

HANKEL DETERMINANTS OF INVERSE FUNCTIONS AND INVERSE LOGARITHMIC COEFFICIENTS FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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Abstract For some subclasses of Starlike with respect to symmetric points, the sharp upper bounds of the third-order Hankel determinant of inverse functions and Hankel determinants of $\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)$ and $\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)$ with logarithmic inverse coefficients are obtained.

Keywords Analytic functions, univalent functions, Hankel determinants, starlike functions with respect to symmetric points, logarithmic coefficients, sigmoid functions, three-leaf-shaped region, inverse functions.

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1. Introduction

Let $\Delta = \{\epsilon : |\epsilon| < 1\}$ be the open unit disk in the complex plane. Let $\mathcal{H}$ be the class of all analytic functions $\Gamma(\epsilon)$ defined in the open unit disc Δ with the power series representation as

$$\Gamma(\epsilon) = \epsilon + \sum_{m=2}^{\infty} c_m \epsilon^m \quad (\epsilon \in \Delta). \quad (1.1)$$

Further, let $\mathcal{S}$ be the subclass of $\mathcal{H}$ consisting of all univalent functions in Δ .

Let $\Gamma \in \mathcal{H}$ with expansion of the form (1.1), then q th Hankel determinant of Γ is defined for $q \geq 1$, and $m \geq 1$ by

$$\mathcal{H}_{q,m}(\Gamma) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+q-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+q} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+q-1} & c_{m+q} & \cdots & c_{m+2q-2} \end{vmatrix}.$$

The upper bound of the modulus of the Hankel determinants has been extensively studied in recent time, mainly the second order case $\mathcal{H}_{2,2} = c_2 c_4 - c_3^2$ and the third order case

$$\mathcal{H}_{3,1}(\Gamma) = 2c_4 c_3 c_2 - c_3^3 + c_5 c_3 - c_5 c_2^2 - c_4^2. \quad (1.2)$$

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This problem, as most others over the class of univalent functions, is difficult to tackle with for the general class, and instead its subclasses are studied. The sharp bounds of the third order Hankel determinant $\mathcal{H}_{3,1}$ of analytic functions were achieved by many authors [1, 4–9, 11–13, 16].

For $\Gamma \in \mathcal{S}$, has an inverse Γ^{-1} given by

$$\begin{aligned} G(w) &= \Gamma^{-1}(w) \\ &= w + \sum_{m=2}^{\infty} A_m w^m \quad |w| < r_0(\Gamma); \quad (r_0(\Gamma) \geq \frac{1}{4}) \\ &= w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2 c_3 + c_4)w^4 \\ &\quad + (14c_2^4 + 3c_3^2 - 21c_2^2 c_3 + 6c_2 c_4 - c_5)w^5 + \cdots \end{aligned} \quad (1.3)$$

From (1.2) and (1.3), we have

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) &= 2A_4 A_3 A_2 - A_3^3 + A_5 A_3 - A_5 A_2^2 - A_4^2 \\ &= -3c_2^4 c_3 + c_2^6 + 3c_2^2 c_3^2 + 2c_2 c_3 c_4 - 2c_3^3 - c_4^2 - c_2^2 c_5 + c_3 c_5. \end{aligned} \quad (1.4)$$

The logarithmic coefficients μ_m of $\Gamma^{-1} \in \mathcal{S}$, denoted by $\mu_m = \mu_m(\Gamma^{-1})$, are described as

$$\log \frac{\Gamma^{-1}(w)}{w} = 2 \sum_{m=1}^{\infty} \mu_m w^m \quad (|w| < \frac{1}{4}). \quad (1.5)$$

Differentiating and using (1.3), we yield

$$\begin{cases} \mu_1 = -\frac{1}{2}c_2, \\ \mu_2 = -\frac{1}{2}(c_3 - \frac{3}{2}c_2^2), \\ \mu_3 = -\frac{1}{2}(c_4 - 4c_2 c_3 + \frac{10}{3}c_2^3), \\ \mu_4 = -\frac{1}{2}(c_5 - 5c_4 c_2 + 15c_3 c_2^2 - \frac{5}{2}c_3^2 - \frac{35}{4}c_2^4). \end{cases} \quad (1.6)$$

Therefore,

$$\mu_3 \mu_1 - \mu_2^2 = \mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2) = \frac{1}{48}(12c_2 c_4 + 13c_2^4 - 12c_2^2 c_3 - 12c_3^2), \quad (1.7)$$

$$\begin{aligned} \mu_2 \mu_4 - \mu_3^2 &= \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) \\ &= \frac{1}{288}(-72c_4^2 + 72c_3 c_5 + 216c_2 c_3 c_4 + 198c_2^2 c_3^2 - 180c_3^3 \\ &\quad - 330c_2^4 c_3 - 108c_2^2 c_5 + 60c_2^3 c_4 + 145c_2^6). \end{aligned} \quad (1.8)$$

The class of starlike functions with respect to symmetric points is introduced by Sakaguchi [10] and is denoted as $\mathcal{SS}^*$. These functions satisfy the analytic condition

$$\mathcal{SS}^* = \left\{ g : g \in \mathcal{S} \quad \text{and} \quad \frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} \prec \frac{1+\epsilon}{1-\epsilon} \quad (\epsilon \in \Delta) \right\}. \quad (1.9)$$

In 2023, Faisal and co-authors [3] introduced a family $\mathcal{SS}_{SG}^*$ of symmetric starlike functions connected with the sigmoid function as follows:

$$\mathcal{SS}_{SG}^* = \left\{ g : g \in \mathcal{S} \quad \text{and} \quad \frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} \prec \frac{2}{1 + e^{-\epsilon}} \quad (\epsilon \in \Delta) \right\}. \quad (1.10)$$

Tang et al. [15] introduced and studied the class $\mathcal{SS}_{3l}^*$ of starlike functions with respect to the symmetric points connected with the three-leaf-shaped region, which is given by

$$\mathcal{SS}_{3l}^* = \left\{ g : g \in \mathcal{S} \quad \text{and} \quad \frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} \prec 1 + \frac{4}{5}\epsilon + \frac{1}{5}\epsilon^4 \quad (\epsilon \in \Delta) \right\}. \quad (1.11)$$

Faisal et al. [3] calculated the sharp bounds of the logarithmic coefficients and some of their combinations (initial coefficients, the Hankel determinants, Zalcman's functional, etc.) for the class $\mathcal{SS}_{SG}^*$. Tang et al. [15] studied the bounds of the Krushkal and Zalcman inequalities, as well as the third-order Hankel determinant for the class $\mathcal{SS}_{3l}^*$. Tang et al. [14] obtained the bounds of the Hankel determinants of inverse functions for the classes $\mathcal{SS}_{SG}^*$ and $\mathcal{SS}_{3l}^*$, and proved that $|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq 0.0317$ for $\Gamma \in \mathcal{SS}_{SG}^*$, and $|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq 0.11600$ for $\Gamma \in \mathcal{SS}_{3l}^*$. In this paper, we found sharp bounds of these classes, also, we obtain the sharp bounds of the second order Hankel determinant of logarithmic inverse functions for these classes.

Lemma 1.1 (Lemma 1.1 [2]). *Let*

$$\varpi(\epsilon) = e_1\epsilon + e_2\epsilon^2 + e_3\epsilon^3 + e_4\epsilon^4 + \dots \quad (1.12)$$

be a Schwarz function. Then

$$e_2 = (1 - |e_1|^2)\nu, \quad (1.13)$$

$$e_3 = \bar{e}_1\nu^2(1 - |e_1|^2) + (1 - |e_1|^2)(1 - |\nu|^2)\tau, \quad (1.14)$$

$$e_4 = \bar{e}_1^2\nu^3(1 - |e_1|^2) - (1 - |e_1|^2)(1 - |\nu|^2)(\bar{\nu}\tau^2 + 2\tau\bar{e}_1\nu) \\ + (1 - |e_1|^2)(1 - |\nu|^2)(1 - |\tau|^2)s, \quad (1.15)$$

for some ν, τ, s with $|\nu| \leq 1$, $|\tau| \leq 1$ and $|s| \leq 1$.

2. The third order Hankel determinants of inverse functions

Theorem 2.1. *If $\Gamma \in \mathcal{SS}_{SG}^*$, then*

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{3}{128}.$$

The bound is sharp.

