

HIGHER-DIMENSIONAL MULTIFRACTAL ANALYSIS ON C^1 NON-CONFORMAL REPELLERS*

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Abstract Let J be a repeller of an expanding and topologically mixing C^1 -map g , and $\Phi = (\phi_1, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$, $\Psi = (\psi_1, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ be continuous functions. For a given compact (and possibly connected) subset C , we denote by $E_{Q \circ (\Phi \times \Psi)}(C)$ and $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$ the two distinct level sets defined in (1.2) and (1.3), respectively. In this paper, we investigate the complexity of $E_{Q \circ (\Phi \times \Psi)}(C)$ and $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$. For average conformal repellers, we study the generalized topological pressure and the Hausdorff dimension of $E_{Q \circ (\Phi \times \Psi)}(C)$. For non-conformal repellers, we analyze $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$ via the topological pressure. Consequently, applying the result to the average conformal case yields a variational principle for the Hausdorff dimension of $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$.

Keywords Dimension theory, multifractal analysis, pressure, non-conformal repeller.

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1. Introduction

The present paper is motivated by [4, 6] and [16, 17]. Let $g : \mathbf{M} \rightarrow \mathbf{M}$ be a C^1 -map of an s_0 -dimension smooth Riemannian manifold $\mathbf{M}$. We consider a compact invariant set $J \subseteq \mathbf{M}$, that is, a compact set $J \subseteq \mathbf{M}$ such that $g(J) = J$. We say that J is a *repeller* of g and that g is *expanding* on J if there exists $C_1 > 1$ such that $\|D_x g(u)\| \geq C_1 \|u\|$ for all $x \in J$ and $u \in T_x \mathbf{M}$, where $\|\cdot\|$ is the norm induced by an adapted Riemannian metric. We call that g is *topologically mixing* on J if given open sets U and W with nonempty intersection with J there exists $m_0 \in \mathbb{N}$ such that

$$g^n(U) \cap W \cap J \neq \emptyset$$

for every $n > m_0$.

A finite closed cover $\mathcal{R} = \{R_0, R_1, \dots, R_{m-1}\}$ of J is called a *Markov partition* of J (with respect to g) if:

- (i) $\overline{\text{int} R_i} = R_i$ for each $i = 0, \dots, m-1$;
- (ii) $\text{int} R_i \cap \text{int} R_j = \emptyset$ for $i \neq j$; and
- (iii) each $g(R_i)$ is the union of a subfamily of $\{R_j\}_{j=0}^{m-1}$.

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It is well known that any repeller J of a continuously differentiable expanding map g has Markov partition of arbitrary small diameter [14].

Let $\mathcal{M}_g(J)$ be the set of all g -invariant Borel probability measures concentrated on J , and $\mathcal{E}_g(J)$ be the set of all ergodic invariant measures supported on J . Let $\Phi = (\phi_1, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$ and $\Psi = (\psi_1, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ be continuous functions. Let

$$\Phi \times \Psi : J \rightarrow \mathbb{R}^d \times (\mathbb{R}^+)^d, \quad (\Phi \times \Psi)(x) := (\phi_1(x), \dots, \phi_d(x), \psi_1(x), \dots, \psi_d(x))$$

and

$$Q : \mathbb{R}^d \times (\mathbb{R}^+)^d \rightarrow \mathbb{R}^d, \quad Q(a_1, \dots, a_d, b_1, \dots, b_d) := \left(\frac{a_1}{b_1}, \dots, \frac{a_d}{b_d} \right) \quad (1.1)$$

be continuous maps such that $Q \circ (\Phi \times \Psi)$ is well defined. For any $x, y \in J$, denote

$$\begin{aligned} \frac{\Phi(x)}{\Psi(x)} &:= \left(\frac{\phi_1(x)}{\psi_1(x)}, \dots, \frac{\phi_d(x)}{\psi_d(x)} \right), \\ (\Phi \times \Psi)(x) + (\Phi \times \Psi)(y) &:= (\Phi(x) + \Phi(y)) \times (\Psi(x) + \Psi(y)), \end{aligned}$$

and

$$\frac{\int \Phi d\mu}{\int \Psi d\mu} := \left(\frac{\int \phi_1 d\mu}{\int \psi_1 d\mu}, \dots, \frac{\int \phi_d d\mu}{\int \psi_d d\mu} \right).$$

We define a function $\mathcal{P} : \mathcal{M}_g(J) \rightarrow \mathbb{R}^d$ by

$$\mathcal{P}(\mu) = \frac{\int \Phi d\mu}{\int \Psi d\mu}.$$

Since $\mathcal{M}_g(J)$ is compact and connected, and $\mathcal{P}$ is continuous, the image $\mathcal{P}(\mathcal{M}_g(J))$ is also compact and connected (see Chapter 10 in [3]).

Let X be any metric space and $\mathcal{A}(x_n)$ be the set of accumulation points of the sequence $(x_n)_n$ in X , i.e.,

$$\mathcal{A}(x_n) = \{x \in X \mid \text{there exists a subsequence } (x_{n_k})_k \text{ such that } x_{n_k} \rightarrow x\}.$$

If C is a compact subset of $\mathbb{R}^d$, define

$$E_{Q \circ (\Phi \times \Psi)}(C) := \left\{ x \in J : \mathcal{A} \left(Q \circ \sum_{j=0}^{n-1} (\Phi \times \Psi)(g^j(x)) \right) \subseteq C \right\} \quad (1.2)$$

and

$$E_{Q \circ (\Phi \times \Psi)}^{equ}(C) := \left\{ x \in J : \mathcal{A} \left(Q \circ \sum_{j=0}^{n-1} (\Phi \times \Psi)(g^j(x)) \right) = C \right\}, \quad (1.3)$$

where

$$Q \circ \sum_{j=0}^{n-1} (\Phi \times \Psi)(g^j(x)) = \left(\frac{S_n \phi_1(x)}{S_n \psi_1(x)}, \dots, \frac{S_n \phi_d(x)}{S_n \psi_d(x)} \right),$$

and $S_n\phi = \phi + \phi \circ g + \dots + \phi \circ g^{n-1}$ for any continuous function $\phi : J \rightarrow \mathbb{R}$.

In recent decades, many experts and scholars have investigated the complexity of level sets $E_{Q_\circ(\Phi \times \Psi)}(C)$ and $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$ for various dynamical systems, in terms of quantities such as topological pressure, entropy, and dimensions. Zhou and Chen [22] studied the dynamical systems satisfying the g -almost product property and the uniform separation property. They established a variational principle for the topological pressure of the level set $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$ with the assumption that $\Psi \equiv (1, 1, \dots, 1)$ and C is compact and connected. If $\Psi \neq (1, 1, \dots, 1)$ and C is closed and convex, they obtained a variational principle for the topological pressure of the level set $E_{Q_\circ(\Phi \times \Psi)}(C)$. They also derived the Hausdorff dimension of $E_{Q_\circ(\Phi \times \Psi)}(C)$ in some C^{1+r} iterated function systems as an application. Yin etc. [20] investigated the lower bound and the upper bound of the topological pressure of the sets $E_{Q_\circ(\Phi \times \Psi)}(C)$ (here C is an open set), in a topological dynamical system satisfying the non-uniform specification. Furthermore if the collection of all recurrent points has the shadowing property, then the upper bound of the topological pressure of $E_{Q_\circ(\Phi \times \Psi)}(C)$ attains equality. Moreover, their results held for the robust classes of multidimensional non-uniformly expanding local diffeomorphisms and Viana maps.

Olsen and Winter [13] studied a graph directed conformal iterated function system (IFS). In the case of $\Psi \equiv (1, 1, \dots, 1)$ and C is compact and connected, they computed a variational principle for the Hausdorff dimension of $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$. If $\Psi \neq (1, 1, \dots, 1)$ and $C = \{\alpha\}$, they also proved a variational principle for the Hausdorff dimension of $E_{Q_\circ(\Phi \times \Psi)}^{equ}(\alpha)$. In [2], Bárány etc investigated self-affine IFSs on the plane satisfying the strong open set condition and the set of corresponding matrices is strongly irreducible such that the generated subgroup of the normalized matrices is not relatively compact. They computed the Hausdorff dimension of $E_{Q_\circ(\Phi \times \Psi)}^{equ}(\alpha)$ for $\Psi \equiv (1, 1, \dots, 1)$. The authors [19] extended their result to $\Psi \neq 1$. Moreover, the conditional variational principle of the Hausdorff dimension of $E_{Q_\circ(\Phi \times \Psi)}^{equ}(\alpha)$ is given for nonlinear skew-product perturbations of general Sierpiński carpets in [19].

Motivated by the study of the multifractal formalism of Birkhoff averages, the level set of the Lyapunov exponents of certain special non-additive potentials on full shifts have been studied. Mohammadpour obtained the entropy spectrum of the level sets for Lyapunov exponents for generic matrix cocycles in the special case of locally constant linear cocycles [9, 10]. However, the derivative cocycles are rarely locally constant for smooth dynamics. Mohammadpour and Varandas [11] developed the topological entropy of the level sets of Lyapunov exponents for typical fiber-bunched linear cocycles recently. Furthermore, they obtained a multifractal formalism for open sets of C^{1+r} repellers and Anosov diffeomorphisms, which is an application to dynamically defined linear cocycles. We also refer the reader to [5, 7, 15] for a detailed description of the recent progress in entropy spectrum of hyperbolic and nonuniformly hyperbolic dynamical systems. Is it possible to prove the dimension spectra of Birkhoff averages for typical fiber-bunched linear cocycles?

By the Oseledec multiplicative ergodic theorem, for any $\mu \in \mathcal{E}_g(J)$, we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \dots \leq \lambda_{s_0}(\mu)$. In [1], the authors introduced a concept of C^1 average conformal repellers which possesses only one positive Lyapunov exponent for any ergodic measure. That is a compact expanding invariant set J is called *an average conformal repeller* if for any $\mu \in \mathcal{E}_g(J)$, $\lambda_1(\mu) = \lambda_2(\mu) = \dots = \lambda_{s_0}(\mu)$. There are average conformal repellers which are not conformal repellers (see an example in [21]). Cao [4] considered asymptotically additive continuous potentials $\{\varphi_n\}$ and $\{\psi_n\}$ ($d = 1$) on C^1 -average conformal repellers and proved the

dimension spectrum of the sets

$$\left\{x \in J : \mathcal{A}\left(\frac{\varphi_n(x)}{\psi_n(x)}\right) \subseteq C\right\}.$$

His result has given an affirmative answer to the problem posed by Olsen [12].

The first part of our main results is to study the complexity of $E_{Q \circ (\Phi \times \Psi)}(C)$ on C^1 -average conformal repellers (J, g) for a compact subset C , which may not be convex. Based on Feng, Lau and Wu's work [6], we define the generalized topological pressure $P_{Q \circ (\Phi \times \Psi)}^*(C, \cdot)$ of C with respect to Φ and $\Psi > 0$ (see (2.1)). We investigate the variational principle for the generalized topological pressure. As an application, we prove the conditional variational principle for the dimension spectra of $E_{Q \circ (\Phi \times \Psi)}(C)$. These results extend the case of C^{1+r} -conformal repellers where $C = \{\alpha\}$ and $\Psi \equiv 1$.

The second part of our main results is to study the topological pressure of $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$ on C^1 non-conformal repellers (J, g) for a compact and connected subset C . We use Thompson's method in [16, 17] to obtain the lower bound, which is the crucial part of the proof. The upper bound can be obtained from Theorem 3.1 (2) in [22]. Moreover, if J is an average conformal repeller, then we derived a conditional variational principle for the dimension spectra of $E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$. These results give a substantial extension of the well-known case that $C = \{\alpha\}$ and $\Psi \equiv 1$.

We arrange the paper as follows. We formulate the main results in Section 2. Section 3 consists of some related definitions and properties. The proofs of Theorem 2.1 and Theorem 2.2 will appear in Section 4. The proofs of Theorem 2.3 and Theorem 2.4 will appear in Section 5.

2. Main results

This section provides the statements of our main results in this paper.

2.1. The generalized topological pressure and the Hausdorff dimension of the level set $E_{Q \circ (\Phi \times \Psi)}(C)$

First of all we define the generalized topological pressure $P_{Q \circ (\Phi \times \Psi)}^*(C, \cdot)$ of C with respect to the continuous potentials Φ and $\Psi > 0$.

The dynamic system (J, g) induces a subshift space of finite type (Σ_A, σ) , where $A = (a_{ij})$ is the transfer matrix of the Markov partition, namely, $a_{ij} = 1$ if $\text{int} R_i \cap g^{-1}(\text{int} R_j) \neq \emptyset$ and $a_{ij} = 0$ otherwise. $\mathcal{M}_\sigma(\Sigma_A)$ denotes the set of all σ -invariant probability measures on Σ_A . The matrix A is primitive, i.e., there is a positive integer M so that $A^M > 0$. For $n \geq 1$, $\Sigma_{A,n}$ denotes the set of finite sequence $w = i_0 i_1 \dots i_{n-1}$ such that $a_{i_j, i_{j+1}} = 1$ for all $0 \leq j \leq n-2$. These sequences w are called words. The length of the word is denoted by $|w|$. For $w = (i_0 i_1 \dots i_{n-1}) \in \Sigma_{A,n}$, the n -cylinder $[w]$ is defined by $\{\mathbf{j} \in \Sigma_A : j_0 = i_0, j_1 = i_1, \dots, j_{n-1} = i_{n-1}\}$. It is clear that there is one-to-one correspondence between $\Sigma_{A,n}$ and the set of n -cylinder. This gives the coding map $\pi : \Sigma_A \rightarrow J$ such that

$$\pi(w) = \bigcap_{n \geq 0} g^{-n}(R_{i_n}), \quad \forall w = (i_0 i_1 \dots)$$

and the following diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \pi \downarrow & & \downarrow \pi \\ J & \xrightarrow{g} & J \end{array}$$

commutes. For each $\omega = (i_0 i_1 \dots i_{n-1}) \in \Sigma_{A,n}$ define

$$R_\omega := R_{i_0} \cap g^{-1}(R_{i_1}) \cap \dots \cap g^{-(n-1)}(R_{i_{n-1}}),$$

which is called a *basic set*.

If $C \subseteq \mathbb{R}^d$ is compact, for any $s \in \mathbb{R}$, any positive integer n and any $\varepsilon > 0$, set

$$F_{Q \circ (\Phi \times \Psi)}(C; n, \varepsilon) := \left\{ [w] : w \in \Sigma_{A,n} \text{ and } d \left(Q \circ \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(x)), C \right) < \varepsilon \right. \\ \left. \text{for some } x \in R_w \right\}.$$

and

$$f_{Q \circ (\Phi \times \Psi)}(C, s; n, \varepsilon) := \sum_{[w] \in F_{Q \circ (\Phi \times \Psi)}(C; n, \varepsilon)} (\text{diam} R_w)^s.$$

Then we define

$$\underline{P}_{Q \circ (\Phi \times \Psi)}(C, s) := \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log f_{Q \circ (\Phi \times \Psi)}(C, s; n, \varepsilon)$$

and

$$\overline{P}_{Q \circ (\Phi \times \Psi)}(C, s) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log f_{Q \circ (\Phi \times \Psi)}(C, s; n, \varepsilon).$$

If $\underline{P}_{Q \circ (\Phi \times \Psi)}(C, s) = \overline{P}_{Q \circ (\Phi \times \Psi)}(C, s)$, put

$$P_{Q \circ (\Phi \times \Psi)}^*(C, s) := \underline{P}_{Q \circ (\Phi \times \Psi)}(C, s) = \overline{P}_{Q \circ (\Phi \times \Psi)}(C, s), \quad (2.1)$$

which is called *the generalized topological pressure of C with respect to Φ and Ψ* . Our first main result is a variational principle for $P_{Q \circ (\Phi \times \Psi)}^*(C, s)$ as follows.

Theorem 2.1. *Let $g : J \rightarrow J$ be expanding, C^1 -average conformal and topological mixing. Assume that $\Phi = (\phi_1, \phi_2, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ are continuous functions, and C is a compact subset of $\mathcal{P}(\mathcal{M}_g(J))$. Then*

$$P_{Q \circ (\Phi \times \Psi)}^*(C, s) = \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\},$$

where $\phi(g, x) = |\det(D_x g)|^{\frac{1}{s_0}}$.

As an application of Theorem 2.1, we establish a conditional variational principle for the dimension spectra of $E_{Q \circ (\Phi \times \Psi)}(C)$.

Theorem 2.2. *Let $g : J \rightarrow J$ be an expanding, C^1 -average conformal and topological mixing map. Assume that $\Phi = (\phi_1, \phi_2, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ are continuous functions, and C is a compact subset of $\mathcal{P}(\mathcal{M}_g(J))$. Then*

$$\dim_H E_{Q_\circ(\Phi \times \Psi)}(C) = \sup \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu(x)} : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\},$$

where $\phi(g, x) = |\det(D_x g)|^{\frac{1}{s_0}}$.

