

A SIMPLIFIED ITERATIVE ALGORITHM WITH RELAXATION FACTOR FOR SOLVING AN ANNULAR INVERSE ELECTROCARDIOGRAPHY PROBLEM

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Abstract This paper presents a simplified iterative algorithm with a relaxation factor for solving the Cauchy problem of Laplace equation in an annular domain, with applications to inverse electrocardiography. Building upon the classical Kozlov-Maz'ya iterative framework, we introduce a new simplified iteration algorithm with the relaxation factor to accelerate convergence. Theoretical analysis and numerical experiments demonstrate that the proposed method maintains computational efficiency while significantly improving reconstruction accuracy and numerical stability.

Keywords Inverse electrocardiography, Cauchy problem for Laplace equation, regularization, relaxation factor, iterative algorithm.

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1. Introduction

Inverse electrocardiography (ECG) constitutes a critical frontier in biomedical engineering and computational cardiology, aiming to noninvasively reconstruct the electrical potential distribution on the epicardial surface (the outer layer of the heart) from measurements obtained on the body surface. Mathematically, this reconstruction is formulated as a Cauchy problem for the Laplace equation defined over an annular domain, where the inner boundary corresponds to the epicardium and the outer boundary to the torso surface [23, 25]. It is well established that such problems are severely ill-posed in the sense of Hadamard [11, 16]: Small perturbations in the measured body-surface data, inevitable in clinical practice due to instrumentation noise or modeling approximations, can lead to arbitrarily large deviations in the reconstructed epicardial potentials, rendering direct inversion unstable and physically meaningless.

Over the past decades, a variety of regularization strategies have been developed to mitigate this inherent instability. Classical approaches encompass a spectrum of techniques, including Tikhonov regularization [24, 26], which imposes a global smoothness constraint via a penalty term; Truncated Singular Value Decomposition (TSVD) [20], which suppresses noise by discarding components associated with small singular values; and iterative methods such as the Landweber iteration [4, 12, 19] or the conjugate gradient method [13], which exhibit implicit regularization when terminated appropriately. Meanwhile, meshless techniques such as the Method

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of Fundamental Solutions (MFS) [27, 28], radial basis function neural network (RBFNN) algorithm [22] and advanced regularization frameworks [3, 5, 17, 29] have demonstrated promising applicability. Recently, advances in computational mathematics have led to the development of various regularization techniques for inverse electrocardiography. Haas et al. [10] developed a finite element-based space-time total variation regularization framework that exploits temporal coherence in ECG signals. Ito et al. [14] developed an iterative direct sampling framework for elliptic inverse problems with limited Cauchy data, showing enhanced robustness against measurement noise. Besides, Wang et al. [9] proposed a generalized Golub–Kahan bidiagonalization (GKB) iterative regularization algorithm for inverse Cauchy problems, combining Krylov subspace projections with a coupled complex boundary method (CCBM) to enhance reconstruction stability and computational efficiency. From a broader PDE perspective, recent analytical studies have further highlighted the importance of boundary observations, unique continuation, and geometric singularities in inverse problems. Diao et al. [8] investigated coupled elliptic PDE systems subject to transparent boundary conditions and revealed their connections with inverse boundary problems based on passive or active boundary measurements. In the context of elasticity, Diao, Liu, and Meng [7] established uniqueness results for recovering dislocation surfaces and jump data from boundary measurements, with particular attention to corner singularities. Moreover, Diao, Fei, and Liu [6] developed a novel unique continuation principle for systems of second-order elliptic equations and applied it to inverse conductive scattering. Although these studies concern mathematical models different from inverse electrocardiography, they illustrate recent developments in the analysis of PDE inverse problems driven by partial or indirect boundary information.

Nevertheless, many of these methods encounter persistent challenges in the context of inverse electrocardiography: High computational complexity due to dense matrix operations (e.g., $\mathcal{O}(N^3)$ for Tikhonov inversion or full SVD computation), a strong dependence on a priori noise estimates for optimal parameter selection (e.g., selecting truncation levels or stopping indices), and difficulties in balancing resolution fidelity with noise suppression. Consequently, efficient iterative regularization algorithms remain attractive for practical ECGI applications.

Alongside optimization-based regularization methods, iterative regularization approaches remain an active research topic for solving elliptic Cauchy problems because of their clear mathematical interpretation and relatively low implementation complexity. The Kozlov–Maz’ya iterative [1, 15] alternating method represents a foundational iterative framework for solving such Cauchy problems by reformulating the ill-posed inverse problem [2] as a sequence of well-posed mixed boundary value problems. While theoretically elegant and equipped with inherent regularization properties akin to the Landweber iteration [21], the classical implementation requires solving two distinct boundary value problems per iteration. This dual-solution structure incurs a significant computational burden, particularly for fine discretizations or real-time clinical applications where rapid reconstruction is desirable.

Building upon the foundational work on a simplified Kozlov–Maz’ya iteration [18], this paper introduces an enhanced computational strategy specifically tailored to the annular Laplace equation arising in inverse electrocardiography. Our primary contributions are twofold.

First, we propose a simplified iterative algorithm with a relaxation factor. By reducing the per-iteration workload to a single forward problem and introducing a scalar relaxation parameter c , we achieve substantial computational savings while enabling precise control over convergence acceleration. A rigorous Fourier analysis yields the admissible theoretical range $-2 < c < 0$ that guarantees error contraction across all frequency modes, providing explicit guidance for param-

ter selection. Numerical experiments demonstrate that, compared with the classical alternating scheme, the proposed method requires fewer iterations to satisfy the stopping criterion. Under a 5% noise level, the total computation time is reduced by approximately 18%, 65%, and 38% for the cosine, multi-frequency, and Gaussian examples, respectively, while maintaining comparable or slightly improved reconstruction accuracy.

Second, we establish a practical stopping criterion based on the Morozov discrepancy principle. In iterative regularization, the iteration index itself functions as a regularization parameter, and improper termination can degrade reconstruction quality. We adapt the discrepancy principle to the proposed scheme, terminating the process when the residual falls below a threshold proportional to the estimated noise level. This provides an automated, data-driven mechanism to halt the iteration near the optimal point without requiring manual tuning.

The remainder of this paper is organized as follows. Section 2 details the mathematical formulation of the annular inverse ECG problem and elucidates its exponential ill-posedness via singular value decomposition. Section 3 presents the proposed simplified iterative algorithm, derives the optimal relaxation factor range, and describes the numerical implementation of the forward solver and stopping rule. Section 4 provides comprehensive numerical experiments on smooth, multi-frequency, and localized (Gaussian peak) epicardial potentials under various noise levels, demonstrating the method's accuracy, efficiency, and robustness. Finally, Section 5 concludes the paper with a summary of findings and directions for future work.

2. Mathematical model

2.1. Annular Laplace equation for inverse electrocardiography

Consider an annular domain $\Omega = \{(r, \theta) | R_1 \leq r \leq R_2, 0 \leq \theta \leq 2\pi\}$, where R_1 corresponds to the epicardial boundary and R_2 corresponds to the body surface boundary. The electrical potential distribution satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (r, \theta) \in \Omega. \quad (2.1)$$

The Cauchy problem is defined as: Given Dirichlet and Neumann data on the body surface boundary $\Gamma_2 = \{(r, \theta) | r = R_2, 0 \leq \theta \leq 2\pi\}$:

$$\frac{\partial u}{\partial r}(R_2, \theta) = 0, \quad (2.2)$$

$$u(R_2, \theta) = g(\theta). \quad (2.3)$$

The spaces $L^2(\Gamma_i)$ are equipped with the standard boundary inner products

$$(\phi, \psi)_{L^2(\Gamma_i)} = \int_{\Gamma_i} \phi \bar{\psi} ds. \quad (2.4)$$

Usually for a practical problem, the exact Cauchy data cannot be obtained and we only know the measured data $g_\delta(\cdot) \in L^2(\Gamma_2)$. We always assume

$$\|g_\delta(\cdot) - g(\cdot)\|_{\Gamma_2} \leq \delta, \quad (2.5)$$

where $\|\cdot\|_{\Gamma_i}$ denotes $L^2(\Gamma_i)$ -norm, and the constant $\delta > 0$ represents a bound on the measurement error. The objective is to reconstruct the potential distribution $f(\theta)$ on the epicardial boundary

$\Gamma_1 = \{(r, \theta) | r = R_1, 0 \leq \theta \leq 2\pi\}$ from the measured Dirichlet data $g_\delta(\theta)$ on the body surface boundary $\Gamma_2 = \{(r, \theta) | r = R_2, 0 \leq \theta \leq 2\pi\}$.

Before we continue we first consider the following boundary value problem, with a solution in the classical sense: Find u satisfying,

$$\begin{aligned}\Delta u &= 0 \quad (r, \theta) \in \Omega, \\ \frac{\partial u}{\partial r}(R_2, \theta) &= 0, \\ u(R_1, \theta) &= f(\theta).\end{aligned}\tag{2.6}$$

This problem is well-posed, and we will call it the auxiliary problem. The inverse problem associated with the Cauchy problem can be formulated in the following way:

Recover the potential $f(\theta)$ on the inner boundary Γ_1 from the measurement $g(\theta)$ on the exterior boundary Γ_2 , where u is the solution to the auxiliary problem (2.6). By using the Fourier transform and the separation of variables, the solution of problem (2.6) can be obtained,

$$u(r, \theta) = \sum_{n=-\infty}^{+\infty} \frac{1 + (\frac{R_2}{r})^{2|n|}}{R_1^{|n|}(1 + (\frac{R_2}{R_1})^{2|n|})} r^{|n|} f_n e^{in\theta},\tag{2.7}$$

where f_n is Fourier coefficient of $f(\theta)$. In the context of the medical application, the function $f(\theta)$ is sought and is to be determined from the boundary values of the potential measured at the surface Γ_2 of the body,

$$g = g(\theta) = u(R_2, \theta) = \sum_{n=-\infty}^{+\infty} \frac{2R_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}} f_n e^{in\theta}.\tag{2.8}$$

The Fourier representations (2.7) and (2.8) are first derived for sufficiently smooth boundary data. Since the corresponding Fourier multiplier sequence is bounded, the forward mapping defined by (2.8) can be uniquely extended by continuity to $L^2(\Gamma_1) \rightarrow L^2(\Gamma_2)$.