Proof. Let $\Gamma \in \mathcal{SS}_{SG}^*$. There exists a Schwarz function ϖ , such that

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = \frac{2}{1 + e^{-\varpi(\epsilon)}}. \quad (2.1)$$

From (1.1), we have

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = 1 + 2c_2\epsilon + 2c_3\epsilon^2 + (-2c_2c_3 + 4c_4)\epsilon^3 + (-2c_3^2 + 4c_5)\epsilon^4 + \dots \quad (2.2)$$

Using (1.12) together with $\frac{2}{1+e^{-\varpi(\epsilon)}}$, we have

$$\frac{2}{1+e^{-\varpi(\epsilon)}} = 1 + \frac{1}{2}e_1\epsilon + \frac{1}{2}e_2\epsilon^2 + \left(-\frac{1}{24}e_1^3 + \frac{1}{2}e_3\right)\epsilon^3 + \left(-\frac{1}{8}e_2e_1^2 + \frac{1}{2}e_4\right)\epsilon^4 + \cdots \quad (2.3)$$

Equating the coefficients of (2.2) and (2.3), we have

$$c_2 = \frac{1}{4}e_1, \quad (2.4)$$

$$c_3 = \frac{1}{4}e_2, \quad (2.5)$$

$$c_4 = -\frac{1}{96}e_1^3 + \frac{1}{32}e_1e_2 + \frac{1}{8}e_3, \quad (2.6)$$

$$c_5 = \frac{1}{8}e_4 - \frac{1}{32}e_1^2e_2 + \frac{1}{32}e_2^2. \quad (2.7)$$

Using (2.4) – (2.7) and (1.4), we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) = & \frac{1}{36864}(-576e_3^2 + 288e_1e_2e_3 + 96e_1^3e_3 + 5e_1^6 - 864e_2^3 + 1152e_2e_4 - 60e_1^4e_2 \\ & + 180e_1^2e_2^2 - 288e_1^2e_4). \end{aligned} \quad (2.8)$$

From Lemma 1.1 and (2.8), we yield

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) = & \frac{1}{36864} [5e_1^6 + 576\bar{e}_1^2(1 - |e_1|^2)^2\nu^4 - 1152\nu\bar{\nu}(1 - |\nu|^2)(1 - |e_1|^2)^2\tau^2 - 288(1 - |e_1|^2)^2 \\ & \times e_1\bar{e}_1\nu^3 - 1152\tau(1 - |e_1|^2)^2(1 - |\nu|^2)\bar{e}_1\nu^2 + 288\nu e_1(1 - |\nu|^2)(1 - |e_1|^2)^2\tau + 1152\nu \\ & \times (1 - |\nu|^2)(1 - |e_1|^2)^2(1 - |\tau|^2)s + 180e_1^2(1 - |e_1|^2)^2\nu^2 - 864(1 - |e_1|^2)^3\nu^3 \\ & - 60(1 - |e_1|^2)e_1^4\nu - 288(1 - |e_1|^2)e_1^2\bar{e}_1^2\nu^3 - 96(1 - |e_1|^2)e_1^3\bar{e}_1\nu^2 - 576\tau^2(1 - |e_1|^2)^2 \\ & \times (1 - |\nu|^2)^2 + 96(1 - |e_1|^2)(1 - |\nu|^2)e_1^3\tau - 288(1 - |e_1|^2)(1 - |\tau|^2)(1 - |\nu|^2)e_1^2s \\ & + 288(1 - |e_1|^2)(1 - |\nu|^2)e_1^2\bar{\nu}\tau^2 + 576\nu(1 - |e_1|^2)(1 - |\nu|^2)e_1^2\bar{e}_1\tau]. \end{aligned}$$

Therefore, we achieve

$$\mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{36864} [\delta_1(e_1, \nu) + \delta_2(e_1, \nu)\tau + \delta_3(e_1, \nu)\tau^2 + \Upsilon_1(e_1, \nu, \tau)s],$$

where $e_1, \nu, \tau \in \bar{\Delta}$ and

$$\begin{aligned} \delta_1(e_1, \nu) = & 5e_1^6 + (1 - |e_1|^2) [(1 - |e_1|^2) (-864 - 576|e_1|^2)\nu^3 + 576\bar{e}_1^2\nu^4 + 180e_1^2\nu^2] - 60e_1^4\nu \\ & - 288e_1^2\bar{e}_1^2\nu^3 - 96e_1^3\bar{e}_1\nu^2, \end{aligned}$$

$$\delta_2(e_1, \nu) = (1 - |e_1|^2) (1 - |\nu|^2) [(288e_1\nu - 1152\bar{e}_1\nu^2) (1 - |e_1|^2) + 96e_1^3 + 576e_1^2\bar{e}_1\nu],$$

$$\delta_3(e_1, \nu) = (1 - |e_1|^2) (1 - |\nu|^2) [(-576|\nu|^2 - 576) (1 - |e_1|^2) + 288\bar{\nu}e_1^2],$$

$$\Upsilon_1(e_1, \nu, \tau) = (1 - |e_1|^2) (1 - |\nu|^2) (1 - |\tau|^2) [1152\nu(1 - |e_1|^2) - 288e_1^2].$$

Setting $|e_1| = e$, $|\nu| = x$, $|\tau| = t$ and utilizing $|s| \leq 1$, we achieve

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{1}{36864} (|\delta_1(e_1, \nu)| + |\delta_2(e_1, \nu)|t + |\delta_3(e_1, \nu)|t^2 + |\Upsilon_1(e_1, \nu, \tau)|)$$

$$\leq \frac{1}{36864} F_1(e, x, t), \quad (2.9)$$

where

$$F_1(e, x, t) = \varsigma_1(e, x) + \varsigma_2(e, x)t + \varsigma_3(e, x)t^2 + \varsigma_4(e, x)(1 - t^2),$$

with

$$\begin{aligned} \varsigma_1(e, x) &= 5e^6 + (1 - e^2) [(864x^3 - 576e^2x^3 + 576e^2x^4 + 180e^2x^2)(1 - e^2) + 60e^4x \\ &\quad + 96e^4x^2 + 288e^4x^3], \\ \varsigma_2(e, x) &= (1 - e^2)(1 - x^2) [(288ex + 1152ex^2)(1 - e^2) + 96e^3 + 576e^3x], \\ \varsigma_3(e, x) &= (1 - e^2)(1 - x^2) [(576 + 576x^2)(1 - e^2) + 288e^2x], \\ \varsigma_4(e, x) &= (1 - x^2)(1 - e^2) [1152x(1 - e^2) + 288e^2]. \end{aligned}$$

We will maximize $F_1(e, x, t)$ in the closed cuboid $\Xi : [0, 1] \times [0, 1] \times [0, 1]$.

By noting that

$$\begin{aligned} \varsigma_3(c, d) &\leq (1 - e^2)(1 - x^2) [(576x^2 + 576)(1 - e^2) + 288e^2] \\ &= \xi_1(e, x). \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_1(e, x, t) &\leq \varsigma_1(e, x) + \varsigma_2(e, x)t + \xi_1(e, x)t^2 + \varsigma_4(e, x)(1 - t^2) \\ &= \psi_1(e, x, t). \end{aligned} \quad (2.10)$$

Differentiating $\psi_1(e, x, t)$ with respect to t , we have

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= \varsigma_2(e, x) + 2t[\xi_1(e, x) - \varsigma_4(e, x)] \\ &= \varsigma_2(e, x) + 1152t(1 - e^2)^2(1 - x^2)(1 - x)^2 \\ &\geq 0. \end{aligned}$$

So, $\psi_1(e, x, t)$ attains its maximum value at $t = 1$. Thus, we yield

$$\begin{aligned} &F_1(e, x, t) \\ &\leq \psi_1(e, x, t) \\ &\leq \psi_1(e, x, 1) \\ &= \varsigma_1(e, x) + \varsigma_2(e, x) + \xi_1(e, x) \\ &= 5e^6 - 96e^5 + 288e^4 + 96e^3 - 864e^2 + 576 + (-60e^6 - 288e^5 + 60e^4 + 288e)x + (84e^6 \\ &\quad + 1248e^5 + 24e^4 - 2400e^3 - 108e^2 + 1152e)x^2 + (-864e^6 + 288e^5 + 2304e^4 - 2304e^2 \\ &\quad - 288e + 864)x^3 + (576e^6 - 1152e^5 - 1728e^4 + 2304e^3 + 1728e^2 - 1152e - 576)x^4 \\ &= \hbar_1(e, x). \end{aligned}$$

The critical points of $\hbar_1(e, x)$ satisfy the system of equations

$$\left\{ \begin{array}{l} \frac{\partial \hbar_1}{\partial e} = 30e^5 - 480e^4 + 1152e^3 + 288e^2 - 1728e + (-360e^5 - 1440e^4 + 240e^3 + 288)x \\ \quad + (504e^5 + 6240e^4 + 96e^3 - 7200e^2 - 216e + 1152)x^2 + (-5184e^5 + 1440e^4 + 9216e^3 \\ \quad - 4608e - 288)x^3 + (3456e^5 - 5760e^4 - 6912e^3 + 6912e^2 + 3456e - 1152)x^4 \\ \quad = 0, \\ \frac{\partial \hbar_1}{\partial x} = 288e + 60e^4 - 288e^5 - 60e^6 + (168e^6 + 2496e^5 + 48e^4 - 4800e^3 - 216e^2 + 2304e)x \\ \quad + (-2592e^6 + 864e^5 + 6912e^4 - 6912e^2 - 864e + 2592)x^2 + (2304e^6 - 4608e^5 \\ \quad - 6912e^4 + 9216e^3 + 6912e^2 - 4608e - 2304)x^3 \\ \quad = 0. \end{array} \right.$$

By calculating, we get

$$\left\{ \begin{array}{l} e_1 \approx 1, \\ x_1 \approx 1.3466, \end{array} \right\} \left\{ \begin{array}{l} e_2 \approx 1, \\ x_2 \approx -1.8253, \end{array} \right\} \left\{ \begin{array}{l} e_3 \approx 1, \\ x_3 \approx -0.5213, \end{array} \right\} \left\{ \begin{array}{l} e_4 = 0, \\ x_4 = 0, \end{array} \right\}$$

$$\left\{ \begin{array}{l} e_5 \approx 0.7136, \\ x_5 \approx 1.3036, \end{array} \right\} \left\{ \begin{array}{l} e_6 \approx 1.3868, \\ x_6 \approx -2.1248, \end{array} \right\} \left\{ \begin{array}{l} e_7 \approx 3.7363, \\ x_7 \approx 2.1503, \end{array} \right\} \left\{ \begin{array}{l} e_8 \approx -0.0024, \\ x_8 \approx -0.0151, \end{array} \right\}$$

$$\left\{ \begin{array}{l} e_9 \approx -0.6715, \\ x_9 \approx -11.5286, \end{array} \right\} \left\{ \begin{array}{l} e_{10} \approx -1.9860, \\ x_{10} \approx 0.4792. \end{array} \right\}$$

Thus, there are no critical points in $(0, 1) \times (0, 1)$.