Remark 2.1. Assume $\psi_i \equiv 1$ for $i = 1, 2, \dots, d$. Then one has

$$\dim_H E_\Phi(C) = \sup \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu(x)} : \mu \in \mathcal{M}_g(J) \text{ and } \int \Phi d\mu \in C \right\},$$

where $E_\Phi(C) := \left\{ x \in J : \mathcal{A} \left(\frac{1}{n} \sum_{j=0}^{n-1} \Phi(g^j(x)) \right) \subseteq C \right\}$.

2.2. The topological pressure and the Hausdorff dimension of the level set $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$.

Our third main result is a variation principle for the topological pressure (see the definition in section 3.3) of the subset $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$, which is f -invariant but not compact.

Theorem 2.3. *Let $g : J \rightarrow J$ be an expanding and topological mixing C^1 -map. Assume that $\Phi = (\phi_1, \phi_2, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$, $\Psi = (\psi_1, \psi_2, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ and $\varphi : J \rightarrow \mathbb{R}$ are continuous functions. Let C be a compact and connected subset of $\mathcal{P}(\mathcal{M}_g(J))$. Then*

$$\begin{aligned} & P \left(E_{Q_\circ(\Phi \times \Psi)}^{equ}(C), \varphi \right) \\ &= \inf_{\alpha \in C} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

Applying Theorem 2.3 to C^1 -average conformal repellers, we establish a conditional variational principle for the dimension spectra of $E_{Q_\circ(\Phi \times \Psi)}^{equ}(C)$ as follows.

Theorem 2.4. *Let $g : J \rightarrow J$ be an expanding, C^1 -average conformal and topological mixing map. Assume that $\Phi = (\phi_1, \phi_2, \dots, \phi_d) : J \rightarrow \mathbb{R}^d$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_d) : J \rightarrow (\mathbb{R}^+)^d$ are continuous functions. Let C be a compact and connected subset of $\mathcal{P}(\mathcal{M}_g(J))$. Then*

$$\begin{aligned} & \dim_H E_{Q_\circ(\Phi \times \Psi)}^{equ}(C) \\ &= \inf_{\alpha \in C} \sup \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu(x)} : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

where $\phi(g, x) = |\det(D_x g)|^{\frac{1}{s_0}}$.

3. Preliminaries

In this section, we will recall some related definitions and propositions which are used in the proofs of the main results.

3.1. Dimension

First we recall the definitions of the Hausdorff dimension and the lower box dimension. For more introduction, one can refer to [3].

Let X be a compact Riemannian manifold with a Riemannian metric. Given a subset Z of X , for $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}_\delta^s(Z) := \inf \left\{ \sum_i (\text{diam } U_i)^s : Z \subset \bigcup_i U_i, \text{diam } U_i \leq \delta \right\}$$

where $\text{diam } U$ denotes the diameter of a subset U . The following quantity

$$\mathcal{H}^s(Z) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(Z)$$

is called the s -dimensional Hausdorff measure of Z . It is easy to show that there is a jump-up value

$$\dim_H Z := \inf \{s : \mathcal{H}^s(Z) = 0\} = \sup \{s : \mathcal{H}^s(Z) = \infty\},$$

which is called the Hausdorff dimension of Z . The lower box dimension of Z is defined by

$$\underline{\dim}_B Z = \liminf_{\varepsilon \rightarrow 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon},$$

where $N(Z, \varepsilon)$ denotes the least number of balls of radius ε that are needed to cover the set Z . It is easy to show that $\dim_H Z \leq \underline{\dim}_B Z$.

Given a Borel probability measure μ on X , the Hausdorff dimension of the measure μ is defined as

$$\dim_H \mu = \inf \{\dim_H Y : Y \subset X, \mu(Y) = 1\}.$$

One can also show that

$$\dim_H \mu = \liminf_{\delta \rightarrow 0} \{\dim_H Z : \mu(Z) \geq 1 - \delta\}.$$

See [3] for more details.

3.2. Property of the basic sets $\{R_w\}$

Recall $\phi(g, x) = |\det(D_x g)|^{\frac{1}{s_0}}$. By Theorem 4.2 in [1], for any $\varepsilon > 0$ there exists a positive integer $\tilde{N}_0$ such that if $n \geq \tilde{N}_0$ and every $x \in J$, then

$$1 \leq \left| \frac{\prod_{i=0}^{n-1} \phi(g, g^i(x))}{m(D_x g^n)} \right| \leq e^{n\varepsilon} \text{ and } e^{-n\varepsilon} \leq \left| \frac{\prod_{i=0}^{n-1} \phi(g, g^i(x))}{\|D_x g^n\|} \right| \leq 1, \quad (3.1)$$

where

$$m(D_x g) = \inf_{0 \neq u \in T_x J} \frac{\|D_x g(u)\|}{\|u\|} \quad \text{and} \quad \|D_x g\| = \sup_{u \in T_x J} \frac{\|D_x g(u)\|}{\|u\|}.$$

Fixed any positive integer $\tilde{N}_1 > \tilde{N}_0$ large enough, for the above $\varepsilon > 0$ there is some $\delta_1 > 0$ such that for every $x, y \in \mathbf{M}$ with $d(x, y) < \delta_1$,

$$e^{-\tilde{N}_1 \varepsilon} \leq \frac{\|D_x g^{\tilde{N}_1}\|}{\|D_y g^{\tilde{N}_1}\|} \leq e^{\tilde{N}_1 \varepsilon}, \quad e^{-\tilde{N}_1 \varepsilon} \leq \frac{m(D_x g^{\tilde{N}_1})}{m(D_y g^{\tilde{N}_1})} \leq e^{\tilde{N}_1 \varepsilon} \quad (3.2)$$

and

$$e^{-\varepsilon} \leq \frac{\phi(g, x)}{\phi(g, y)} \leq e^{\varepsilon}. \quad (3.3)$$

Then we have the following proposition.

Proposition 3.1. *For any integer $n \geq 0$, the interiors of distinct R_w , $w = (i_0 i_1 \dots i_n) \in \Sigma_{A, n+1}$ are disjoint. And for any $\varepsilon > 0$ there is a positive integer $\tilde{N}_1$ such that for each $n \geq \tilde{N}_1$ large enough (Here $\tilde{N}_1$ is as in (3.2).), $\{R_w\}_{w \in \Sigma_{A, n+1}}$ satisfies the following properties.*

- (i) *Each R_w contains a ball of diameter $\underline{r}_w$ and is contained in a ball of diameter $\bar{r}_w$.*
- (ii) *There exists $K(n, \varepsilon) > 0$ with $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log K(n, \varepsilon)}{n} = 0$ such that*

$$K^{-1}(n, \varepsilon) \prod_{j=0}^{n-1} \phi(g, g^j(x))^{-1} \leq \underline{r}_w \leq \bar{r}_w \leq K(n, \varepsilon) \prod_{j=0}^{n-1} \phi(g, g^j(x))^{-1} \quad (3.4)$$

for every $x \in R_w$.

Proof. For any $\varepsilon > 0$, any $n \geq \tilde{N}_1$, each $w = (i_0 i_1 \dots i_n) \in \Sigma_{A, n+1}$ and every $x, y \in R_w$, by (3.3) we obtain

$$\left| \frac{\phi(g^n, x)}{\phi(g^n, y)} \right| = \prod_{i=0}^{n-1} \left| \frac{\phi(g, g^i(x))}{\phi(g, g^i(y))} \right| \leq e^{n\varepsilon}. \quad (3.5)$$

Set $h = (g^n|_{R_w})^{-1}$. For each $x, y \in R_w$, it follows from (3.1), (3.2) and (3.5) that there exists a constant $K_1 > 0$,

$$\begin{aligned} d(x, y) &\geq d(g^n(x), g^n(y)) \cdot m(D_{\tilde{z}}h) \quad (\text{for some } \tilde{z} \in R_{i_n}) \\ &= d(g^n(x), g^n(y)) \cdot m((D_{h(\tilde{z})}h^{-1})^{-1}) \\ &= d(g^n(x), g^n(y)) \cdot m((D_z g^n)^{-1}) \quad (\text{here } z = h(\tilde{z}) \in R_w) \\ &= d(g^n(x), g^n(y)) \cdot \|D_z g^n\|^{-1} \\ &\geq K_1 e^{-3n\varepsilon} \cdot d(g^n(x), g^n(y)) \cdot \prod_{i=0}^{n-1} \phi(g, g^i(x))^{-1}. \end{aligned}$$

This yields that there exists $\underline{r}_w > 0$ such that R_w contains a ball of diameter $\underline{r}_w$. On the other hand similarly there also exists some constant $K_2 > 1$ such that

$$\begin{aligned} \text{diam} R_w &= \sup_{x, y \in R_w} d(x, y) \\ &\leq \sup_{x, y \in R_w} d(g^n(x), g^n(y)) \cdot \sup_{\tilde{z} \in R_{i_n}} \|D_{\tilde{z}}h\| \\ &= \sup_{x, y \in R_w} d(g^n(x), g^n(y)) \cdot \sup_{\tilde{z} \in R_{i_n}} \|(D_{h(\tilde{z})}h^{-1})^{-1}\| \\ &= \sup_{x, y \in R_w} d(g^n(x), g^n(y)) \cdot \sup_{z \in R_w} \|(D_z g^n)^{-1}\| \\ &= \sup_{x, y \in R_w} d(g^n(x), g^n(y)) \cdot \sup_{z \in R_w} m(D_z g^n)^{-1} \\ &\leq K_2 e^{3n\varepsilon} \cdot \text{diam } \mathcal{R} \cdot \prod_{i=0}^{n-1} \phi(g, g^i(x))^{-1} \end{aligned} \quad (3.6)$$

for every $x \in R_w$. This implies that there exists $\bar{r}_w > 0$ such that R_w contains in a ball of diameter $\bar{r}_w$. Moreover there is some $K > 1$ such that

$$K^{-1} \cdot e^{-3n\varepsilon} \cdot \prod_{i=0}^{n-1} \phi(g, g^i x)^{-1} \leq \underline{r}_w \leq \bar{r}_w \leq K \cdot e^{3n\varepsilon} \cdot \prod_{i=0}^{n-1} \phi(g, g^i x)^{-1}$$

for every $x \in R_w$ with $n \geq \tilde{N}_1$. Taking $K(n, \varepsilon) = K \cdot e^{3n\varepsilon}$, then we complete the proofs of Statements (i) and (ii). $\square$

3.3. Topological pressure

Let $(\mathbf{M}, d)$ be a compact metric space and consider a map $g : \mathbf{M} \rightarrow \mathbf{M}$. Now fix a potential function $\varphi : \mathbf{M} \rightarrow \mathbb{R}$. Given $Z \subseteq \mathbf{M}$, $\delta > 0$ and $N \in \mathbb{N}$, let $\mathcal{P}(Z, N, \delta)$ be the collection of countable sets $\{(x_i, n_i) \subseteq Z \times \{N, N+1, \dots\}\}$ such that $Z \subseteq \bigcup_i B_{n_i}(x_i, \delta)$, where

$$B_{n_i}(x_i, \varepsilon) := \{y \in \mathbf{M} : d(g^j(x_i), g^j(y)) < \varepsilon \text{ for all } j = 0, 1, \dots, n_i - 1\}.$$

For each $s \in \mathbb{R}$, consider the set functions

$$m_p(Z, s, \varphi, N, \delta) = \inf_{\mathcal{C} \in \mathcal{P}(Z, N, \delta)} \sum_{(x_i, n_i) \in \mathcal{C}} \exp \left(-n_i s + \sup_{x \in B_{n_i}(x_i, \delta)} S_{n_i} \varphi(x) \right),$$

$$m_p(Z, s, \varphi, \delta) = \lim_{N \rightarrow \infty} m_p(Z, s, \varphi, N, \delta).$$

This function is non-increasing in s , and takes values ∞ and 0 at all but at most one value of s . Denoting the critical value of s by

$$P(Z, \varphi, \delta) = \inf\{s \in \mathbb{R} | m_p(Z, s, \varphi, \delta) = 0\}$$

$$= \sup\{s \in \mathbb{R} | m_p(Z, s, \varphi, \delta) = \infty\}.$$

The topological pressure of φ on Z is $P(Z, \varphi) := \lim_{\delta \rightarrow 0} P(Z, \varphi, \delta)$.

It is obvious that the following hold:

- (1) $P(Z_1, \varphi) \leq P(Z_2, \varphi)$ for any $Z_1 \subseteq Z_2 \subseteq \mathbf{M}$;
- (2) $P(Z, \varphi) = \sup_i P(Z_i, \varphi)$, where $Z = \bigcup_i Z_i \subseteq \mathbf{M}$.

3.4. The generalized pressure distribution principle

Let $(\mathbf{M}, d)$ be a compact metric space and let $g : \mathbf{M} \rightarrow \mathbf{M}$ be a continuous transformation. Given $\varepsilon > 0$, $n \in \mathbb{N}$ and a point $x \in \mathbf{M}$, define a new metric as follows:

$$d_n(x, y) := \max \{d(f^i(x), f^i(y)) : i = 0, 1, \dots, n-1\}.$$

Let $Z \subseteq \mathbf{M}$. We say that a set $S \subseteq Z$ is an (n, ε) spanning set for Z if, for every $z \in Z$, there exists $x \in S$ with $d_n(x, z) \leq \varepsilon$. We say that a set $\mathcal{S} \subseteq Z$ is an (n, ε) separated set for Z if, for every $x, y \in \mathcal{S}$, we have $d_n(x, y) > \varepsilon$. See [18] for the basic properties of spanning sets and separated sets.

The proof of the lower bound in Theorem 2.3 requires the following proposition. For details we refer the reader to [17] and [8].

Proposition 3.2. *Let $g : \mathbf{M} \rightarrow \mathbf{M}$ be a continuous transformation and $\varphi : \mathbf{M} \rightarrow \mathbb{R}$ be a continuous function. Let $F \subseteq \mathbf{M}$ be an arbitrary Borel set. Suppose there exists $\varepsilon > 0$ and $s \geq 0$ such that one can find a sequence of Borel probability measures μ_k , a constant $K > 0$ and an integer N satisfying*

$$\limsup_{k \rightarrow \infty} \mu_k(B_n(x, \varepsilon)) \leq K \exp \left\{ -ns + \sum_{i=0}^{n-1} \varphi(g^i(x)) \right\}$$

for every ball $B_n(x, \varepsilon)$ such that $B_n(x, \varepsilon) \cap F \neq \emptyset$ and $n \geq N$. Furthermore, assume that at least one limit measure ν of the sequence μ_k satisfies $\nu(F) > 0$. Then $P(F, \varphi, \varepsilon) \geq s$.

3.5. The specification property

A continuous map $g : \mathbf{M} \rightarrow \mathbf{M}$ satisfies the *specification property* if, for all $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that, for any collection $\{I_j = [a_j, b_j] \subseteq \mathbb{N} : j = 1, 2, \dots, k\}$ of finite intervals with $a_{j+1} - b_j \geq m(\varepsilon)$ for $j = 1, 2, \dots, k-1$ and any $x_1, x_2, \dots, x_k$ in M , there exists a point $x \in M$ such that

$$d(f^{p+a_j}(x), f^p(x_j)) < \varepsilon \quad \text{for all } p = 0, 1, \dots, b_j - a_j \quad \text{and every } j = 1, 2, \dots, k.$$

4. Proofs of Theorem 2.1 and Theorem 2.2

In this section, we have proved Theorems 2.1 and Theorem 2.2 respectively.

