\alpha) \widehat{F}_m(x) dx (= I),$$

$$\|T\| := \sup_{\widehat{F}_m (\neq 0) \in \widehat{L}(\mathbf{R}_+)} \frac{\|T\widehat{F}_m\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi_m}}.$$

By Theorem 4.1-Theorem 4.2, we have

Theorem 4.3. With regards to the assumption of Theorem 3.2, if $\widehat{F}_m(\geq 0) \in \widehat{L}(\mathbf{R}_+)$, $b(\geq 0) \in l_{q,\psi}$, $\|b\|_{q,\psi} > 0$, $\|f\|_{p,\phi_m} > 0$, then the following equivalent operator inequalities hold:

$$(T\widehat{F}_m, b) < \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \|f\|_{p,\phi_m} \|b\|_{q,\psi}, \quad (4.7)$$

$$\|T\widehat{F}_m\|_{p,\psi^{1-p}} < \prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma) \|f\|_{p,\phi_m}. \quad (4.8)$$

Furthermore, if $\sigma_1 = \sigma (< 1)$, there exists constants $b \in (\sigma, 1)$ and $M_b > 0$, such that (3.3) holds, then the constant factor

$$\prod_{i=1}^m \frac{1}{i - \widehat{\sigma}} \left(\frac{\Gamma^n(\frac{1}{\beta}) K(\sigma_1)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K^{\frac{1}{q}}(\sigma)$$

in (4.7) and (4.8) is the best possible, namely,

$$\|T\| = \prod_{i=1}^m \frac{1}{i - \sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} K(\sigma).$$

In contrast, if $0 \leq \sigma_1 - \sigma < p(1 - \sigma)$, the same constant factor in (4.7) or (4.8) is the best possible, then we have $\sigma_1 = \sigma$.

5. Some examples

Example 5.1. Some particular kernels are provided in the following:

(i) Setting $h(xy) = \frac{1}{(xy+1)^s}$ ($0 < \eta < s$), we find

$$K(\eta) = \int_0^\infty \frac{t^{\eta-1}}{(t+1)^s} dt = B(\eta, s - \eta) \in \mathbf{R}_+.$$

(ii) Setting $h(xy) = \frac{\ln(xy)}{(xy)^{s-1}}$ ($0 < \eta < s$), we find

$$K(\eta) = \int_0^\infty \frac{t^{\eta-1} \ln t}{t^s - 1} dt = \left[\frac{\pi}{s \sin(\pi\eta/s)} \right]^2 \in \mathbf{R}_+.$$

(iii) Setting $h(xy) = \frac{1}{(\max\{xy, 1\})^s}$ ($0 < \eta < s$), we find

$$\begin{aligned} K(\eta) &= \int_0^\infty \frac{t^{\eta-1}}{(\max\{t, 1\})^s} dt \\ &= \int_0^1 t^{\eta-1} dt + \int_1^\infty \frac{t^{\eta-1}}{t^s} dt \\ &= \frac{s}{\eta(s - \eta)} \\ &\in \mathbf{R}_+. \end{aligned}$$

Remark 5.1. For any $h(t)$ in Example 5.1 (i)-(iii), we can find that for any $b \in (0, s)$, $h(t)t^b \rightarrow 0$ ($t \rightarrow 0^+$ or $t \rightarrow \infty$). Since $h(t)$ ($t \in (0, \infty)$) is continuous, it follows that there exists a constant $M_b > 0$, such that $h(t)t^b \leq M_b$ ($t \in (0, \infty)$), namely, (3.3) holds. In Example 5.1 (i)-(iii), setting $s = \lambda > 0, \eta = \sigma_1, \sigma \in (0, \lambda)$, for $0 < \sigma < \min\{1, \lambda\}$, there exists a positive constant $b \in (\sigma, 1)$, satisfying (3.3), and for $\sigma_1 \leq \frac{n}{\alpha}$, $h(t)t^{\sigma_1 - \frac{n}{\alpha}}$ is nonnegative decreasing in $\mathbf{R}_+$ and strictly decreasing in an interval $I \subset \mathbf{R}_+$.

Example 5.2. By using Example 5.1 (i)-(iii) with Remark 5.1 to (3.2) and (4.2), in view of the assumption (C₁), $\sigma \in (0, \min\{1, \lambda\}) \cap (0, \frac{n}{\alpha}]$, we obtain the following three couples equivalent inequalities with the best possible constant factors:

$$\begin{aligned} & \sum_k \int_0^\infty \frac{\widehat{F}_m(x) b_k dx}{(1 + x \|k\|_\beta^\alpha)^\lambda} \\ & < \prod_{i=1}^m \frac{1}{i - \sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} \\ & \times B(\sigma, \lambda - \sigma) \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k \|k\|_\beta^{q(n-\alpha\sigma)-n} b_k^q \right]^{\frac{1}{q}}, \quad (5.1) \\ & \left\{ \sum_k \|k\|_\beta^{p\alpha\sigma-n} \left[\int_0^\infty \frac{\widehat{F}_m(x) dx}{(1 + x \|k\|_\beta^\alpha)^\lambda} \right]^p \right\}^{\frac{1}{p}} \end{aligned}$$

$$< \prod_{i=1}^m \frac{1}{i-\sigma} \times \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\sigma, \lambda - \sigma) \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (5.2)$$