(1) For $e = 1$,

$$\hbar_1(1, x) = 5.$$

(2) For $e = 0$,

$$\hbar_1(0, x) = 576 + 864x^3 - 576x^4 \leq 576.$$

(3) For $x = 0$,

$$\hbar_1(e, 0) = 576 - 864e^2 + 96e^3 + 288e^4 - 96e^5 + 5e^6 \leq 576.$$

(4) For $x = 1$,

$$\hbar_1(e, 1) = 864 - 1548e^2 + 948e^4 - 259e^6 \leq 864.$$

Therefore, we have

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{864}{36864} = \frac{3}{128}.$$

Equality will be achieved by using (2.1) together with

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = \frac{2}{1 + \epsilon^2} = 1 - \frac{1}{2}\epsilon^2 + \frac{1}{24}\epsilon^6 + \dots. \quad (2.11)$$

□

Theorem 2.2. *If $\Gamma \in \mathcal{SS}_{3l}^*$, then*

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{12}{125} = 0.096.$$

The bound is sharp.

Proof. Let $g \in \mathcal{SS}_{3l}^*$. There exists a Schwarz function ϖ such that

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = 1 + \frac{4}{5}\varpi(\epsilon) + \frac{1}{5}(\varpi(\epsilon))^4. \quad (2.12)$$

Utilizing (1.1), we have

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = 1 + 2c_2\epsilon + 2c_3\epsilon^2 + (4c_4 - 2c_2c_3)\epsilon^3 + (4c_5 - 2c_3^2)\epsilon^4 + \dots \quad (2.13)$$

Using the series expansion of (1.12), we get

$$1 + \frac{4}{5}\varpi(\epsilon) + \frac{1}{5}(\varpi(\epsilon))^4 = 1 + \frac{4}{5}e_1\epsilon + \frac{4}{5}e_2\epsilon^2 + \frac{4}{5}e_3\epsilon^3 + \left(\frac{4}{5}e_4 + \frac{1}{5}e_1^4\right)\epsilon^4 + \dots \quad (2.14)$$

By comparing (2.13) and (2.14), we have

$$c_2 = \frac{2}{5}e_1, \quad (2.15)$$

$$c_3 = \frac{2}{5}e_2, \quad (2.16)$$

$$c_4 = \frac{2}{25}e_1e_2 + \frac{1}{5}e_3, \quad (2.17)$$

$$c_5 = \frac{1}{5}e_4 + \frac{2}{25}e_2^2 + \frac{1}{20}e_1^4. \quad (2.18)$$

Using (2.15) – (2.18) and (1.4), we have

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{31250} & (-1250e_3^2 + 1000e_1e_2e_3 - 122e_1^6 - 3000e_2^3 - 335e_1^4e_2 + 2600e_1^2e_2^2 \\ & - 1000e_1^2e_4 + 2500e_2e_4). \end{aligned} \quad (2.19)$$

From Lemma 1.1 and (2.19), we obtain

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{31250} & \left[-122e_1^6 + 1250(1 - |e_1|^2)^2\bar{e}_1^2\nu^4 - 2500(1 - |\nu|^2)(1 - |e_1|^2)^2\nu\bar{\nu}\tau^2 \right. \\ & - 2500(1 - |e_1|^2)^2(1 - |\nu|^2)\bar{e}_1\nu^2\tau + 2500(1 - |\nu|^2)(1 - |e_1|^2)^2(1 - |\tau|^2)\nu s \\ & - 1000(1 - |e_1|^2)^2e_1\bar{e}_1\nu^3 + 2600e_1^2(1 - |e_1|^2)^2\nu^2 + 1000e_1(1 - |e_1|^2)^2 \\ & \times (1 - |\nu|^2)\nu\tau - 3000(1 - |e_1|^2)^3\nu^3 - 335e_1^4(1 - |e_1|^2)\nu - 1000\nu^3(1 - |e_1|^2)e_1^2\bar{e}_1^2 \\ & + 1000(1 - |\nu|^2)(1 - |e_1|^2)e_1^2\bar{\nu}\tau^2 + 2000\nu(1 - |e_1|^2)(1 - |\nu|^2)e_1^2\bar{e}_1\tau \\ & \left. - 1000e_1^2(1 - |e_1|^2)(1 - |\nu|^2)(1 - |\tau|^2)s - 1250\tau^2(1 - |e_1|^2)^2(1 - |\nu|^2)^2 \right]. \end{aligned}$$

Thus, we obtain

$$\mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{31250} [\delta_4(e_1, \nu) + \delta_5(e_1, \nu)\tau + \delta_6(e_1, \nu)\tau^2 + \Upsilon_2(e_1, \nu, \tau)s],$$

where

$$\begin{aligned}\delta_4(e_1, \nu) &= -122e_1^6 + (1 - |e_1|^2)[(-3000\nu^3 + 2000|e_1|^2\nu^3 + 2600e_1^2\nu^2 + 1250\bar{e}_1^2\nu^4)(1 - |e_1|^2) \\ &\quad - 335e_1^4\nu - 1000\bar{e}_1^2e_1^2\nu^3], \\ \delta_5(e_1, \nu) &= (1 - |\nu|^2)(1 - |e_1|^2)[(-2500\bar{e}_1\nu^2 + 1000e_1\nu)(1 - |e_1|^2) + 2000e_1^2\bar{e}_1\nu], \\ \delta_6(e_1, \nu) &= (1 - |e_1|^2)(1 - |\nu|^2)[(-1250 - 1250|\nu|^2)(1 - |e_1|^2) + 1000c_1^2\nu], \\ \Upsilon_2(e_1, \nu, \tau) &= (1 - |e_1|^2)(1 - |\tau|^2)(1 - |\nu|^2)[2500\nu(1 - |e_1|^2) - 1000e_1^2].\end{aligned}$$

Setting $e = |e_1|$, $x = |\nu|$, $t = |\tau|$ and upon using $|s| \leq 1$, we obtain

$$\begin{aligned}|\mathcal{H}_{3,1}(\Gamma^{-1})| &\leq \frac{1}{31250} (|\delta_4(e_1, \nu)| + |\delta_5(e_1, \nu)|t + |\delta_6(e_1, \nu)|t^2 + |\Upsilon_2(e_1, \nu, \tau)|) \\ &\leq \frac{1}{31250} F_2(e, x, t),\end{aligned}$$

where

$$F_2(e, x, t) = \varsigma_5(e, x) + \varsigma_6(e, x)t + \varsigma_7(e, x)t^2 + \varsigma_8(e, x)(1 - t^2),$$

with

$$\begin{aligned}\varsigma_5(e, x) &= 122e^6 + (1 - e^2)[(3000x^3 - 2000e^2x^3 + 2600e^2x^2 + 1250e^2x^4)(1 - e^2) \\ &\quad + 335xe^4 + 1000x^3e^4], \\ \varsigma_6(e, x) &= (1 - e^2)(1 - x^2)[(2500ex^2 + 1000ex)(1 - e^2) + 2000e^3x], \\ \varsigma_7(e, x) &= (1 - x^2)(1 - e^2)[(1250 + 1250x^2)(1 - e^2) + 1000e^2x], \\ \varsigma_8(e, x) &= (1 - e^2)(1 - x^2)[1000e^2 + 2500x(1 - e^2)].\end{aligned}$$

Note that

$$\varsigma_7(e, x) \leq (1 - e^2)(1 - x^2)[(1 - e^2)(1250x^2 + 1250) + 1000e^2] = \xi_2(e, x),$$

thus, we get

$$F_2(e, x, t) \leq \varsigma_5(e, x) + \varsigma_6(e, x)t + \xi_2(e, x)t^2 + \varsigma_8(e, x)(1 - t^2) = \psi_2(e, x, t).$$

Differentiating $\psi_2(e, x, t)$ with respect to t , we yield

$$\frac{\partial \psi_2}{\partial t} = \varsigma_6(e, x) + 2t[\xi_2(e, x) - \varsigma_8(e, x)].$$

Upon $\varsigma_6(e, x) \geq 0$ and

$$\xi_2(e, x) - \varsigma_8(e, x) = 1250(1 - x^2)(1 - e^2)^2(x - 1)^2 \geq 0,$$

we have $\frac{\partial \psi_2}{\partial t} \geq 0$. Therefore, $\psi_2(e, x, t)$ attains its maximum value at $t = 1$. Thus, we achieve

$$\begin{aligned}F_2(e, x, t) &\leq \psi_2(e, x, t) \\ &\leq \psi_2(e, x, 1) \\ &= \varsigma_5(e, x) + \varsigma_6(e, x) + \xi_2(e, x) \\ &= 122e^6 + 250e^4 - 1500e^2 + 1250 + (-335e^6 - 1000e^5 + 335e^4 + 1000e)x\end{aligned}$$

$$\begin{aligned}
& +(2600e^6 + 2500e^5 - 4200e^4 - 5000e^3 + 1600e^2 + 2500e)x^2 + (-3000e^6 + 1000e^5 \\
& + 8000e^4 - 8000e^2 - 1000e + 3000)x^3 + (1250e^6 - 2500e^5 - 3750e^4 + 5000e^3 \\
& + 3750e^2 - 2500e - 1250)x^4 \\
& = \hbar_2(e, x).
\end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \hbar_2}{\partial e} = 732e^5 + 1000e^3 - 3000e + (-2010e^5 - 5000e^4 + 1340e^3 + 1000)x + (15600e^5 + 12500e^4 \\ - 16800e^3 - 15000e^2 + 3200e + 2500)x^2 + (-18000e^5 + 5000e^4 + 32000e^3 - 16000e \\ - 1000)x^3 + (7500e^5 - 12500e^4 - 15000e^3 + 15000e^2 + 7500e - 2500)x^4 \\ = 0, \\ \frac{\partial \hbar_2}{\partial x} = 1000e + 335e^4 - 1000e^5 - 335e^6 + (5200e^6 + 5000e^5 - 8400e^4 - 10000e^3 + 3200e^2 \\ + 5000e)x + (-9000e^6 + 3000e^5 + 24000e^4 - 24000e^2 - 3000e + 9000)x^2 \\ + (5000e^6 - 10000e^5 - 15000e^4 + 20000e^3 + 15000e^2 - 10000e - 5000)x^3 \\ = 0. \end{array} \right.$$