4.1. Proof of Theorem 2.1

For a compact subset $\tilde{C} \subseteq \mathbb{R}^d \times (\mathbb{R}^+)^d$, any positive integer n , any $\varepsilon > 0$ and any $s \in \mathbb{R}$, set

$$F_{\Phi \times \Psi}(\tilde{C}; n, \varepsilon) := \left\{ [w] : w \in \Sigma_{A,n} \text{ and } d\left(\frac{1}{n} \sum_{j=0}^{n-1} (\Phi \times \Psi)(g^j(x)), \tilde{C}\right) < \varepsilon \text{ for some } x \in R_w \right\}$$

and

$$f_{\Phi \times \Psi}(\tilde{C}, s; n, \varepsilon) := \sum_{[w] \in F_{\Phi \times \Psi}(\tilde{C}; n, \varepsilon)} (\text{diam} R_w)^s.$$

Then we define

$$P_{\Phi \times \Psi}(\tilde{C}, s) := \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log f_{\Phi \times \Psi}(\tilde{C}, s; n, \varepsilon)$$

and

$$\overline{P}_{\Phi \times \Psi}(\tilde{C}, s) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log f_{\Phi \times \Psi}(\tilde{C}, s; n, \varepsilon).$$

4.1.1. Lower bound on $\underline{P}_{Q \circ (\Phi \times \Psi)}(C, s)$

The proof of the lower bound was inspired by [6]. Let $\tilde{\mathcal{P}} : \mathcal{M}_g(J) \rightarrow \mathbb{R}^d \times (\mathbb{R}^+)^d$,

$$\tilde{\mathcal{P}}(\mu) = \left(\int \Phi d\mu, \int \Psi d\mu \right).$$

Then $Q(\tilde{\mathcal{P}}(\mathcal{M}_g(J))) = \mathcal{P}(\mathcal{M}_g(J))$. Taking $\tilde{C}_0 = \left(Q|_{\tilde{\mathcal{P}}(\mathcal{M}_g(J))} \right)^{-1}(C)$. Since $C \subseteq \mathcal{P}(\mathcal{M}_g(J))$ is compact, then $\tilde{C}_0$ is compact and $Q(\tilde{C}_0) = C$. Combined with the continuity of Q , one has for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F_{\Phi \times \Psi}(\tilde{C}_0; n, \delta) \subseteq F_{Q \circ (\Phi \times \Psi)}(C; n, \varepsilon).$$

Therefore

$$\underline{P}_{Q \circ (\Phi \times \Psi)}(C, s) \geq \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s). \quad (4.1)$$

Lemma 4.1. *For each $s \in \mathbb{R}$ and any $\eta > 0$, there is $\varepsilon > 0$ such that for each compact subset $\tilde{E} \subset \mathbb{R}^{2d}$ with $D(\tilde{E}, \tilde{C}_0) < \frac{\varepsilon}{3}$, then*

$$\underline{P}_{\Phi \times \Psi}(\tilde{E}, s) \leq \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) + \eta.$$

(Here the metric D is the Hausdorff metric.)

Proof. For any $\eta > 0$ there is $\varepsilon > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\log f_{\Phi \times \Psi}(\tilde{C}_0, s; n, \varepsilon)}{n} < \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) + \eta.$$

Let $\tilde{E} \subseteq \mathbb{R}^{2d}$ be compact with $D(\tilde{C}_0, \tilde{E}) < \frac{\varepsilon}{3}$. If $F_{\Phi \times \Psi}(\tilde{E}; n, \frac{\varepsilon}{3}) = \emptyset$, then $F_{\Phi \times \Psi}(\tilde{E}; n, \frac{\varepsilon}{3}) \subseteq F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon)$ is trivial. If $F_{\Phi \times \Psi}(\tilde{E}; n, \frac{\varepsilon}{3}) \neq \emptyset$, then for each $[w] \in F_{\Phi \times \Psi}(\tilde{E}; n, \frac{\varepsilon}{3})$ there exists $x \in R_w$ such that

$$d\left(\frac{1}{n} \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(x)), \tilde{E}\right) < \frac{\varepsilon}{3}.$$

Hence

$$d\left(\frac{1}{n} \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(x)), \tilde{C}_0\right) \leq d\left(\frac{1}{n} \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(x)), \tilde{E}\right) + D(\tilde{E}, \tilde{C}_0) < \varepsilon$$

and $[w] \in F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon)$. This implies that $F_{\Phi \times \Psi}(\tilde{E}; n, \frac{\varepsilon}{3}) \subseteq F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon)$. Thus

$$f_{\Phi \times \Psi}\left(\tilde{E}, s; n, \frac{\varepsilon}{3}\right) \leq f_{\Phi \times \Psi}\left(\tilde{C}_0, s; n, \varepsilon\right).$$

Therefore

$$\begin{aligned} \underline{P}_{\Phi \times \Psi}(\tilde{E}, s) &\leq \liminf_{n \rightarrow \infty} \frac{\log f_{\Phi \times \Psi}(\tilde{E}, s; n, \frac{\varepsilon}{3})}{n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log f_{\Phi \times \Psi}(\tilde{C}_0, s; n, \varepsilon)}{n} \\ &< \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) + \eta. \end{aligned}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2.

$$\begin{aligned} & \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \\ & \geq \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \tilde{C}_0 \right\} \\ & = \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\}. \end{aligned}$$

Proof. Let μ be any g -invariant measure on J with $(\int \Phi d\mu, \int \Psi d\mu) \in \tilde{C}_0$. Proposition 2.6 in [6] told us that there exists $\nu \in \mathcal{M}_\sigma(\Sigma_A)$ such that

$$\mu = \nu \circ \pi^{-1} \text{ and } h_\sigma(\nu) = h_g(\mu).$$

Let us assume at first that ν is ergodic. Fixed $\varepsilon > 0$ and for any $n \in \mathbb{N}$, let

$$F_n \left(\subseteq F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon) \right)$$

be the collection of all the n -cylinders $[w]$ in Σ_A such that there exists $y \in R_w$ satisfying

$$d \left(\frac{1}{n} \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(y)), \tilde{C}_0 \right) < \varepsilon \quad (4.2)$$

and

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \log \phi(g, g^i(y)) - \int \log \phi(g, x) d\mu(x) \right| < \varepsilon. \quad (4.3)$$

Claim 4.1. The sequence $\{F_n\}$ satisfies $\liminf_{n \rightarrow \infty} \frac{\log \#F_n}{n} \geq h_g(\mu)$.

Proof. Let

$$E_j := \bigcup \{[w] : [w] \in F_j\}.$$

It follows from (4.2), (4.3) and the Birkhoff ergodic theorem that

$$\nu \left(\bigcup_{n \geq 0} \bigcap_{j \geq n} E_j \right) = 1.$$

Hence there exists an integer N such that $\nu \left(\bigcap_{j \geq N} E_j \right) > 0$, which implies that

$$\underline{\dim}_B \left(\bigcap_{j \geq N} E_j \right) \geq \dim_H \left(\bigcap_{j \geq N} E_j \right) \geq \dim_H \nu \geq \frac{h_\sigma(\nu)}{\log m},$$

where m is the cardinality of the Markov partition $\mathcal{R}$. And by the definition of the lower-box dimension,

$$\liminf_{n \rightarrow \infty} \frac{\log \#F_n}{n} \geq (\log m) \cdot \underline{\dim}_B \left(\bigcap_{j \geq N} E_j \right) \geq h_\sigma(\nu) = h_g(\mu)$$

and the claim follows. $\square$

By (3.4) and (4.3), it is easy to see that for n large enough and any $[w] \in F_n$, we can choose some $y \in R_w$ such that

$$\begin{aligned} & \left| \frac{1}{n} \log \text{diam} R_w + \int \log \phi(g, x) d\mu(x) \right| \\ & \leq \left| \frac{1}{n} \log \text{diam} R_w + \frac{1}{n} \sum_{i=0}^{n-1} \log \phi(g, y) \right| + \left| \int \log \phi(g, x) d\mu(x) - \frac{1}{n} \sum_{i=0}^{n-1} \log \phi(g, g^i(y)) \right| \\ & \leq 2\varepsilon. \end{aligned}$$

Recall that

$$\underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) := \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log f_{\Phi \times \Psi}(\tilde{C}_0, s; n, \varepsilon)}{n}$$

with

$$f_{\Phi \times \Psi}(\tilde{C}_0, s; n, \varepsilon) := \sum_{[w] \in F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon)} (\text{diam } R_w)^s.$$

Applying Claim 4.1 and the above inequality one has

$$\begin{aligned} \underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) & \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log \#F_n}{n} - s \int \log \phi(g, x) d\mu(x) - 2s\varepsilon \\ & \geq h_g(\mu) - s \int \log \phi(g, x) d\mu(x) - 2s\varepsilon. \end{aligned}$$

Let $\varepsilon \rightarrow 0$, we have

$$\underline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \geq h_g(\mu) - s \int \log \phi(g, x) d\mu(x).$$

To complete the proof of Lemma 4.2, we consider now ν without assuming the ergodicity. By Corollary 2.5 in [6], there is a sequence of ergodic measures $\{\nu_n\}$ which converges to ν in the weak* topology and satisfies $\lim_{n \rightarrow \infty} h_\sigma(\nu_n) = h_\sigma(\nu)$. Set $\mu_n := \nu_n \circ \pi^{-1}$ and then $\lim_{n \rightarrow \infty} h_g(\mu_n) = h_g(\mu)$. Let $\tilde{\alpha} = (\int \Phi d\mu, \int \Psi d\mu)$ and $\tilde{C}_0(\tilde{\alpha})$ be a connected component in $\tilde{C}_0$ containing the point $\tilde{\alpha}$. Let $\tilde{E}_n = \overline{c\tilde{o}} \left(\tilde{C}_0(\tilde{\alpha}) \cup \left\{ (\int \Phi d\mu_n, \int \Psi d\mu_n) \right\} \right) \cup \tilde{C}_0$, where $\overline{c\tilde{o}}(\tilde{E})$ denote the closed convex hull of $\tilde{E}$. By a similar argument as above one can show that

$$\underline{P}_{\Phi \times \Psi}(\tilde{E}_n, s) \geq h_g(\mu_n) - s \int \log \phi(g, x) d\mu_n(x).$$

Let $n \rightarrow \infty$, $D(\tilde{E}_n, \tilde{C}_0) \rightarrow 0$ and by Lemma 4.1, we have the desired inequality again. $\square$

According to (4.1) and Lemma 4.2, we can conclude that

$$\underline{P}_{Q \circ (\Phi \times \Psi)}(C, s) \geq \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\}.$$

4.1.2. Upper bound on $\overline{P}_{Q \circ (\Phi \times \Psi)}(C, s)$

The proof of the upper bound was inspired by Thompson [16]. Recall $\tilde{C}_0 = \left(Q|_{\tilde{\mathcal{P}}(\mathcal{M}_g(J))} \right)^{-1}(C)$ and $\tilde{C}_0$ is compact satisfying $Q(\tilde{C}_0) = C$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F_{Q \circ (\Phi \times \Psi)}(C; n, \delta) \subseteq F_{\Phi \times \Psi}(\tilde{C}_0; n, \varepsilon).$$

Thus

$$\overline{P}_{Q \circ (\Phi \times \Psi)}(C, s) \leq \overline{P}_{\Phi \times \Psi}(\tilde{C}_0, s). \quad (4.4)$$

Our main goal in this section is to prove the following lemma.

Lemma 4.3.

$$\begin{aligned} & \overline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \\ & \leq \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \tilde{C}_0 \right\} \\ & = \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\}. \end{aligned}$$

Proof. Suppose $\tilde{C}_1, \tilde{C}_2$ are closed subsets of $\mathbb{R}^d \times (\mathbb{R}^+)^d$, then obviously

$$F_{\Phi \times \Psi}(\tilde{C}_1 \cup \tilde{C}_2; n, \varepsilon) = F_{\Phi \times \Psi}(\tilde{C}_1; n, \varepsilon) \cup F_{\Phi \times \Psi}(\tilde{C}_2; n, \varepsilon).$$

Therefore

$$f_{\Phi \times \Psi}(\tilde{C}_1 \cup \tilde{C}_2, s; n, \varepsilon) \leq 2 \max \left\{ f_{\Phi \times \Psi}(\tilde{C}_1, s; n, \varepsilon), f_{\Phi \times \Psi}(\tilde{C}_2, s; n, \varepsilon) \right\}.$$

This implies that

$$\overline{P}_{\Phi \times \Psi}(\tilde{C}_1 \cup \tilde{C}_2, s) \leq \max \left\{ \overline{P}_{\Phi \times \Psi}(\tilde{C}_1, s), \overline{P}_{\Phi \times \Psi}(\tilde{C}_2, s) \right\}.$$

Hence for any small $r > 0$, we have

$$\overline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \leq \sup_{\alpha \in \tilde{C}_0} \overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s), \quad (4.5)$$

where $\overline{B}_r(\alpha) \subseteq \mathbb{R}^d \times (\mathbb{R}^+)^d$ is the closed ball of radius r with the center α . In the following, we derive an upper bound for $\overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s)$.

Proposition 4.1. For each $\alpha \in \tilde{C}_0$ one has

$$\begin{aligned} & \overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s) \\ & \leq \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \overline{B}_r(\alpha) \right\}. \end{aligned}$$

Proof. Given any $\gamma > 0$, by the definition of $\overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log f_{\Phi \times \Psi}(\overline{B}_r(\alpha), s; n, \varepsilon) > \overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s) - \frac{1}{2}\gamma.$$

Taking $\varepsilon_k = \frac{\varepsilon}{2^k}$, $k \geq 1$. Set $N_0 = 0$, then pick a sequence N_k , $k \geq 1$, so that

$$N_{k+1} > \exp \left[\sum_{i=0}^k (N_i + M) \right] \quad (4.6)$$

(Recall that the positive integer M is defined in Section 2.1.) and

$$f_{\Phi \times \Psi}(\overline{B}_r(\alpha), s; N_k, \varepsilon_k) > \exp [N_k (\overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s) - \gamma)]. \quad (4.7)$$

Let $F_k := F_{\Phi \times \Psi}(\overline{B}_r(\alpha); N_k, \varepsilon_k)$. Since $A^M > 0$, for any $w \in \Sigma_{A,n}$ and any $z \in \{0, 1, \dots, m-1\}$, there exists $\sigma_{wz} = (y_1, \dots, y_M) \in \Sigma_{A,M}$ such that $w\sigma_{wz}z \in \Sigma_{A,n+M+1}$. Define

$$\Gamma_A := \{w_1\sigma_1w_2\sigma_2\dots \mid w_j \in F_j, \sigma_j \in \Sigma_{A,M} \text{ for } j \geq 1\}$$

and $Z := \pi(\Gamma_A)$. Let $t_1 = N_1$ and $t_k = t_{k-1} + M + N_k$ for $k \geq 2$. By (4.6) we have

$$\lim_{k \rightarrow \infty} \frac{t_{k-1}}{t_k} = 0 \text{ and } \lim_{k \rightarrow \infty} \frac{N_k}{t_k} = 1.$$

Claim 4.2. *There exists $\{\varepsilon'_k\}_{k \geq 1}$ such that if $p \in Z$ then*

$$d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \overline{B}_r(\alpha)\right) < \varepsilon'_k$$

and $\lim_{k \rightarrow \infty} \varepsilon'_k = 0$.

Proof. For any $p \in Z$,

$$\begin{aligned} & \frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p) \\ &= \frac{1}{t_k} \left[\sum_{i=0}^{N_1-1} (\Phi \times \Psi)(g^i(p)) + \sum_{i=0}^{N_2-1} (\Phi \times \Psi)(g^{t_1+M+i}(p)) + \dots + \sum_{i=0}^{N_k-1} (\Phi \times \Psi)(g^{t_{k-1}+M+i}(p)) \right. \\ & \quad \left. + \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_1+i}(p)) + \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_2+i}(p)) + \dots + \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_{k-1}+i}(p)) \right] \\ &= \frac{N_1 + \dots + N_k}{t_k} \cdot \sum_{j=0}^{k-1} \frac{N_{j+1}}{N_1 + \dots + N_k} \cdot \frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^{t_j+M+i}(p)) \\ & \quad + \frac{M}{t_k} \cdot \sum_{j=1}^{k-1} \frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_j+i}(p)) \\ &= u_k \cdot \Lambda_k + \Delta_k, \end{aligned}$$

where

$$\begin{aligned} t_0 &:= -M, \\ u_k &:= \frac{N_1 + \dots + N_k}{t_k}, \\ \Lambda_k &:= \sum_{j=0}^{k-1} \frac{N_{j+1}}{N_1 + \dots + N_k} \cdot \frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^{t_j+M+i}(p)), \\ \Delta_k &:= \frac{M}{t_k} \cdot \sum_{j=1}^{k-1} \frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_j+i}(p)). \end{aligned}$$

Observation 4.1. *We have $\lim_{k \rightarrow \infty} u_k = 1$ because $1 - u_k = \frac{(k-1)M}{t_k} \leq \frac{(k-1)M}{N_k} \rightarrow 0$ (as $k \rightarrow \infty$).*

Observation 4.2. We have $\lim_{k \rightarrow \infty} d(0, \Delta_k) = 0$. In fact

$$\begin{aligned} & d(0, \Delta_k) \\ &= d\left(0, \frac{M}{t_k} \left[\frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_1+i}(p)) + \dots + \frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_{k-1}+i}(p)) \right] \right) \\ &\leq \frac{M}{t_k} \cdot d\left(0, \frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_1+i}(p))\right) + \dots + \frac{M}{t_k} \cdot d\left(0, \frac{1}{M} \sum_{i=0}^{M-1} (\Phi \times \Psi)(g^{t_{k-1}+i}(p))\right) \\ &\leq \frac{kM}{t_k} \cdot \|\Phi \times \Psi\|_\infty. \end{aligned}$$

The desired result follows from the above inequality since $\lim_{k \rightarrow \infty} \frac{kM}{t_k} \|\Phi \times \Psi\|_\infty = 0$.