For real analytic Dirichlet data $g(\theta)$, the Cauchy–Kovalevsky theorem states that (2.1)–(2.3) admits a (unique) local solution near Γ_2 . However, such a local solution need not extend to the entire annular domain, and its continuation from Γ_2 to the inaccessible boundary Γ_1 is highly unstable with respect to perturbations in the measured data. In fact, a comparison of (2.6) and (2.7) reveals that for

$$g(\theta) = \sum_{n=-\infty}^{+\infty} g_n e^{in\theta},\tag{2.9}$$

the solution of the Cauchy problem (2.1)–(2.3) is given by

$$\begin{aligned}u(r, \theta) &= \sum_{n=-\infty}^{+\infty} \frac{(\frac{r}{R_2})^n + (\frac{R_2}{r})^n}{2} g_n e^{in\theta} \\ &= \sum_{n=-\infty}^{+\infty} \cosh(n \ln \frac{R_2}{r}) g_n e^{in\theta},\end{aligned}\tag{2.10}$$

this solution exists (i.e., this series converges absolutely) in the annulus $\frac{R_2^2}{R} < r < R$, where $R = R_2(\lim_{|n| \rightarrow \infty} \sup |g_n|^{\frac{1}{|n|}})^{-1}$, but in no bigger annulus. In the case when the data $g = Kf$ were

given exactly, we conclude from (2.7) that

$$R = \left(\lim_{|n| \rightarrow \infty} \sup |f_n|^{\frac{1}{|n|}} \right)^{-1} \geq R_2, \quad (2.11)$$

the solution of the Cauchy problem (2.1)-(2.3) indeed exists in all of Ω . A more careful analysis shows that it extends continuously (in L^2) to Γ_1 , and hence we could recover $f(\theta)$ from (2.9) by evaluating the series for $r = R_1$, which gives

$$f = f(\theta) = \sum_{n=-\infty}^{+\infty} \cosh(n \ln \frac{R_2}{R_1}) g_n e^{in\theta}. \quad (2.12)$$

2.2. Ill-posedness of the problem

The Cauchy problem for Laplace's equation in an annular domain is a classical ill-posed problem. Based on separation of variables from Section 2.1, the Fourier coefficients of the solution exhibit exponential growth due to the divergence of $(\frac{R_2}{R_1})^{|n|}$ as $|n| \rightarrow +\infty$, causing measurement noise to be dramatically amplified.

We can more precisely characterize the ill-posedness of this Cauchy problem from the perspective of singular value decomposition (SVD). From $f = u|_{\Gamma_1}$ and defining a linear operator:

$$\begin{aligned} K : \mathcal{X} &\longrightarrow \mathcal{Y}, \\ f &\longrightarrow g = u|_{\Gamma_2}, \end{aligned} \quad (2.13)$$

with $\mathcal{X} = L^2(\Gamma_1)$, $\mathcal{Y} = L^2(\Gamma_2)$ and where u is the solution of (2.6), then the well-posed problem (2.6) can be rewritten as

$$Kf = g, \quad (2.14)$$

solving the inverse problem (2.1)-(2.3) is thus equivalent to looking for a solution f of the operator equation (2.13). Let (σ_n, u_n, v_n) be the singular system for the compact operator K . We know $u_n = u_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi R_1}}$, $v_n = v_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi R_2}}$, are the orthonormal bases of $\mathcal{X} = L^2(\Gamma_1)$, $\mathcal{Y} = L^2(\Gamma_2)$, and the singular values σ_n are calculated by

$$\begin{aligned} \sigma_n &= \sqrt{\frac{R_1}{R_2} \frac{2R_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}}} \\ &= \sqrt{\frac{R_1}{R_2} \frac{2}{(\frac{R_1}{R_2})^{|n|} + (\frac{R_2}{R_1})^{|n|}}}, \quad n \in \mathbb{Z}. \end{aligned} \quad (2.15)$$

Note that $\sigma_{-n} = \sigma_n$, and as $|n| \rightarrow \infty$, since $0 < R_1 < R_2$, we have $\sigma_n \sim 2 \left(\frac{R_1}{R_2} \right)^{|n|+1/2}$. This indicates that the singular values decay exponentially with increasing frequency $|n|$, with a decay rate of $(R_1/R_2)^{|n|+1/2}$. Therefore, the operator K is compact.

The singular values of the inverse operator K^{-1} are σ_n^{-1} , whose asymptotic behavior is: $\frac{1}{\sigma_n} \sim \frac{1}{2} \left(\frac{R_2}{R_1} \right)^{|n|+1/2}$ ($|n| \rightarrow \infty$). This means that the singular values of the inverse operator grow exponentially, with a growth base of $R_2/R_1 > 1$, so K^{-1} is an unbounded operator.

When the measured data g_δ contains noise, its Fourier coefficients $g_{\delta,n}$ include errors. When attempting to reconstruct $f(\theta)$ via $f_n = \sigma_n^{-1} g_{\delta,n}$, the noise in the high-frequency components

will be amplified by a factor of $\sigma_n^{-1} \sim (R_2/R_1)^{|n|+1/2}$. Even if the measurement error δ is small, high-frequency noise can cause significant deviations in the reconstruction result under the L^2 norm, meaning the solution does not depend continuously on the data.

Therefore, from the perspective of singular value decomposition (SVD), this Cauchy problem is exponentially ill-posed: The singular values of the operator K decay exponentially, resulting in an unbounded inverse operator and rendering the solution highly sensitive to measurement errors. Consequently, regularization methods must be employed to suppress high-frequency noise and obtain stable approximate solutions.

3. Methodology

Traditional regularization methods, such as Tikhonov regularization and Truncated Singular Value Decomposition (TSVD), while capable of suppressing solution instability, exhibit notable limitations in practical applications. The Tikhonov method requires the inversion of a dense matrix, resulting in a computational complexity as high as $\mathcal{O}(N^3)$. Moreover, the selection of its regularization parameter α is highly dependent on prior noise estimates. The TSVD method, on the other hand, suffers from spectral information loss due to its hard truncation, making it difficult to adaptively balance truncation error against noise amplification. Therefore, it is necessary to resort to alternative methods for approximating the function f corresponding to the given data g . Subsequently, we will introduce a simplified iterative method with a relaxation factor, an improvement upon the Kozlov-Maz'ya iteration, to address the inverse problem (2.1)-(2.3).

We first introduce the framework of the traditional Kozlov-Maz'ya iteration. Its fundamental idea is to gradually approximate the solution of the original ill-posed problem by alternately solving two well-posed boundary value problems. For the Cauchy problem of Laplace's equation, the complete iterative procedure is as follows:

1. **Initial step:** Provide an initial guess $f^{(0)}$ (defined on the inaccessible boundary Γ_1)
2. **First subproblem:** Solve the well-posed boundary value problem

$$\begin{cases} \Delta w^{(k+\frac{1}{2})} = 0, & \text{in } \Omega, \\ \frac{\partial w^{(k+\frac{1}{2})}}{\partial r}(R_2, \theta) = 0, & \text{on } \Gamma_2, \\ w^{(k+\frac{1}{2})}(R_1, \theta) = f^{(k+\frac{1}{2})}(\theta), & \text{on } \Gamma_1, \end{cases} \quad (3.1)$$

to obtain the solution $w^{(k+\frac{1}{2})}(r, \theta)$ and compute its derivative $\frac{\partial w^{(k+\frac{1}{2})}}{\partial r}(R_1, \theta)$.

3. **Second subproblem:** Solve the Neumann problem

$$\begin{cases} \Delta w^{(k+1)} = 0, & \text{in } \Omega, \\ w^{(k+1)}(R_2, \theta) = g(\theta), \\ \frac{\partial w^{(k+1)}}{\partial r}(R_1, \theta) = \frac{\partial w^{(k+\frac{1}{2})}}{\partial r}(R_1, \theta). \end{cases} \quad (3.2)$$

4. **Update step:** Update the guess on the inaccessible boundary.

Let $d_{k+1} \triangleq w^{(k+1)} - w^{(k+\frac{1}{2})}$ satisfies the residual equation

$$\begin{cases} \Delta d^{(k+1)} = 0, & \text{in } \Omega, \\ d^{(k+1)} = g - Kf^{(k)} \triangleq e^{(k)} & \text{on } \Gamma_2, \\ \frac{\partial d^{(k+1)}}{\partial r} = 0, & \text{on } \Gamma_1. \end{cases} \quad (3.3)$$

For (3.1), we introduce the operator $K^* : e^{(k)} \longrightarrow d^{(k+1)}|_{\Gamma_1}$. Due to $f^{(k+1)} = w^{(k+1)}|_{\Gamma_1}$, $f^{(k)} = w^{(k+\frac{1}{2})}|_{\Gamma_1}$, then there is $f^{(k+1)} - f^{(k)} = d^{(k+1)}|_{\Gamma_1}$. From the definition of K^* , we can obtain

$$f^{(k+1)} = f^{(k)} + K^*(g - Kf^{(k)}). \quad (3.4)$$

This iteration has been proven equivalent to the implementation of the Landweber iteration in specific function spaces, possessing regularization properties. when the iteration count k is appropriate, $f^{(k)}$ can stably approximate the true solution. However, as k increases, the error may regrow due to noise amplification, exhibiting typical semi-convergence behavior.

Nevertheless, the traditional Kozlov-Maz'ya iteration requires solving two boundary value problems per step, resulting in relatively high computational cost. Considering the characteristics of the inverse electrocardiography problem which features complex geometry, large-scale computation, and real-time clinical requirements, we propose a simplified version: Solving only one forward problem per iteration while introducing a relaxation factor and a modified update strategy. This simplification not only preserves the adaptive filtering properties of the original iteration but also provides additional convergence control through the relaxation factor, forming the new algorithmic framework to be detailed in the following.