By calculating, we yield

$$\begin{array}{llll}
\left\{ \begin{array}{l} e_1 \approx 1, \\ x_1 \approx 1.2576, \end{array} \right. & \left\{ \begin{array}{l} e_2 \approx 1, \\ x_2 \approx -2.0063, \end{array} \right. & \left\{ \begin{array}{l} e_3 \approx 1, \\ x_3 \approx -0.2513, \end{array} \right. & \left\{ \begin{array}{l} e_4 = 0, \\ x_4 = 0, \end{array} \right. \\
\left\{ \begin{array}{l} e_5 \approx 0.6412, \\ x_5 \approx 1.6980, \end{array} \right. & \left\{ \begin{array}{l} e_6 \approx 1.4100, \\ x_6 \approx -3.2088, \end{array} \right. & \left\{ \begin{array}{l} e_7 \approx -0.0089, \\ x_7 \approx -0.0289, \end{array} \right. & \left\{ \begin{array}{l} e_8 \approx -0.6707, \\ x_8 \approx -18.3099. \end{array} \right.
\end{array}$$

Therefore, there are no critical points in $(0, 1) \times (0, 1)$.

(5) For $e = 1$,

$$\hbar_2(1, x) = 122.$$

(6) For $e = 0$,

$$\hbar_2(0, x) = 1250 + 3000x^3 - 1250x^4 \leq 3000.$$

(7) For $x = 0$,

$$\hbar_2(e, 0) = 1250 - 1500e^2 + 250e^4 + 122e^6 \leq 1250.$$

(8) For $x = 1$,

$$\hbar_2(e, 1) = 3000 - 4150e^2 + 635e^4 + 637e^6 \leq 3000.$$

Thus, we have

$$|\mathcal{H}_{3,1}(g)| \leq \frac{3000}{31250} = \frac{12}{125}.$$

Equality will be obtained by using (2.12) together with

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = 1 + \frac{4}{5}\epsilon^2 + \frac{1}{5}\epsilon^8. \quad (2.20)$$

□

Theorem 2.3. *If $\Gamma \in \mathcal{SS}^*$, then*

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq 1.$$

The bound is sharp.

Proof. For $\Gamma \in \mathcal{SS}^*$, there exists a Schwarz function ϖ such that

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = \frac{1 + \varpi(\epsilon)}{1 - \varpi(\epsilon)} \Leftrightarrow 2\epsilon g'(\epsilon)(1 - \varpi(\epsilon)) = (g(\epsilon) - g(-\epsilon))(1 + \varpi(\epsilon)). \quad (2.21)$$

Using the series representation for g and w in (2.21), a simple calculation gives

$$c_2 = e_1, \quad (2.22)$$

$$c_3 = e_1^2 + e_2, \quad (2.23)$$

$$c_4 = \frac{1}{2}(2e_1^3 + 3e_1e_2 + e_3), \quad (2.24)$$

$$c_5 = \frac{1}{2}(e_4 + 5e_1^2e_2 + 2e_2^2 + 2e_1^4 + e_1e_3). \quad (2.25)$$

Using (2.22) – (2.25) and (1.4), we yield

$$\mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{64}(-16e_3^2 - 64e_2^3 + 32e_2e_4 + 16e_1^2e_2^2). \quad (2.26)$$

From (2.26) and Lemma 1.1, we have

$$\begin{aligned} \mathcal{H}_{3,1}(\Gamma^{-1}) = & \frac{1}{36864} [16\bar{e}_1^2\nu^4(1 - |e_1|^2)^2 - 32\bar{e}_1(1 - |\nu|^2)\nu^2(1 - |e_1|^2)^2\tau - 64\nu^3(1 - |e_1|^2)^3 \\ & + 16e_1^2\nu^2(1 - |e_1|^2)^2 - 32(1 - |\nu|^2)(1 - |e_1|^2)^2\nu\bar{\nu}\tau^2 - 16\tau^2(1 - |\nu|^2)^2(1 - |e_1|^2)^2 \\ & + 32s\nu(1 - |\tau|^2)(1 - |e_1|^2)^2(1 - |\nu|^2)]. \end{aligned}$$

So, we have

$$\mathcal{H}_{3,1}(\Gamma^{-1}) = \frac{1}{64} [\delta_7(e_1, \nu) + \delta_8(e_1, \nu)\tau + \delta_9(e_1, \nu)\tau^2 + \Upsilon_3(e_1, \nu, \tau)s],$$

where

$$\begin{aligned} \delta_7(e_1, \nu) &= (1 - |e_1|^2)^2 [16e_1^2\nu^2 - 64(1 - |e_1|^2)\nu^3 + 16\bar{e}_1^2\nu^4], \\ \delta_8(e_1, \nu) &= -32(1 - |e_1|^2)^2(1 - |\nu|^2)\bar{e}_1\nu^2, \\ \delta_9(e_1, \nu) &= -16(1 - |e_1|^2)^2(1 - |\nu|^4), \\ \Upsilon_3(e_1, \nu, \tau) &= 32(1 - |e_1|^2)^2(1 - |\tau|^2)(1 - |\nu|^2)\nu. \end{aligned}$$

Setting $|\tau| = t$, $|\nu| = x$, $|e_1| = e$, and utilizing $|s| \leq 1$, we have

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{1}{64} (|\delta_7(e_1, \nu)| + |\delta_8(e_1, \nu)|t + |\delta_9(e_1, \nu)|t^2 + |\Upsilon_3(e_1, \nu, \tau)|)$$

$$\leq \frac{1}{64} F_3(e, x, t), \quad (2.27)$$

where

$$F_3(e, x, t) = \varsigma_9(e, x) + \varsigma_{10}(e, x)t + \varsigma_{11}(e, x)t^2 + \varsigma_{12}(e, x)(1 - t^2),$$

with

$$\begin{aligned} \varsigma_9(e, x) &= (1 - e^2)^2 [16e^2x^2 + 64(1 - e^2)x^3 + 16e^2x^4], \\ \varsigma_{10}(e, x) &= 32(1 - e^2)^2(1 - x^2)ex^2, \\ \varsigma_{11}(e, x) &= 16(1 - e^2)^2(1 - x^4), \\ \varsigma_{12}(e, x) &= 32x(1 - x^2)(1 - e^2)^2. \end{aligned}$$

Differentiating $F_3(e, x, t)$ with respect to t , we achieve

$$\begin{aligned} \frac{\partial F_3}{\partial t} &= \varsigma_{10}(e, x) + 2t[\varsigma_{11}(e, x) - \varsigma_{12}(e, x)] \\ &= \varsigma_{10}(e, x) + 32(1 - e^2)^2(1 - x^2)(x^2 - 2x + 1)t \\ &\geq 0. \end{aligned}$$

Thus, $F_3(e, x, t)$ attains its maximum value at $t = 1$. Thus, we obtain

$$\begin{aligned} F_3(e, x, t) &\leq F_3(e, x, 1) \\ &= \varsigma_9(e, x) + \varsigma_{10}(e, x) + \varsigma_{11}(e, x) \\ &= 16e^4 - 32e^2 + 16 + (16e^6 + 32e^5 - 32e^4 - 64e^3 + 16e^2 + 32e)x^2 + (-64e^6 \\ &\quad + 192e^4 - 192e^2 + 64)x^3 + (16e^6 - 32e^5 - 48e^4 + 64e^3 + 48e^2 - 32e - 16)x^4 \\ &= \hbar_3(e, x). \end{aligned}$$

The critical points of $\hbar_3(e, x)$ satisfy the system of equations

$$\left\{ \begin{aligned} \frac{\partial \hbar_3}{\partial e} &= 64e^3 - 64e + (96e^5 + 160e^4 - 128e^3 - 192e^2 + 32e + 32)x^2 + (-384e^5 \\ &\quad + 768e^3 - 384e)x^3 + (96e^5 - 160e^4 - 192e^3 + 192e^2 + 96e - 32)x^4 \\ &= 0, \\ \frac{\partial \hbar_3}{\partial x} &= (32e^6 + 64e^5 - 64e^4 - 128e^3 + 32e^2 + 64e)x + (-192e^6 + 576e^4 - 576e^2 + 192)x^2 \\ &\quad + (64e^6 - 128e^5 - 192e^4 + 256e^3 + 192e^2 - 128e - 64)x^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we yield

$$\left\{ \begin{aligned} e_1 &= 0, & \left\{ \begin{aligned} e_2 &\approx 1, \\ x_2 &\approx 0, \end{aligned} \right. & \left\{ \begin{aligned} e_3 &\approx -1, \\ x_3 &\approx x_3, \end{aligned} \right. & \left\{ \begin{aligned} e_4 &\approx -1.2901, \\ x_4 &\approx x_4, \end{aligned} \right. & \left\{ \begin{aligned} e_5 &\approx 3.4401, \\ x_5 &\approx 0.7924, \end{aligned} \right. & \left\{ \begin{aligned} e_6 &\approx -0.8686, \\ x_6 &\approx -0.8718. \end{aligned} \right. \end{aligned}$$

Therefore, there are no critical points in $(0, 1) \times (0, 1)$.