Observation 4.3. We have $\lim_{k \rightarrow \infty} d(\Lambda_k, \overline{B}_r(\alpha)) = 0$. Indeed, since $p \in Z$, then there exists $w = (w_1 \sigma_1 w_2 \sigma_2 \dots) \in \Gamma_A$ such that $p = \pi(w)$. For each $j \in \{1, \dots, k\}$, there exists $x_j \in R_{w_j}$ and $y_j \in \overline{B}_r(\alpha)$ such that

$$d\left(\frac{1}{N_j} \sum_{i=0}^{N_j-1} (\Phi \times \Psi)(g^i(x_j)), y_j\right) < \varepsilon_j.$$

It follows that

$$\begin{aligned} & d\left(\Lambda_k, \frac{N_1}{N_1 + \dots + N_k} y_1 + \dots + \frac{N_k}{N_1 + \dots + N_k} y_k\right) \\ &\leq \sum_{j=0}^{k-1} \frac{N_{j+1}}{N_1 + \dots + N_k} d\left(\frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^{t_j+M+i}(p)), y_{j+1}\right) \\ &\leq \sum_{j=0}^{k-1} \frac{N_{j+1}}{N_1 + \dots + N_k} d\left(\frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^{t_j+M+i}(p)), \frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^i(x_{j+1}))\right) \\ &\quad + \sum_{j=0}^{k-1} \frac{N_{j+1}}{N_1 + \dots + N_k} d\left(\frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} (\Phi \times \Psi)(g^i(x_{j+1})), y_{j+1}\right) \\ &\leq \frac{\sum_{i=1}^{N_1} \text{var}_i(\Phi \times \Psi) + \dots + \sum_{i=1}^{N_k} \text{var}_i(\Phi \times \Psi)}{N_1 + \dots + N_k} + \frac{N_1 \varepsilon_1 + \dots + N_k \varepsilon_k}{N_1 + \dots + N_k} \rightarrow 0, \end{aligned}$$

since $\lim_{k \rightarrow \infty} N_k = \infty$, $\lim_{k \rightarrow \infty} \text{var}_{N_k}(\Phi \times \Psi) = 0$ and $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, where

$$\text{var}_i(\Phi \times \Psi) := \sup \{d((\Phi \times \Psi)(x), (\Phi \times \Psi)(y)) : x, y \in R_w, w \in \Sigma_{A,i}\}.$$

This completes the proof of Observation 4.3, because $\overline{B}_r(\alpha)$ is convex.

Observation 4.1, 4.2 and 4.3 imply that there is $\{\varepsilon'_k\}_{k \geq 1}$ with $\lim_{k \rightarrow \infty} \varepsilon'_k = 0$ such that

$$d\left(\frac{1}{t_k} \sum_{i=0}^{t_k-1} (\Phi \times \Psi)(g^i(p)), \overline{B}_r(\alpha)\right) = d(u_k \Lambda_k + \Delta_k, \overline{B}_r(\alpha))$$

$$\begin{aligned}
&= d(\Lambda_k + (u_k - 1)\Lambda_k + \Delta_k, \bar{B}_r(\alpha)) \\
&\leq d(\Lambda_k, \bar{B}_r(\alpha)) + d((u_k - 1)\Lambda_k, 0) + d(\Delta_k, 0) \\
&\leq \varepsilon'_k.
\end{aligned}$$

It completes the proof of Claim 4.2. $\square$

To finish the proof of Proposition 4.1, we need to define $\underline{CP}_Z(-s \log \phi(g, \cdot))$. Let

$$Q_{t_k}(Z, -s \log \phi(g, \cdot), \delta) := \inf \left\{ \sum_{x \in S} \exp[-s(S_{t_k} \log \phi(g, x))] : S \text{ is a } (t_k, \delta) \text{ spanning set for } Z \right\}$$

and

$$P_{t_k}(Z, -s \log \phi(g, \cdot), \delta) := \sup \left\{ \sum_{x \in S} \exp[-s(S_{t_k} \log \phi(g, x))] : S \text{ is a } (t_k, \delta) \text{ separated set for } Z \right\}.$$

We have

$$Q_{t_k}(Z, -s \log \phi(g, \cdot), \delta) \leq P_{t_k}(Z, -s \log \phi(g, \cdot), \delta), \quad (4.8)$$

and see [18] for more details. Define

$$\underline{CP}_Z(-s \log \phi(g, \cdot), \delta) := \liminf_{k \rightarrow \infty} \frac{1}{t_k} \log Q_{t_k}(Z, -s \log \phi(g, \cdot), \delta)$$

and

$$\underline{CP}_Z(-s \log \phi(g, \cdot)) := \lim_{\delta \rightarrow 0} \underline{CP}_Z(-s \log \phi(g, \cdot), \delta).$$

Claim 4.3. $\underline{CP}_Z(-s \log \phi(g, \cdot)) \geq \bar{P}_{\Phi \times \Psi}(\bar{B}_r(\alpha), s) - \gamma$.

Proof. For any $\varepsilon \in (0, \varepsilon_0)$, there exists $0 < \delta_0 < \delta_1$ such that for any $\delta \in (0, \delta_0)$, if $d(x, y) < \delta$ then

$$e^{-\varepsilon} < \frac{\phi(g, x)}{\phi(g, y)} < e^{\varepsilon}.$$

(Here δ_1 is the same as in Section 3.2.) Take a (t_k, δ) spanning set T_k of Z satisfying

$$\sum_{x \in T_k} \exp[-s(S_{t_k} \log \phi(g, x))] \leq Q_{t_k}(Z, -s \log \phi(g, \cdot), \delta) \cdot e^{\varepsilon}. \quad (4.9)$$

It follows that $g^{t_{k-1}+M}(T_k)$ is a (N_k, δ) spanning set for $g^{t_{k-1}+M}(Z) = \pi(\sigma^{t_{k-1}+M}(\Gamma_A))$. Recall $F_k := F_{\Phi \times \Psi}(\bar{B}_r(\alpha); N_k, \varepsilon_k)$ and $R_w := \pi([w])$ for any $[w] \in F_k$. Then we have $R_w \cap g^{t_{k-1}+M}(Z) \neq \emptyset$ for any $[w] \in F_k$. Since

$$\bigcup_{y \in g^{t_{k-1}+M}(T_k)} B_{N_k}(y, \delta) \supseteq g^{t_{k-1}+M}(Z),$$

then for any $[w] \in F_k$ there exists $y \in g^{t_{k-1}+M}(T_k)$ such that

$$R_w \cap B_{N_k}(y, \delta) \neq \emptyset.$$

Fixed any $y \in g^{t_{k-1}+M}(T_k)$, denote

$$\mathcal{B}_y := \{[w] : [w] \in F_k \text{ and } R_w \cap B_{N_k}(y, \delta) \neq \emptyset\}.$$

For each $[w] \in \mathcal{B}_y$ and any $x \in R_w \cap B_{N_k}(y, \delta)$, Proposition 3.1 told us that there exists z such that

$$B\left(z, \frac{1}{2}K^{-1}(N_k, \varepsilon) \prod_{j=0}^{N_k-1} \phi(g, g^j(x))^{-1}\right) \subseteq R_w$$

and $\text{diam} R_w \leq K(N_k, \varepsilon) \cdot \prod_{j=0}^{N_k-1} \phi(g, g^j(x))^{-1}$. Similarly to (3.6) there is a constant $K_3 > 1$ such that

$$\text{diam} B_{N_k}(y, \delta) \leq 2K_3 e^{3N_k \varepsilon} \delta \cdot \prod_{j=0}^{N_k-1} \phi(g, g^j(y))^{-1}, \quad (4.10)$$

where $N_k > \tilde{N}_1$ is large enough and $\tilde{N}_1$ is the same as in (3.2). Therefore

$$B_{N_k}(y, \delta) \subseteq B\left(y, (2K_3 e^{3N_k \varepsilon} \delta + e^{N_k \varepsilon} \cdot K(N_k, \varepsilon)) \cdot \prod_{i=0}^{N_k-1} \phi(g, g^i(y))^{-1}\right)$$

and

$$R_w \subseteq B\left(y, (2K_3 e^{3N_k \varepsilon} \delta + e^{N_k \varepsilon} \cdot K(N_k, \varepsilon)) \cdot \prod_{i=0}^{N_k-1} \phi(g, g^i(y))^{-1}\right).$$

This yields that

$$\sharp \mathcal{B}_y \leq (4K_3 \delta e^{4N_k \varepsilon} K(N_k, \varepsilon) + 2e^{2N_k \varepsilon} K^2(N_k, \varepsilon))^{s_0}.$$

So

$$\begin{aligned} & \sum_{[w] \in F_k} (\text{diam} R_w)^s \\ & \leq \sum_{y \in g^{t_{k-1}+M}(T_k)} \sum_{[w] \in \mathcal{B}_y} (\text{diam} R_w)^s \\ & \leq \sum_{y \in g^{t_{k-1}+M}(T_k)} \sum_{[w] \in \mathcal{B}_y} (K(N_k, \varepsilon))^s \cdot \left(\prod_{i=0}^{N_k-1} \phi(g, g^i(x))^{-1} \right)^s \quad (\text{where } x \in R_w \cap B_{N_k}(y, \delta)) \\ & \leq \sum_{y \in g^{t_{k-1}+M}(T_k)} \sum_{[w] \in \mathcal{B}_y} (e^{N_k \varepsilon} K(N_k, \varepsilon))^s \cdot \left(\prod_{i=0}^{N_k-1} \phi(g, g^i(y))^{-1} \right)^s \\ & \leq (e^{N_k \varepsilon} K(N_k, \varepsilon))^s \cdot [4K_3 \delta e^{4N_k \varepsilon} K(N_k, \varepsilon) + 2(e^{N_k \varepsilon} K(N_k, \varepsilon))^2]^{s_0} \\ & \quad \times \sum_{y \in g^{t_{k-1}+M}(T_k)} \left(\prod_{i=0}^{N_k-1} \phi(g, g^i(y))^{-1} \right)^s. \end{aligned}$$

Combining with (4.7) one has

$$\begin{aligned} & \sum_{z \in T_k} \exp[-s(S_{t_k} \log \phi(g, z))] \\ & = \sum_{z \in T_k} \exp[-s(S_{t_{k-1}+M} \log \phi(g, z)) - s(S_{N_k} \log \phi(g, g^{t_{k-1}+M}(z)))] \end{aligned}$$

$$\begin{aligned}
&\geq \exp \left[(t_{k-1} + M) \cdot \inf_{x \in J} (-s \log \phi(g, x)) \right] \cdot \sum_{z \in T_k} \exp [-s(S_{N_k} \log \phi(g, g^{t_{k-1}+M}(z)))] \\
&= \exp \left[(t_{k-1} + M) \cdot \inf_{x \in J} (-s \log \phi(g, x)) \right] \sum_{y \in g^{t_{k-1}+M}(T_k)} \exp [-s(S_{N_k} \log \phi(g, y))] \\
&\geq \exp \left[(t_{k-1} + M) \inf_{x \in J} (-s \log \phi(g, x)) - (s + 2s_0)N_k \varepsilon \right] \cdot (K(N_k, \varepsilon))^{-(s+s_0)} \\
&\quad \times (4K_3 \delta e^{2N_k \varepsilon} + 2K(N_k, \varepsilon))^{-s_0} \cdot \sum_{[w] \in F_k} (\text{diam } R_w)^s \\
&\geq (4K_3 \delta e^{2N_k \varepsilon} + 2(K(N_k, \varepsilon)))^{-s_0} \cdot (K(N_k, \varepsilon))^{-(s+s_0)} \\
&\quad \times \exp \left[(t_{k-1} + M) \inf_{x \in J} (-s \log \phi(g, x)) - (s + 2s_0)N_k \varepsilon + N_k \cdot (\bar{P}_{\Phi \times \Psi}(\bar{B}_r(\alpha), s) - \gamma) \right].
\end{aligned}$$

It follows from (4.9), the definition of $\underline{CP}_Z(-s \log \phi(g, \cdot), \delta)$, $\lim_{k \rightarrow \infty} \frac{N_k}{t_k} = 1$, $\lim_{k \rightarrow \infty} \frac{t_{k-1}}{t_k} = 0$ and $\lim_{n \rightarrow \infty} \frac{\log K(n, \varepsilon)}{n} = 3\varepsilon$ that

$$\underline{CP}_Z(-s \log \phi(g, \cdot), \delta) \geq \bar{P}_{\Phi \times \Psi}(\bar{B}_r(\alpha), s) - \gamma - (s + 7s_0)\varepsilon.$$

Let $\varepsilon \rightarrow 0$, then $\delta \rightarrow 0$. It follows that

$$\underline{CP}_Z(-s \log \phi(g, \cdot)) \geq \bar{P}_{\Phi \times \Psi}(\bar{B}_r(\alpha), s) - \gamma.$$

This completes the proof of Claim 4.3. $\square$

Proceed with the proof of Proposition 4.1. We fix $\gamma > 0$ and find some $\delta > 0$ such that

$$\underline{CP}_Z(-s \log \phi(g, \cdot), \delta) > \underline{CP}_Z(-s \log \phi(g, \cdot)) - \gamma. \quad (4.11)$$

Let D_k be a (t_k, δ) separated set for Z such that

$$\sum_{x \in D_k} \exp [-s(S_{t_k} \log \phi(g, x))] \geq P_{t_k}(Z, -s \log \phi(g, \cdot), \delta) \cdot e^{-\gamma},$$

and write $P_{t_k} := \sum_{x \in D_k} \exp [-s(S_{t_k} \log \phi(g, x))]$. Define

$$\sigma_{t_k} := \frac{1}{P_{t_k}} \sum_{x \in D_k} \exp [-s(S_{t_k} \log \phi(g, x))] \cdot \delta_x$$

and let $\mu_{t_k} := \frac{1}{t_k} \sum_{i=0}^{t_k-1} \sigma_{t_k} \circ g^{-i}$.

Let t_{k_j} be a sequence of numbers so that $\mu_{t_{k_j}}$ converges, and μ be the limit measure. We have $\mu \in \mathcal{M}_g(J)$ and we will verify that $\int (\Phi \times \Psi) d\mu \in \bar{B}_r(\alpha)$ (here $\int (\Phi \times \Psi) d\mu := (\int \Phi d\mu, \int \Psi d\mu)$), because

$$\int (\Phi \times \Psi) d\mu_{t_{k_j}} = \frac{1}{t_{k_j}} \sum_{i=0}^{t_{k_j}-1} \int (\Phi \times \Psi) d(\sigma_{t_{k_j}} \circ g^{-i})$$

$$\begin{aligned}
&= \frac{1}{t_{k_j}} \sum_{i=0}^{t_{k_j}-1} \int (\Phi \times \Psi) \circ g^i d\sigma_{t_{k_j}} \\
&= \frac{1}{t_{k_j}} \sum_{i=0}^{t_{k_j}-1} \frac{1}{P_{t_{k_j}}} \sum_{x \in D_{k_j}} \left\{ \exp \left[-s(S_{t_{k_j}} \log \phi(g, x)) \right] \cdot (\Phi \times \Psi)(g^i x) \right\} \\
&= \frac{1}{P_{t_{k_j}}} \sum_{x \in D_{k_j}} \left\{ \exp \left[-s(S_{t_{k_j}} \log \phi(g, x)) \right] \cdot \left[\frac{1}{t_{k_j}} \sum_{i=0}^{t_{k_j}-1} (\Phi \times \Psi)(g^i x) \right] \right\}
\end{aligned}$$

and $\frac{1}{t_{k_j}} \sum_{i=0}^{t_{k_j}-1} (\Phi \times \Psi)(g^i x) \in B(\overline{B}_r(\alpha), \varepsilon'_{k_j})$ (see Claim 4.2) and $\overline{B}_r(\alpha)$ is convex and closed.

