

FROM POINCARÉ TO CAUCHY–SCHWARZ INEQUALITIES: A UNIFIED APPROACH TO PDE SOLUTIONS

Abdon Atangana^{1,2,†}

Abstract We study energy estimates for classical and fractional Dirichlet problems through a functional setting that records the regularity of a solution and its nonlinear image. For the spectral and restricted Dirichlet fractional Laplacians on bounded Lipschitz domains, we equip the domain $\{u \in D(B^{1/2}) : f(u) \in D(B^{1/2})\}$ with a complete graph metric whenever f is continuous and fixes zero. The associated squared functional is lower semicontinuous in L^2 and has compact sublevel sets. For the restricted operator of order $0 < \alpha < 2$ and the power map $f_p(t) = |t|^{p-1}t$, with $p > 1$ and $n > \alpha$, we establish a Hölder inclusion into L^{q_*} , where $q_* = 2np/(n - \alpha)$, and compactness below this exponent. A concentration sequence proves optimality of both thresholds. Dirichlet form estimates and power dissipation identify the nonlinear transformation whose fractional regularity is controlled by an absorption equation. Fractional wave energies are treated under an explicit potential coercivity condition. Supporting results include variational estimates, parabolic regularity in the Dirichlet operator domain, and stability and second-order convergence of a classical wave discretization under a fixed strict CFL condition and smooth-solution assumptions.

Keywords Fractional Laplacian, nonlinear graph spaces, Sobolev embeddings, compactness, energy estimates.

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1. Introduction

Energy estimates connect the differential structure of a partial differential equation with the function spaces in which its solutions can be studied. For Dirichlet problems, the Poincaré inequality controls the magnitude of a function through its spatial variation, while the Cauchy–Schwarz inequality estimates the pairings that arise in weak formulations. These principles underlie many coercivity, stability, and regularity arguments [6, 10, 19]. Their effective use requires a compatible choice of operator domain, boundary condition, and solution regularity. The variational framework provides this compatibility and allows energy identities to be justified without assuming classical derivatives or boundary traces that the solution may not possess [13, 14, 25].

Fractional diffusion introduces an additional dependence on the realization of the spatial operator. Although several definitions of the fractional Laplacian agree on the whole space, they give different Dirichlet operators on bounded domains [12, 15]. The spectral realization is

[†]The corresponding author.

¹Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa

²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, China

Email: AtanganaA@ufs.ac.za (A. Atangana)

obtained by taking fractional powers of the classical Dirichlet Laplacian, whereas the restricted integral realization uses an exterior condition on the complement of the domain. Their operator domains, boundary behavior, and spectral constants must therefore be specified separately. Positive self-adjoint operators and their associated Dirichlet forms provide a common language for the energy estimates that remain valid in both settings [9, 22].

A second issue concerns the interaction between fractional regularity and nonlinear composition. If a solution is controlled in a Sobolev space, a nonlinear transformation of that solution need not have the same regularity. Conditions guaranteeing such composition properties are central to the theory of superposition operators [2, 5]. For fractional spaces, the relevant questions involve both integrability and nonlocal differences [18]. This leads to a natural analytical requirement: When an equation involves a nonlinear image $f(u)$, the functional setting should identify precisely which properties of $f(u)$ are controlled, how these properties behave under convergence, and whether they follow from the equation's dissipation.

The purpose of this paper is to develop these connections for classical and fractional Dirichlet problems. The elementary Poincaré–Cauchy–Schwarz estimates serve as preliminary tools. The principal focus is the quantitative information obtained by controlling a function and its nonlinear image in the same fractional form domain. This approach combines an operator formulation of the energy estimates with a metric formulation of nonlinear composition.

Let B be either of the two fractional Dirichlet operators, let $\mathcal{V}_B = D(B^{1/2})$, and write $\mathfrak{e}_B(u) = \|B^{1/2}u\|_2^2$. For a continuous scalar function f satisfying $f(0) = 0$, we consider the graph domain

$$\mathcal{A}_{f,B}(\Omega) = \{u \in \mathcal{V}_B : f(u) \in \mathcal{V}_B\}$$

with the metric

$$d_{f,B}(u, v)^2 = \|u - v\|_2^2 + \mathfrak{e}_B(u - v) + \mathfrak{e}_B(f(u) - f(v)).$$

Theorem 5.2 proves completeness, lower semicontinuity of the associated squared functional, and compactness of its sublevel sets in $L^2(\Omega)$. Bounded sequences also have strongly convergent nonlinear images in $L^2(\Omega)$. The metric measures the difference $f(u) - f(v)$, which is essential to the completeness argument. The distance from the origin generally fails the linear norm axioms, as the power example demonstrates.

For power nonlinearities, the graph condition yields an explicit integrability scale. If B is the restricted fractional Laplacian of order α , $n > \alpha$, and $f_p(t) = |t|^{p-1}t$ with $p > 1$, the fractional Sobolev inequality applied to $f_p(u)$ gives

$$\|u\|_{L^{q_*}(\Omega)}^p \leq C_S \mathfrak{e}_B(f_p(u))^{1/2}, \quad q_* = \frac{2np}{n - \alpha}.$$

Theorem 5.3 establishes Hölder continuity of this inclusion with respect to the graph metric and compactness on bounded graph sublevels for every $1 \leq q < q_*$. A concentrating family has bounded graph functional and converges to zero in the underlying fractional energy space, while retaining a positive L^{q_*} norm. The same family has unbounded L^q norms for $q > q_*$. These calculations identify the optimal exponent and distinguish convergence in the graph metric from convergence in the underlying form norm.

Evolution estimates involve the nonlinear form interaction $\mathfrak{e}_B(f(u), u)$. We derive quantitative bounds under Lipschitz and secant assumptions and use the established Stroock–Varopoulos mechanism for power nonlinearities [3]. For an absorption equation with reaction $|u|^{r-1}u$, $r > 1$, the dissipation controls the form energy of $|u|^{(r-1)/2}u$. For strong solutions satisfying the stated integrability hypotheses, this gives an integrated graph estimate and, in the restricted setting,

a corresponding improvement of spatial integrability. The nonlinear transformation selected by the estimate is thus determined by the reaction. The fractional wave analysis complements this dissipative setting: It identifies the conserved energy and the potential lower bound needed for that energy to control displacement and velocity.

The classical analysis provides the supporting operator and evolution framework. We formulate the Dirichlet Laplacian on its variational domain, derive elliptic and parabolic estimates, and state the additional boundary regularity needed for H^2 conclusions [10, 11, 14]. Applications include nonlinear diffusion and dissipative reaction equations. We also examine the classical one-dimensional wave discretization through its exact staggered energy. A fixed strict CFL condition gives uniform stability, and consistency with a sufficiently smooth exact solution yields second-order convergence in discrete displacement, gradient, and velocity norms. This part uses the standard energy approach to finite-difference analysis [17, 23].

The paper is organized as follows. Section 2 establishes the variational setting and basic inequalities. Section 3 develops applications to linear and nonlinear boundary-value and evolution problems. Section 4 treats parabolic regularity, Sobolev embeddings, and continuous and discrete wave energies. Section 5 develops the fractional Dirichlet form estimates, nonlinear graph spaces, sharp power embeddings, and fractional evolution applications.

2. Analytical setting and basic inequalities

This section fixes the functional setting and records the classical estimates used in the subsequent analysis. Particular attention is given to operator domains, the interpretation of boundary conditions, and the integrability of nonlinear expressions. The inequalities below are standard consequences of variational identities, the Poincaré inequality, and the Cauchy–Schwarz inequality. Their role is to provide precise hypotheses and consistent constants for the later evolution problems. We use the Sobolev and operator theory developed in [1, 6, 10, 13, 25].