(9) For $e = 1$,

$$\hbar_3(1, x) = 0.$$

(10) For $e = 0$,

$$\hbar_3(0, x) = 16 + 64x^3 - 16x^4 \leq 64.$$

(11) For $x = 0$,

$$\hbar_3(e, 0) = 16 - 32e^2 + 16e^4 \leq 16.$$

(12) For $x = 1$,

$$\hbar_3(e, 1) = 64 - 160e^2 + 128e^4 - 32e^6 \leq 64.$$

Thus, we achieve

$$|\mathcal{H}_{3,1}(\Gamma^{-1})| \leq \frac{64}{64} = 1.$$

Equality will be obtained by using (2.21) together with

$$\frac{2\epsilon g'(\epsilon)}{g(\epsilon) - g(-\epsilon)} = \frac{1 + \epsilon^2}{1 - \epsilon^2} = 1 + 2\epsilon^2 + 2\epsilon^4 + \dots \quad (2.28)$$

□

3. Hankel determinants of logarithmic coefficients of inverse functions

Theorem 3.1. *If $\Gamma \in \mathcal{SS}_{SG}^*$, then*

$$|\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)| \leq \frac{1}{64}.$$

The result is sharp for the function given by (2.11).

Proof. Let $\Gamma \in \mathcal{SS}_{SG}^*$. From (2.4) – (2.6) and (1.7), we have

$$H_{2,1}(G_{\Gamma^{-1}}/2) = \frac{1}{12288}(96e_1e_3 - 24e_1^2e_2 - 192e_2^2 + 5e_1^4). \quad (3.1)$$

From (3.1) and Lemma 1.1, we obtain

$$\begin{aligned} \mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2) = \frac{1}{12288} [& 96(1 - |e_1|^2)e_1\bar{e}_1\nu^2 + 96e_1(1 - |e_1|^2)(1 - |\nu|^2)\tau - 24e_1^2(1 - |e_1|^2)\nu \\ & + 5e_1^4 - 192(1 - |e_1|^2)^2\nu^2]. \end{aligned}$$

By putting $e = |e_1|, |\nu| = x$ and upon utilizing $|\tau| \leq 1$, we achieve

$$|H_{2,1}(G_{\Gamma^{-1}}/2)| \leq \frac{1}{12288} [96(1 - e^2)e^2x^2 + 96e(1 - e^2)(1 - x^2) + 24e^2(1 - e^2)x + 5e^4]$$

$$+192(1-e^2)^2x^2] \\ = \hbar_4(e, x).$$

Differentiating $\hbar_4(e, x)$ with respect to x , we obtain

$$\frac{\partial \hbar_4}{\partial x} = \frac{1}{512}[e^2(1-e^2) + 8x(1-e^2)(1-e)(2+e)] \geq 0.$$

Therefore, $\hbar_4(e, x)$ attains its maximum value at $x = 1$. Thus, we achieve

$$|H_{2,1}(G_{\Gamma^{-1}}/2)| \leq \hbar_4(e, 1) = \frac{1}{12288}[192 - 264e^2 + 77e^4] \leq \frac{192}{12288} = \frac{1}{64}.$$

□

Theorem 3.2. *If $\Gamma \in \mathcal{SS}_{SG}^*$, then*

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{1}{128}.$$

The outcome is sharp for the function given by (2.11).

Proof. Let $g \in \mathcal{SS}_{SG}^*$. By using (2.4) – (2.7) and (1.8), we obtain

$$\begin{aligned} \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) &= \frac{1}{1179648}(-4608e_3^2 + 4608e_3e_2e_1 + 1248e_1^3e_3 + 73e_1^6 - 9216e_2^3 - 720e_1^4e_2 \\ &\quad + 1440e_1^2e_2^2 - 3456e_1^2e_4 + 9216e_2e_4). \end{aligned} \quad (3.2)$$

From (3.2) and Lemma 1.1, we have

$$\begin{aligned} \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) &= \frac{1}{1179648} \left[73e_1^6 + 4608\bar{e}_1^2\nu^4(1-|e_1|^2)^2 - 9216\tau^2\nu\bar{\nu}(1-|\nu|^2)(1-|e_1|^2)^2 \right. \\ &\quad - 9216\tau\bar{e}_1\nu^2(1-|\nu|^2)(1-|e_1|^2)^2 + 9216s\nu(1-|\nu|^2)(1-|e_1|^2)^2(1-|\tau|^2) \\ &\quad - 4608e_1\bar{e}_1\nu^3(1-|e_1|^2)^2 + 1440\nu^2e_1^2(1-|e_1|^2)^2 + 4608\tau\nu e_1(1-|e_1|^2)^2(1-|\nu|^2) \\ &\quad - 9216\nu^3(1-|e_1|^2)^3 - 720e_1^4\nu(1-|e_1|^2) - 3456e_1^2\bar{e}_1^2\nu^3(1-|e_1|^2) + 3456\tau^2e_1^2\bar{\nu} \\ &\quad \times (1-|\nu|^2)(1-|e_1|^2) + 6912\tau e_1^2\bar{e}_1\nu(1-|e_1|^2)(1-|\nu|^2) - 3456se_1^2(1-|\tau|^2) \\ &\quad \times (1-|e_1|^2)(1-|\nu|^2) - 1248\nu^2e_1^3\bar{e}_1(1-|e_1|^2) - 4608\tau^2(1-|\nu|^2)^2(1-|e_1|^2)^2 \\ &\quad \left. + 1248\tau e_1^3(1-|e_1|^2)(1-|\nu|^2) \right]. \end{aligned}$$

Thus, we have

$$\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) = \frac{1}{1179648} [\delta_{10}(e_1, \nu) + \delta_{11}(e_1, \nu)\tau + \delta_{12}(e_1, \nu)\tau^2 + \Upsilon_4(e_1, \nu, \tau)s],$$

where

$$\begin{aligned} \delta_{10}(e_1, \nu) &= 73e_1^6 + (1-|e_1|^2)[(4608\bar{e}_1^2\nu^4 + 4608|e_1|^2\nu^3 + 1440e_1^2\nu^2 - 9216\nu^3)(1-|e_1|^2) \\ &\quad - 720e_1^4\nu - 3456e_1^2\bar{e}_1^2\nu^3 - 1248\bar{e}_1e_1^3\nu^2], \\ \delta_{11}(e_1, \nu) &= (1-|\nu|^2)(1-|e_1|^2)[(-9216\nu^2\bar{e}_1 + 4608\nu e_1)(1-|e_1|^2) + 6912\nu e_1^2\bar{e}_1 + 1248e_1^3], \\ \delta_{12}(e_1, \nu) &= (1-|e_1|^2)(1-|\nu|^2)[3456\bar{\nu}e_1^2 + (1-|e_1|^2)(-4608 - 4608|\nu|^2)], \end{aligned}$$

$$\Upsilon_4(e_1, \nu, \tau) = (1 - |\nu|^2)(1 - |\tau|^2)(1 - |e_1|^2)[-3456e_1^2 + 9216\nu(1 - |e_1|^2)].$$

Setting $t = |\tau|$, $x = |\nu|$, $e = |e_1|$ and upon using $|s| \leq 1$, we yield

$$\begin{aligned} |\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| &\leq \frac{1}{1179648} (|\delta_{10}(e_1, \nu)| + |\delta_{11}(e_1, \nu)|t + |\delta_{12}(e_1, \nu)|t^2 + |\Upsilon_4(e_1, \nu, \tau)|) \\ &\leq \frac{1}{1179648} F_4(e, x, t), \end{aligned}$$

where

$$F_4(e, x, t) = \varsigma_{13}(e, x) + \varsigma_{14}(e, x)t + \varsigma_{15}(e, x)t^2 + \varsigma_{16}(e, x)(1 - t^2),$$

with

$$\begin{aligned} \varsigma_{13}(e, x) &= 73e^6 + (1 - e^2)[(4608e^2x^4 + 4608e^2x^3 + 1440e^2x^2 + 9216x^3)(1 - e^2) + 720xe^4 \\ &\quad + 3456x^3e^4 + 1248e^4x^2], \\ \varsigma_{14}(e, x) &= (1 - x^2)(1 - e^2)[(9216ex^2 + 4608ex)(1 - e^2) + 6912e^3x + 1248e^3], \\ \varsigma_{15}(e, x) &= (1 - x^2)(1 - e^2)[(4608 + 4608x^2)(1 - e^2) + 3456xe^2], \\ \varsigma_{16}(e, x) &= (1 - e^2)(1 - x^2)[3456e^2 + 9216(1 - e^2)x]. \end{aligned}$$

Note that

$$\varsigma_{15}(e, x) \leq (1 - e^2)(1 - x^2)[(1 - e^2)(4608x^2 + 4608) + 3456e^2] = \xi_3(e, x),$$

thus, we get

$$F_4(e, x, t) \leq \varsigma_{13}(e, x) + \varsigma_{14}(e, x)t + \xi_3(e, x)t^2 + \varsigma_{16}(e, x)(1 - t^2) = \psi_3(e, x, t).$$

Differentiating $\psi_3(e, x, t)$ with respect to t , we have

$$\frac{\partial \psi_3}{\partial t} = \varsigma_{14}(e, x) + 2t[\xi_3(e, x) - \varsigma_{16}(e, x)].$$