To show that

$$\liminf_{j \rightarrow \infty} \frac{1}{t_{k_j}} \log P_{t_{k_j}} \leq h_g(\mu) - s \int \log \phi(g, \cdot) d\mu,$$

we recall some key ingredients of the proof of the variational principle. (See Theorem 9.10 in [18] for more notations and details.) Let ξ be a partition of J with diameter less than ε and $\mu(\partial\xi) = 0$. Then we obtain

$$H_{\sigma_{t_k}} \left(\bigvee_{i=0}^{t_k} g^{-i} \xi \right) - s \int S_{t_k} \log \phi(g, \cdot) d\sigma_{t_k} = \log P_{t_k}. \quad (\text{see page 220 in [18]})$$

Since $\mu(\partial\xi) = 0$, we have for any $q \in \mathbb{N}$,

$$\lim_{j \rightarrow \infty} H_{\mu_{t_{k_j}}} \left(\bigvee_{i=0}^{q-1} g^{-i} \xi \right) = H_\mu \left(\bigvee_{i=0}^{q-1} g^{-i} \xi \right). \quad (\text{see page 220 in [18]})$$

For a fixed t_k and $1 < q < t_k$, we have

$$\frac{q}{t_k} \log P_{t_k} \leq H_{\mu_{t_k}} \left(\bigvee_{i=0}^{q-1} g^{-i} \xi \right) - sq \int \log \phi(g, \cdot) d\mu_{t_k} + 2 \frac{q^2}{t_k} \log \sharp \xi. \quad (\text{see } (\star) \text{ in page 220 of [18]})$$

Replacing t_k by t_{k_j} and taking $j \rightarrow \infty$, we obtain

$$q \liminf_{j \rightarrow \infty} \frac{1}{t_{k_j}} \log P_{t_{k_j}} \leq H_\mu \left(\bigvee_{i=0}^{q-1} g^{-i} \xi \right) - qs \int \log \phi(g, \cdot) d\mu.$$

Dividing by q and letting $q \rightarrow \infty$, one has

$$\liminf_{j \rightarrow \infty} \frac{1}{t_{k_j}} \log P_{t_{k_j}} \leq h_g(\mu) - s \int \log \phi(g, \cdot) d\mu.$$

Combining with (4.8) we obtain

$$\begin{aligned}
\underline{CP}_Z(-s \log \phi(g, \cdot), \delta) &\leq \liminf_{k \rightarrow \infty} \frac{1}{t_k} P_{t_k}(Z, -s \log \phi(g, \cdot), \delta) \\
&\leq \liminf_{j \rightarrow \infty} \frac{1}{t_{k_j}} \log P_{t_{k_j}} + \gamma
\end{aligned}$$

$$\begin{aligned}
&\leq h_\mu(g, \xi) - s \int \log \phi(g, \cdot) d\mu + \gamma \\
&\leq h_g(\mu) - s \int \log \phi(g, \cdot) d\mu + \gamma.
\end{aligned} \tag{4.12}$$

It follows from Claim 4.3, (4.11) and (4.12) that

$$\begin{aligned}
\overline{P}_{\Phi \times \Psi}(\overline{B}_r(\alpha), s) &\leq \underline{CP}_Z(-s \log \phi(g, \cdot)) + \gamma \\
&\leq \underline{CP}_Z(-s \log \phi(g, \cdot), \delta) + 2\gamma \\
&\leq h_g(\mu) - s \int \log \phi(g, \cdot) d\mu + 3\gamma.
\end{aligned}$$

This completes the proof of Proposition 4.1. $\square$

Proceed with the proof of Lemma 4.3. It follows from (4.5) and Proposition 4.1 that

$$\begin{aligned}
&\overline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \\
&\leq \liminf_{r \rightarrow 0} \sup_{\alpha \in \tilde{C}_0} \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \right. \\
&\quad \left. \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \overline{B}_r(\alpha) \right\}.
\end{aligned} \tag{4.13}$$

Let us show that the right-hand side of (4.13) does not exceed

$$H_{\Phi \times \Psi}(\tilde{C}_0) := \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \tilde{C}_0 \right\}.$$

Assume the contrary. Then for any arbitrary sequence $r_n \rightarrow 0$, there exists $\gamma > 0$ and a sequence $\alpha_n \in \tilde{C}_0$ such that

$$\begin{aligned}
&\sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \left(\int \Phi d\mu, \int \Psi d\mu \right) \in \overline{B}_{r_n}(\alpha_n) \right\} \\
&\geq H_{\Phi \times \Psi}(\tilde{C}_0) + \gamma.
\end{aligned}$$

Hence there exists a sequence of invariant measures $\{\mu_n\}$ such that $(\int \Phi d\mu_n, \int \Psi d\mu_n) \in \overline{B}_{r_n}(\alpha_n)$ and

$$h_g(\mu_n) - s \int \log \phi(g, x) d\mu_n \geq H_{\Phi \times \Psi}(\tilde{C}_0) + \frac{\gamma}{2}.$$

The sequence $\{\alpha_n\}$ is drawn from a compact set $\tilde{C}_0$ and hence has a convergent subsequence $\alpha_{n_k} \rightarrow \alpha \in \tilde{C}_0$. Without loss of generality, we may assume that a sequence of measures μ_{n_k} converges in the weak* topology to an invariant measure μ . (Otherwise choose a convergent subsequence.) It is clear that

$$\left(\int \Phi d\mu_{n_k}, \int \Psi d\mu_{n_k} \right) = \alpha_{n_k} + O(r_{n_k}) \rightarrow \alpha \text{ as } k \rightarrow \infty$$

and μ is such that $(\int \Phi d\mu, \int \Psi d\mu) = \alpha \in \tilde{C}_0$. Since $g : J \rightarrow J$ is expanding and J is compact, then the entropy map is upper semi-continuous (Theorem 8.2, [18]). It follows that

$$h_g(\mu) - s \int \log \phi(g, x) d\mu \geq \limsup_{k \rightarrow \infty} h_{\mu_{n_k}}(g) - s \lim_{k \rightarrow \infty} \int \log \phi(g, x) d\mu_{n_k}$$

$$\geq H_{\Phi \times \Psi}(\tilde{C}_0) + \frac{\gamma}{2}.$$

But $h_g(\mu) - s \int \log \phi(g, x) d\mu \leq H_{\Phi \times \Psi}(\tilde{C}_0)$. Hence we have arrived at a contradiction. So $\overline{P}_{\Phi \times \Psi}(\tilde{C}_0, s) \leq H_{\Phi \times \Psi}(\tilde{C}_0)$ which finishes the proof of Lemma 4.3. $\square$

From (4.4) and Lemma 4.3, it follows that

$$\overline{P}_{Q_0(\Phi \times \Psi)}(C, s) \leq \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\}.$$

This completes the proof of Theorem 2.1.

4.2. Proof of Theorem 2.2

First of all, we present the property of the pressure function $P_{Q_0(\Phi \times \Psi)}^*(C, \cdot)$ for any compact subset $C \subseteq \mathcal{P}(\mathcal{M}_g(J))$.

Proposition 4.2. *For any compact subset $C \subseteq \mathcal{P}(\mathcal{M}_g(J))$, the pressure function $P_{Q_0(\Phi \times \Psi)}^*(C, \cdot)$ is continuous and strictly decreasing on s . Moreover, $\Lambda_{Q_0(\Phi \times \Psi)}^*(C)$ is the unique root of $P_{Q_0(\Phi \times \Psi)}^*(C, s) = 0$.*

Proof. For any $s, t \in \mathbb{R}$ with $t > s$, Theorem 2.1 told us that

$$\begin{aligned} & P_{Q_0(\Phi \times \Psi)}^*(C, s) \\ &= \sup \left\{ h_g(\mu) - s \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\} \\ &= \sup \left\{ h_g(\mu) - t \int \log \phi(g, x) d\mu + (t - s) \int \log \phi(g, x) d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) \in C \right\}. \end{aligned} \quad (4.14)$$

Thus

$$\lambda_2 \cdot (t - s) \leq P_{Q_0(\Phi \times \Psi)}^*(C, s) - P_{Q_0(\Phi \times \Psi)}^*(C, t) \leq \lambda_1 \cdot (t - s),$$

where

$$\lambda_1 := \max_{x \in J} \{\log \phi(g, x)\} \quad \text{and} \quad \lambda_2 := \min_{x \in J} \{\log \phi(g, x)\}.$$

This implies that $P_{Q_0(\Phi \times \Psi)}^*(C, \cdot)$ is continuous and strictly decreasing on s . By (4.14) one has $P_{Q_0(\Phi \times \Psi)}^*(C, 0) \geq 0$. It follows from g is C^1 average conformal and Theorem A in [4] that $P_{Q_0(\Phi \times \Psi)}^*(C, s_0) \leq 0$, where $\dim \mathbf{M} = s_0$. Thus the desired results follow. $\square$

Then we will utilize the unique root of $P_{Q_0(\Phi \times \Psi)}^*(C, s) = 0$ to give an upper bound for the Hausdorff dimension of the subset $E_{Q_0(\Phi \times \Psi)}(C)$.

Lemma 4.4. *For any compact subset $C \subseteq \mathcal{P}(\mathcal{M}_g(J))$, we obtain*

$$\dim_H E_{Q_0(\Phi \times \Psi)}(C) \leq \Lambda_{Q_0(\Phi \times \Psi)}^*(C),$$

where $\Lambda_{Q_0(\Phi \times \Psi)}^*(C)$ is the unique root of $P_{Q_0(\Phi \times \Psi)}^*(C, s) = 0$.

Proof. Let $t > \Lambda_{Q_0(\Phi \times \Psi)}^*(C)$, then $P_{Q_0(\Phi \times \Psi)}^*(C, t) < 0$. It follows that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{\log f_{Q_0(\Phi \times \Psi)}(C, t; n, \varepsilon)}{n} < -\delta.$$

Therefore there is a number $N = N(\varepsilon, \delta)$ such that for $n > N$,

$$f_{Q \circ (\Phi \times \Psi)}(C, t; n, \varepsilon) \leq e^{-n\delta}.$$

Let

$$G(C, k, \varepsilon) := \bigcap_{n=k}^{\infty} \left\{ [w] \in \Sigma_{A,n} : \text{there exists } x \in R_w \text{ satisfying} \right. \\ \left. d \left(Q \circ \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i x), C \right) < \varepsilon \right\}.$$

It is clear that for any $\varepsilon > 0$,

$$E_{Q \circ (\Phi \times \Psi)}(C) \subseteq \bigcup_{k=1}^{\infty} \pi(G(C, k, \varepsilon)).$$

We show below that for each k ,

$$\dim_H \pi(G(C, k, \varepsilon)) \leq t.$$

Let $\mathcal{F}$ be a collection of disjoint cylinders $[w]$, $w \in \Sigma_{A,n}$, $n \geq \max\{k, N\}$ such that

$$R_w \cap \pi(G(C, k, \varepsilon)) \neq \emptyset.$$

Thus each $[w]$ in $\mathcal{F}$ is contained in $F_{Q \circ (\Phi \times \Psi)}(C; n, \varepsilon)$ for some $n \geq N$. It follows that

$$\begin{aligned} \sum_{[w] \in \mathcal{F}} (\text{diam } R_w)^t &\leq \sum_{n \geq N} \sum_{[w] \in F_{Q \circ (\Phi \times \Psi)}(C; n, \varepsilon)} (\text{diam } R_w)^t \\ &= \sum_{n \geq N} f_{Q \circ (\Phi \times \Psi)}(C, t; n, \varepsilon) \\ &\leq \sum_{n \geq N} e^{-n\delta} \\ &\leq \frac{e^{-\delta}}{1 - e^{-\delta}} \\ &< +\infty. \end{aligned}$$

This implies that

$$\dim_H \pi(G(C, k, \varepsilon)) \leq t.$$

By the property of Hausdorff dimension, we have

$$\dim_H E_{Q \circ (\Phi \times \Psi)}(C) \leq t$$

as desired. □

It follows from Lemma 4.4 that

$$\dim_H E_{Q \circ (\Phi \times \Psi)}(C) \leq \Lambda_{Q \circ (\Phi \times \Psi)}^*(C),$$

where $\Lambda_{Q_\circ(\Phi \times \Psi)}^*(C)$ is the unique solution that satisfies $P_{Q_\circ(\Phi \times \Psi)}^*(C, s) = 0$. Combing with Theorem 2.1, one has

$$\dim_H E_{Q_\circ(\Phi \times \Psi)}(C) \leq \sup_{\mu \in \mathcal{M}_g(J)} \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu} : \mathcal{P}(\mu) \in C \right\}.$$

On the other hand we will show

$$\dim_H E_{Q_\circ(\Phi \times \Psi)}(C) \geq \sup_{\mu \in \mathcal{M}_g(J)} \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu} : \mathcal{P}(\mu) \in C \right\}.$$

For every $\mu \in \mathcal{M}_g(J)$ with $\mathcal{P}(\mu) \in C$, define

$$G_\mu := \left\{ x \in J : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{g^i(x)} \rightarrow \mu \text{ in the weak* topological as } n \rightarrow \infty \right\},$$

where δ_y denotes the probability measure whose support is the single point y . It follows from Proposition 3.1 in [4] and Lemma 4.2 in [4] that

$$\dim_H G_\mu = \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu}.$$

(If (J, g) satisfies the specification property, then (J, g) is saturated, that is, $G_\mu \neq \emptyset$ and $h_{\text{top}}(G_\mu) = h_\mu(f)$.) Since $G_\mu \subseteq E_{Q_\circ(\Phi \times \Psi)}(C)$, then

$$\dim_H E_{Q_\circ(\Phi \times \Psi)}(C) \geq \sup_{\mu \in \mathcal{M}_g(J)} \left\{ \frac{h_g(\mu)}{\int \log \phi(g, x) d\mu} : \mathcal{P}(\mu) \in C \right\}.$$

This completes the proof of Theorem 2.2.

5. Proofs of Theorem 2.3 and Theorem 2.4

5.1. Proof of Theorem 2.3

Recall $\tilde{\mathcal{P}} : \mathcal{M}_g(J) \rightarrow \mathbb{R}^d \times (\mathbb{R}^+)^d$, $\tilde{\mathcal{P}}(\mu) = (\int \Phi d\mu, \int \Psi d\mu)$, and $Q(\tilde{\mathcal{P}}(\mathcal{M}_g(J))) = \mathcal{P}(\mathcal{M}_g(J))$.

5.1.1. Upper bound on $P(E_{Q_\circ(\Phi \times \Psi)}^{\text{equ}}(C), \varphi)$

For each $x \in E_{Q_\circ(\Phi \times \Psi)}^{\text{equ}}(C)$, by (1.3) one has $\mathcal{A}\left(\frac{S_n \Phi(x)}{S_n \Psi(x)}\right) = C$. It follows from Lemma 7.5 in [13] that

$$\tilde{C}_1(x) := \mathcal{A}\left(\frac{1}{n} S_n(\Phi \times \Psi)(x)\right)$$

is continuum. Thus $\tilde{C}_1(x)$ is compact and connected. This implies that

$$E_{Q_\circ(\Phi \times \Psi)}^{\text{equ}}(C) \subseteq \bigcup_{x \in E_{Q_\circ(\Phi \times \Psi)}^{\text{equ}}(C)} E_{\Phi \times \Psi}^{\text{equ}}(\tilde{C}_1(x)), \quad (5.1)$$

where $E_{\Phi \times \Psi}^{\text{equ}}(\tilde{C}_1(x)) := \left\{ x \in J : \mathcal{A}\left(\frac{1}{n} \sum_{i=0}^{n-1} (\Phi \times \Psi)(g^i(x))\right) = \tilde{C}_1(x) \right\}$. Then we have the following lemma.

Lemma 5.1. For each $x \in E_{Q \circ (\Phi \times \Psi)}^{equ}(C)$, we obtain

$$(1) \quad Q(\tilde{C}_1(x)) = C;$$

$$(2)$$

$$\begin{aligned} & P\left(E_{\Phi \times \Psi}^{equ}(\tilde{C}_1(x)), \varphi\right) \\ & \leq \inf_{\tilde{\alpha} \in \tilde{C}_1(x)} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \tilde{\mathcal{P}}(\mu) = \tilde{\alpha} \right\} \\ & \leq \inf_{\alpha \in C} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

Proof. (1) For each $\tilde{\alpha} \in \tilde{C}_1(x)$, there is a subsequence $\{n_k\}$ such that

$$\tilde{\alpha} = \lim_{n_k \rightarrow \infty} \frac{1}{n_k} S_{n_k}(\Phi \times \Psi)(x).$$

Thus there exists a g -invariant probability measure $\mu \in \mathcal{A}(\frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{g^i(x)})$ such that $\tilde{\alpha} = (\int \Phi d\mu, \int \Psi d\mu)$. Therefore

$$Q(\tilde{\alpha}) = \frac{\int \Phi d\mu}{\int \Psi d\mu} = \lim_{n_k \rightarrow \infty} \frac{S_{n_k} \Phi(x)}{S_{n_k} \Psi(x)} \in C.$$

This implies that $Q(\tilde{C}_1(x)) \subseteq C$.

For each $\alpha \in C$, there is a subsequence $\{m_k\}$ such that $\lim_{k \rightarrow \infty} \frac{S_{m_k} \Phi(x)}{S_{m_k} \Psi(x)} = \alpha$. Taking $\tilde{\alpha} \in \mathcal{A}(\frac{1}{m_k} S_{m_k}(\Phi \times \Psi)(x)) \subseteq \tilde{C}_1(x)$ such that

$$\tilde{\alpha} = \lim_{j \rightarrow \infty} \frac{1}{m_{k_j}} S_{m_{k_j}}(\Phi \times \Psi)(x) = \left(\int \Phi d\mu, \int \Psi d\mu \right)$$

for some subsequence $\{m_{k_j}\}$ of $\{m_k\}$ and some $\mu \in \mathcal{M}_g(J)$. Thus $\alpha = Q(\tilde{\alpha}) \in Q(\tilde{C}_1(x))$. Therefore $C \subseteq Q(\tilde{C}_1(x))$.