3.1. Simplified iterative algorithm with relaxation factor

The iterative algorithm for solving (2.1)-(2.3) in Ω can be constructed as follows:

- **Step 1.** Provide an initial guess $f^{(0)}$ ($k = 0$) for $f^{(k)}$ defined on the inaccessible boundary Γ_1 .
- **Step 2.** If $f^{(k)}$ has been constructed, solve the well-posed boundary value problem

$$\begin{cases} \Delta v^{(k)} = 0, & \text{in } \Omega, \\ \frac{\partial v^{(k)}}{\partial r}(R_2, \theta) = 0, \\ v^{(k)}(R_1, \theta) = f^{(k)}(\theta), \end{cases} \quad (3.5)$$

to determine the k -th approximation $v^{(k)}(R_2, \theta)$.

- **Step 3.** Update boundary value

$$f^{(k+1)}(\theta) = f^{(k)}(\theta) + c(v^{(k)}(R_2, \theta) - g(\theta)), \quad (3.6)$$

where c is a relaxation factor that makes the algorithm convergent.

- **Step 4.** Repeat Steps 2 and 3 until a prescribed stopping criterion is satisfied.

Clearly, it suffices to solve only the forward problem (3.5) in each iteration, in contrast to classical methods. In Step 3, the iteration scheme is constructed by utilizing the Dirichlet data $v^{(k)}(R_2, \theta)$ to fit the given data $g(\theta)$.

3.2. Relaxation factor selection and convergence analysis

For the convergence analysis, boundary functions are identified with their 2π -periodic angular representations, and we work in $H = L^2(0, 2\pi)$. Since $\|h\|_{\Gamma_i} = \sqrt{R_i}\|h\|_{L^2(0, 2\pi)}$, convergence in $L^2(0, 2\pi)$ is equivalent to convergence in $L^2(\Gamma_i)$ for each fixed R_i .

Next, based on Fourier analysis, we derive the theoretical selection range for the relaxation factor. Let us define the error function $E^{(k)}(r, \theta) := v^{(k)}(r, \theta) - u(r, \theta)$, where $u(r, \theta)$ is the solution of the forward problem (2.6). Due to (2.6) and (3.5), we have for $E^{(k)}(r, \theta)$:

$$\begin{cases} \Delta E^{(k)} = 0, & \text{in } \Omega, \\ \frac{\partial E^{(k)}}{\partial r}(R_2, \theta) = 0, \\ E^{(k)}(R_1, \theta) = \varepsilon^{(k)}(\theta), \end{cases} \quad (3.7)$$

where $\varepsilon^{(k)} := f^{(k)}(\theta) - f(\theta)$. (3.6) becomes

$$\varepsilon^{(k+1)} = \varepsilon^{(k)} + cE^{(k)}(R_2, \theta). \quad (3.8)$$

Using the equation (2.7), we can obtain the solution of (3.7) in the frequency domain:

$$E^{(k)}(r, \theta) = \sum_{n=-\infty}^{+\infty} \frac{1 + (\frac{R_2}{r})^{2|n|}}{R_1^{|n|}(1 + (\frac{R_2}{R_1})^{2|n|})} r^{|n|} \varepsilon_n^{(k)} e^{in\theta}. \quad (3.9)$$

From (3.5), we have

$$E^{(k)}(R_2, \theta) = \sum_{n=-\infty}^{+\infty} \frac{2R_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}} \varepsilon_n^{(k)} e^{in\theta}. \quad (3.10)$$

Consider (3.8) in the frequency domain, we have

$$\varepsilon_n^{(k+1)} = (1 + \frac{2cR_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}}) \varepsilon_n^{(k)}. \quad (3.11)$$

In order to guarantee the convergence of the algorithm, we should select the parameter c , which should satisfy $|1 + \frac{2cR_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}}| < 1$. Define the positive coefficient $\lambda_n := \frac{2R_1^{|n|}R_2^{|n|}}{R_1^{2|n|} + R_2^{2|n|}}$, it is easy to verify that $\lambda_n \in (0, 1]$ for all integers n . Specifically, $\lambda_0 = 1$, and λ_n decreases monotonically to 0 as $|n| \rightarrow \infty$. Using this definition, equation (3.11) simplifies to: $\varepsilon_n^{(k+1)} = (1 + c\lambda_n)\varepsilon_n^{(k)}$.

For each fixed Fourier mode n , the error relation

$$\varepsilon_n^{(k+1)} = (1 + c\lambda_n)\varepsilon_n^{(k)}$$

shows that the corresponding modal error converges to zero if

$$|1 + c\lambda_n| < 1.$$

Since $\lambda_n > 0$, this condition is equivalent to

$$-2 < c\lambda_n < 0.$$

The inequality $c\lambda_n < 0$ implies $c < 0$. Moreover, since

$$0 < \lambda_n \leq \lambda_0 = 1,$$

the condition $c\lambda_n > -2$ is satisfied for every n whenever $c > -2$. Consequently, the admissible range of the relaxation factor is

$$-2 < c < 0. \quad (3.12)$$

Therefore, for every fixed $n \in \mathbb{Z}$,

$$|1 + c\lambda_n| < 1,$$

and hence

$$\varepsilon_n^{(k)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It should be emphasized, however, that the above estimate does not imply a uniform contraction in the infinite-dimensional space. Indeed, since

$$\lambda_n \rightarrow 0 \quad \text{as } |n| \rightarrow \infty,$$

we have

$$|1 + c\lambda_n| \rightarrow 1,$$

and therefore

$$\sup_{n \in \mathbb{Z}} |1 + c\lambda_n| = 1.$$

Thus, the preceding argument establishes modewise convergence only. The strong convergence in $L^2(\Gamma_1)$ for exact data is proved below by means of Parseval's identity and the dominated convergence theorem.

For every fixed Fourier mode n , the condition $-2 < c < 0$ implies $|1 + c\lambda_n| < 1$. Hence, each fixed Fourier component of the error converges to zero as the iteration number tends to infinity. However, since $\lambda_n \rightarrow 0$ as $|n| \rightarrow \infty$, $|1 + c\lambda_n| \rightarrow 1$ and consequently $\sup_{n \in \mathbb{Z}} |1 + c\lambda_n| = 1$. Therefore, the iteration is not a uniform contraction in the infinite-dimensional L^2 space. The convergence established above is first modewise, while the strong L^2 -convergence for exact data is proved below.

Theorem 3.1. *Let $f \in L^2(0, 2\pi)$ be the exact solution and let $f^{(0)} \in L^2(0, 2\pi)$ be an arbitrary initial guess. Suppose that the exact data $g = Kf$ are available. If the relaxation factor satisfies $-2 < c < 0$, then the iterates generated by (3.6) converge strongly to f in $L^2(\Gamma_1)$, namely,*

$$\lim_{k \rightarrow \infty} \|f^{(k)} - f\|_{\Gamma_1} = 0. \quad (3.13)$$

Proof. Let

$$\varepsilon^{(k)} = f^{(k)} - f$$

and denote its Fourier coefficients by $\varepsilon_n^{(k)}$. From (3.11), we have

$$\varepsilon_n^{(k)} = (1 + c\lambda_n)^{(k)} \varepsilon_n^{(0)}.$$

Since $0 < \lambda_n \leq 1$ and $-2 < c < 0$, it follows that

$$|1 + c\lambda_n| < 1$$

for every fixed n . Therefore,

$$(1 + c\lambda_n)^{(k)} \varepsilon_n^{(0)} \rightarrow 0 \quad (k \rightarrow \infty)$$

for each $n \in \mathbb{Z}$.

By Parseval's identity,

$$\|f^{(k)} - f\|_{L^2(0,2\pi)}^2 = \sum_{n \in \mathbb{Z}} |1 + c\lambda_n|^{2k} |\varepsilon_n^{(0)}|^2.$$

Moreover,

$$|1 + c\lambda_n|^{2k} |\varepsilon_n^{(0)}|^2 \leq |\varepsilon_n^{(0)}|^2,$$

and

$$\sum_{n \in \mathbb{Z}} |\varepsilon_n^{(0)}|^2 < \infty.$$

Hence, the dominated convergence theorem yields

$$\lim_{k \rightarrow \infty} \|f^{(k)} - f\|_{L^2(0,2\pi)} = 0.$$

Since $\|h\|_{\Gamma_1} = \sqrt{R_1} \|h\|_{L^2(0,2\pi)}$, it follows that

$$\lim_{k \rightarrow \infty} \|f^{(k)} - f\|_{\Gamma_1} = 0.$$

This completes the proof. $\square$

Remark 3.1. Although strong L^2 -convergence holds for exact data, the iteration is not a uniform contraction because

$$\sup_{n \in \mathbb{Z}} |1 + c\lambda_n| = 1.$$

Consequently, no geometric convergence rate that is uniform with respect to all Fourier modes can be obtained without imposing additional smoothness or source conditions on the exact solution.