Upon $\varsigma_{13}(e, x) \geq 0$ and

$$\xi_3(e, x) - \varsigma_{16}(e, x) = 4608(x - 1)^2(1 - x^2)(1 - e^2)^2 \geq 0,$$

we have $\frac{\partial \psi_3}{\partial t} \geq 0$. Thus, $\psi_3(e, x, t)$ attains its maximum value at $t = 1$. Thus, we yield

$$\begin{aligned} F_4(e, x, t) &\leq \psi_3(e, x, t) \\ &\leq \psi_3(e, x, 1) \\ &= \varsigma_{13}(e, x) + \varsigma_{14}(e, x) + \xi_3(e, x) \\ &= 73e^6 - 1248e^5 + 1152e^4 + 1248e^3 - 5760e^2 + 4608 + (-720e^6 - 2304e^5 + 720e^4 \\ &\quad - 2304e^3 + 4608e)x + (192e^6 + 10464e^5 + 1824e^4 - 19680e^3 - 2016e^2 + 9216e)x^2 \\ &\quad + (1152e^6 + 2304e^5 + 3456e^4 + 2304e^3 - 13824e^2 - 4608e + 9216)x^3 + (4608e^6 \\ &\quad - 9216e^5 - 13824e^4 + 18432e^3 + 13824e^2 - 9216e - 4608)x^4 \\ &= h_5(e, x). \end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \hbar_5}{\partial e} = 438e^5 - 6240e^4 + 4608e^3 + 3744e^2 - 11520e + (-4320e^5 - 11520e^4 + 2880e^3 - 6912e^2 \\ \quad + 4608)x + (1152e^5 + 52320e^4 + 7296e^3 - 59040e^2 - 4032e + 9216)x^2 + (6912e^5 \\ \quad + 11520e^4 + 13824e^3 + 6912e^2 - 27648e - 4608)x^3 + (27648e^5 - 46080e^4 - 55296e^3 \\ \quad + 55296e^2 + 27648e - 9216)x^4 \\ \quad = 0, \\ \frac{\partial \hbar_5}{\partial x} = 4608e - 2304e^3 + 720e^4 - 2304e^5 - 720e^6 + (384e^6 + 20928e^5 + 3648e^4 - 39360e^3 \\ \quad - 4032e^2 + 18432e)x + (3456e^6 + 6912e^5 + 10368e^4 + 6912e^3 - 41472e^2 - 13824e \\ \quad + 27648)x^2 + (18432e^6 - 36864e^5 - 55296e^4 + 73728e^3 + 41472e^2 - 36864e - 18432)x^3 \\ \quad = 0. \end{array} \right.$$

By calculating, we yield

$$\left\{ \begin{array}{l} e_1 = 0, \\ x_1 = 0, \end{array} \right. \quad \left\{ \begin{array}{l} e_2 \approx 1.0574, \\ x_2 \approx -0.4220, \end{array} \right. \quad \left\{ \begin{array}{l} e_3 \approx 1.2451, \\ x_3 \approx 0.7568. \end{array} \right.$$

Therefore, there are no critical points in $(0, 1) \times (0, 1)$.

(13) For $e = 1$,

$$\hbar_5(1, x) = 73.$$

(14) For $e = 0$,

$$\hbar_5(0, x) = 4608 + 9216x^3 - 4608x^4 \leq 9216.$$

(15) For $x = 0$,

$$\hbar_5(e, 0) = 4608 - 5760e^2 + 1248e^3 + 1152e^4 - 1248e^5 + 73e^6 \leq 9216.$$

(16) For $x = 1$,

$$\hbar_5(e, 1) = 9216 - 7776e^2 - 6672e^4 + 5305e^6 \leq 9216.$$

Thus, we have

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{9216}{1179648} = \frac{1}{128}.$$

□

Theorem 3.3. *If $\Gamma \in \mathcal{SS}_{3l}^*$ is of the form (1.1), then*

$$|\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)| \leq \frac{1}{25}.$$

The inequality is sharp for the function given by (2.20).

Proof. Let $\Gamma \in \mathcal{SS}_{3l}^*$. From (2.15) – (2.17) and (1.7), we obtain

$$\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2) = \frac{1}{3750}(75e_1e_3 - 30e_1^2e_2 - 150e_2^2 + 26e_1^4). \quad (3.3)$$

Using Lemma 1.1 and (3.3), we achieve

$$\begin{aligned} \mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2) &= \frac{1}{3750}[75(1 - |e_1|^2)e_1\bar{e}_1\nu^2 + 75(1 - |\nu|^2)(1 - |e_1|^2)e_1\tau - 30e_1^2\nu(1 - |e_1|^2) \\ &\quad - 150(1 - |e_1|^2)^2\nu^2 + 26e_1^4]. \end{aligned}$$

Setting $|\nu| = x$, $|e_1| = e$ and upon using $|\tau| \leq 1$, we have

$$\begin{aligned} |\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)| &\leq \frac{1}{3750}[75(1 - e^2)e^2x^2 + 75(1 - x^2)(1 - e^2)e + 30e^2x(1 - e^2) + 150(1 - e^2)^2x^2] \\ &= h_6(e, x). \end{aligned}$$

Differentiating $h_6(e, x)$ with respect to x , we get

$$\frac{\partial h_6}{\partial x} = \frac{1}{3750}[30(1 - e^2)e^2 + 150x(1 - e^2)(-e^2 - e + 2)] \geq 0.$$

Thus, $h_6(e, x)$ attains its maximum value at $x = 1$. Therefore, we yield

$$|\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)| \leq h_6(e, 1) = \frac{1}{3750}[150 - 198e^2 + 74e^4] \leq \frac{150}{3750} = \frac{1}{25}.$$

□

Theorem 3.4. Let $\Gamma \in \mathcal{SS}_{3l}^*$. Then

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{4}{125}.$$

The bound is sharp the function given by (2.20).

Proof. Let $\Gamma \in \mathcal{SS}_{3l}^*$. By using (2.15) – (2.18) and (1.8), we yield

$$\begin{aligned} \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) &= \frac{1}{225000}(-2250e_3^2 + 3600e_1e_2e_3 + 4680e_1^2e_2^2 + 4500e_2e_4 - 7200e_2^3 - 1275e_1^4e_2 \\ &\quad - 2700e_1^2e_4 - 211e_1^6 + 600e_1^3e_3). \end{aligned} \quad (3.4)$$

From (3.4) and Lemma 1.1, we get

$$\begin{aligned} &\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) \\ &= \frac{1}{225000} \left[-211e_1^6 + 2250\nu^4\bar{e}_1^2(1 - |e_1|^2)^2 - 4500\nu\bar{\nu}(1 - |\nu|^2)(1 - |e_1|^2)^2\tau^2 \right. \\ &\quad - 4500\bar{e}_1\nu^2(1 - |e_1|^2)^2(1 - |\nu|^2)\tau + 4500\nu s(1 - |\tau|^2)(1 - |\nu|^2)(1 - |e_1|^2)^2 \\ &\quad - 3600e_1\bar{e}_1\nu^3(1 - |e_1|^2)^2 + 3600\tau\nu e_1(1 - |\nu|^2)(1 - |e_1|^2)^2 + 4608e_1^2\nu^2(1 - |e_1|^2)^2 \\ &\quad - 7200(1 - |e_1|^2)^3\nu^3 - 1275e_1^4(1 - |e_1|^2)\nu - 2700\nu^3e_1^2\bar{e}_1^2(1 - |e_1|^2) + 2700e_1^2 \\ &\quad \times (1 - |\nu|^2)(1 - |e_1|^2)\bar{\nu}\tau^2 + 5400\tau e_1^2\bar{e}_1\nu(1 - |e_1|^2)(1 - |\nu|^2) + 600\tau e_1^3(1 - |\nu|^2) \\ &\quad \times (1 - |e_1|^2) - 2700se_1^2(1 - |\tau|^2)(1 - |\nu|^2)(1 - |e_1|^2) - 2250\tau^2(1 - |e_1|^2)^2(1 - |\nu|^2)^2 \end{aligned}$$

$$-600e_1^3\bar{e}_1(1-|e_1|^2)\nu^2\Big].$$

Thus, we have

$$\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) = \frac{1}{225000} [\delta_{13}(e_1, \nu) + \delta_{14}(e_1, \nu)\tau + \delta_{15}(e_1, \nu)\tau^2 + \Upsilon_5(e_1, \nu, \tau)s],$$

where

$$\begin{aligned}\delta_{13}(e_1, \nu) &= -211e_1^6 + (1-|e_1|^2)[(2250\bar{e}_1^2\nu^4 + 3600|e_1|^2\nu^3 - 7200\nu^3 + 4680e_1^2\nu^2)(1-|e_1|^2) \\ &\quad - 1275e_1^4\nu - 600e_1^3\bar{e}_1\nu^2 - 2700e_1^2\bar{e}_1^2\nu^3], \\ \delta_{14}(e_1, \nu) &= (1-|\nu|^2)(1-|e_1|^2)[(-4500\bar{e}_1\nu^2 + 3600e_1\nu)(1-|e_1|^2) + 5400e_1^2\bar{e}_1\nu + 600e_1^3], \\ \delta_{15}(e_1, \nu) &= (1-|e_1|^2)(1-|\nu|^2)[(1-|e_1|^2)(-2250 - 2250|\nu|^2) + 2700e_1^2\bar{\nu}], \\ \Upsilon_5(e_1, \nu, \tau) &= (1-|e_1|^2)(1-|\tau|^2)(1-|\nu|^2)[4500\nu(1-|e_1|^2) - 2700e_1^2].\end{aligned}$$