2.1. Variational framework and the Dirichlet Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain, where $n \geq 1$. All functions are real-valued. We write $H = L^2(\Omega)$, with inner product

$$(u, v)_H = \int_{\Omega} uv \, dx,$$

and set $V = H_0^1(\Omega)$. The notation $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm, including its vector-valued version for gradients. By the trace theorem, membership in V expresses the homogeneous Dirichlet boundary condition in the Sobolev sense.

Define the first Dirichlet eigenvalue variationally by

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|\nabla v\|_2^2}{\|v\|_2^2} > 0, \quad C_P = \lambda_1^{-1/2}. \quad (2.1)$$

Thus C_P denotes the optimal Dirichlet Poincaré constant throughout the paper, and

$$\|v\|_2 \leq C_P \|\nabla v\|_2, \quad v \in V. \quad (2.2)$$

The historical development of this inequality goes back to [19]; its variational formulation and spectral interpretation are treated in [6, 10].

We equip V with the Hilbert norm $\|v\|_V = \|\nabla v\|_2$. This convention is equivalent to the full Sobolev norm because

$$\|v\|_V \leq \|v\|_{H^1(\Omega)} \leq (1 + C_P^2)^{1/2} \|v\|_V, \quad v \in V. \quad (2.3)$$

Let $V' = H^{-1}(\Omega)$ be the continuous dual of V , with the specific norm

$$\|\ell\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|\langle \ell, v \rangle_{V', V}|}{\|v\|_V}. \quad (2.4)$$

Identifying $h \in H$ with the functional $v \mapsto (h, v)_H$ gives the continuous dense embeddings $V \hookrightarrow H \hookrightarrow V'$ and the estimate

$$\|h\|_{V'} \leq C_P \|h\|_2. \quad (2.5)$$

The bilinear form associated with homogeneous Dirichlet diffusion is

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in V. \quad (2.6)$$

It is symmetric and continuous, and its coercivity is expressed by

$$|a(u, v)| \leq \|u\|_V \|v\|_V, \quad a(v, v) = \|v\|_V^2 \geq C_P^{-2} \|v\|_2^2. \quad (2.7)$$

In particular, a is an inner product on V .

The Dirichlet Laplacian in H is defined by

$$\begin{aligned} D(A) &= \{u \in V : \text{there exists } h \in H \text{ such that } a(u, v) = (h, v)_H \text{ for every } v \in V\}, \\ Au &= h, \quad u \in D(A). \end{aligned} \quad (2.8)$$

The function h is unique because V is dense in H . Equivalently,

$$D(A) = \{u \in H_0^1(\Omega) : -\Delta u \in L^2(\Omega) \text{ in the distributional sense}\}, \quad Au = -\Delta u. \quad (2.9)$$

Indeed, testing (2.8) with smooth compactly supported functions gives the distributional identity. Conversely, that identity extends from $C_c^\infty(\Omega)$ to V by density, since both $v \mapsto a(u, v)$ and $v \mapsto (h, v)_H$ are continuous on V . Consequently,

$$(Au, v)_H = a(u, v), \quad u \in D(A), \quad v \in V. \quad (2.10)$$

This is the integration-by-parts identity used below; it requires no classical boundary trace of Δu or of the normal derivative.

The operator A is positive and self-adjoint, has compact inverse, and satisfies

$$D(A^{1/2}) = V, \quad \|A^{1/2}v\|_2 = \|\nabla v\|_2, \quad v \in V. \quad (2.11)$$

These standard consequences of the closed form a and the compact embedding $V \hookrightarrow H$ provide the operator setting for fractional powers [6, 25]. In particular, $H^2(\Omega) \cap H_0^1(\Omega) \subset D(A)$. Equality holds when, for example, Ω has a $C^{1,1}$ boundary; no such equality is assumed for a general Lipschitz domain [10, 11].

For comparison, the Neumann form is defined on $H^1(\Omega)$. Since its kernel consists of constants on the connected domain Ω , the normalization

$$\int_{\Omega} v \, dx = 0 \quad (2.12)$$

gives a Poincaré–Wirtinger inequality

$$\|v\|_2 \leq C_N \|\nabla v\|_2, \quad v \in H^1(\Omega), \quad \int_{\Omega} v \, dx = 0, \quad (2.13)$$

where C_N is the Neumann constant. The zero-mean condition removes the constant functions; it does not itself impose a Neumann boundary condition. A weak homogeneous Neumann problem must use the form on $H^1(\Omega)$ and require its source to annihilate constants. The subsequent analysis uses the Dirichlet realization (2.8).

2.2. Energy estimates and nonlinear source terms

The basic product estimate underlying the energy method is

$$|(h, v)_H| \leq \|h\|_2 \|v\|_2 \leq C_P \|h\|_2 \|\nabla v\|_2, \quad h \in H, \quad v \in V. \quad (2.14)$$

For every $\varepsilon > 0$, Young's inequality therefore gives

$$|(h, v)_H| \leq \varepsilon \|\nabla v\|_2^2 + \frac{C_P^2}{4\varepsilon} \|h\|_2^2. \quad (2.15)$$

The free parameter ε allows the gradient contribution to be absorbed into a coercive diffusion term. Its choice will be made explicitly in each evolution estimate.

Theorem 2.1 (Dirichlet energy and Laplacian estimates). *For every $w \in D(A)$,*

$$\left| \int_{\Omega} w(-\Delta w) \, dx \right| \leq C_P \|\nabla w\|_2 \|\Delta w\|_2, \quad (2.16)$$

and

$$\int_{\Omega} w(-\Delta w) \, dx = \int_{\Omega} |\nabla w|^2 \, dx. \quad (2.17)$$

Consequently,

$$\|\nabla w\|_2^2 \leq C_P \|\nabla w\|_2 \|\Delta w\|_2, \quad (2.18)$$

$$\|\nabla w\|_2 \leq C_P \|\Delta w\|_2, \quad (2.19)$$

and

$$\|w\|_2 \leq C_P^2 \|\Delta w\|_2. \quad (2.20)$$

The constants C_P in (2.19) and C_P^2 in (2.20) are optimal.

Proof. Membership in $D(A)$ ensures that $w \in V$ and $Aw \in H$, so every term is well defined. Applying (2.14) with $h = Aw$ gives

$$|(Aw, w)_H| \leq \|Aw\|_2 \|w\|_2 \leq C_P \|Aw\|_2 \|\nabla w\|_2,$$

which is (2.16). Taking $v = w$ in (2.10) gives

$$(Aw, w)_H = a(w, w) = \|\nabla w\|_2^2,$$

and proves (2.17) and (2.18).

If $\|\nabla w\|_2 > 0$, division yields (2.19). If $\|\nabla w\|_2 = 0$, the Poincaré inequality implies $w = 0$, and the same conclusion holds. A further application of (2.2) proves (2.20).

To verify optimality, compactness of $V \hookrightarrow H$ implies that the infimum in (2.1) is attained by a function $\phi_1 \in V$ with $\|\phi_1\|_2 = 1$. The Euler–Lagrange equation for this minimization is

$$a(\phi_1, v) = \lambda_1(\phi_1, v)_H, \quad v \in V.$$

Hence $\phi_1 \in D(A)$, $A\phi_1 = \lambda_1\phi_1$, and $\|\nabla\phi_1\|_2 = \lambda_1^{1/2}$. It follows that

$$\frac{\|\nabla\phi_1\|_2}{\|A\phi_1\|_2} = \lambda_1^{-1/2} = C_P, \quad \frac{\|\phi_1\|_2}{\|A\phi_1\|_2} = \lambda_1^{-1} = C_P^2.$$

Thus neither constant can be decreased. $\square$

For an arbitrary $h \in V$, the same variational identity also gives

$$|(h, Aw)_H| = |a(h, w)| \leq \|\nabla h\|_2 \|\nabla w\|_2, \quad w \in D(A). \quad (2.21)$$

This estimate will be useful when h is a nonlinear image of w . Its applicability depends on proving that this image belongs to V , so that it is an admissible variational test function.