Proposition 3.1. *Let $f^{(k)}$ and $f^{(k),\delta}$ denote the iterates generated from the exact data g and the noisy data g_δ , respectively, with the same initial guess $f^{(0)}$. Assume that g and g_δ satisfy the equation (2.5) and that the relaxation factor satisfies $-2 < c < 0$, then*

$$\|f^{(k),\delta} - f^{(k)}\|_{\Gamma_1} \leq |c|k\delta. \quad (3.14)$$

Proof. For the convergence analysis, the boundary functions are identified with their 2π -periodic angular representations in

$$H = L^2(0, 2\pi),$$

and the corresponding forward operator is denoted by

$$\mathcal{K}: H \rightarrow H.$$

Its Fourier multiplier representation is

$$\widehat{\mathcal{K}h_n} = \lambda_n \widehat{h_n}, \quad 0 < \lambda_n \leq 1.$$

Let

$$q^{(k)} := f^{(k),\delta} - f^{(k)}.$$

Subtracting the exact-data iteration from the noisy-data iteration gives

$$q^{(k+1)} = (I + c\mathcal{K})q^{(k)} - c(g_\delta - g), \quad q^{(0)} = 0.$$

Therefore,

$$q^{(k)} = -c \sum_{j=0}^{k-1} (I + c\mathcal{K})^j (g_\delta - g).$$

Since $-2 < c < 0$ and $0 < \lambda_n \leq 1$, we have

$$|1 + c\lambda_n| \leq 1$$

for all $n \in \mathbb{Z}$. Hence,

$$\|I + c\mathcal{K}\|_{H \rightarrow H} = \sup_{n \in \mathbb{Z}} |1 + c\lambda_n| = 1.$$

It follows that

$$\begin{aligned} \|q^{(k)}\|_H &\leq |c| \sum_{j=0}^{k-1} \|(I + c\mathcal{K})^j (g_\delta - g)\|_H \\ &\leq |c|k \|g_\delta - g\|_H. \end{aligned}$$

For a boundary function represented by $h(\theta)$, the boundary L^2 -norm satisfies

$$\sqrt{R_i} |h|_H, i = 1, 2.$$

Therefore,

$$\begin{aligned} |f^{(k),\delta} - f^{(k)}|_{\Gamma_1} &= \sqrt{R_1} \|q^{(k)}\|_H \\ &\leq |c|k \sqrt{R_1} \|g_\delta - g\|_H \\ &= |c|k \sqrt{\frac{R_1}{R_2}} \|g_\delta - g\|_{\Gamma_2} \\ &\leq |c|k \sqrt{\frac{R_1}{R_2}} \delta. \end{aligned}$$

Since $R_1 < R_2$, we further obtain

$$\|f^{(k),\delta} - f^{(k)}\|_{\Gamma_1} \leq |c|k\delta.$$

This completes the proof. $\square$

The above estimate shows that the iteration number plays the role of a regularization parameter. Indeed, by the triangle inequality,

$$\|f^{(k),\delta} - f\|_{\Gamma_1} \leq \|f^{(k),\delta} - f^{(k)}\|_{\Gamma_1} + \|f^{(k)} - f\|_{\Gamma_1}.$$

The second term is the approximation error for exact data and tends to zero as $k \rightarrow \infty$, according to Theorem 3.1. In contrast, the first term represents the propagation of data noise and is bounded by a quantity proportional to $k\delta$. Therefore, increasing the number of iterations

improves the approximation of the exact-data solution but simultaneously increases sensitivity to measurement noise.

Consequently, if the stopping index $k = k(\delta)$ is chosen such that

$$k(\delta) \rightarrow \infty, \quad \delta k(\delta) \rightarrow 0 \quad \delta \rightarrow 0,$$

then

$$\|f^{(k(\delta)),\delta} - f\|_{\Gamma_1} \rightarrow 0 \quad (\delta \rightarrow 0).$$

Thus, the proposed iteration constitutes an iterative regularization method when it is terminated appropriately. In particular, early stopping is essential for preventing excessive propagation of measurement noise.

3.3. The initial guess

An arbitrary function $f^{(0)}$ may be selected as an initial guess for the epicardial boundary $v((R_1, \theta))$ of inverse electrocardiography problem.

3.4. The convergence error

In order to investigate the convergence of the described algorithm, at every iteration, we evaluate the convergence error defined by

$$r_k = \|f^{(k)} - f\|_{\Gamma_1}. \quad (3.15)$$

$f^{(k)}$ is the computation value after k iterations, and f is the exact value. However, the error r_k can only be computed if the exact solution f is known, and such a solution is not available in practice. Therefore, a stopping criterion must be developed in order to cease the iterative process when convergence has been achieved.

3.5. Stopping criterion

For noisy data, the iteration cannot in general be continued indefinitely. As shown in Proposition 3.1, the perturbation caused by the measurement noise satisfies

$$\|f^{(k),\delta} - f^{(k)}\|_{\Gamma_1} \leq |c|k\sqrt{\frac{R_1}{R_2}}\delta.$$

Thus, although the approximation error associated with the exact data decreases as the iteration proceeds, the influence of the data perturbation may increase with k . The iteration number therefore acts as a regularization parameter, and an appropriate early-stopping rule is required to balance approximation accuracy and noise propagation.

For the proposed simplified iterative algorithm, we adopt the Morozov discrepancy principle. The iteration is terminated at the first index $k = k^*$ such that

$$\|Kf^{(k),\delta} - g_\delta\|_{\Gamma_2} = \|v^{(k),\delta}(R_2, \cdot) - g_\delta\|_{\Gamma_2} \leq \tau\delta, \quad (3.16)$$

where:

- δ is the noise level satisfying $\|g_\delta - g\|_{\Gamma_2} \leq \delta$ (see (2.5)).

- $Kf^{(k),\delta} = v^{(k),\delta}|_{\Gamma_2}$ is the computed Dirichlet trace on the accessible boundary.
- $\tau > 1$ is a safety factor that prevents stopping too early due to statistical fluctuations in the noise. Typical choices are $\tau = 1.1, 1.5$, or 2 , depending on the reliability of the noise estimate.

To verify that the discrepancy stopping index is well defined, define the noisy-data residual by

$$r^{(k),\delta} := \mathcal{K}f^{(k),\delta} - g_\delta$$

in $H = L^2(0, 2\pi)$. From the iteration formula

$$f^{(k+1),\delta} = f^{(k),\delta} + c \left(\mathcal{K}f^{(k),\delta} - g_\delta \right),$$

we obtain

$$r^{(k+1),\delta} = (I + c\mathcal{K})r^{(k),\delta}.$$

Hence, for each Fourier mode,

$$\widehat{r}_n^{(k+1),\delta} = (1 + c\lambda_n)\widehat{r}_n^{(k),\delta},$$

and therefore

$$\widehat{r}_n^{(k),\delta} = (1 + c\lambda_n)^k \widehat{r}_n^{(0),\delta}.$$

For every fixed n , the condition $-2 < c < 0$ implies

$$|1 + c\lambda_n| < 1,$$

so that

$$\widehat{r}_n^{(k),\delta} \rightarrow 0 \quad (k \rightarrow \infty).$$

Moreover,

$$\left| (1 + c\lambda_n)^k \widehat{r}_n^{(0),\delta} \right|^2 \leq \left| \widehat{r}_n^{(0),\delta} \right|^2,$$

and

$$\sum_{n \in \mathbb{Z}} \left| \widehat{r}_n^{(0),\delta} \right|^2 < \infty.$$

By Parseval's identity and the dominated convergence theorem,

$$\|r^{(k),\delta}\|_H \rightarrow 0 \quad (k \rightarrow \infty).$$

Since

$$\|r^{(k),\delta}\|_{\Gamma_2} = \sqrt{R_2} \|r^{(k),\delta}\|_H,$$

we also have

$$\|Kf^{(k),\delta} - g_\delta\|_{\Gamma_2} \rightarrow 0.$$

Consequently, for every fixed $\delta > 0$ and $\tau > 1$, there exists a finite iteration index k^* for which the discrepancy condition (3.16) is satisfied.

The discrepancy principle therefore provides a practical early-stopping mechanism for the noisy-data iteration. Once the residual has reached the prescribed noise level, further reduction of the residual may correspond primarily to fitting components of the measurement noise rather than improving the reconstruction of the exact solution. Combined with the stability estimate in Proposition 1, this explains why the iteration should be terminated before excessive noise propagation occurs.

3.6. Solve the forward problem

For the proposed iteration method, how to solve the forward problem (3.5) is the key problem. For convenience, we drop the index k in (3.5). Before conducting the numerical experiment, we first give the finite difference scheme for solving the forward problem.

We concentrate on the following forward problem for the given data $f(\theta)$:

$$\begin{cases} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial r}(R_2, \theta) = 0, \\ v(R_1, \theta) = f(\theta). \end{cases} \quad (3.17)$$

Using an unconditionally stable implicit finite difference scheme, we can obtain the data $u|_{\Gamma_2} = g(\theta)$.

For the variable r, θ , we take the equal step size, $h_r = \frac{R_2 - R_1}{N_r - 1}$ and $h_\theta = \frac{2\pi}{N_\theta}$, respectively.

Let

$$\begin{aligned} r_i &= R_1 + (i - 1)h_r, \quad i = 1, 2, \dots, N_r, \\ \theta_j &= (j - 1)h_\theta, \quad j = 1, 2, \dots, N_\theta. \end{aligned} \quad (3.18)$$

For any given point (r_i, θ_j) , central differencing which is second-order accurate is utilized for the discretization of radial and circumferential terms derivatives,

$$\begin{aligned} \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right]_{(r_i, \theta_j)} &\approx \frac{1}{r_i} \frac{r_{i+\frac{1}{2}} v_{i+1,j} - (r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}) v_{i,j} + r_{i-\frac{1}{2}} v_{i-1,j}}{h_r^2}, \\ \left[\frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right] &\approx \frac{1}{r_i^2} \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_\theta^2} \quad (v_{i,0} = v_{i,N_\theta}, \quad v_{i,N_\theta+1} = v_{i,1}). \end{aligned} \quad (3.19)$$

By substituting (3.19) to (3.17), it can be approximated by the following difference equation for $2 \leq i \leq N_r - 1$:

$$\frac{r_{i+\frac{1}{2}} v_{i+1,j} - (r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}) v_{i,j} + r_{i-\frac{1}{2}} v_{i-1,j}}{h_r^2} + \frac{1}{r_i} \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_\theta^2} = 0. \quad (3.20)$$

At the inner boundary $r = R_1$ (i.e. $i = 1$) the Dirichlet condition is imposed directly:

$$u_{1,j} = f(\theta_j). \quad (3.21)$$

For the handling of the Neumann condition on the outer boundary $r = R_2$ (i.e. $i = N_r$), we use backward difference discretization:

$$\frac{\partial u(R_2, \theta)}{\partial r} \approx \frac{u_{N_r, j} - u_{N_r-1, j}}{h_r} = 0. \quad (3.22)$$

From the above, formula we get $u_{N_r, j} = u_{N_r-1, j}$.