By writing $|\nu| = x$, $e = |e_1|$, $|\tau| = t$ and using $|s| \leq 1$, and using the triangle inequality, we yield

$$\begin{aligned}|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| &\leq \frac{1}{225000} (|\delta_{13}(e_1, \nu)| + |\delta_{14}(e_1, \nu)|t + |\delta_{15}(e_1, \nu)|t^2 + |\Upsilon_5(e_1, \nu, \tau)|) \\ &\leq \frac{1}{225000} F_5(e, x, t),\end{aligned}\tag{3.5}$$

where

$$F_5(e, x, t) = \varsigma_{17}(e, x) + \varsigma_{18}(e, x)t + \varsigma_{19}(e, x)t^2 + \varsigma_{20}(e, x)(1-t^2),$$

with

$$\begin{aligned}\varsigma_{17}(c, d) &= 211e^6 + (1-e^2)[(2250e^2x^4 + 7200x^3 - 3600e^2x^3 + 4680e^2x^2)(1-e^2) + 1275e^4x \\ &\quad + 600e^4x^2 + 2700e^4x^3], \\ \varsigma_{18}(e, x) &= 600(1-e^2)(1-x^2)[(75ex^2 + 6ex)(1-e^2) + 9e^3x + e^3], \\ \varsigma_{19}(e, x) &= 450(1-x^2)(1-e^2)[(5+5x^2)(1-e^2) + 6xe^2], \\ \varsigma_{20}(e, x) &= 900(1-e^2)(1-x^2)[3e^2 + 5x(1-e^2)].\end{aligned}$$

It is clear that for $x \in [0, 1]$,

$$\varsigma_{19}(e, x) \leq 450(1-e^2)(1-x^2)[(5+5x^2)(1-e^2) + 6e^2] = \xi_4(e, x).$$

Then, we have

$$F_5(e, x, t) \leq \varsigma_{17}(e, x) + \varsigma_{18}(e, x)t + \xi_4(e, x)t^2 + \varsigma_{20}(e, x)(1-t^2) = \psi_4(e, x, t).$$

It is easy to see that

$$\begin{aligned}\frac{\partial \psi_4}{\partial t} &= \varsigma_{18}(e, x) + 2t[\xi_4(e, x) - \varsigma_{20}(e, x)] \\ &= \varsigma_{18}(e, x) + 4500t(1-x^2)(x^2 - 2x + 1)(1-e^2)^2 \\ &\geq 0.\end{aligned}$$

So, $\psi_4(e, x, t)$ attains its maximum value at $t = 1$. Thus, we yield

$$\begin{aligned}
 F_5(e, x, t) &\leq \psi_4(e, x, t) \\
 &\leq \psi_4(e, x, 1) \\
 &= \varsigma_{17}(e, x) + \varsigma_{18}(e, x) + \xi_4(e, x) \\
 &= 211e^6 - 600e^5 - 450e^4 + 600e^3 - 1800e^2 + 2250 + (-1275e^6 - 1800e^5 + 1275e^4 \\
 &\quad - 1800e^3 + 3600e)x + (4080e^6 + 5100e^5 - 6060e^4 - 9600e^3 + 1980e^2 + 4500e)x^2 \\
 &\quad + (-6300e^6 + 1800e^5 + 17100e^4 + 1800e^3 - 18000e^2 - 3600e + 7200)x^3 \\
 &\quad + (2250e^6 - 4500e^5 - 6750e^4 + 9000e^3 + 6750e^2 - 4500e - 2250)x^4 \\
 &= \hbar_7(e, x).
 \end{aligned}$$

The optimal points of $\hbar_7$ has to satisfy

$$\left\{ \begin{aligned} \frac{\partial \hbar_7}{\partial e} &= 1266e^5 - 3000e^4 - 1800e^3 + 1800e^2 - 3600e + (-7650e^5 - 9000e^4 + 5100e^3 - 5400e^2 \\ &\quad + 3600)x + (24480e^5 + 25500e^4 - 24240e^3 - 28800e^2 + 3960e + 4500)x^2 + (-37800e^5 \\ &\quad + 9000e^4 + 68400e^3 + 5400e^2 - 36000e - 3600)x^3 + (13500e^5 - 22500e^4 - 27000e^3 \\ &\quad + 27000e^2 + 13500e - 4500)x^4 \\ &= 0, \\ \frac{\partial \hbar_7}{\partial x} &= 3600e - 1800e^3 + 1275e^4 - 1800e^5 - 1275e^6 + (8160e^6 + 10200e^5 - 12120e^4 - 19200e^3 \\ &\quad + 3960e^2 + 9000e)x + (-18900e^6 + 5400e^5 + 51300e^4 + 5400e^3 - 54000e^2 - 10800e \\ &\quad + 21600)x^2 + (9000e^6 - 18000e^5 - 27000e^4 + 36000e^3 + 27000e^2 - 18000e - 9000)x^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we have

$$\begin{aligned}
 &\left\{ \begin{aligned} e_1 &\approx 1, \\ x_1 &\approx 1.3574, \end{aligned} \right. & \left\{ \begin{aligned} e_2 &\approx 1, \\ x_2 &\approx -1.9921, \end{aligned} \right. & \left\{ \begin{aligned} e_3 &\approx 1, \\ x_3 &\approx -0.3653, \end{aligned} \right. & \left\{ \begin{aligned} e_4 &\approx -1, \\ x_4 &\approx -0.7744, \end{aligned} \right. \\
 &\left\{ \begin{aligned} e_5 &= 0, \\ x_5 &= 0, \end{aligned} \right. & \left\{ \begin{aligned} e_6 &\approx 3.5894, \\ x_6 &\approx 4.0582, \end{aligned} \right. & \left\{ \begin{aligned} e_7 &\approx -0.0842, \\ x_7 &\approx -0.1003, \end{aligned} \right. & \left\{ \begin{aligned} e_8 &\approx -0.7811, \\ x_8 &\approx -4.9377, \end{aligned} \right. \\
 &\left\{ \begin{aligned} e_9 &\approx -1.0053, \\ x_9 &\approx 0.4785, \end{aligned} \right. & \left\{ \begin{aligned} e_{10} &\approx -1.2931, \\ x_{10} &\approx -0.2609, \end{aligned} \right. & \left\{ \begin{aligned} e_{11} &\approx -1.5374, \\ x_{11} &\approx 1.3718. \end{aligned} \right.
 \end{aligned}$$

So, there is no solution of this system in $(0, 1) \times (0, 1)$.

(17) For $e = 1$,

$$\hbar_7(1, x) = 211.$$

(18) For $e = 0$,

$$h_7(0, x) = 2250 + 7200x^3 - 2250x^4 \leq 7200.$$

(19) For $x = 0$,

$$h_7(e, 0) = 2250 - 1800e^2 + 600e^3 - 450e^4 - 600e^5 + 211e^6 \leq 2250.$$

(20) For $x = 1$,

$$h_7(e, 1) = 7200 - 11070e^2 + 5115e^4 - 1034e^6 \leq 7200.$$

Thus, we yield

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{7200}{225000} = \frac{4}{125}.$$

□

Theorem 3.5. *If $\Gamma \in \mathcal{SS}^*$, then*

$$|\mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2)| \leq \frac{1}{4}.$$

The bound is sharp for the function given by (2.28).

Proof. Let $\Gamma \in \mathcal{SS}^*$. From (2.22) – (2.24) and (1.7), we achieve

$$H_{2,1}(G_{\Gamma^{-1}}/2) = \frac{1}{48}(6e_1e_3 - 12e_2^2 - 18e_1^2e_2 + e_1^4). \quad (3.6)$$

From (3.6) and Lemma 1.1, we get

$$\begin{aligned} \mathcal{H}_{2,1}(G_{\Gamma^{-1}}/2) &= \frac{1}{48}[6e_1\bar{e}_1\nu^2(1 - |e_1|^2) - 18e_1^2\nu(1 - |e_1|^2) + 6e_1(1 - |\nu|^2)(1 - |e_1|^2)\tau \\ &\quad + e_1^4 - 12\nu^2(1 - |e_1|^2)^2]. \end{aligned}$$

Setting $|e_1| = e$, $|\nu| = x$, and upon using $|\tau| \leq 1$, we yield

$$\begin{aligned} |H_{2,1}(G_{\Gamma^{-1}}/2)| &\leq \frac{1}{48}[6(1 - e^2)x^2e^2 + 18xe^2(1 - e^2) + 6(1 - e^2)e(1 - x^2) + e^4 + 12(1 - e^2)^2x^2] \\ &= h_8(e, x). \end{aligned}$$

By differentiating $h_8(e, x)$ with respect to x , we achieve

$$\frac{\partial h_8}{\partial x} = \frac{1}{48}[18(1 - e^2)e^2 + 12x(1 - e^2)(-e^2 - e + 1)] \geq 0.$$

So, $h_8(e, x)$ attains its maximum value at $x = 1$. Thus, we have

$$|H_{2,1}(G_{\Gamma^{-1}}/2)| \leq h_8(e, 1) = \frac{1}{48}[12 - 11e^4] \leq \frac{12}{48} = \frac{1}{4}.$$

□

Theorem 3.6. *If $\Gamma \in \mathcal{SS}^*$, then*

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{3}{8}.$$

The equality holds for the function given by (2.28).