Theorem 2.2 (Mixed gradient estimate). *Let $w \in V$ and assume $\nabla w \in L^4(\Omega; \mathbb{R}^n)$. Then $w|\nabla w|^2 \in L^1(\Omega)$ and*

$$\left| \int_{\Omega} w |\nabla w|^2 dx \right| \leq C_P \|\nabla w\|_2 \left(\int_{\Omega} |\nabla w|^4 dx \right)^{1/2}. \quad (2.22)$$

Equivalently,

$$\left| \int_{\Omega} w |\nabla w|^2 dx \right| \leq C_P \|\nabla w\|_2 \|\nabla w |\nabla w|\|_2. \quad (2.23)$$

Proof. The assumption on ∇w implies

$$\| |\nabla w|^2 \|_2 = \left(\int_{\Omega} |\nabla w|^4 dx \right)^{1/2} = \|\nabla w\|_{L^4(\Omega)}^2. \quad (2.24)$$

Since $w \in L^2(\Omega)$, the Cauchy–Schwarz inequality first establishes the absolute integrability of the product and then gives

$$\left| \int_{\Omega} w |\nabla w|^2 dx \right| \leq \int_{\Omega} |w| |\nabla w|^2 dx \leq \|w\|_2 \| |\nabla w|^2 \|_2. \quad (2.25)$$

Substitution of (2.2) and (2.24) proves (2.22). The pointwise equality $|\nabla w| |\nabla w|^2 = |\nabla w|^4$ proves (2.23). $\square$

The hypothesis $\nabla w \in L^4$ is an additional integrability condition. The theorem estimates the indicated mixed integral under that condition; obtaining the condition from a PDE requires a separate regularity argument.

We next formulate the elliptic source estimate at the weak level. For a prescribed $f \in H$, consider

$$-\Delta w = f \quad \text{in } \Omega, \quad (2.26)$$

with

$$w = 0 \quad \text{on } \partial\Omega. \quad (2.27)$$

A weak solution is a function $w \in V$ satisfying

$$a(w, v) = (f, v)_H, \quad v \in V. \quad (2.28)$$

Proposition 2.1 (Weak elliptic solvability and source estimates). *For each $\ell \in V'$ there exists a unique $w \in V$ such that*

$$a(w, v) = \langle \ell, v \rangle_{V', V}, \quad v \in V. \quad (2.29)$$

With the dual norm (2.4),

$$\|w\|_V = \|\ell\|_{V'}. \quad (2.30)$$

In particular, for $\ell = f \in H$, the solution belongs to $D(A)$, satisfies $Aw = f$, and obeys

$$\|\nabla w\|_2 \leq C_P \|f\|_2, \quad (2.31)$$

and

$$\|w\|_2 \leq C_P^2 \|f\|_2. \quad (2.32)$$

Proof. The form a is continuous and coercive on the Hilbert space V , so the Lax–Milgram theorem gives existence and uniqueness [6, 10]. Taking $v = w$ in (2.29) yields

$$\|w\|_V^2 = \langle \ell, w \rangle_{V', V} \leq \|\ell\|_{V'} \|w\|_V.$$

This proves $\|w\|_V \leq \|\ell\|_{V'}$, including the case $w = 0$. Conversely,

$$|\langle \ell, v \rangle_{V', V}| = |a(w, v)| \leq \|w\|_V \|v\|_V$$

for every $v \in V$, and taking the supremum gives the reverse inequality in (2.30).

If $\ell = f \in H$, the definition of $D(A)$ implies $w \in D(A)$ and $Aw = f$. Testing by w gives the energy balance

$$\int_{\Omega} w(-\Delta w) dx = \|\nabla w\|_2^2 = \int_{\Omega} w f dx. \quad (2.33)$$

Consequently,

$$\|\nabla w\|_2^2 \leq C_P \|\nabla w\|_2 \|f\|_2. \quad (2.34)$$

The same zero-solution argument used in Theorem 2.1 proves (2.31); applying (2.2) then gives (2.32). $\square$

When the source depends on the solution, write

$$f(x) = F(x, w(x)). \quad (2.35)$$

The composition must be measurable and have the integrability required by the weak formulation. One useful sufficient assumption is that $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function: It is measurable in x for every fixed s , and continuous in s for almost every fixed x . Suppose also that, for a nonnegative $a_0 \in L^2(\Omega)$, a constant $b \geq 0$, and an exponent $q \geq 1$,

$$|F(x, s)| \leq a_0(x) + b|s|^q \quad \text{for almost every } x \in \Omega \text{ and every } s \in \mathbb{R}. \quad (2.36)$$

If $w \in V \cap L^{2q}(\Omega)$, then

$$\|F(\cdot, w)\|_2 \leq \|a_0\|_2 + b\|w\|_{L^{2q}(\Omega)}^q < \infty. \quad (2.37)$$

For $n \geq 3$, the Sobolev embedding guarantees this membership for every $w \in V$ when $1 \leq q \leq n/(n-2)$. In dimension $n = 2$ every finite $q \geq 1$ is admissible, while in dimension $n = 1$ the embedding into $L^\infty(\Omega)$ also covers every finite q . These are sufficient conditions for an L^2 source; weak formulations with sources in V' can allow broader growth [1, 2, 6].

For any weak solution satisfying $F(\cdot, w) \in H$, the preceding argument therefore gives

$$\|\nabla w\|_2 \leq C_P \|F(\cdot, w)\|_2. \quad (2.38)$$

This is a conditional estimate because its right-hand side contains the unknown solution. A bound in terms of the data follows if an additional sign or growth condition allows that dependence to be absorbed.

Proposition 2.2 (An absorption estimate for nonlinear sources). *Let $w \in V$ be a weak solution of $-\Delta w = F(x, w)$, with $F(\cdot, w) \in H$. Assume there are a nonnegative $a_1 \in L^2(\Omega)$ and a constant $0 \leq \gamma < \lambda_1$ such that*

$$F(x, s)s \leq a_1(x)|s| + \gamma s^2 \quad \text{for almost every } x \in \Omega \text{ and every } s \in \mathbb{R}. \quad (2.39)$$

Then

$$\|\nabla w\|_2 \leq \frac{C_P}{1 - \gamma C_P^2} \|a_1\|_2, \quad \|w\|_2 \leq \frac{C_P^2}{1 - \gamma C_P^2} \|a_1\|_2. \quad (2.40)$$

Proof. The weak formulation permits w as a test function, and hence

$$\|\nabla w\|_2^2 = \int_{\Omega} F(x, w)w \, dx \leq \int_{\Omega} a_1|w| \, dx + \gamma \|w\|_2^2.$$

Cauchy–Schwarz and (2.2) give

$$\|\nabla w\|_2^2 \leq C_P \|a_1\|_2 \|\nabla w\|_2 + \gamma C_P^2 \|\nabla w\|_2^2.$$

The assumption $\gamma < \lambda_1 = C_P^{-2}$ makes $1 - \gamma C_P^2$ strictly positive. Moving the last term to the left and dividing by $\|\nabla w\|_2$ when this norm is nonzero proves the first bound. When it vanishes, both bounds hold because $w = 0$. The second bound follows from (2.2). $\square$

This proposition is an a priori estimate for a weak solution; existence for a specified nonlinearity requires a separate argument. The strict restriction on γ cannot be relaxed to equality in this statement. Indeed, for $F(x, s) = \lambda_1 s$ and $a_1 = 0$, every multiple of the first eigenfunction solves the equation, so there is no uniform bound on the size of all solutions.