(2) Since $\tilde{C}_1(x)$ is a non-empty closed set, combining with Theorem 3.1 (2) in [22] and $Q(\tilde{C}_1(x)) = C$, we get

$$\begin{aligned} & P\left(E_{\Phi \times \Psi}^{equ}(\tilde{C}_1(x)), \varphi\right) \\ & \leq \inf_{\tilde{\alpha} \in \tilde{C}_1(x)} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \tilde{\mathcal{P}}(\mu) = \tilde{\alpha} \right\} \\ & \leq \inf_{\alpha \in C} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

This completes the proof of Lemma 5.1. □

It follows from (5.1) and Lemma 5.1 that

$$\begin{aligned} P\left(E_{Q^\circ(\Phi \times \Psi)}^{equ}(C), \varphi\right) &\leq P\left(\bigcup_{x \in E_{Q^\circ(\Phi \times \Psi)}^{equ}(C)} E_{\Phi \times \Psi}^{equ}(\tilde{C}_1(x)), \varphi\right) \\ &\leq \sup_{x \in E_{Q^\circ(\Phi \times \Psi)}^{equ}(C)} P\left(E_{\Phi \times \Psi}^{equ}(\tilde{C}_1(x)), \varphi\right) \\ &\leq \inf_{\alpha \in C} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

5.1.2. Lower bound on $P\left(E_{Q^\circ(\Phi \times \Psi)}^{equ}(C), \varphi\right)$

Since the expanding and topological mixing C^1 -map $g : J \rightarrow J$ has the specification property, then the lower bound can be proved following Thompson's method in [16, 17]. For each $\alpha \in C$, the upper semi-continuity of the map $\mu \mapsto h_g(\mu)$ implies that there exists μ_α with $\mathcal{P}(\mu_\alpha) = \alpha$ such that

$$\sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\} = h_g(\mu_\alpha) + \int \varphi d\mu_\alpha. \quad (5.2)$$

Set

$$\tilde{C} := \overline{\left\{ \tilde{\alpha} = \left(\int \Phi d\mu_\alpha, \int \Psi d\mu_\alpha \right) : \alpha \in C \text{ and } \mu_\alpha \text{ satisfying (5.2)} \right\}}.$$

Since C is compact and connected, then $\tilde{C}$ is compact and connected. Combining with the definition of Q one has $Q(\tilde{C}) = C$ and

$$\begin{aligned} &\inf_{\tilde{\alpha} \in \tilde{C}} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \tilde{\mathcal{P}}(\mu) = \tilde{\alpha} \right\} \\ &\geq \inf_{\alpha \in C} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

Since $Q(\tilde{C}) = C$, then $E_{\Phi \times \Psi}^{equ}(\tilde{C}) \subseteq E_{Q^\circ(\Phi \times \Psi)}^{equ}(C)$. This implies that

$$P\left(E_{\Phi \times \Psi}^{equ}(\tilde{C}), \varphi\right) \leq P\left(E_{Q^\circ(\Phi \times \Psi)}^{equ}(C), \varphi\right).$$

Thus, the proof of the lower bound reduces to proving the following proposition.

Proposition 5.1.

$$P\left(E_{\Phi \times \Psi}^{equ}(\tilde{C}), \varphi\right) \geq \inf_{\tilde{\alpha} \in \tilde{C}} \sup \left\{ h_g(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_g(J) \text{ and } \tilde{\mathcal{P}}(\mu) = \tilde{\alpha} \right\}.$$

Proof. Denote the right-hand side by $\mathcal{I}_{\tilde{C}}$. The proof of this proposition is divided into four parts.

Step 1. Construction of the intermediate sets $S_{n,i}$ for each $n \in \mathbb{N}$ and $i = 1, 2, \dots, M_n$. Since $\tilde{C}$ is compact and connected, then for each $n \in \mathbb{N}$ we may choose $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,M_n} \in \tilde{C}$ such that

$$\tilde{C} \subseteq \bigcup_{i=1}^{M_n} B\left(\alpha_{n,i}, \frac{1}{n}\right) \quad \text{and} \quad d(\alpha_{n,M_n}, \alpha_{n+1,1}) \leq \frac{1}{n}.$$

Fixed an arbitrary $\gamma > 0$, there exist $\mu_{n,1}, \mu_{n,2}, \dots, \mu_{n,M_n} \in \mathcal{M}_g(J)$ such that $\int \Phi \times \Psi d\mu_{n,i} = \alpha_{n,i}$ and

$$h_g(\mu_{n,i}) + \int \varphi d\mu_{n,i} \geq \mathcal{I}_{\tilde{C}} - \gamma$$

for each $n \in \mathbb{N}$ and $i = 1, 2, \dots, M_n$.

Lemma 5.2 (Lemma 3.5, [16]). *For each $n \in \mathbb{N}$, $i \in \{1, 2, \dots, M_n\}$ and $\delta_{n,i} = \frac{1}{n}$, there exists $\eta_{n,i} \in \mathcal{M}_g(J)$ such that $\eta_{n,i} = \sum_{j=1}^{j(n,i)} \lambda_j^{n,i} \eta_j^{n,i}$, where $\sum_{j=1}^{j(n,i)} \lambda_j^{n,i} = 1$ and $\eta_j^{n,i} \in \mathcal{E}_g(J)$ satisfying*

$$\left| \int \Phi \times \Psi d\mu_{n,i} - \int \Phi \times \Psi d\eta_{n,i} \right| < \frac{1}{n}$$

and $h_g(\eta_{n,i}) \geq h_g(\mu) - \frac{1}{n}$.

Fixed any small $\gamma \in (0, 1)$, choose a strictly increasing sequence $\ell_n \rightarrow \infty$ so that each of the sets

$$Y_j^{n,i} := \left\{ x \in J : \left| \frac{1}{m} S_m(\Phi \times \Psi)(x) - \int \Phi \times \Psi d\eta_j^{n,i} \right| < \frac{1}{n} \text{ for all } m \geq \ell_n \right\}$$

satisfies $\eta_j^{n,i}(Y_j^{n,i}) > 1 - \gamma$ for $j \in \{1, 2, \dots, j(n,i)\}$. This is possible by Birkhoff's ergodic theorem.

Lemma 5.3 (Lemma 3.6, [16]). *Let $\gamma' \in (0, 1)$ be sufficiently small. For each $n \in \mathbb{N}$, $i \in \{1, 2, \dots, M_n\}$, $j \in \{1, 2, \dots, j(n,i)\}$ and any sufficiently small $\varepsilon > 0$, we can find a sequence $\hat{m}_{n,i}$ with $[\lambda_j^{n,i} \hat{m}_{n,i}] \geq \ell_n$ and find sets $S_j^{n,i}$, so that each $S_j^{n,i}$ is $([\lambda_j^{n,i} \hat{m}_{n,i}], 5\varepsilon)$ separated set for $Y_j^{n,i}$ and*

$$\sum_{x \in S_j^{n,i}} \exp \left\{ \sum_{k=0}^{[\lambda_j^{n,i} \hat{m}_{n,i}] - 1} \varphi(g^k(x)) \right\} \geq \exp \left\{ [\lambda_j^{n,i} \hat{m}_{n,i}] \left(h_g(\eta_j^{n,i}) + \int \varphi d\eta_j^{n,i} - \frac{4\gamma'}{j(n,i)} \right) \right\}.$$

Furthermore the sequence $\hat{m}_{n,i}$ can be choose such that $\hat{m}_{n,i} > 2^{s_{n,i}}$, where

$$s_{n,i} = s \left(\frac{\varepsilon}{2^{\sum_{j=0}^{n-1} M_j + i}} \right)$$

is similar to ' $m(\varepsilon)$ ' which is defined in Section 3.5 (set $M_0 = 0$).

We choose ε sufficiently small so that Lemma 5.3 applies and $\text{var}(\varphi, 2\varepsilon) < \gamma$, $\text{var}(\Phi \times \Psi, 2\varepsilon) < \gamma$, where $\text{var}(\varphi, 2\varepsilon) := \sup \{ |\varphi(x) - \varphi(y)| : d(x, y) < 2\varepsilon \}$ and $\text{var}(\Phi \times \Psi, 2\varepsilon) := \sup \{ d((\Phi \times \Psi)(x), (\Phi \times \Psi)(y)) : d(x, y) < 2\varepsilon \}$. Let $y_j \in S_j^{n,i}$ for $j = 1, 2, \dots, j(n,i)$. We use the specification property to define $x = x(y_1, y_2, \dots, y_{j(n,i)})$ such that

$$d_{[\lambda_l^{n,i} \hat{m}_{n,i}]}(y_l, f^{a_l}(x)) < \frac{\varepsilon}{2^{\sum_{j=0}^{n-1} M_j + i}}$$

for all $l \in \{1, 2, \dots, j(n,i)\}$, where $a_1 = 0$ and $a_l = \sum_{j=1}^{l-1} [\lambda_j^{n,i} \hat{m}_{n,i}] + (l-1)s_{n,i}$ for $l \in \{2, 3, \dots, j(n,i)\}$. Let

$$S_{n,i} = \left\{ x = x(y_1, \dots, y_{j(n,i)}) : y_j \in S_j^{n,i} \text{ and } j = 1, 2, \dots, j(n,i) \right\}$$

and $m_{n,i} = \sum_{j=1}^{j(n,i)} [\lambda_j^{n,i} \hat{m}_{n,i}] + (j(n,i) - 1)s_{n,i}$.

Lemma 5.4. *For each $n \in \mathbb{N}$ and $i \in \{1, 2, \dots, M_n\}$ one has*

- (1) $\lim_{n \rightarrow \infty} \frac{\hat{m}_{n,i}}{m_{n,i}} = 1$;
- (2) $S_{n,i}$ is a $(m_{n,i}, 4\varepsilon)$ separated set;
- (3) $\#S_{n,i} = \#S_1^{n,i} \cdot \#S_2^{n,i} \cdot \dots \cdot \#S_{j(n,i)}^{n,i}$.

Proof. (1) Since $\hat{m}_{n,i} \geq 2^{s_{n,i}}$ and $\sum_{j=1}^{j(n,i)} \lambda_j^{n,i} = 1$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{m_{n,i}}{\hat{m}_{n,i}} &= \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{j(n,i)} [\lambda_j^{n,i} \hat{m}_{n,i}] + (j(n,i) - 1)s_{n,i}}{\hat{m}_{n,i}} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{j(n,i)} [\lambda_j^{n,i} \hat{m}_{n,i}]}{\hat{m}_{n,i}} \\ &= 1. \end{aligned}$$

(2) For each $x = x(y_1, \dots, y_{j(n,i)}) \in S_{n,i}$ and $z = z(w_1, \dots, w_{j(n,i)}) \in S_{n,i}$ with $x \neq z$, where $y_j, w_j \in S_j^{n,i}$ and $j = 1, 2, \dots, j(n,i)$. Since $x \neq z$, then there is $\ell_0 \in \{1, 2, \dots, j(n,i)\}$ such that $y_{\ell_0} \neq w_{\ell_0}$. We obtain

$$d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(y_{\ell_0}, f^{a_{\ell_0}}(x)) < \frac{\varepsilon}{2^{\sum_{j=0}^{n-1} M_j + i}}, \quad d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(w_{\ell_0}, f^{a_{\ell_0}}(z)) < \frac{\varepsilon}{2^{\sum_{j=0}^{n-1} M_j + i}},$$

and $d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(y_{\ell_0}, w_{\ell_0}) > 5\varepsilon$. Combining these inequalities we have

$$\begin{aligned} d_{m_{n,i}}(x, z) &\geq d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(f^{a_{\ell_0}}(x), f^{a_{\ell_0}}(z)) \\ &\geq d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(y_{\ell_0}, w_{\ell_0}) - d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(y_{\ell_0}, f^{a_{\ell_0}}(x)) - d_{[\lambda_{\ell_0} \hat{m}_{n,i}]}(w_{\ell_0}, f^{a_{\ell_0}}(z)) \\ &\geq 5\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= 4\varepsilon. \end{aligned}$$

The proof of (3) is trivial. Then we complete the proof of the lemma. $\square$

Lemma 5.5 (Lemma 3.7, [16]). *Let $\gamma > 0$ be sufficiently small as in Lemma 5.3. Then*

- (1) *for sufficiently large n and each $i \in \{1, 2, \dots, M_n\}$, we obtain*

$$\prod_{j=1}^{j(n,i)} \sum_{x \in S_j^{n,i}} \exp \left(\sum_{k=0}^{[\lambda_j^{n,i} \hat{m}_{n,i}] - 1} \varphi(g^k(x)) \right) \geq \exp \left[m_{n,i} \left(h_g(\mu_{n,i}) + \int \varphi d\mu_{n,i} - \gamma \right) \right];$$

- (2) *if $x \in S_{n,i}$, then*

$$d \left(\frac{1}{m_{n,i}} S_{m_{n,i}}(\Phi \times \Psi)(x), \alpha_{n,i} \right) < \frac{2}{n} + \text{var} \left(\Phi \times \Psi, \frac{\varepsilon}{2^{\sum_{j=0}^{n-1} M_j + i}} \right).$$

Step 2. Construction of the intermediate sets $\{\mathcal{C}_k\}_{k \in \mathbb{N}}$. Define

$$\begin{aligned}\{S_k\}_k &= \{S_{1,1}, S_{1,2}, \dots, S_{1,M_1}, S_{2,1}, S_{2,2}, \dots, S_{2,M_2}, \dots, S_{n,1}, S_{n,2}, \dots, S_{n,M_n}, \dots\}, \\ \{m_k\}_k &= \{m_{1,1}, m_{1,2}, \dots, m_{1,M_1}, m_{2,1}, m_{2,2}, \dots, m_{2,M_2}, \dots, m_{n,1}, m_{n,2}, \dots, m_{n,M_n}, \dots\}, \\ \{\alpha_k\}_k &= \{\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,M_1}, \alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,M_2}, \dots, \alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,M_n}, \dots\}, \\ \{\mu_k\}_k &= \{\mu_{1,1}, \mu_{1,2}, \dots, \mu_{1,M_1}, \mu_{2,1}, \mu_{2,2}, \dots, \mu_{2,M_2}, \dots, \mu_{n,1}, \mu_{n,2}, \dots, \mu_{n,M_n}, \dots\}, \\ \{s_k\}_k &= \{s_{1,1}, s_{1,2}, \dots, s_{1,M_1}, s_{2,1}, s_{2,2}, \dots, s_{2,M_2}, \dots, s_{n,1}, s_{n,2}, \dots, s_{n,M_n}, \dots\}.\end{aligned}$$

We choose a sequence N_k that increases to ∞ sufficiently quickly so that

$$\lim_{k \rightarrow \infty} \frac{m_{k+1} + s_{k+1}}{N_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{N_1(m_1 + s_1) + \dots + N_k(m_k + s_k)}{N_{k+1}} = 0. \quad (5.3)$$

Let $x_1, x_2, \dots, x_{N_k} \in S_k$. We use the specification property to define $y = y(x_1, x_2, \dots, x_{N_k})$ such that $d_{m_k}(x_l, f^{a_l}(y)) < \frac{\varepsilon}{2^k}$ for all $l \in \{1, 2, \dots, N_k\}$ and $a_l = (l-1)(m_k + s_k)$. Define

$$\mathcal{C}_k = \{y(x_1, x_2, \dots, x_{N_k}) : x_l \in S_k \text{ for } l = 1, 2, \dots, N_k\}.$$

Let $c_k = N_k m_k + (N_k - 1)s_k$.

Lemma 5.6. $\mathcal{C}_k$ is a $(c_k, 3\varepsilon)$ separated set and $\sharp \mathcal{C}_k = (\sharp S_k)^{N_k}$.