$$\begin{aligned} & \sum_k \int_0^\infty \frac{\ln(x||k||_\beta^\alpha) \widehat{F}_m(x) b_k}{x^\lambda ||k||_\beta^{\alpha\lambda} - 1} dx \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \\ & \quad \times \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k ||k||_\beta^{q(n-\alpha\sigma)-n} b_k^q \right]^{\frac{1}{q}}, \end{aligned} \quad (5.3)$$

$$\begin{aligned} & \left\{ \sum_k ||k||_\beta^{p\alpha\sigma-n} \left[\int_0^\infty \frac{\ln(x||k||_\beta^\alpha) \widehat{F}_m(x) dx}{x^\lambda ||k||_\beta^{\alpha\lambda} - 1} \right]^p \right\}^{\frac{1}{p}} \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \times \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} & \sum_k \int_0^\infty \frac{\widehat{F}_m(x) b_k dx}{(\max\{1, x||k||_\beta^\alpha\})^\lambda} \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} \\ & \quad \times \frac{\lambda}{\sigma(\lambda - \sigma)} \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_k ||k||_\beta^{q(n-\alpha\sigma)-n} b_k^q \right]^{\frac{1}{q}}, \end{aligned} \quad (5.5)$$

$$\begin{aligned} & \left\{ \sum_k ||k||_\beta^{p\alpha\sigma-n} \left[\int_0^\infty \frac{\widehat{F}_m(x) dx}{(\max\{1, x||k||_\beta^\alpha\})^\lambda} \right]^p \right\}^{\frac{1}{p}} \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} \frac{\lambda}{\sigma(\lambda - \sigma)} \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (5.6)$$

Example 5.3. For $n = \alpha = 1$ in Example 5.2, $\sigma \in (0, \min\{1, \lambda\}) \cap (0, 1]$, we have the following three couples equivalent inequalities with the best possible constant factors:

$$\begin{aligned} & \sum_{k=1}^\infty \int_0^\infty \frac{\widehat{F}_m(x) b_k}{(1+xk)^\lambda} dx \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} B(\sigma, \lambda - \sigma) \times \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{k=1}^\infty k^{q(1-\sigma)-1} b_k^q \right]^{\frac{1}{q}}, \end{aligned} \quad (5.7)$$

$$\begin{aligned} & \left\{ \sum_{k=1}^\infty k^{p\sigma-1} \left[\int_0^\infty \frac{\widehat{F}_m(x) dx}{(1+xk)^\lambda} \right]^p \right\}^{\frac{1}{p}} \\ & < \prod_{i=1}^m \frac{1}{i-\sigma} \times B(\sigma, \lambda - \sigma) \left[\int_0^\infty x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \end{aligned} \quad (5.8)$$

$$\sum_{k=1}^{\infty} \int_0^{\infty} \frac{\ln(xk)}{x^{\lambda} k^{\lambda} - 1} \widehat{F}_m(x) b_k dx$$

$$< \prod_{i=1}^m \frac{1}{i - \sigma} \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \times \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} k^{q(1-\sigma)-1} b_k^q \right]^{\frac{1}{q}}, \quad (5.9)$$

$$\left\{ \sum_{k=1}^{\infty} k^{p\sigma-1} \left[\int_0^{\infty} \frac{\ln(xk) \widehat{F}_m(x) dx}{x^{\lambda} k^{\lambda} - 1} \right]^p \right\}^{\frac{1}{p}}$$

$$< \prod_{i=1}^m \frac{1}{i - \sigma} \times \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (5.10)$$

$$\sum_{k=1}^{\infty} \int_0^{\infty} \frac{\widehat{F}_m(x) b_k}{(\max\{1, xk\})^{\lambda}} dx$$

$$< \prod_{i=1}^m \frac{1}{i - \sigma} \frac{\lambda}{\sigma(\lambda - \sigma)} \times \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} k^{q(1-\sigma)-1} b_k^q \right]^{\frac{1}{q}}, \quad (5.11)$$

$$\left\{ \sum_{k=1}^{\infty} k^{p\sigma-1} \left[\int_0^{\infty} \frac{\widehat{F}_m(x) dx}{(\max\{1, xk\})^{\lambda}} \right]^p \right\}^{\frac{1}{p}}$$

$$< \prod_{i=1}^m \frac{1}{i - \sigma} \frac{\lambda}{\sigma(\lambda - \sigma)} \left[\int_0^{\infty} x^{p(1+m-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (5.12)$$

6. Conclusion

In this paper, by means of the weight functions, the idea of introduced parameters and Hardy's integral inequality, an extended half-discrete multidimensional Hilbert-type inequality with the general nonhomogeneous kernel as $h(x||k||_{\beta}^{\alpha})$ ($\alpha > 0, \beta \in (0, 1]$) and a multiple lower limit function is obtained in Theorem 3.1. The equivalent statements of the best value in the new inequality related to several parameters are considered in Theorem 3.2-Theorem 3.3. The equivalent forms and the operator expressions are provided in Theorem 4.1-Theorem 4.3. Some particular cases are deduced in Corollary 3.1-Corollary 3.2, Corollary 4.1-Corollary 4.2, and Example 5.2-Example 5.3.

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