Finally, nonlinear testing against Aw requires control of a Sobolev derivative of the nonlinear image. For example, let $G \in C^1(\mathbb{R})$ satisfy $G(0) = 0$ and $\|G'\|_{L^\infty(\mathbb{R})} \leq L$. The Sobolev chain rule and trace compatibility imply, for every $w \in V$,

$$G(w) \in V, \quad \nabla G(w) = G'(w)\nabla w, \quad \|\nabla G(w)\|_2 \leq L \|\nabla w\|_2 \quad (2.41)$$

[1, 16]. Thus, when $w \in D(A)$,

$$\begin{aligned} \int_{\Omega} G(w)Aw \, dx &= a(G(w), w) = \int_{\Omega} G'(w)|\nabla w|^2 \, dx, \\ \left| \int_{\Omega} G(w)Aw \, dx \right| &\leq \|\nabla G(w)\|_2 \|\nabla w\|_2 \leq L \|\nabla w\|_2^2. \end{aligned} \quad (2.42)$$

If $G' \geq 0$, the first integral is nonnegative. This calculation identifies the composition regularity and boundary compatibility needed for nonlinear energy testing, and provides the classical formulation to which the subsequent fractional analysis will be related.

3. Applications and examples

We apply the variational framework of Section 2 to elliptic, parabolic, and reaction–diffusion problems. The estimates are derived in spaces that justify the test functions and the time derivatives appearing in the energy balances. The constants are expressed in terms of $C_P = \lambda_1^{-1/2}$ and, for nonlinear diffusion, the lower bound on the diffusion coefficient. These examples illustrate standard energy and monotonicity arguments [6, 10, 13, 14].

We retain $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, the gradient norm on V , its corresponding dual norm on V' , and the Dirichlet operator A introduced previously. Let $T > 0$ and $Q_T = \Omega \times (0, T)$. Unless otherwise specified, the evolution data satisfy

$$u_0 \in H, \quad f \in L^2(0, T; H). \quad (3.1)$$

3.1. Linear boundary-value and evolution problems

Example 3.1 (The Dirichlet Poisson problem). Given $f \in H$, consider

$$-\Delta w = f \quad \text{in } \Omega, \quad (3.2)$$

with

$$w = 0 \quad \text{on } \partial\Omega. \quad (3.3)$$

Proposition 2.1 gives a unique weak solution $w \in V$. Since $f \in H$, this solution belongs to $D(A)$ and satisfies $Aw = f$. Its energy identity is

$$\|\nabla w\|_2^2 = a(w, w) = (f, w)_H. \quad (3.4)$$

The source estimate (2.14) therefore gives

$$\|\nabla w\|_2 \leq C_P \|f\|_2, \quad \|w\|_2 \leq C_P^2 \|f\|_2. \quad (3.5)$$

Both bounds include the zero solution, and neither requires H^2 regularity of w .

The same estimates quantify the dependence on the data. Let w_1, w_2 correspond to sources $f_1, f_2 \in H$. Their difference $z = w_1 - w_2$ satisfies

$$a(z, v) = (f_1 - f_2, v)_H, \quad v \in V.$$

Taking $v = z$ and applying the source estimate yields

$$\|\nabla z\|_2^2 \leq C_P \|f_1 - f_2\|_2 \|\nabla z\|_2.$$

Division when $\|\nabla z\|_2 > 0$, together with the immediate zero case, proves

$$\|\nabla(w_1 - w_2)\|_2 \leq C_P \|f_1 - f_2\|_2, \quad \|w_1 - w_2\|_2 \leq C_P^2 \|f_1 - f_2\|_2. \quad (3.6)$$

Thus the inverse Dirichlet operator depends continuously on the source in both the energy and L^2 norms.

For the parabolic problems, define

$$\mathcal{W}(0, T) = \{u \in L^2(0, T; V) : u_t \in L^2(0, T; V')\}, \quad (3.7)$$

where u_t denotes the distributional time derivative. Every $u \in \mathcal{W}(0, T)$ has a representative in $C([0, T]; H)$, and its squared H norm is absolutely continuous, with

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 = \langle u_t(t), u(t) \rangle_{V', V} \quad \text{for almost every } t \in (0, T). \quad (3.8)$$

This Hilbert-space energy identity follows by temporal regularization and passage to the limit in the V', V pairing [6, 14]. It gives the integrated identity for every pair of times in $[0, T]$ and gives a precise meaning to the initial value. In particular, no assumption that u_t belongs to $L^2(0, T; H)$ is needed.

Example 3.2 (The linear heat equation). Consider

$$u_t - \Delta u = f \quad \text{in } Q_T, \quad (3.9)$$

with

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3.10)$$

and

$$u(0) = u_0 \quad \text{in } H. \quad (3.11)$$

For the data (3.1), the standard variational theory provides a weak solution $u \in \mathcal{W}(0, T)$ [10, 14]. Its weak formulation is

$$\langle u_t(t), v \rangle_{V', V} + a(u(t), v) = (f(t), v)_H, \quad v \in V, \quad (3.12)$$

for almost every $t \in (0, T)$. Taking $v = u(t)$ and using (3.8) gives

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 = (f(t), u(t))_H. \quad (3.13)$$

By Cauchy–Schwarz, Poincaré, and Young,

$$|(f(t), u(t))_H| \leq C_P \|f(t)\|_2 \|\nabla u(t)\|_2 \leq \frac{1}{2} \|\nabla u(t)\|_2^2 + \frac{C_P^2}{2} \|f(t)\|_2^2. \quad (3.14)$$

Absorbing the gradient term and multiplying by two yields

$$\frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq C_P^2 \|f(t)\|_2^2. \quad (3.15)$$

After integration, we obtain the estimate in terms of the data

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|u_0\|_2^2 + C_P^2 \int_0^t \|f(s)\|_2^2 ds, \quad 0 \leq t \leq T. \quad (3.16)$$

Writing

$$D_T = \|u_0\|_2^2 + C_P^2 \int_0^T \|f(s)\|_2^2 ds,$$

we obtain, separately,

$$\sup_{0 \leq t \leq T} \|u(t)\|_2^2 \leq D_T, \quad \int_0^T \|\nabla u(s)\|_2^2 ds \leq D_T. \quad (3.17)$$

Their sum is consequently bounded by $2D_T$.

The spectral gap also gives a bound that retains the time at which the forcing acts. Since $\|\nabla u(t)\|_2^2 \geq \lambda_1 \|u(t)\|_2^2$, equation (3.15) implies

$$\frac{d}{dt} \|u(t)\|_2^2 + \lambda_1 \|u(t)\|_2^2 \leq C_P^2 \|f(t)\|_2^2.$$

Multiplication by $e^{\lambda_1 t}$ and integration give

$$\|u(t)\|_2^2 \leq e^{-\lambda_1 t} \|u_0\|_2^2 + C_P^2 \int_0^t e^{-\lambda_1(t-s)} \|f(s)\|_2^2 ds. \quad (3.18)$$

If $f = 0$, the exact identity (3.13) yields the stronger estimate

$$\|u(t)\|_2^2 \leq e^{-2\lambda_1 t} \|u_0\|_2^2. \quad (3.19)$$

For two solutions with data $(u_{0,1}, f_1)$ and $(u_{0,2}, f_2)$, the difference satisfies the same weak equation with the corresponding differences of the data. Applying (3.16) to that equation gives

$$\begin{aligned} & \|u_1(t) - u_2(t)\|_2^2 + \int_0^t \|\nabla(u_1 - u_2)(s)\|_2^2 ds \\ & \leq \|u_{0,1} - u_{0,2}\|_2^2 + C_P^2 \int_0^t \|f_1(s) - f_2(s)\|_2^2 ds. \end{aligned} \quad (3.20)$$

This proves uniqueness in $\mathcal{W}(0, T)$ and continuous dependence on the initial value and forcing.