Proof. For each $y = y(x_1, \dots, x_{N_k}) \in \mathcal{C}_k$ and $z = z(w_1, \dots, w_{N_k}) \in \mathcal{C}_k$ with $y \neq z$, where $x_j, w_j \in S_k$ and $j = 1, 2, \dots, N_k$. Since $y \neq z$, then there is $\ell_0 \in \{1, 2, \dots, N_k\}$ such that $x_{\ell_0} \neq w_{\ell_0}$. Thus

$$d_{m_k}(x_{\ell_0}, f^{a_{\ell_0}}(y)) < \frac{\varepsilon}{2^k}, \quad d_{m_k}(w_{\ell_0}, f^{a_{\ell_0}}(z)) < \frac{\varepsilon}{2^k} \quad \text{and} \quad d_{m_k}(x_{\ell_0}, w_{\ell_0}) > 4\varepsilon.$$

Combining these inequalities we have

$$\begin{aligned}d_{c_k}(y, z) &\geq d_{m_k}(f^{a_{\ell_0}}(y), f^{a_{\ell_0}}(z)) \\ &\geq d_{m_k}(x_{\ell_0}, w_{\ell_0}) - d_{m_k}(x_{\ell_0}, f^{a_{\ell_0}}(y)) - d_{m_k}(w_{\ell_0}, f^{a_{\ell_0}}(z)) \\ &\geq 4\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= 3\varepsilon.\end{aligned}$$

The statement $\sharp \mathcal{C}_k = (\sharp S_k)^{N_k}$ is trivial. Thus we finish the proof of the lemma. $\square$

Step 3. Construction of the intermediate sets $\{\mathcal{T}_k\}_{k \in \mathbb{N}}$. We define $\mathcal{T}_k$ inductively. Let $\mathcal{T}_1 = \mathcal{C}_1$. We construct $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$ as follows. Let $x \in \mathcal{T}_k$ and $y \in \mathcal{C}_{k+1}$. Let

$$t_1 = c_1 \quad \text{and} \quad t_{k+1} = t_k + s_{k+1} + c_{k+1}.$$

Using specification we can find a point $z = z(x, y)$ that satisfies the following

$$d_{t_k}(x, z) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{c_{k+1}}(y, f^{t_k + s_{k+1}}(z)) < \frac{\varepsilon}{2^{k+1}}.$$

Define

$$\mathcal{T}_{k+1} = \{z(x, y) : x \in \mathcal{T}_k \text{ and } y \in \mathcal{C}_{k+1}\}.$$

Lemma 5.7. $\mathcal{T}_k$ is a $(t_k, 2\varepsilon)$ separated set and $\sharp\mathcal{T}_k = \sharp\mathcal{C}_1 \cdot \dots \cdot \sharp\mathcal{C}_k$. In particular if $z, z' \in \mathcal{T}_k$, then we have

$$\overline{B}_{t_k}\left(z, \frac{\varepsilon}{2^k}\right) \cap \overline{B}_{t_k}\left(z', \frac{\varepsilon}{2^k}\right) = \emptyset.$$

Proof. If $k = 1$, then $\mathcal{T}_1 = \mathcal{C}_1$. The conclusion follows from Lemma 5.6. If $k > 1$ for each $p, q \in \mathcal{T}_k$, there exist $x, y \in \mathcal{T}_{k-1}$ and $z, w \in \mathcal{C}_k$ such that $p = p(x, z)$, $q = q(y, w)$ and

$$\begin{aligned} d_{t_{k-1}}(p, x) &< \frac{\varepsilon}{2^k}, & d_{c_k}(f^{t_{k-1}+s_k}(p), z) &< \frac{\varepsilon}{2^k}, \\ d_{t_{k-1}}(q, y) &< \frac{\varepsilon}{2^k}, & d_{c_k}(f^{t_{k-1}+s_k}(q), w) &< \frac{\varepsilon}{2^k} \end{aligned}$$

and $d_{c_k}(z, w) \geq 3\varepsilon$. This implies that

$$\begin{aligned} d_{t_k}(p, q) &\geq d_{c_k}(f^{t_{k-1}+s_k}(p), f^{t_{k-1}+s_k}(q)) \\ &\geq d_{c_k}(z, w) - d_{c_k}(z, f^{t_{k-1}+s_k}(p)) - d_{c_k}(f^{t_{k-1}+s_k}(q), w) \\ &\geq 3\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= 2\varepsilon. \end{aligned}$$

The second statement is a straightforward consequence of the first statement. $\square$

Lemma 5.8. Let $z = z(x, y) \in \mathcal{T}_{k+1}$. Then one has

$$\overline{B}_{t_{k+1}}\left(z, \frac{\varepsilon}{2^k}\right) \subseteq \overline{B}_{t_k}\left(x, \frac{\varepsilon}{2^{k-1}}\right).$$

Proof. Let $z' \in \overline{B}_{t_{k+1}}\left(z, \frac{\varepsilon}{2^k}\right)$. Then

$$\begin{aligned} d_{t_k}(z', x) &\leq d_{t_k}(z', z) + d_{t_k}(z, x) \\ &\leq d_{t_{k+1}}(z', z) + d_{t_k}(z, x) \\ &\leq \frac{\varepsilon}{2^k} + \frac{\varepsilon}{2^{k+1}} \\ &\leq \frac{\varepsilon}{2^{k-1}}. \end{aligned}$$

$\square$

Step 4. Construction of the fractal F and a special sequence of measures μ_k .

Let $F_k = \bigcup_{x \in \mathcal{T}_k} \overline{B}_{t_k}(x, \frac{\varepsilon}{2^{k-1}})$. By Lemma 5.8, we have $F_{k+1} \subseteq F_k$. Since we have a decreasing sequence of compact sets, the intersection $F = \bigcap_k F_k$ is non-empty. Further, every point $p \in F$ can be uniquely represented by a sequence $\underline{p} = (\underline{p}_1, \underline{p}_2, \underline{p}_3, \dots)$, where $\underline{p}_i = (p_1^i, \dots, p_{N_i}^i) \in \{1, 2, \dots, \sharp S_i\}^{N_i}$. Recall each $y(\underline{p}_i) = y(x_{p_1^i}^i, \dots, x_{p_{N_i}^i}^i) \in \mathcal{C}_i$ be defined as in Step 2 where $x_{p_\ell^i}^i \in S_i$ for $\ell = 1, 2, \dots, N_i$. Let $z_1(\underline{p}) = y(\underline{p}_1)$ and, proceeding inductively, let $z_{i+1}(\underline{p}) = z(z_i(\underline{p}), y(\underline{p}_{i+1})) \in \mathcal{T}_{i+1}$ be defined as in Step 3, here $z_i(\underline{p}) \in \mathcal{T}_i$ and $y(\underline{p}_{i+1}) \in \mathcal{C}_{i+1}$. We can also write $z_i(\underline{p})$ as $z(\underline{p}_1, \dots, \underline{p}_i)$. Each point in $\mathcal{T}_i$ can be uniquely represented by a finite word $(\underline{p}_1, \dots, \underline{p}_i)$. Then define $p := \pi(\underline{p})$ by

$$p = \bigcap_{i \in \mathbb{N}} \overline{B}_{t_i}\left(z_i(\underline{p}), \frac{\varepsilon}{2^{i-1}}\right).$$

It is clear from our construction that we can uniquely represent every point in F in this way.

Lemma 5.9 (Lemma 3.11 in [16]). *Given $z = z(\underline{p}_1, \dots, \underline{p}_k) \in \mathcal{T}_k$, for all $i \in \{1, 2, \dots, k\}$ and all $l \in \{1, 2, \dots, N_i\}$, we have*

$$d_{m_i} \left(x_{p_l^i}^i, f^{t_{i-1}+s_{i-1}+(l-1)(s_i+m_i)}(z) \right) < 2\varepsilon.$$

We now define the measures on F that yield the required estimates for the pressure distribution principle. For each $z \in \mathcal{T}_k$, we associate a number $\mathcal{L}(z) \in (0, +\infty)$. Using these numbers as weights, we define, for each k , an atomic measure centered on $\mathcal{T}_k$. Precisely, if $z = z(\underline{p}_1, \dots, \underline{p}_k)$, then we define

$$\mathcal{L}(z) := \mathcal{L}(\underline{p}_1) \cdot \dots \cdot \mathcal{L}(\underline{p}_k),$$

where if $\underline{p}_i = (p_1^i, \dots, p_{N_i}^i) \in \{1, 2, \dots, \#S_i\}^{N_i}$, then

$$\mathcal{L}(\underline{p}_i) := \prod_{l=1}^{N_i} \exp \left\{ S_{m_i} \log \phi \left(g, x_{p_l^i}^i \right) \right\}.$$

We define $\nu_k := \sum_{z \in \mathcal{T}_k} \mathcal{L}(z) \cdot \delta_z$. We normalise ν_k to obtain a sequence of probability measures μ_k . More precisely, we let $\mu_k := (1/\kappa_k) \nu_k$, where κ_k is the normalising constant given by

$$\kappa_k := \sum_{z \in \mathcal{T}_k} \mathcal{L}(z).$$

Lemma 5.10. (1) *Suppose that ν is a limit measure of the sequence of probability measures μ_k . Then $\nu(F) = 1$.*

(2) *$F \subseteq E_{\Phi \times \Psi}^{equ}(\tilde{C})$.*

(3) *For sufficiently large n , let $B_n(q, \frac{\varepsilon}{2})$ be an arbitrary ball which intersects F . Then*

$$\limsup_{k \rightarrow \infty} \mu_k(B_n(q, \frac{\varepsilon}{2})) \leq \exp \left\{ -n (\mathcal{I}_{\tilde{C}} - 2\text{var}(\varphi, 2\varepsilon) - 3\gamma) + \sum_{i=0}^{n-1} \log \phi(g, g^i(x)) \right\}.$$

Proof. The proof of statement (1) is provided in Lemma 3.10 in [17] or Lemma 3.13 in [16]. We refer to Lemma 3.17 in [17] or Lemma 3.20 in [16] for the proof of (3). So it suffices to prove (2). Let us choose a point $p \in F$. There exists $\underline{p} = (\underline{p}_1, \underline{p}_2, \dots)$ such that $\pi(\underline{p}) = p$. Let $y_k := y(\underline{p}_k) \in \mathcal{C}_k$ and $z_k := z_k(\underline{p}) = z(\underline{p}_1, \dots, \underline{p}_k) \in \mathcal{T}_k$. We will prove $p \in E_{\Phi \times \Psi}^{equ}(\tilde{C})$, that is $\mathcal{A}(\frac{1}{n} S_n(\Phi \times \Psi)(p)) = \tilde{C}$.

Part I. Our first step is to prove $\mathcal{A}(\frac{1}{n} S_n(\Phi \times \Psi)(p)) \subseteq \tilde{C}$. The next lemma provides the upper bound of $d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right)$. Recall $t_1 = c_1$ and $t_k = t_{k-1} + s_k + c_k$ for each $k > 1$.

Lemma 5.11. *Let $\gamma > 0$ be as above. Then for each sufficiently large $k \in \mathbb{N}$, one has*

$$d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) \leq 4\gamma (\|\Phi \times \Psi\| + 1) + 3\gamma^2.$$

Proof. Let $a_j = (j-1)(m_k + s_k)$ and $y_k = y(\underline{p}_k) = y(x_{i_1}^k, \dots, x_{i_{N_k}}^k) \in \mathcal{C}_k$, where $x_{i_j}^k \in S_k$ for $j = 1, 2, \dots, N_k$. It follows from $d_{m_k}(x_{i_j}^k, f^{a_j}(y_k)) < \frac{\varepsilon}{2k}$ and Lemma 5.5 (2) that

$$d\left(S_{c_k}(\Phi \times \Psi)(y_k), c_k \int \Phi \times \Psi d\mu_k\right)$$

$$\begin{aligned}
&= d \left(\sum_{j=1}^{N_k} S_{m_k}(\Phi \times \Psi)(f^{a_j}(y_k)) + \sum_{j=1}^{N_k-1} S_{s_k}(\Phi \times \Psi)(f^{a_j+m_k}(y_k)), c_k \int \Phi \times \Psi d\mu_k \right) \\
&\leq d \left(\sum_{j=1}^{N_k} S_{m_k}(\Phi \times \Psi)(f^{a_j}(y_k)), c_k \int \Phi \times \Psi d\mu_k \right) + (N_k - 1)s_k \cdot \|\Phi \times \Psi\| \\
&\leq d \left(\sum_{j=1}^{N_k} S_{m_k}(\Phi \times \Psi)(f^{a_j}(y_k)), \sum_{j=1}^{N_k} S_{m_k}(\Phi \times \Psi)(x_{i_j}^k) \right) \\
&\quad + d \left(\sum_{j=1}^{N_k} S_{m_k}(\Phi \times \Psi)(x_{i_j}^k), c_k \int \Phi \times \Psi d\mu_k \right) + (N_k - 1)s_k \cdot \|\Phi \times \Psi\| \\
&\leq \sum_{j=1}^{N_k} d \left(S_{m_k}(\Phi \times \Psi)(f^{a_j}(y_k)), S_{m_k}(\Phi \times \Psi)(x_{i_j}^k) \right) \\
&\quad + \sum_{j=1}^{N_k} d \left(S_{m_k}(\Phi \times \Psi)(x_{i_j}^k), m_k \int \Phi \times \Psi d\mu_k \right) + 2s_k(N_k - 1) \cdot \|\Phi \times \Psi\| \\
&\leq m_k N_k \left(2\text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^k}) + \frac{2}{n(k)} \right) + 2s_k(N_k - 1) \cdot \|\Phi \times \Psi\|
\end{aligned}$$

for some integer $n(k) \rightarrow \infty$ if $k \rightarrow \infty$. This implies that

$$\begin{aligned}
&d \left(\frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(y_k), \int \Phi \times \Psi d\mu_k \right) \\
&\leq \frac{m_k N_k}{c_k} \left(2\text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^k}) + \frac{2}{n(k)} \right) + \frac{s_k(N_k - 1)}{c_k} \cdot 2\|\Phi \times \Psi\|.
\end{aligned}$$

Let $p' = f^{t_k - c_k}(p)$ and $z'_k = f^{t_k - c_k}(z_k)$. Using $d_{t_k}(p, z_k) \leq \frac{\varepsilon}{2^{k-1}}$, one has

$$\begin{aligned}
d_{c_k}(p', y_k) &\leq d_{c_k}(p', z'_k) + d_{c_k}(z'_k, y_k) \\
&\leq d_{t_k}(p, z_k) + \frac{\varepsilon}{2^k} \\
&\leq \frac{\varepsilon}{2^{k-1}} + \frac{\varepsilon}{2^k} \\
&\leq \frac{\varepsilon}{2^{k-2}}.
\end{aligned}$$

Recall $\alpha_k := \int \Phi \times \Psi d\mu_k$. Therefore

$$\begin{aligned}
&d \left(\frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p'), \alpha_k \right) \\
&\leq d \left(\frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p'), \frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(y_k) \right) + d \left(\frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(y_k), \alpha_k \right) \\
&\leq \text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^{k-2}}) + \frac{m_k N_k}{c_k} \left(2\text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^k}) + \frac{2}{n(k)} \right) + \frac{s_k(N_k - 1)}{c_k} \cdot 2\|\Phi \times \Psi\|.
\end{aligned}$$

From the assumptions of (5.3) and $m_k \geq 2^{s_k}$, we can verify that

$$\lim_{k \rightarrow \infty} \frac{c_k}{t_k} = 1, \lim_{k \rightarrow \infty} \frac{m_k N_k}{c_k} = 1 \text{ and } \lim_{k \rightarrow \infty} \frac{s_k(N_k - 1)}{c_k} = 0.$$

Thus for arbitrary $\gamma > 0$ and sufficiently large k , we have

$$\left| \frac{c_k}{t_k} - 1 \right| < \gamma, \quad \left| \frac{m_k N_k}{c_k} - 1 \right| < \gamma, \quad \left| \frac{s_k(N_k - 1)}{c_k} \right| < \gamma, \quad \frac{2}{n(k)} < \gamma$$

and $\text{var}(\Phi \times \Psi, 2\varepsilon) < \gamma$. Therefore

$$\begin{aligned} & d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p')\right) \\ &= d\left(\frac{1}{t_k} S_{t_k - c_k}(\Phi \times \Psi)(p) + \frac{1}{t_k} S_{c_k}(\Phi \times \Psi)(p'), \frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p')\right) \\ &\leq d\left(\frac{1}{t_k} S_{t_k - c_k}(\Phi \times \Psi)(p), 0\right) + d\left(\frac{1}{t_k} S_{c_k}(\Phi \times \Psi)(p'), \frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p')\right) \\ &\leq \frac{t_k - c_k}{t_k} \|\Phi \times \Psi\| + \left| \frac{c_k}{t_k} - 1 \right| \cdot \|\Phi \times \Psi\| \\ &\leq 2\gamma \|\Phi \times \Psi\|. \end{aligned}$$

Hence

$$\begin{aligned} & d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) \\ &\leq d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p')\right) + d\left(\frac{1}{c_k} S_{c_k}(\Phi \times \Psi)(p'), \alpha_k\right) \\ &\leq 4\gamma \|\Phi \times \Psi\| + \text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^{k-2}}) + (1 + \gamma) \left(2\text{var}(\Phi \times \Psi, \frac{\varepsilon}{2^k}) + \frac{2}{n(k)} \right) \\ &\leq 4\gamma (\|\Phi \times \Psi\| + 1) + 3\gamma^2. \end{aligned}$$

Then we finished the proof of Lemma 5.11. $\square$

To prove $\mathcal{A}(\frac{1}{n} S_n(\Phi \times \Psi)(p)) \subseteq \tilde{C}$, we must prove that $d\left(\frac{1}{n} S_n(\Phi \times \Psi)(p), \tilde{C}\right) \rightarrow 0$. For each large integer n , let k be the unique number which satisfies $t_k \leq n < t_{k+1}$. Let $j \in \{0, 1, \dots, N_{k+1} - 1\}$ be the unique number so that

$$t_k + (m_{k+1} + s_{k+1})j \leq n < t_k + (m_{k+1} + s_{k+1})(j + 1).$$

Then $n = t_k + (m_{k+1} + s_{k+1})j + q$, where $0 \leq q < m_{k+1} + s_{k+1}$. Let $a = (m_{k+1} + s_{k+1})j$. We have

$$\begin{aligned} & d\left(\frac{1}{n} S_n(\Phi \times \Psi)(p), \tilde{C}\right) \\ &\leq d\left(\frac{1}{n} S_n(\Phi \times \Psi)(p), \frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p)\right) + d\left(\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p), \alpha_{k+1}\right) \\ &\quad + d(\alpha_{k+1}, \tilde{C}) \\ &= d\left(\frac{1}{n} S_n(\Phi \times \Psi)(p), \frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p)\right) + d\left(\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p), \alpha_{k+1}\right). \end{aligned}$$

Lemma 5.12 is to estimate the upper bound of $d\left(\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p), \alpha_{k+1}\right)$. And Lemma 5.13 is to estimate the upper bound of $d\left(\frac{1}{n} S_n(\Phi \times \Psi)(p), \frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p)\right)$.