Proof. Let $\Gamma \in \mathcal{SS}^*$. From (2.22) – (2.25) and (1.8), we obtain

$$\begin{aligned} \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) = & \frac{1}{288} (36e_3e_2e_1 - 18e_3^2 - 36e_2^2e_1^2 + 48e_1^3e_3 - 78e_1^4e_2 + e_1^6 - 108e_2^3 \\ & + 36e_2e_4 - 18e_1^2e_4). \end{aligned} \quad (3.7)$$

From (3.7) and Lemma 1.1, we get

$$\begin{aligned} & \mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) \\ = & \frac{1}{288} \left[e_1^6 + 18\bar{e}_1^2\nu^4(1 - |e_1|^2)^2 - 36\bar{\nu}\nu(1 - |e_1|^2)^2(1 - |\nu|^2)\tau^2 - 36\nu^2\bar{e}_1(1 - |e_1|^2)^2 \right. \\ & \times (1 - |\nu|^2)\tau + 36s\nu(1 - |e_1|^2)^2(1 - |\tau|^2)(1 - |\nu|^2) - 36\nu^3e_1\bar{e}_1(1 - |e_1|^2)^2 + 36\tau\nu e_1 \\ & \times (1 - |e_1|^2)^2(1 - |\nu|^2) - 36(1 - |e_1|^2)^2\nu^2e_1^2 - 108(1 - |e_1|^2)^3\nu^3 - 78e_1^4(1 - |e_1|^2)\nu \\ & - 18\bar{e}_1^2\nu^3e_1^2(1 - |e_1|^2) + 18e_1^2\bar{\nu}(1 - |\nu|^2)(1 - |e_1|^2)\tau^2 + 36\tau\bar{e}_1e_1^2\nu(1 - |e_1|^2)(1 - |\nu|^2) \\ & - 48e_1^3(1 - |e_1|^2)\bar{e}_1\nu^2 - 18e_1^2(1 - |\tau|^2)(1 - |e_1|^2)(1 - |\nu|^2)s - 18\tau^2(1 - |\nu|^2)^2(1 - |e_1|^2)^2 \\ & \left. + 48e_1^3(1 - |\nu|^2)(1 - |e_1|^2)\tau \right]. \end{aligned}$$

Therefore, we achieve

$$\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2) = \frac{1}{288} [\delta_{16}(e_1, \nu) + \delta_{17}(e_1, \nu)\tau + \delta_{18}(e_1, \nu)\tau^2 + \Upsilon_6(e_1, \nu, \tau)s],$$

where

$$\begin{aligned} \delta_{16}(e_1, \nu) &= e_1^6 + (1 - |e_1|^2)[(18\nu^4\bar{e}_1^2 - (108 - 72|e_1|^2)\nu^3 - 36\nu^2e_1^2)(1 - |e_1|^2) - 78\nu e_1^4 \\ & \quad - 48\nu^2e_1^3\bar{e}_1 - 18\nu^3e_1^2\bar{e}_1^2], \\ \delta_{17}(e_1, \nu) &= (1 - |\nu|^2)(1 - |e_1|^2)[(-36\nu^2\bar{e}_1 + 36\nu e_1)(1 - |e_1|^2) + 36\nu e_1^2\bar{e}_1 + 48e_1^3], \\ \delta_{18}(e_1, \nu) &= (1 - |e_1|^2)(1 - |\nu|^2)[(1 - |e_1|^2)(-18 - 18|\nu|^2) + 18\bar{\nu}e_1^2], \\ \Upsilon_6(e_1, \nu, \tau) &= (1 - |e_1|^2)(1 - |\tau|^2)(1 - |\nu|^2)[36(1 - |e_1|^2)\nu - 18e_1^2]. \end{aligned}$$

By writing $|\tau| = t, e = |e_1|, |\nu| = x$ and using $|s| \leq 1$, we have

$$\begin{aligned} |\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| &\leq \frac{1}{288} (|\delta_{16}(e_1, \nu)| + |\delta_{17}(e_1, \nu)|t + |\delta_{18}(e_1, \nu)|t^2 + |\Upsilon_6(e_1, \nu, \tau)|) \\ &\leq \frac{1}{225000} F_6(e, x, t), \end{aligned} \quad (3.8)$$

where

$$F_6(e, x, t) = \varsigma_{21}(e, x) + \varsigma_{22}(e, x)t + \varsigma_{23}(e, x)t^2 + \varsigma_{24}(e, x)(1 - t^2),$$

with

$$\varsigma_{21}(c, d) = e^6 + (1 - e^2)[(18e^2x^4 + 108x^3 - 72e^2x^3 + 36e^2x^2)(1 - e^2) + 78e^4x$$

$$\begin{aligned}
& +48e^4x^2 + 18e^4x^3], \\
\varsigma_{22}(e, x) &= 12(1 - e^2)(1 - x^2) [(3ex^2 + 3ex)(1 - e^2) + 3e^3x + 4e^3], \\
\varsigma_{23}(e, x) &= 18(1 - x^2)(1 - e^2) [(1 + x^2)(1 - e^2) + xe^2], \\
\varsigma_{24}(e, x) &= 18(1 - e^2)(1 - x^2) [e^2 + 2x(1 - e^2)].
\end{aligned}$$

It is clear that for $x \in [0, 1]$,

$$\varsigma_{23}(e, x) \leq 18(1 - e^2)(1 - x^2) [(x^2 + 1)(1 - e^2) + e^2] = \xi_5(e, x).$$

Then, we have

$$F_6(e, x, t) \leq \varsigma_{21}(e, x) + \varsigma_{22}(e, x)t + \xi_5(e, x)t^2 + \varsigma_{24}(e, x)(1 - t^2) = \psi_5(e, x, t).$$

It is easy to see that

$$\begin{aligned}
\frac{\partial \psi_5}{\partial t} &= \varsigma_{22}(e, x) + 2t[\xi_5(e, x) - \varsigma_{24}(e, x)] \\
&= \varsigma_{22}(e, x) + 36t(x^2 - 2x + 1)(1 - x^2)(1 - e^2)^2 \\
&\geq 0.
\end{aligned}$$

So, $\psi_5(e, x, t)$ attains its maximum value at $t = 1$. Thus, we have

$$\begin{aligned}
F_6(e, x, t) &\leq \psi_5(e, x, t) \\
&\leq \psi_5(e, x, 1) \\
&= \varsigma_{21}(e, x) + \varsigma_{22}(e, x) + \xi_5(e, x) \\
&= e^6 - 48e^5 + 48e^3 - 18e^2 + 18 + (-78e^6 + 78e^4 - 36e^3 + 36e)x + (-12e^6 + 84e^5 \\
&\quad - 6e^4 - 120e^3 + 18e^2 + 36e)x^2 + (-90e^6 + 270e^4 + 36e^3 - 288e^2 - 36e + 108)x^3 \\
&\quad + (18e^6 - 36e^5 - 54e^4 + 72e^3 + 54e^2 - 36e - 18)x^4 \\
&= \hbar_9(e, x).
\end{aligned}$$

The optimal points of $\hbar_9$ has to satisfy

$$\left\{ \begin{aligned} \frac{\partial \hbar_9}{\partial e} &= 6e^5 - 240e^4 + 144e^2 - 36e + (-468e^5 + 312e^3 - 108e^2 + 36)x + (-72e^5 + 420e^4 - 24e^3 \\ &\quad - 360e^2 + 36e + 36)x^2 + (-540e^5 + 1080e^3 + 108e^2 - 576e - 36)x^3 + (108e^5 - 180e^4 \\ &\quad - 216e^3 + 216e^2 + 108e - 36)x^4 \\ &= 0, \\ \frac{\partial \hbar_9}{\partial x} &= 36e - 36e^3 + 78e^4 - 78e^6 + (-24e^6 + 168e^5 - 12e^4 - 240e^3 + 36e^2 + 72e)x + (-270e^6 \\ &\quad + 810e^4 + 108e^3 - 864e^2 - 108e + 324)x^2 + (72e^6 - 144e^5 - 216e^4 + 288e^3 + 216e^2 \\ &\quad - 144e - 72)x^3 \\ &= 0. \end{aligned} \right.$$

By calculating, we have

$$\left\{ \begin{aligned} e_1 &= 1, \\ x_1 &= 2.3421, \end{aligned} \right\} \quad \left\{ \begin{aligned} e_2 &= 1, \\ x_2 &= -2.8104, \end{aligned} \right\} \quad \left\{ \begin{aligned} e_3 &= 1, \\ x_3 &= -0.5317, \end{aligned} \right\} \quad \left\{ \begin{aligned} e_4 &= -1, \\ x_4 &= 0.4020, \end{aligned} \right\} \quad \left\{ \begin{aligned} e_5 &= 0, \\ x_5 &= 0, \end{aligned} \right\}$$

$$\begin{cases} e_6 = 3.3422, \\ x_6 = 10.7582, \end{cases} \quad \begin{cases} e_7 = -0.0540 \\ x_7 = -0.0710, \end{cases} \quad \begin{cases} e_8 = -0.8563, \\ x_8 = -5.4291, \end{cases} \quad \begin{cases} e_9 = -1.1744, \\ x_9 = 3.9895. \end{cases}$$

So, there is no solution of this system in $(0, 1) \times (0, 1)$.

(21) For $e = 1$,

$$\hbar_9(1, x) = 1.$$

(22) For $e = 0$,

$$\hbar_9(0, x) = 18 + 108x^3 - 18x^4 \leq 108.$$

(23) For $x = 0$,

$$\hbar_9(e, 0) = 18 - 18e^2 + 48e^3 - 48e^5 + e^6 \leq 18.$$

(24) For $x = 1$,

$$\hbar_9(e, 1) = 108 - 234e^2 + 288e^4 - 161e^6 \leq 108.$$

Thus, we yield

$$|\mathcal{H}_{2,2}(G_{\Gamma^{-1}}/2)| \leq \frac{108}{288} = \frac{3}{8}.$$

□

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