3.2. Nonlinear diffusion and dissipative reactions

In the next two examples, the diffusion coefficient $b : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies

$$0 < b_0 \leq b(s) \leq b_1 < \infty, \quad s \in \mathbb{R}. \quad (3.21)$$

The lower bound provides coercivity, while the upper bound ensures that $b(u)\nabla u \in L^2(Q_T; \mathbb{R}^n)$ whenever $u \in L^2(0, T; V)$. In particular,

$$\|b(u)\nabla u\|_{L^2(Q_T)} \leq b_1 \|\nabla u\|_{L^2(Q_T)}. \quad (3.22)$$

These hypotheses hold, for example, for $b(s) = \kappa + \beta \tanh s$ with $\kappa > |\beta|$.

Example 3.3 (Uniformly positive nonlinear diffusion). Consider

$$u_t - \nabla \cdot (b(u)\nabla u) = f \quad \text{in } Q_T, \quad (3.23)$$

with

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3.24)$$

and

$$u(0) = u_0 \quad \text{in } H. \quad (3.25)$$

Let $u \in \mathcal{W}(0, T)$ be a weak solution, meaning that

$$\langle u_t(t), v \rangle_{V', V} + \int_{\Omega} b(u(t)) \nabla u(t) \cdot \nabla v \, dx = (f(t), v)_H, \quad v \in V, \quad (3.26)$$

for almost every t . The flux estimate (3.22) makes this formulation well defined. Testing with $v = u(t)$ gives

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \int_{\Omega} b(u(t)) |\nabla u(t)|^2 dx = (f(t), u(t))_H. \quad (3.27)$$

The diffusion term is evaluated directly through its divergence form. This calculation requires no differentiation of b .

The coefficient bound and the source estimate give

$$\int_{\Omega} b(u) |\nabla u|^2 dx \geq b_0 \|\nabla u\|_2^2, \quad (3.28)$$

and

$$|(f, u)_H| \leq C_P \|f\|_2 \|\nabla u\|_2 \leq \frac{b_0}{2} \|\nabla u\|_2^2 + \frac{C_P^2}{2b_0} \|f\|_2^2. \quad (3.29)$$

Substitution into (3.27) yields

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \frac{b_0}{2} \|\nabla u(t)\|_2^2 \leq \frac{C_P^2}{2b_0} \|f(t)\|_2^2. \quad (3.30)$$

Therefore, for every $t \in [0, T]$,

$$\|u(t)\|_2^2 + b_0 \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|u_0\|_2^2 + \frac{C_P^2}{b_0} \int_0^t \|f(s)\|_2^2 ds. \quad (3.31)$$

When $f = 0$, the exact energy identity gives

$$\|u(t)\|_2^2 \leq e^{-2b_0\lambda_1 t} \|u_0\|_2^2. \quad (3.32)$$

A comparison estimate for this nonlinear equation is naturally obtained in V' . Define

$$B(s) = \int_0^s b(r) dr. \quad (3.33)$$

The function B is globally Lipschitz, $B(0) = 0$, and the Sobolev chain rule gives $B(u) \in L^2(0, T; V)$ with $\nabla B(u) = b(u) \nabla u$. Moreover,

$$(B(r) - B(s))(r - s) \geq b_0 |r - s|^2, \quad r, s \in \mathbb{R}. \quad (3.34)$$

Let $R : V' \rightarrow V$ be the variational inverse defined by

$$a(R\ell, v) = \langle \ell, v \rangle_{V', V}, \quad v \in V.$$

By Proposition 2.1, $\|R\ell\|_V = \|\ell\|_{V'}$ and $\langle \ell, R\ell \rangle_{V', V} = \|\ell\|_{V'}^2$.

Suppose $u_1, u_2 \in \mathcal{W}(0, T)$ are solutions with the same forcing, and set $z = u_1 - u_2$. Subtracting their weak formulations and testing with $Rz(t)$ gives

$$\langle z_t, Rz \rangle_{V', V} + a(B(u_1) - B(u_2), Rz) = 0.$$

Since $z \in H^1(0, T; V')$, the Hilbert-space chain rule in V' applies to the first term. The definition of R evaluates the second term, leading to

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{V'}^2 + \int_{\Omega} (B(u_1) - B(u_2)) z dx = 0. \quad (3.35)$$

Using (3.34) and $\|z\|_{V'} \leq C_P \|z\|_2$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{V'}^2 + b_0 \lambda_1 \|z(t)\|_{V'}^2 \leq 0.$$

Integration proves

$$\|u_1(t) - u_2(t)\|_{V'}^2 \leq e^{-2b_0 \lambda_1 t} \|u_1(0) - u_2(0)\|_{V'}^2. \quad (3.36)$$

Thus weak solutions in this class are unique for fixed data, and perturbations of the initial value decay in the dual norm. The argument uses the monotonicity of the primitive B ; it requires only the bounds (3.21), rather than any bound on the derivative of b .

Example 3.4 (A dissipative reaction–diffusion equation). Let $1 \leq p < \infty$ and consider

$$u_t - \nabla \cdot (b(u) \nabla u) + |u|^{p-1} u = f \quad \text{in } Q_T, \quad (3.37)$$

with

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3.38)$$

and

$$u(0) = u_0 \quad \text{in } H. \quad (3.39)$$

The reaction $|s|^{p-1}s$ is real-valued and continuous for every $s \in \mathbb{R}$, and its product with s is $|s|^{p+1}$. Thus its contribution to the energy balance is nonnegative for sign-changing solutions as well as for nonnegative solutions.

Assume that u is a weak solution with

$$u \in C([0, T]; H) \cap L^2(0, T; V) \cap L^{p+1}(Q_T). \quad (3.40)$$

Set $r = p + 1$ and $r' = r/(r - 1)$. The diffusion and forcing define

$$g_1 = f + \nabla \cdot (b(u) \nabla u) \in L^2(0, T; V'),$$

whereas the reaction defines

$$g_2 = -|u|^{p-1} u \in L^{r'}(Q_T).$$

Indeed,

$$\|g_1(t)\|_{V'} \leq C_P \|f(t)\|_2 + b_1 \|\nabla u(t)\|_2, \quad \|g_2\|_{L^{r'}(Q_T)}^{r'} = \|u\|_{L^r(Q_T)}^r. \quad (3.41)$$

The equation is understood as $u_t = g_1 + g_2$ in distributions, with spatial test functions in $V \cap L^r(\Omega)$. This is the dual-space formulation appropriate to the arbitrary finite exponent p .

To justify testing with u , regularize this distributional equation in time on a compact subinterval of $(0, T)$. For the regularized functions, the squared H norm has the usual derivative, and the equation can be paired with the regularized solution. As the regularization parameter tends to zero, the solution converges in both $L^2(0, T; V)$ and $L^r(Q_T)$ on the subinterval. The two terms in its time derivative converge in $L^2(0, T; V')$ and $L^{r'}(Q_T)$, respectively. Hölder's inequality therefore passes both pairings to the limit in L^1 in time. It follows that $\|u(t)\|_2^2$ has an integrable distributional derivative. The continuity assumed in (3.40) identifies its absolutely continuous representative and extends the integrated identity to the endpoints 0 and T .