4. Numerical experiments

This section provides several representative numerical examples, implemented in MATLAB under IEEE double-precision arithmetic, to empirically demonstrate the effectiveness of the proposed method.

4.1. Experimental setup

In the computations, the inner and outer radii are fixed as $R_1 = 1$ and $R_2 = 2$, respectively. The computational domain is discretized with $N_r = 51$ grid points in the radial direction and $N_\theta = 100$ grid points in the angular direction. The value of the relaxation factor varies depending on the specific example. Meanwhile, we can obtain the Cauchy data $g(\theta)$ in the form of vector denoted as G . The noisy Cauchy data of G_δ is generated as:

$$G_\delta = G + \max(G) \cdot \varepsilon \cdot \text{rand}(\text{size}(G)), \quad (4.1)$$

where

$$G = (g(\theta_1), g(\theta_2), \dots, g(\theta_{N_\theta}))^T, \quad \theta_i = (i-1)\Delta\theta, \Delta\theta = \frac{2\pi}{N_\theta}, i = 1, 2, \dots, N_\theta, \quad (4.2)$$

$$\delta = \|G_\delta - G\|_{l_2(\theta)} = \sqrt{\sum_{i=1}^{N_\theta} (G_\delta(\theta_i) - G(\theta_i))^2}, \quad (4.3)$$

in which $\max(G)$ denotes the maximum of the vector G , ε is a prescribed noise-amplitude parameter controlling the magnitude of the perturbation. The actual noise magnitude used in the discrepancy principle is evaluated by $\delta = \|G_\delta - G\|_{l_2}$. $\text{rand}(i)$ denotes a random number from the uniform distribution of interval $(0, 1)$. “ $\text{rand}(\text{size}(G))$ ” returns an array of random entries that is the same size as G .

In order to test the accuracy of the approximate solution, the relative root mean square error (RRMSE) is defined as

$$\text{RRMSE}(f^{(k)} - f) = \sqrt{\frac{\sum_{i=1}^{N_\theta} (f^{(k)}(\theta_i) - f(\theta_i))^2}{\sum_{i=1}^{N_\theta} (f(\theta_i))^2}}, \quad (4.4)$$

where $f^{(k)}(\theta_i)$ is the reconstructed epicardial potential at the k -th iteration and $f(\theta_i)$ is the exact solution.

The iteration is terminated according to the following discrete form of formula (3.16):

$$\sqrt{\sum_{i=1}^{N_\theta} (v^{(k)}(R_2, \theta_i) - G_\delta(\theta_i))^2} \leq \tau\delta, \quad (4.5)$$

with $\tau = 1.1$ in the following experiments.

4.2. Numerical examples and analysis

4.2.1. Numerical examples

Example 4.1. Smooth Cosine-Type Solution: We consider a smooth cosine-type exact solution on the epicardial boundary Γ_1 ,

$$f(\theta) = \left(R_1 + \frac{R_2^2}{R_1}\right) \cos \theta, \quad \theta \in [0, 2\pi]. \quad (4.6)$$

First, we solve the forward problem (3.17) with this boundary value by using the finite difference method in Section 3.6. It can be verified that the corresponding Cauchy data on the body surface Γ_2 is given by $g(\theta) = 2R_2 \cos \theta$.

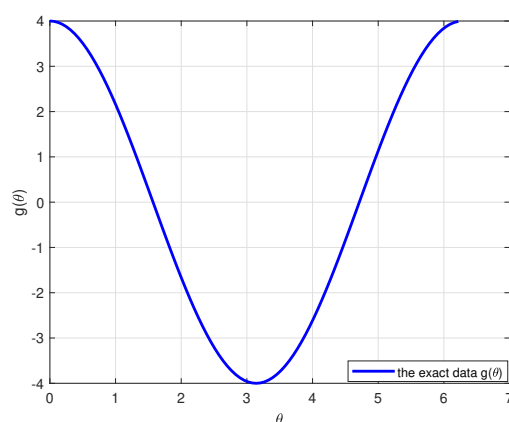


Figure 1. The input data $g(\theta)$ for Example 4.1.

Figure 1 displays the input data $g(\theta)$. Now we use the iterative algorithm with the initial guess $f^{(0)} = 0$ for reconstructing the solution $f(\theta)$. The relaxation factor is set to $c = -1.1$.

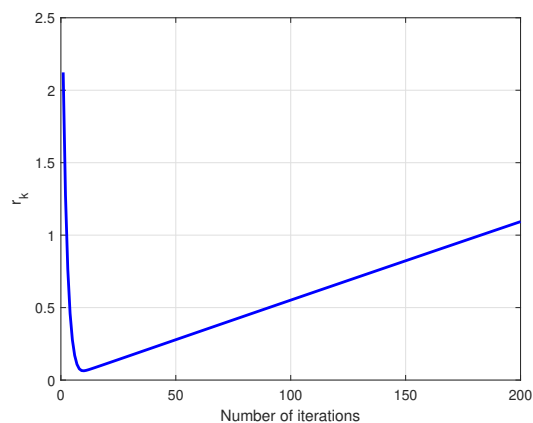


Figure 2. The convergence error vs number of iterations for Example 4.1.

Figure 2 presents the convergence error of the iteration number k for the noise-amplitude parameter $\varepsilon = 0.01$. The semi-convergence phenomenon is clearly observed: The error decreases

rapidly during the initial iterations, reaching a minimum at an optimal iteration count, and then increases gradually due to noise amplification. This behavior confirms the regularization property of the proposed iterative algorithm.

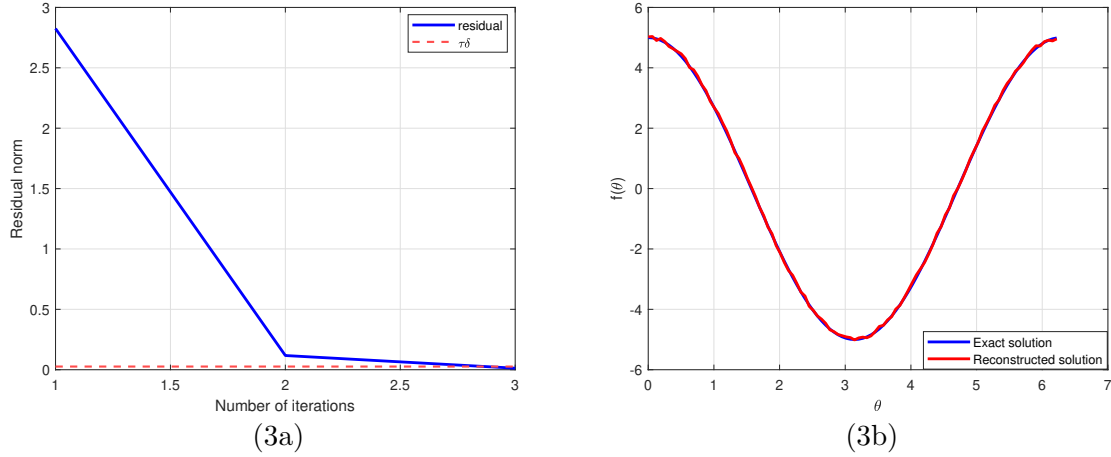


Figure 3. (3a) The residual norm during the iterative process with $\varepsilon = 0.01$ for Example 4.1; (3b) reconstruction result with $\varepsilon = 0.01$ where RRMSE=0.0131 and iterative step is $k = 3$ for Example 4.1.

In Figure 3, (3a) illustrates the residual $\|v^{(k),\delta}(R_2, \cdot) - g_\delta\|_{\Gamma_2}$ as a function of the iteration number for $\varepsilon = 0.01$. The horizontal dashed line represents the stopping threshold $\tau\delta$ with $\tau = 1.1$. The iteration is terminated at $k = 3$, where the residual first falls below the threshold. This stopping index is very close to the optimal iteration number observed in Figure 2, validating the effectiveness of the Morozov discrepancy principle as a stopping criterion for the proposed method. (3b) shows the reconstruction result when the noise-amplitude parameter is $\varepsilon = 0.01$. It can be observed that the reconstructed solutions closely match the exact solution, demonstrating the stability of the method.

Since $R_1 = 1$ and $R_2 = 2$, the first Fourier mode has $\lambda_1 = \frac{4}{5}$. With $c = -1.1$, the corresponding exact-data error factor is $|1 + c\lambda| = 0.12$. Therefore, starting from $f(0) = 0$, the relative error contributed by the iteration after three steps is $0.12^3 = 1.728 \times 10^{-3}$. The larger RRMSE observed for the noisy-data experiment is consequently attributable to measurement noise and discretization errors rather than insufficient convergence of the iteration.

Figures 4 and 5 show the results when the noise-amplitude parameters are 0.05 and 0.1 respectively. It can be seen that the proposed algorithm effectively recovers the epicardial potential distribution, even when the measured data contains significant noise.

Furthermore, the optimal stopping iteration number decreases as the noise level increases, which is consistent with the theory of iterative regularization methods. For $\varepsilon = 0.01$, the optimal iteration is approximately $k = 3$; for $\varepsilon = 0.05$, it is around $k = 2$; and for $\varepsilon = 0.10$, it is about $k = 2$.