Lemma 5.12. *For sufficiently large $k \in \mathbb{N}$ one has*

$$d\left(\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p), \alpha_{k+1}\right) \leq 5\gamma \|\Phi \times \Psi\| + 9\gamma + 3\gamma^2.$$

Proof. Since clearly

$$\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p) = \frac{t_k}{t_k + a} \frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p) + \frac{a}{t_k + a} \frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)),$$

then one has

$$\begin{aligned} & d\left(\frac{1}{t_k + a} S_{t_k + a}(\Phi \times \Psi)(p), \alpha_{k+1}\right) \\ &= d\left(\frac{t_k}{t_k + a} \frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p) + \frac{a}{t_k + a} \frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)), \frac{t_k}{t_k + a} \alpha_{k+1} + \frac{a}{t_k + a} \alpha_{k+1}\right) \\ &\leq d\left(\frac{t_k}{t_k + a} \frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p) + \frac{a}{t_k + a} \frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)), \frac{t_k}{t_k + a} \alpha_k + \frac{a}{t_k + a} \alpha_{k+1}\right) \\ &\quad + d\left(\frac{t_k}{t_k + a} \alpha_k + \frac{a}{t_k + a} \alpha_{k+1}, \frac{t_k}{t_k + a} \alpha_{k+1} + \frac{a}{t_k + a} \alpha_{k+1}\right) \\ &\leq \frac{t_k}{t_k + a} d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) + \frac{a}{t_k + a} d\left(\frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)), \alpha_{k+1}\right) \\ &\quad + \frac{t_k}{t_k + a} d(\alpha_k, \alpha_{k+1}) \\ &\leq d\left(\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) + d\left(\frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)), \alpha_{k+1}\right) + d(\alpha_k, \alpha_{k+1}). \end{aligned}$$

Note that

$$\begin{aligned} & \frac{1}{a} S_a(\Phi \times \Psi)(f^{t_k}(p)) \\ &= \frac{1}{a} \sum_{l=1}^j S_{s_{k+1}}(\Phi \times \Psi)(f^{t_k + (s_{k+1} + m_{k+1})(l-1)}(p)) \\ &\quad + \frac{1}{a} \sum_{l=1}^j S_{m_{k+1}}(\Phi \times \Psi)(f^{t_k + s_{k+1} + (s_{k+1} + m_{k+1})(l-1)}(p)) \\ &= \frac{1}{a} \sum_{l=1}^j S_{s_{k+1}}(\Phi \times \Psi)(f^{t_k + (s_{k+1} + m_{k+1})(l-1)}(p)) \\ &\quad + \frac{1}{j} \sum_{l=1}^j \frac{j m_{k+1}}{a} \frac{1}{m_{k+1}} S_{m_{k+1}}(\Phi \times \Psi)(f^{t_k + s_{k+1} + (s_{k+1} + m_{k+1})(l-1)}(p)) \end{aligned}$$

and

$$\alpha_{k+1} = \underbrace{\frac{1}{j} \alpha_{k+1} + \frac{1}{j} \alpha_{k+1} + \dots + \frac{1}{j} \alpha_{k+1}}_j.$$

Thus

$$\begin{aligned} & d\left(\frac{1}{a}S_a(\Phi \times \Psi)(f^{t_k}(p)), \alpha_{k+1}\right) \\ & \leq \frac{1}{a} \sum_{l=1}^j s_{k+1} \cdot d\left(\frac{1}{s_{k+1}}S_{s_{k+1}}(\Phi \times \Psi)(f^{t_k+(s_{k+1}+m_{k+1})(l-1)}(p)), 0\right) \\ & \quad + \frac{1}{j} \sum_{l=1}^j d\left(\frac{jm_{k+1}}{a} \frac{1}{m_{k+1}}S_{m_{k+1}}(\Phi \times \Psi)(f^{t_k+(s_{k+1}+m_{k+1})(l-1)+s_{k+1}}(p)), \alpha_{k+1}\right). \end{aligned}$$

Combining with $\alpha_{k+1} = \frac{jm_{k+1}}{a}\alpha_{k+1} + \frac{js_{k+1}}{a}\alpha_{k+1}$, Lemma 5.9 and Lemma 5.5, one has

$$\begin{aligned} & d\left(\frac{1}{a}S_a(\Phi \times \Psi)(f^{t_k}(p)), \alpha_{k+1}\right) \\ & \leq 2\frac{js_{k+1}}{a}\|\Phi \times \Psi\| + \frac{1}{j} \sum_{l=1}^j d\left(\frac{1}{m_{k+1}}S_{m_{k+1}}(\Phi \times \Psi)(f^{t_k+(s_{k+1}+m_{k+1})(l-1)}(p)), \alpha_{k+1}\right) \\ & \leq 2\frac{js_{k+1}}{a}\|\Phi \times \Psi\| + 3\text{var}(\Phi \times \Psi, 2\varepsilon) + \frac{2}{n(k)} \end{aligned}$$

for some integer $n(k) \rightarrow \infty$ if $k \rightarrow \infty$. Since $\lim_{k \rightarrow \infty} \frac{2js_{k+1}}{a} = 0$, then for sufficiently large k

$$\left|\frac{2js_{k+1}}{a}\right| < \gamma \text{ and } 0 < \frac{2}{n(k)} < \gamma.$$

This, together with $\text{var}(\Phi \times \Psi, 2\varepsilon) < \gamma$ and Lemma 5.11, shows

$$d\left(\frac{1}{t_k+a}S_{t_k+a}(\Phi \times \Psi)(p), \alpha_{k+1}\right) \leq 5\gamma\|\Phi \times \Psi\| + 9\gamma + 3\gamma^2.$$

Thus, the desired result follows. $\square$

Lemma 5.13. *For sufficiently large $n \in \mathbb{N}$ one has*

$$d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \frac{1}{t_k+a}S_{t_k+a}(\Phi \times \Psi)(p)\right) \leq \gamma\|\Phi \times \Psi\|.$$

Proof. Since

$$\begin{aligned} & d\left(\frac{1}{n+1}S_{n+1}(\Phi \times \Psi)(p), \frac{1}{n}S_n(\Phi \times \Psi)(p)\right) \\ & = d\left(\frac{1}{n+1}S_n(\Phi \times \Psi)(p) + \frac{1}{n+1}(\Phi \times \Psi)(g^n(p)), \frac{1}{n}S_n(\Phi \times \Psi)(p)\right) \\ & \leq d\left(\frac{1}{n+1}S_n(\Phi \times \Psi)(p), \frac{1}{n}S_n(\Phi \times \Psi)(p)\right) + \frac{1}{n+1}\|\Phi \times \Psi\| \\ & \leq \sum_{l=0}^{n-1} \frac{1}{n} d\left(\frac{n}{n+1}(\Phi \times \Psi)(g^l(p)), (\Phi \times \Psi)(g^l(p))\right) + \frac{1}{n+1}\|\Phi \times \Psi\| \\ & \leq \frac{2}{n+1}\|\Phi \times \Psi\|, \end{aligned}$$

then

$$\begin{aligned}
& d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \frac{1}{t_k+a}S_{t_k+a}(\Phi \times \Psi)(p)\right) \\
& \leq d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \frac{1}{n-1}S_{n-1}(\Phi \times \Psi)(p)\right) \\
& \quad + d\left(\frac{1}{n-1}S_{n-1}(\Phi \times \Psi)(p), \frac{1}{n-2}S_{n-2}(\Phi \times \Psi)(p)\right) + \dots \\
& \quad + d\left(\frac{1}{t_k+a+1}S_{t_k+a+1}(\Phi \times \Psi)(p), \frac{1}{t_k+a}S_{t_k+a}(\Phi \times \Psi)(p)\right) \\
& \leq \frac{2}{n}\|\Phi \times \Psi\| + \frac{2}{n-1}\|\Phi \times \Psi\| + \dots + \frac{2}{t_k+a+1}\|\Phi \times \Psi\| \\
& \leq \frac{n-t_k-a}{t_k+a}2\|\Phi \times \Psi\| \\
& \leq \frac{m_{k+1}+s_{k+1}}{t_k+a} \cdot 2\|\Phi \times \Psi\|.
\end{aligned}$$

Combining with $\lim_{k \rightarrow \infty} \frac{m_{k+1}+s_{k+1}}{t_k+a} = \lim_{k \rightarrow \infty} \frac{m_{k+1}+s_{k+1}}{N_k} = 0$, then for sufficiently large $k \in \mathbb{N}$ one has

$$d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \frac{1}{t_k+a}S_{t_k+a}(\Phi \times \Psi)(p)\right) \leq \gamma\|\Phi \times \Psi\|.$$

Thus Lemma 5.13 follows. □

Letting $n \rightarrow \infty$, $k \rightarrow \infty$. It follows from Lemma 5.12 and Lemma 5.13 that

$$d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \tilde{C}\right) \leq 6\gamma\|\Phi \times \Psi\| + 9\gamma + 3\gamma^2$$

for sufficiently large n . Since γ is arbitrary,

$$\lim_{n \rightarrow \infty} d\left(\frac{1}{n}S_n(\Phi \times \Psi)(p), \tilde{C}\right) = 0.$$

Therefore we obtain $\mathcal{A}(\frac{1}{n}S_n(\Phi \times \Psi)(p)) \subseteq \tilde{C}$.

Step II. We will prove $\tilde{C} \subseteq \mathcal{A}(\frac{1}{n}S_n(\Phi \times \Psi)(p))$. Let $\tilde{\alpha} \in \tilde{C}$. Fix $n \in \mathbb{N}$ large enough. Since $\tilde{\alpha} \in \tilde{C} \subseteq \bigcup_{k=1}^{M_n} B(\alpha_{n,k}, \frac{1}{n})$, there exists an integer q_n such that $\tilde{\alpha} \in B(\alpha_{n,q_n}, \frac{1}{n})$. Now let $k = M_1 + M_2 + \dots + M_{n-1} + q_n$. It follows immediately from Lemma 5.11 that

$$d\left(\frac{1}{t_k}S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) \leq 4\gamma\|\Phi \times \Psi\| + 4\gamma + 3\gamma^2,$$

whence

$$\begin{aligned}
d\left(\frac{1}{t_k}S_{t_k}(\Phi \times \Psi)(p), \tilde{\alpha}\right) & \leq d\left(\frac{1}{t_k}S_{t_k}(\Phi \times \Psi)(p), \alpha_k\right) + d(\alpha_k, \tilde{\alpha}) \\
& \leq 4\gamma\|\Phi \times \Psi\| + 4\gamma + 3\gamma^2 + \frac{1}{n}.
\end{aligned}$$

Letting $n \rightarrow \infty$, $k \rightarrow \infty$. Thus if $\gamma \rightarrow 0$ then $\frac{1}{t_k} S_{t_k}(\Phi \times \Psi)(p) \rightarrow \tilde{\alpha}$. This shows that $\tilde{\alpha} \in \mathcal{A}(\frac{1}{n} S_n(\Phi \times \Psi)(p))$. Then we complete the proof of Lemma 5.10. $\square$

Applying Proposition 3.2, we have $P(F, \varphi) \geq \mathcal{I}_{\tilde{C}} - 2\text{var}(\varphi, 2\varepsilon) - 3\gamma$. Recall that ε was chosen sufficiently small, so that $\text{var}(\varphi, 2\varepsilon) < \gamma$. It follows that

$$P(E_{\Phi \times \Psi}^{equ}(\tilde{C}), \varphi) \geq P(F, \varphi) \geq \mathcal{I}_{\tilde{C}} - 5\gamma.$$

Since γ was arbitrary, the proof of Proposition 5.1 is complete. $\square$

5.2. Proof of Theorem 2.4

We first examine the proof regarding lower bounds. Theorem B in [4] told us that $\dim_H E_{Q \circ (\Phi \times \Psi)}^{equ}(C) = s_*$, where s_* is the unique root of the equation

$$P(E_{Q \circ (\Phi \times \Psi)}^{equ}(C), -s \log \phi) = 0.$$

Combining with Theorem 2.3 that

$$\inf_{\alpha \in C} \sup \left\{ h_g(\mu) - s_* \int \log \phi(g, x) d\mu(x) : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\} = 0.$$

For each $\varepsilon > 0$, there is some $\alpha_0 \in C$ such that

$$\sup \left\{ h_g(\mu) - s_* \int \log \phi(g, x) d\mu(x) : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha_0 \right\} \leq \varepsilon.$$

This implies that for each $\mu \in \mathcal{M}_g(J)$ with $\mathcal{P}(\mu) = \alpha_0$,

$$h_g(\mu) - s_* \int \log \phi d\mu \leq \varepsilon.$$

Thus

$$s_* \geq \frac{h_g(\mu) - \varepsilon}{\int \log \phi d\mu}$$

for each $\mu \in \mathcal{M}_g(J)$ with $\mathcal{P}(\mu) = \alpha_0$. Therefore

$$\begin{aligned} s_* &\geq \sup \left\{ \frac{h_g(\mu) - \varepsilon}{\int \log \phi d\mu} : \mu \in \mathcal{M}_g \text{ and } \mathcal{P}(\mu) = \alpha_0 \right\} \\ &\geq \inf_{\alpha \in C} \sup \left\{ \frac{h_g(\mu) - \varepsilon}{\int \log \phi d\mu} : \mu \in \mathcal{M}_g \text{ and } \mathcal{P}(\mu) = \alpha \right\}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$s_* \geq \inf_{\alpha \in C} \sup \left\{ \frac{h_g(\mu)}{\int \log \phi d\mu} : \mu \in \mathcal{M}_g \text{ and } \mathcal{P}(\mu) = \alpha \right\}.$$

We now proceed to prove the upper bound. It follows from Theorem 2.3 that

$$\inf_{\alpha \in C} \sup \left\{ h_g(\mu) - s_* \int \log \phi(g, x) d\mu(x) : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\} = 0.$$

For each $\alpha \in C$ one has

$$\sup \left\{ h_g(\mu) - s_* \int \log \phi(g, x) d\mu(x) : \mu \in \mathcal{M}_g(J) \text{ and } \mathcal{P}(\mu) = \alpha \right\} \geq 0.$$

This implies that there exists $\mu_* \in \mathcal{M}_g(J)$ with $\mathcal{P}(\mu_*) = \alpha$, and

$$h_g(\mu_*) - s_* \int \log \phi d\mu_* \geq 0.$$

Thus

$$s_* \leq \frac{h_g(\mu_*)}{\int \log \phi d\mu_*}$$

for some $\mu_* \in \mathcal{M}_g(J)$ with $\mathcal{P}(\mu_*) = \alpha$. Therefore

$$s_* \leq \sup \left\{ \frac{h_g(\mu)}{\int \log \phi d\mu} : \mu \in \mathcal{M}_g \text{ and } \mathcal{P}(\mu) = \alpha \right\}$$

for each $\alpha \in C$. This yields that

$$s_* \leq \inf_{\alpha \in C} \sup \left\{ \frac{h_g(\mu)}{\int \log \phi d\mu} : \mu \in \mathcal{M}_g \text{ and } \mathcal{P}(\mu) = \alpha \right\}.$$

It completes the proof of Theorem 2.4.

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