Consequently, for almost every $t \in (0, T)$,

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \int_{\Omega} b(u(t)) |\nabla u(t)|^2 dx + \|u(t)\|_{L^{p+1}(\Omega)}^{p+1} = (f(t), u(t))_H. \quad (3.42)$$

Using (3.21) and (3.29) gives

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \frac{b_0}{2} \|\nabla u(t)\|_2^2 + \|u(t)\|_{L^{p+1}(\Omega)}^{p+1} \leq \frac{C_P^2}{2b_0} \|f(t)\|_2^2. \quad (3.43)$$

Multiplying by two and integrating yields

$$\begin{aligned} & \|u(t)\|_2^2 + b_0 \int_0^t \|\nabla u(s)\|_2^2 ds + 2 \int_0^t \|u(s)\|_{L^{p+1}(\Omega)}^{p+1} ds \\ & \leq \|u_0\|_2^2 + \frac{C_P^2}{b_0} \int_0^t \|f(s)\|_2^2 ds, \quad 0 \leq t \leq T. \end{aligned} \quad (3.44)$$

This estimate controls the solution magnitude, the diffusion dissipation, and the space–time integrability associated with the reaction, with explicit dependence on the data.

Dropping the nonnegative reaction contribution and using the Poincaré inequality in (3.43), we also obtain

$$\|u(t)\|_2^2 \leq e^{-b_0 \lambda_1 t} \|u_0\|_2^2 + \frac{C_P^2}{b_0} \int_0^t e^{-b_0 \lambda_1 (t-s)} \|f(s)\|_2^2 ds. \quad (3.45)$$

For $f = 0$, the exact balance (3.42) instead gives

$$\|u(t)\|_2^2 \leq e^{-2b_0 \lambda_1 t} \|u_0\|_2^2. \quad (3.46)$$

When $p = 1$, the reaction term equals u and contributes $\|u(t)\|_2^2$ directly to the dissipation. Retaining this term improves the unforced bound to

$$\|u(t)\|_2^2 \leq e^{-2(b_0 \lambda_1 + 1)t} \|u_0\|_2^2. \quad (3.47)$$

The estimates apply to every weak solution in (3.40). They provide the uniform bounds needed in approximation arguments; constructing such a solution also requires the corresponding compactness and limit-passage steps.

4. Regularity, Sobolev embeddings, and numerical stability

The energy estimates of Section 3 provide a starting point for regularity and numerical analysis. We first derive parabolic regularity and the associated Sobolev bounds in the variational setting of Section 2. We then examine continuous and discrete energy identities, including the dependence of the discrete estimates on the time step. Throughout, A denotes the positive Dirichlet Laplacian, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $C_P = \lambda_1^{-1/2}$.

4.1. Parabolic regularity and Sobolev embeddings

Consider the linear problem

$$\begin{cases} u_t - \Delta u = f & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } H, \end{cases} \quad (4.1)$$

where $Q_T = \Omega \times (0, T)$ and

$$f \in L^2(0, T; H), \quad (4.2)$$

$$u_0 \in H. \quad (4.3)$$

The weak solution belongs to $\mathcal{W}(0, T)$ defined in (3.7). The energy calculation in Example 3.2 gives, for every $t \in [0, T]$,

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|u_0\|_2^2 + C_P^2 \int_0^t \|f(s)\|_2^2 ds. \quad (4.4)$$

Define

$$D_T = \|u_0\|_2^2 + C_P^2 \|f\|_{L^2(0, T; H)}^2.$$

The supremum of the first term and the full dissipation integral are each bounded by D_T . In particular,

$$\sup_{0 \leq t \leq T} \|u(t)\|_2^2 + \int_0^T \|\nabla u(t)\|_2^2 dt \leq 2D_T. \quad (4.5)$$

These estimates quantify the norms in

$$u \in C([0, T]; H) \cap L^2(0, T; V) \subset L^\infty(0, T; H) \cap L^2(0, T; V). \quad (4.6)$$

All time differentiation here is interpreted through the V', V identity (3.8).

Stronger initial regularity gives square-integrable time and operator derivatives. The following standard Hilbert-space regularity result is stated with the precise operator domain [6, 10, 14].

Theorem 4.1 (Parabolic regularity in the Dirichlet operator domain). *Assume $u_0 \in V$ and $f \in L^2(0, T; H)$. The weak solution of (4.1) satisfies*

$$u \in C([0, T]; V) \cap H^1(0, T; H) \cap L^2(0, T; D(A)). \quad (4.7)$$

Here $D(A)$ carries its operator graph norm. For every $t \in [0, T]$,

$$\|\nabla u(t)\|_2^2 + \int_0^t (\|u_t(s)\|_2^2 + \|Au(s)\|_2^2) ds = \|\nabla u_0\|_2^2 + \int_0^t \|f(s)\|_2^2 ds. \quad (4.8)$$

If, in addition, Ω has a $C^{1,1}$ boundary, then

$$u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \quad (4.9)$$

and

$$\int_0^T \|u(t)\|_{H^2(\Omega)}^2 dt \leq C_\Omega^2 \left(\|\nabla u_0\|_2^2 + \|f\|_{L^2(0, T; H)}^2 \right). \quad (4.10)$$

Proof. Let $(\phi_j)_{j \geq 1}$ be an orthonormal eigenbasis of A in H , with $A\phi_j = \lambda_j \phi_j$, and let P_N be the orthogonal projection onto the first N eigenfunctions. Solve the finite-dimensional system

$$u'_N + Au_N = P_N f, \quad u_N(0) = P_N u_0.$$

Its coefficient functions are absolutely continuous. Squaring the equation in H and using

$$2(u'_N, Au_N)_H = \frac{d}{dt} \|A^{1/2} u_N\|_2^2$$

gives

$$\|A^{1/2} u_N(t)\|_2^2 + \int_0^t (\|u'_N\|_2^2 + \|Au_N\|_2^2) ds = \|A^{1/2} P_N u_0\|_2^2 + \int_0^t \|P_N f\|_2^2 ds. \quad (4.11)$$

For $M > N$, the difference $u_M - u_N$ satisfies the same identity with data $(P_M - P_N)u_0$ and $(P_M - P_N)f$. The right-hand side tends to zero because $u_0 \in D(A^{1/2}) = V$ and $f \in L^2(0, T; H)$. Consequently (u_N) is Cauchy in $C([0, T]; V)$, its time derivatives are Cauchy in $L^2(0, T; H)$, and (Au_N) is Cauchy in $L^2(0, T; H)$.

Closedness of A and passage to the limit in the distributional time derivative identify the limit as a function satisfying (4.7) and $u_t + Au = f$. The initial value is inherited from convergence in $C([0, T]; V)$. Uniqueness of the weak heat solution identifies this limit with u . Strong convergence of every term in (4.11) now proves (4.8), including both endpoints.

For a $C^{1,1}$ domain, the Dirichlet elliptic regularity estimate is

$$\|w\|_{H^2(\Omega)} \leq C_\Omega \|Aw\|_2, \quad w \in D(A),$$

and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ [10, 11]. Applying this inequality for almost every time and integrating proves (4.9)–(4.10). $\square$

In particular,

$$-\Delta u(t) = f(t) - u_t(t) \in H \quad \text{for almost every } t \in (0, T) \quad (4.12)$$

is a consequence of the theorem. On a Lipschitz domain the regularity conclusion remains (4.7); the identification with H^2 uses the additional boundary assumption stated above.

Spatial integrability follows from the Sobolev embedding theorem [1, 6]. For $n \geq 3$,

$$V \hookrightarrow L^{2^*}(\Omega), \quad 2^* = \frac{2n}{n-2}, \quad (4.13)$$

with

$$\|u(t)\|_{L^{2^*}(\Omega)} \leq C_S \|\nabla u(t)\|_2 \quad \text{for almost every } t. \quad (4.14)$$

The separate dissipation bound from (4.4) therefore gives

$$\int_0^T \|u(t)\|_{L^{2^*}(\Omega)}^2 dt \leq C_S^2 \int_0^T \|\nabla u(t)\|_2^2 dt \leq C_S^2 D_T. \quad (4.15)$$

For $n = 2$, the same conclusion holds with every fixed finite spatial exponent $r \geq 2$, with a constant depending on r . For $n = 1$, it holds with the spatial L^∞ norm.