Example 4.2. Multi-Frequency Smooth Solution: To further validate the effectiveness of the proposed algorithm for more complex epicardial potential distributions, we consider a multi-frequency smooth exact solution on the inner boundary Γ_1 :

$$f(\theta) = \left(R_1 + \frac{4}{R_1}\right) \cos(\theta) + \left(R_1^2 + \frac{16}{R_1^2}\right) \sin(2\theta) + \left(R_1^3 + \frac{64}{R_1^3}\right) \cos(3\theta), \quad (4.7)$$

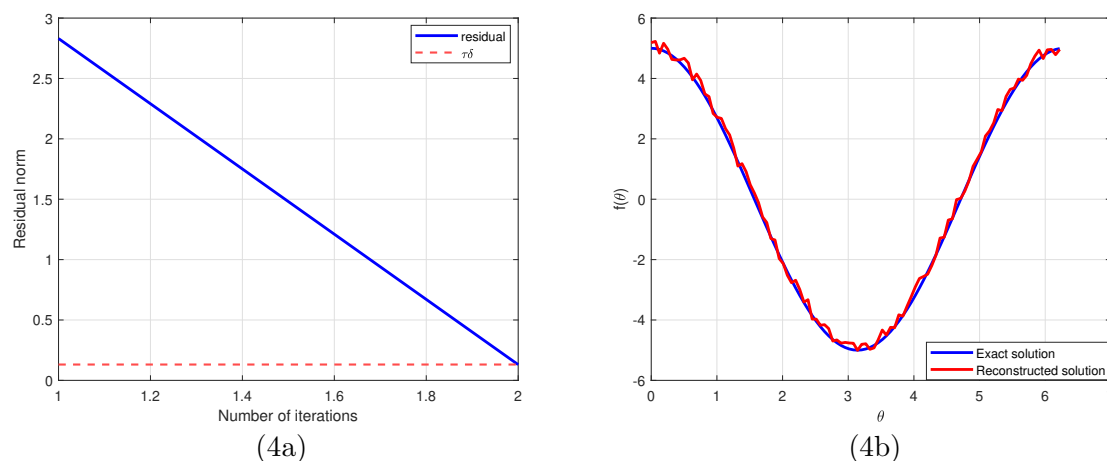


Figure 4. (4a) The residual norm during the iterative process with $\varepsilon = 0.05$ for Example 4.1; (4b) reconstruction result with $\varepsilon = 0.05$ where RRMSE=0.0486 and iterative step is $k = 2$ for Example 4.1.

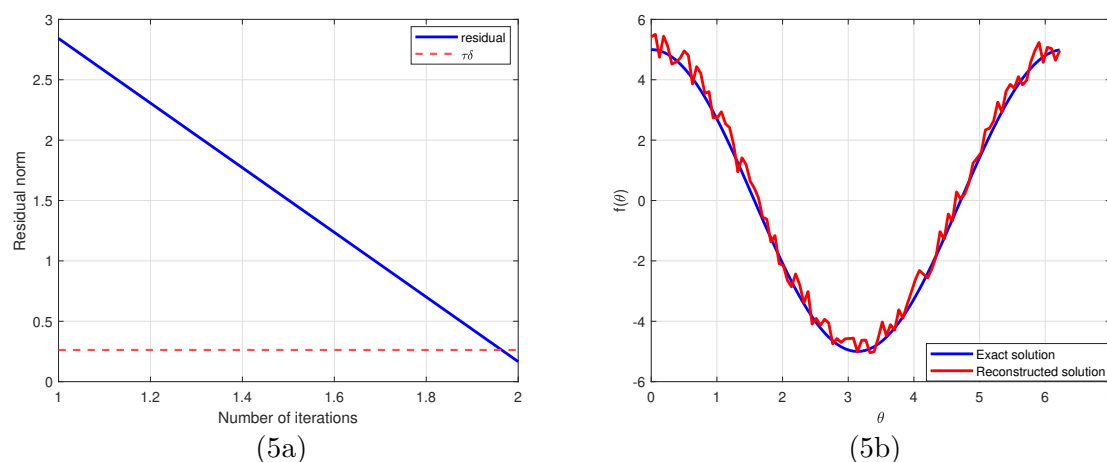


Figure 5. (5a) The convergence error during the iterative process with $\varepsilon = 0.1$ for Example 4.1; (5b) reconstruction result with $\varepsilon = 0.10$ where RRMSE=0.0967 and iterative step is $k = 2$ for Example 4.1.

where $R_1 = 1$.

This solution contains three distinct frequency components, providing a more stringent test for the iterative reconstruction. It can be verified that the corresponding exact Cauchy data on the body surface Γ_2 (with $R_2 = 2$) is given by:

$$g(\theta) = \left(R_2 + \frac{4}{R_2}\right) \cos(\theta) + \left(R_2^2 + \frac{16}{R_2^2}\right) \sin(2\theta) + \left(R_2^3 + \frac{64}{R_2^3}\right) \cos(3\theta). \quad (4.8)$$

Figure 6 displays the input data $g(\theta)$ for Example 4.2. Compared with the single-frequency case in Example 4.1, the data exhibits more oscillations, making the inverse problem more challenging.

We use the same initial guess $f^{(0)}(\theta) = 0$ and apply the proposed iterative algorithm. The iteration is terminated according to the Morozov discrepancy principle (4.5) with $\tau = 1.1$.

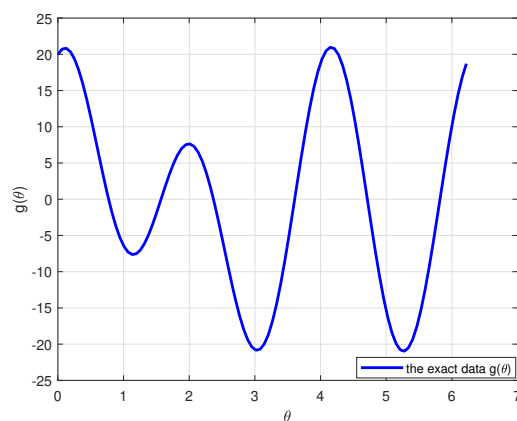
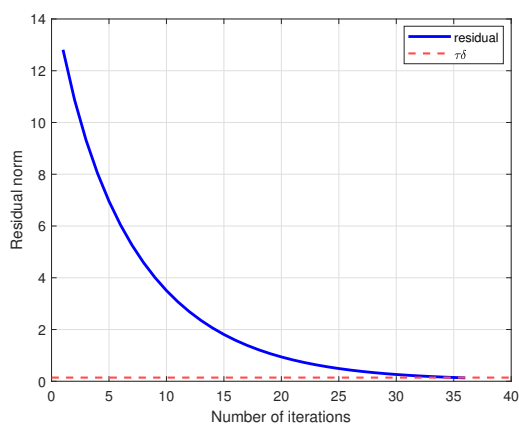
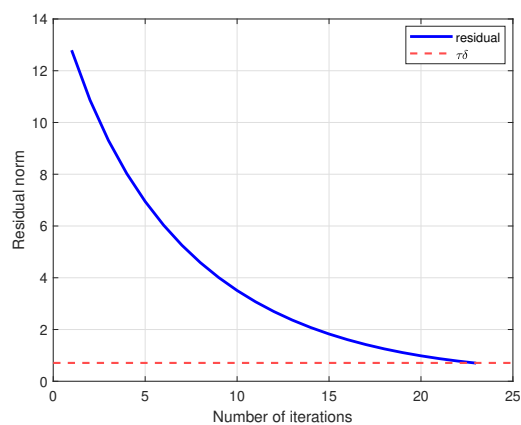


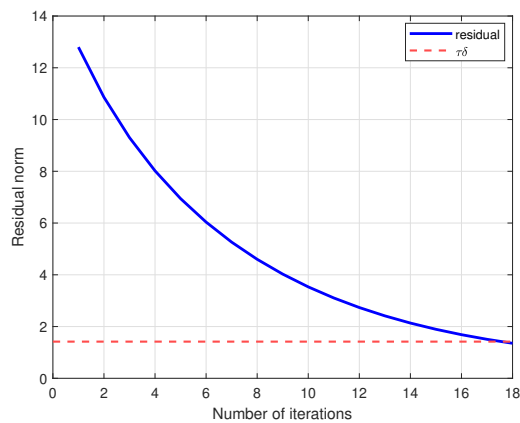
Figure 6. The input data $g(\theta)$ for Example 4.2.



(7a)



(7b)



(7c)

Figure 7. The residual norm during the iterative process for Example 4.2 with relaxation factor $c = -0.5$ under different noise-amplitude parameters: (7a) $\varepsilon = 0.01$, (7b) $\varepsilon = 0.05$, (7c) $\varepsilon = 0.10$. Stopping iterations: $k = 36, 23, 18$, respectively.

Figure 7 shows the variation of the residual norm with the number of iterations under the condition of a fixed relaxation factor $c = -0.5$, for three different noise-amplitude parameters $\varepsilon = 0.01, 0.05, 0.10$. At this time, the RRMSE of the results are 0.0264, 0.0902, and 0.1529 respectively.

Figure 8 systematically investigates the influence of the relaxation factor c on the final reconstruction accuracy. The RRMSE is plotted against different values of c within the theoretically admissible range $(-2, 0)$, for three noise-amplitude parameters $\varepsilon = 0.01, 0.05, 0.10$. It can be seen from Figure 8 that within the test range, when $c = -1.8$, the RRMSE of the results is superior to that of $c = -0.5$ and other values regardless of the noise level, indicating that its reconstruction accuracy is higher.

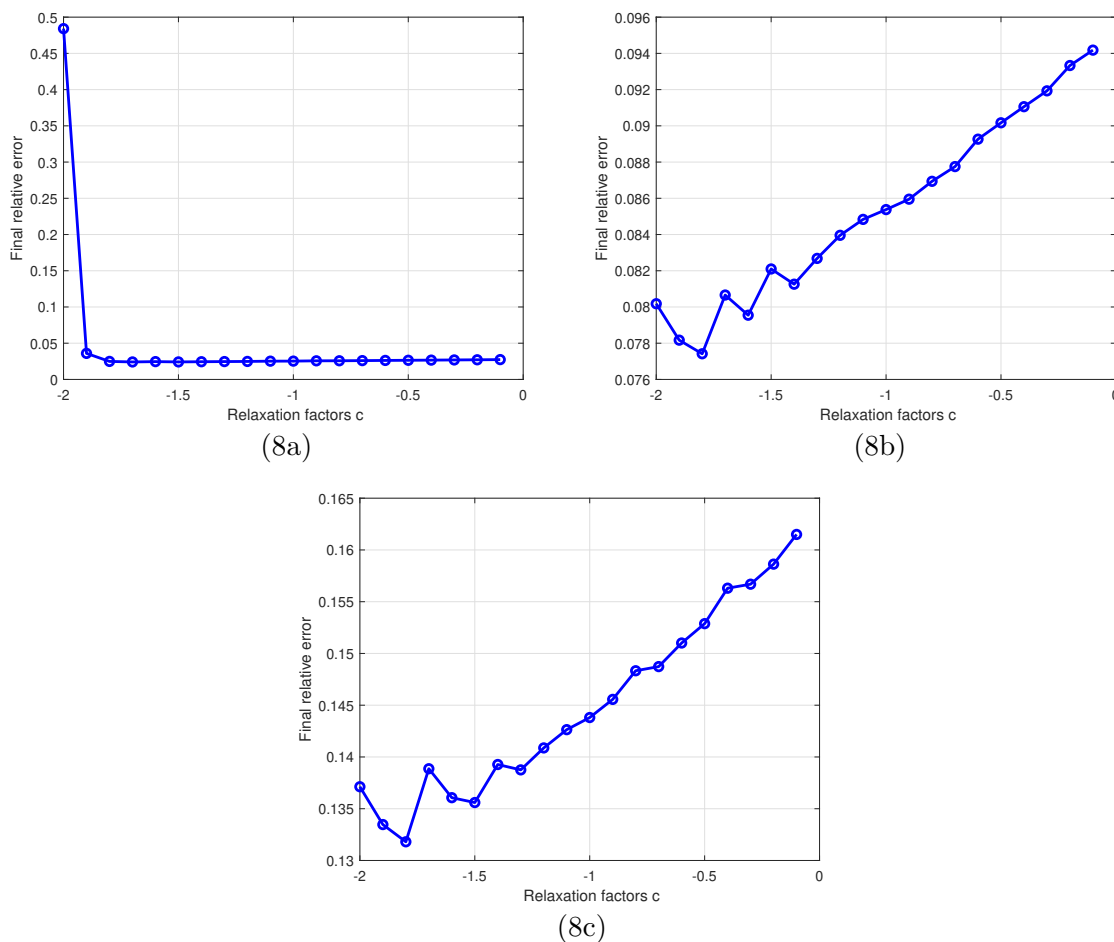


Figure 8. Final relative error (RRMSE) as a function of the relaxation factor c for different noise-amplitude parameters: (8a) $\varepsilon = 0.01$, (8b) $\varepsilon = 0.05$, (8c) $\varepsilon = 0.10$.

Figure 9 shows the results when the larger magnitude relaxation factor $c = -1.8$ is used. Subfigures (9a), (9c), and (9e) present the convergence error curves for $\varepsilon = 0.01, 0.05, 0.1$, respectively. The stopping iterations are significantly reduced compared to Figure 7: $\varepsilon = 0.01, k = 10$, RRMSE = 0.0249; $\varepsilon = 0.05, k = 6$, RRMSE = 0.0774; $\varepsilon = 0.10, k = 5$, RRMSE = 0.1318. Subfigures (9b), (9d), and (9f) display the corresponding reconstructed epicardial potential distributions at the optimal stopping iterations. In all cases, the reconstructed solutions closely

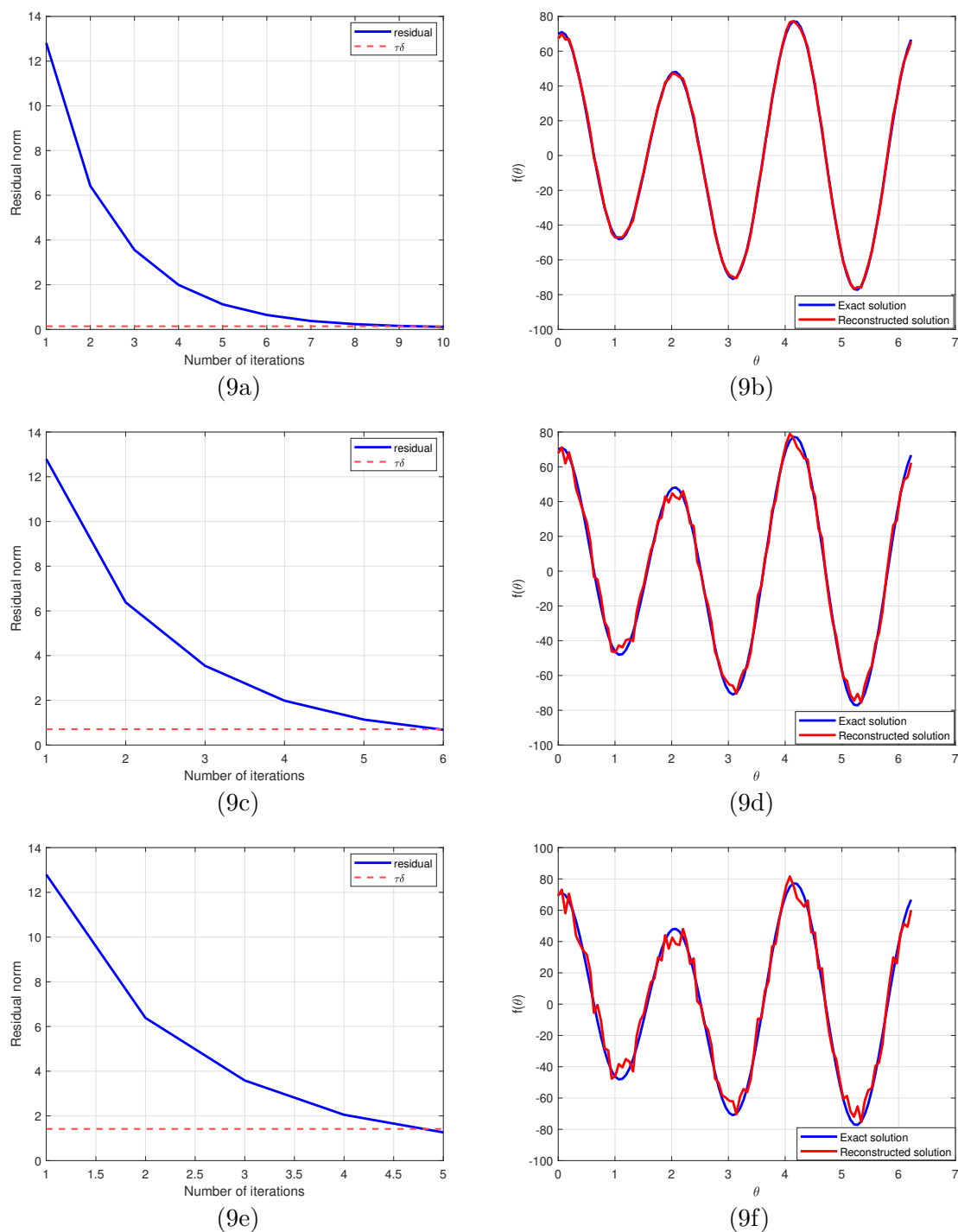


Figure 9. Results for Example 4.2 with relaxation factor $c = -1.8$. (a,c,e) Convergence error versus iteration number for $\varepsilon = 0.01, 0.05, 0.10$; (b,d,f) corresponding reconstructed epicardial potential distributions at the optimal stopping iterations.

match the exact solution, demonstrating that the algorithm effectively recovers the multi-frequency potential distribution even under significant noise.

Comparing Figures 7 and 9, we observe that increasing $|c|$ from 0.5 to 1.8 accelerates convergence substantially, reducing the required number of iterations by approximately a factor of three, while maintaining similar or slightly better reconstruction accuracy. The larger magnitude relaxation factor provides stronger correction per iteration, leading to faster convergence and reduced final error.

Example 4.3. Gaussian peak function: To further evaluate the performance of the proposed algorithm for localized features, we consider a Gaussian peak function on the epicardial boundary Γ_1 :

$$f(\theta) = \exp\left(-\frac{(\theta - \pi)^2}{2\sigma^2}\right), \quad \theta \in [0, 2\pi], \quad (4.9)$$

with $\sigma = \frac{1}{2}$.

This function attains its maximum at $\theta = \pi$ and decays smoothly to near zero elsewhere, simulating a localized electrical activity such as a cardiac R-wave.

The corresponding exact Cauchy data $G = g(\theta)$, as shown in Figure 10, on the body surface Γ_2 is obtained by solving the forward problem (3.5) with the finite difference scheme described in Section 3.6.

The noisy data G^δ is generated according to (4.1) with noise-amplitude parameters $\varepsilon = 0.01, 0.05, 0.10$. The iteration is initialized with $f^{(0)}(\theta) = 0$ and terminated using the Morozov discrepancy principle (3.16) with $\tau = 1.1$. The relaxation factor is set to $c = -1.3$.