The energy norms also give a space–time estimate in every dimension $n \geq 1$. Set $r_c = 2(n+2)/n$. The Gagliardo–Nirenberg inequality for zero extensions of functions in V gives

$$\|w\|_{L^{r_c}(\Omega)}^{r_c} \leq C \|w\|_2^{4/n} \|\nabla w\|_2^2, \quad w \in V.$$

Applying this inequality at almost every time yields

$$\|u\|_{L^{r_c}(Q_T)}^{r_c} \leq C \left(\sup_{0 \leq t \leq T} \|u(t)\|_2^2 \right)^{2/n} \int_0^T \|\nabla u(t)\|_2^2 dt \leq C D_T^{1+2/n}. \quad (4.16)$$

This conclusion uses the time and spatial energy bounds together.

Example 4.1 (Regularity with a nonlinear source). Consider

$$u_t + Au = F(x, t, u), \quad u(0) = u_0, \quad (4.17)$$

with the homogeneous Dirichlet condition encoded by V . Initially, this equation is understood variationally as

$$\langle u_t, \varphi \rangle_{V', V} + a(u, \varphi) = (F(\cdot, t, u), \varphi)_H, \quad \varphi \in V.$$

Assume F is measurable in (x, t) and continuous in its last argument for almost every (x, t) . Suppose

$$|F(x, t, s)| \leq a_F(x, t) + b_F |s|^q, \quad a_F \in L^2(Q_T), \quad a_F \geq 0, \quad b_F \geq 0, \quad (4.18)$$

where $1 \leq q \leq (n+2)/n$. If u belongs to the energy class, the first inequality in (4.16) and the finite measure of Q_T imply

$$F(\cdot, \cdot, u) \in L^2(Q_T), \quad \|F(\cdot, \cdot, u)\|_{L^2(Q_T)} \leq \|a_F\|_{L^2(Q_T)} + b_F \|u\|_{L^{2q}(Q_T)}^q. \quad (4.19)$$

This establishes integrability of the evaluated source, without asserting a bound independent of the solution.

For linear growth, $q = 1$, such a bound follows directly. Let $u \in \mathcal{W}(0, T)$ be a weak solution and write $y(t) = \|u(t)\|_2^2$. Testing the equation by $u(t)$ gives

$$\frac{1}{2} y'(t) + \|\nabla u(t)\|_2^2 = (F(\cdot, t, u(t)), u(t))_H. \quad (4.20)$$

The growth condition and Young's inequality imply

$$y'(t) + \|\nabla u(t)\|_2^2 \leq C_P^2 \|a_F(t)\|_2^2 + 2b_F y(t).$$

Set $Z(t) = y(t) + \int_0^t \|\nabla u(s)\|_2^2 ds$. Since $y(t) \leq Z(t)$, multiplication of $Z' \leq C_P^2 \|a_F(t)\|_2^2 + 2b_F Z$ by $e^{-2b_F t}$ and integration give

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \leq e^{2b_F t} \left(\|u_0\|_2^2 + C_P^2 \int_0^t e^{-2b_F s} \|a_F(s)\|_2^2 ds \right). \quad (4.21)$$

Thus the nonlinear source has an $L^2(Q_T)$ bound determined by the data and T . If $u_0 \in V$, Theorem 4.1 applies with this evaluated source and gives (4.7). These are regularity statements for a weak solution; its construction requires the appropriate existence argument.

4.2. Continuous energies and finite-difference stability

The continuous identities below follow from the operator equation and its solution regularity. The numerical identities are derived from the difference scheme itself. We use the finite-difference framework described in [17, 23].

Example 4.2 (Energy of a semilinear heat equation). Consider

$$v_t + Av = g(v), \quad v(0) = v_0 \in V, \quad (4.22)$$

where $g \in C^1(\mathbb{R})$, $g(0) = 0$, and $|g'(s)| \leq L_g$ for every $s \in \mathbb{R}$. For any weak solution in the energy class, $\|g(v)\|_2 \leq L_g \|v\|_2$, so Theorem 4.1 gives

$$v \in H^1(0, T; H) \cap L^2(0, T; D(A)) \cap C([0, T]; V).$$

Define

$$\mathcal{E}_{\text{heat}}(t) = \frac{1}{2} (\|A^{1/2} v(t)\|_2^2 + \|v(t)\|_2^2). \quad (4.23)$$

The form chain rule, justified by the same spectral approximation used above, yields

$$\mathcal{E}'_{\text{heat}}(t) = (Av(t) + v(t), v_t(t))_H \quad (4.24)$$

for almost every t . Substituting the equation gives

$$\mathcal{E}'_{\text{heat}} = -\|Av\|_2^2 - \|A^{1/2}v\|_2^2 + (g(v), Av + v)_H. \quad (4.25)$$

Every pairing is well defined under the stated regularity.

The composition result (2.41) gives $g(v) \in V$ and

$$(g(v), Av)_H = a(g(v), v) \leq L_g \|\nabla v\|_2^2, \quad (g(v), v)_H \leq L_g \|v\|_2^2.$$

Moreover, (2.19) gives $\|Av\|_2^2 \geq \lambda_1 \|\nabla v\|_2^2$. Combining these facts with the Poincaré inequality proves

$$\mathcal{E}'_{\text{heat}}(t) \leq -2(\lambda_1 - L_g)\mathcal{E}_{\text{heat}}(t), \quad \mathcal{E}_{\text{heat}}(t) \leq e^{-2(\lambda_1 - L_g)t} \mathcal{E}_{\text{heat}}(0). \quad (4.26)$$

In particular, the energy decays exponentially when $L_g < \lambda_1$. The derivation uses the variational operator identity and does not require a boundary trace of Δv .

Example 4.3 (Continuous wave energy). Let $c > 0$ and consider

$$v_{tt} + c^2 Av = G(v), \quad (4.27)$$

with homogeneous Dirichlet data and initial values

$$v(0) = v_0, \quad v_t(0) = v_1. \quad (4.28)$$

For the direct calculation, assume

$$v \in C([0, T]; D(A)) \cap C^1([0, T]; V) \cap C^2([0, T]; H),$$

and $G(v) \in C([0, T]; H)$. In particular, the initial values satisfy $v_0 \in D(A)$ and $v_1 \in V$. The wave energy is

$$\mathcal{E}_w(t) = \frac{1}{2} \|v_t(t)\|_2^2 + \frac{c^2}{2} \|A^{1/2}v(t)\|_2^2. \quad (4.29)$$

Pairing (4.27) with v_t gives

$$\mathcal{E}'_w(t) = (G(v(t)), v_t(t))_H, \quad |\mathcal{E}'_w(t)| \leq \|G(v(t))\|_2 \sqrt{2\mathcal{E}_w(t)}. \quad (4.30)$$

For $\varepsilon > 0$, division by $2\sqrt{\mathcal{E}_w} + \varepsilon$ and integration give a bound for $\sqrt{\mathcal{E}_w} + \varepsilon$. Letting $\varepsilon \downarrow 0$ proves

$$\sqrt{\mathcal{E}_w(t)} \leq \sqrt{\mathcal{E}_w(0)} + \frac{1}{\sqrt{2}} \int_0^t \|G(v(s))\|_2 ds. \quad (4.31)$$

This proof also covers times at which the energy vanishes. If $\|G(v(t))\|_2 \leq F_*$ on $[0, T]$, then

$$\mathcal{E}_w(t) \leq \left(\sqrt{\mathcal{E}_w(0)} + \frac{F_* t}{\sqrt{2}} \right)^2. \quad (4.32)$$

For a globally Lipschitz G with Lipschitz constant L_G and $G(0) = 0$, a bound independent of the unknown forcing norm is also available. Indeed,

$$|(G(v), v_t)_H| \leq L_G C_P \|\nabla v\|_2 \|v_t\|_2 \leq \frac{L_G C_P}{c} \mathcal{E}_w.$$

Therefore

$$\mathcal{E}_w(t) \leq e^{L_G C_P t/c} \mathcal{E}_w(0). \quad (4.33)$$

The same calculation for the difference of two strong solutions gives this estimate for their difference energy, and hence continuous dependence and uniqueness in the stated class.