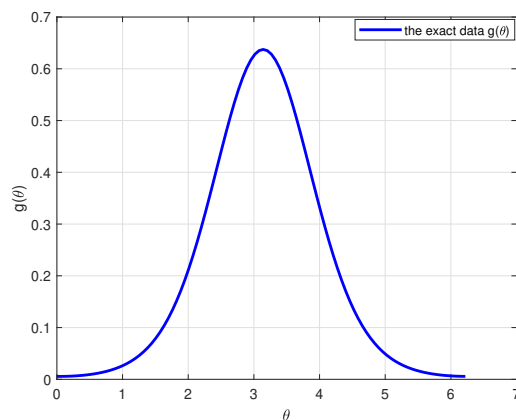
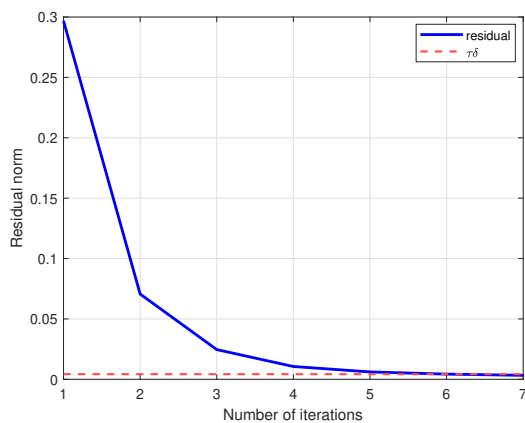


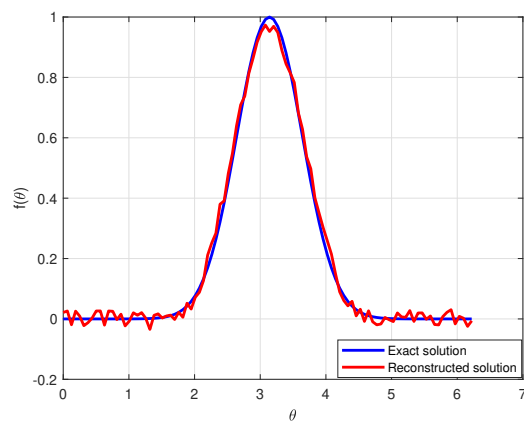
Figure 10. The input data $g(\theta)$ for Example 4.3.

Figure 11 presents the reconstruction results for the Gaussian peak function at different noise levels. Subfigures (a), (c), and (e) show the residual as a function of the iteration number for $\varepsilon = 0.01, 0.05, 0.10$, respectively. Subfigures (b), (d), and (f) display the corresponding reconstructed epicardial potential distributions at the optimal stopping iterations, compared with the exact solution. The optimal stopping iterations and RRMSE are summarized in Table 1. According to Figure 11 and Table 1, this algorithm is capable of successfully recovering the position and approximate amplitude of the Gaussian peak for all noise-amplitude parameters, which indicates its effectiveness in identifying local features.

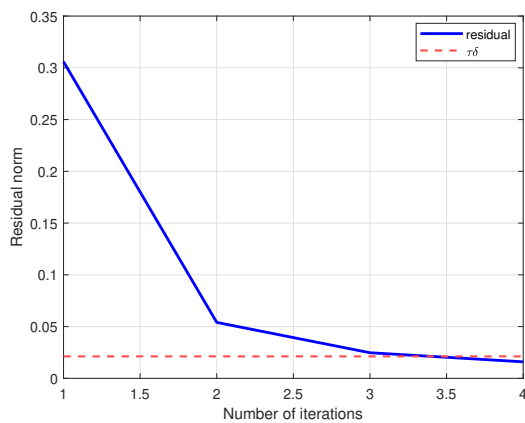
Some underestimation of the peak height is observed at higher noise-amplitude parameters ($\varepsilon = 0.10$), which is expected due to the low-pass nature of the inverse problem: High-frequency components that contribute to the sharpness of the peak are attenuated in the measured data.



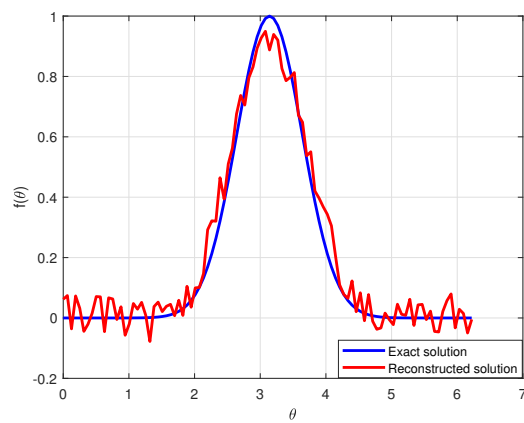
(11a)



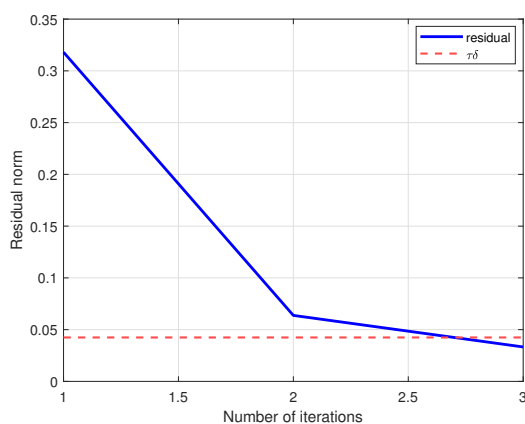
(11b)



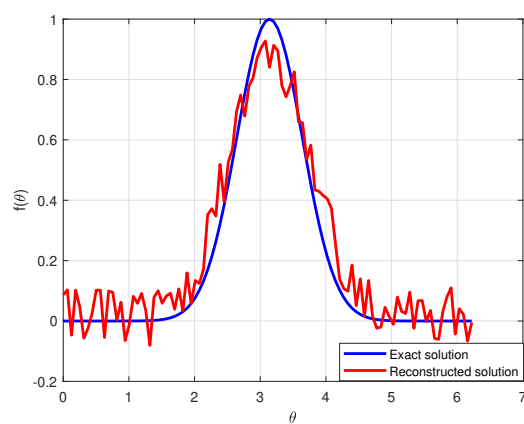
(11c)



(11d)



(11e)



(11f)

Figure 11. Results for Example 4.3 with relaxation factor $c = -1.3$. (a,c,e) Convergence error versus iteration number for $\varepsilon = 0.01, 0.05, 0.10$; (b,d,f) corresponding reconstructed epicardial potential distributions at the optimal stopping iterations.

For the Gaussian peak function with $\varepsilon = 0.10$, the proposed algorithm achieves a relative reconstruction error of 0.2193. This corresponds to an error amplification factor of approximately

Table 1. Optimal stopping iterations and final reconstruction errors for Example 4.3.

noise-amplitude parameter ε	Stopping iteration k^*	Final RRMSE
0.01	7	0.0592
0.05	4	0.1475
0.10	3	0.2193

2.2, which is significantly smaller than the theoretical upper bound for such ill-posed problems. Considering the severe ill-posedness of the inverse electrocardiography problem (singular values decay as $(R_1/R_2)^{|n|} = 0.5^{|n|}$), this level of accuracy is considered acceptable for clinical applications, where the primary interest lies in the location rather than the exact magnitude of epicardial potentials. Moreover, the reconstruction successfully captures the position and approximate shape of the Gaussian peak, demonstrating the practical utility of the proposed method.

The Gaussian peak example provides additional evidence of the practical utility of the proposed method. Localized electrical activity, such as that occurring in cardiac ischemia or infarction, is of primary clinical interest in inverse electrocardiography. The ability to reconstruct the location and approximate magnitude of such features from noisy body surface measurements is a key requirement for clinical applications.

4.2.2. Comparison with the classical Kozlov–Maz’ya method

To further assess the computational efficiency and reconstruction accuracy of the proposed algorithm, we present a comprehensive comparison with the classical Kozlov–Maz’ya alternating scheme across all three benchmark examples. The comparison is conducted under a fixed noise-amplitude parameter of $\varepsilon = 0.05$, and both methods are terminated using the same discrepancy principle with $\tau = 1.1$. The initial guess $f^{(0)}(\theta) = 0$ is employed for both algorithms to ensure a fair comparison.

Table 2 summarizes the relative root mean square error (RRMSE), the number of iterations required to satisfy the stopping criterion, and the total CPU time for each test case.

Table 2. Comprehensive comparison of accuracy, iterations, and per-case time ($\varepsilon = 0.05$).

Method	Ex 4.1 (Cosine)			Ex 4.2 (Multi.)			Ex 4.3 (Gauss.)		
	RRMSE	Iter	Time(s)	RRMSE	Iter	Time(s)	RRMSE	Iter	Time(s)
Proposed method	0.0486	2	3.8569	0.0774	6	6.0270	0.1475	4	6.8201
Kozlov–Maz’ya	0.0506	3	4.6909	0.0824	20	17.3555	0.1620	8	10.9386

Several observations can be drawn from Table 2. First, in terms of reconstruction accuracy, the proposed method consistently achieves slightly lower RRMSE values than the classical Kozlov–Maz’ya scheme across all three examples, demonstrating that the introduction of the relaxation factor does not compromise solution fidelity. Second, and more notably, the proposed algorithm requires substantially fewer iterations to reach the stopping threshold. The reduction is particularly pronounced for the multi-frequency example (Example 4.2), where the iteration count decreases from 20 to 6. Third, the total computation time is reduced by approximately

18% for the simple cosine case, 65% for the multi-frequency case, and 38% for the Gaussian peak case. The significant time savings stem from the combined effect of solving only one forward problem per iteration and the accelerated convergence induced by the relaxation factor.

These results confirm that the proposed simplified iterative algorithm not only preserves the regularizing properties of the original Kozlov–Maz’ya framework but also offers substantial improvements in computational efficiency, making it better suited for practical inverse electrocardiography applications where rapid reconstruction is desirable.

5. Conclusion

This paper presents a simplified iterative algorithm with a relaxation factor for solving the Cauchy problem of Laplace’s equation in an annular domain, with applications to inverse electrocardiography. The main conclusions are as follows:

1. The proposed simplified iterative algorithm reduces computational complexity compared to traditional methods while maintaining reconstruction accuracy.
2. The theoretically derived relaxation factor selection range and improved stopping criterion ensure algorithm stability.
3. Numerical experiments validate the effectiveness and robustness of the method under different noise levels.

Future work will extend the method to 2D real electrocardiography data and investigate machine learning-based adaptive parameter selection strategies.

CRedit authorship contribution statement. **Yu Shen:** Writing - review & editing, Writing - original draft, Visualization, Validation, Methodology, Conceptualization. **Xiangtuan Xiong:** Writing - review & editing, Validation, Supervision, Methodology, Conceptualization.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability. No external datasets were used in this study. All numerical data were generated using the computational methods described in this manuscript.

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