For the numerical analysis, take $\Omega = (0, \ell)$ and define $h = \ell/M$, $M \geq 2$, and $t_m = mk$, where $k > 0$, $N \geq 1$ is an integer, and $Nk \leq T$. A grid function U satisfies $U_0 = U_M = 0$. Use the inner products

$$(U, V)_h = h \sum_{j=1}^{M-1} U_j V_j, \quad (\xi, \zeta)_{h,e} = h \sum_{j=0}^{M-1} \xi_j \zeta_j, \quad (4.34)$$

with their associated norms. Define

$$(D_h U)_j = \frac{U_{j+1} - U_j}{h}, \quad (A_h U)_j = \frac{2U_j - U_{j-1} - U_{j+1}}{h^2}. \quad (4.35)$$

Discrete summation by parts gives

$$(A_h U, V)_h = (D_h U, D_h V)_{h,e}. \quad (4.36)$$

The discrete Poincaré and inverse inequalities are

$$\|U\|_h \leq C_{P,h} \|D_h U\|_{h,e}, \quad C_{P,h} = \frac{h}{2 \sin(\pi/(2M))} \leq \frac{\ell}{2} =: C_*, \quad \|D_h U\|_{h,e} \leq \frac{2}{h} \|U\|_h. \quad (4.37)$$

The Poincaré constant follows from the smallest eigenvalue $4h^{-2} \sin^2(\pi/(2M))$ of A_h ; the inverse bound follows from $|U_{j+1} - U_j|^2 \leq 2|U_{j+1}|^2 + 2|U_j|^2$.

We approximate the wave equation by

$$\frac{U^{m+1} - 2U^m + U^{m-1}}{k^2} + c^2 A_h U^m = G(U^m) + \mathcal{R}^m, \quad 1 \leq m \leq N-1. \quad (4.38)$$

The function G acts componentwise. The grid vector $\mathcal{R}^m$ represents an added forcing or a perturbation; the approximation of (4.27) uses $\mathcal{R}^m = 0$. Given U^0 and an initial velocity vector P^0 , a consistent starting value is

$$U^1 = U^0 + kP^0 + \frac{k^2}{2} (-c^2 A_h U^0 + G(U^0) + \mathcal{R}^0). \quad (4.39)$$

All endpoint values remain zero. Introduce

$$q^{m+1/2} = \frac{U^{m+1} - U^m}{k}, \quad E^{m+1/2} = \frac{1}{2} \|q^{m+1/2}\|_h^2 + \frac{c^2}{2} (D_h U^{m+1}, D_h U^m)_{h,e}. \quad (4.40)$$

The spatial term couples the two adjacent time levels. Its positivity is established under the following CFL condition.

Theorem 4.2 (Discrete energy balance and stability). *Assume*

$$\frac{ck}{h} \leq \rho < 1, \quad \theta = 1 - \rho > 0, \quad (4.41)$$

where ρ is fixed independently of the grid. For either $W = U^m$ or $W = U^{m+1}$,

$$\frac{\theta}{2}(\|q^{m+1/2}\|_h^2 + c^2\|D_h W\|_{h,e}^2) \leq E^{m+1/2} \leq \frac{1+\rho}{2}(\|q^{m+1/2}\|_h^2 + c^2\|D_h W\|_{h,e}^2). \quad (4.42)$$

Writing $S^m = G(U^m) + \mathcal{R}^m$, the exact balance is

$$E^{m+1/2} - E^{m-1/2} = k \left(S^m, \frac{q^{m+1/2} + q^{m-1/2}}{2} \right)_h. \quad (4.43)$$

Consequently,

$$\left| \sqrt{E^{m+1/2}} - \sqrt{E^{m-1/2}} \right| \leq \frac{k}{\sqrt{2\theta}} \|S^m\|_h. \quad (4.44)$$

If G is globally Lipschitz with constant L_G and $G(0) = 0$, then, with $K = L_G C_*/(c\theta)$,

$$\sqrt{E^{m+1/2}} \leq e^{Kt_m} \left(\sqrt{E^{1/2}} + \frac{k}{\sqrt{2\theta}} \sum_{j=1}^m \|\mathcal{R}^j\|_h \right). \quad (4.45)$$

All constants in this estimate are independent of h and k subject to (4.41).

Proof. Since $U^{m+1} = U^m + kq^{m+1/2}$, the energy equals

$$E^{m+1/2} = \frac{1}{2}\|q^{m+1/2}\|_h^2 + \frac{c^2}{2}\|D_h U^m\|_{h,e}^2 + \frac{c^2 k}{2}(D_h q^{m+1/2}, D_h U^m)_{h,e}.$$

By the inverse inequality and the CFL condition, the absolute value of the last term is bounded by

$$\rho c \|q^{m+1/2}\|_h \|D_h U^m\|_{h,e} \leq \frac{\rho}{2}(\|q^{m+1/2}\|_h^2 + c^2\|D_h U^m\|_{h,e}^2).$$

This proves (4.42) for $W = U^m$. Writing $U^m = U^{m+1} - kq^{m+1/2}$ gives the other endpoint estimate by the same argument.

Pair the scheme with $(q^{m+1/2} + q^{m-1/2})/2 = (U^{m+1} - U^{m-1})/(2k)$. The time term becomes

$$\frac{1}{2k}(\|q^{m+1/2}\|_h^2 - \|q^{m-1/2}\|_h^2).$$

Summation by parts transforms the spatial term into

$$\frac{c^2}{2k}((D_h U^{m+1}, D_h U^m)_{h,e} - (D_h U^m, D_h U^{m-1})_{h,e}).$$

Their sum gives (4.43) exactly.

Put $e_+ = \sqrt{E^{m+1/2}}$ and $e_- = \sqrt{E^{m-1/2}}$. Coercivity controls both velocities, so Cauchy-Schwarz gives

$$|e_+^2 - e_-^2| \leq \frac{k}{\sqrt{2\theta}} \|S^m\|_h (e_+ + e_-).$$

If $e_+ + e_- > 0$, division proves (4.44); if this sum vanishes, that inequality holds immediately.

For the nonlinear bound, the endpoint estimate and discrete Poincaré inequality give

$$\|G(U^m)\|_h \leq L_G \|U^m\|_h \leq \frac{\sqrt{2}L_G C_*}{c\sqrt{\theta}} \sqrt{E^{m-1/2}}.$$

Substitution into (4.44) yields

$$\sqrt{E^{m+1/2}} \leq (1 + kK)\sqrt{E^{m-1/2}} + \frac{k}{\sqrt{2\theta}}\|\mathcal{R}^m\|_h.$$

Iteration and $(1 + kK)^m \leq e^{Kt_m}$ establish (4.45). $\square$

The starting energy is also controlled by the initial data. Setting $S^0 = G(U^0) + \mathcal{R}^0$ in (4.39) and expanding the energy gives

$$E^{1/2} = \frac{1}{2} \left\| P^0 + \frac{k}{2} S^0 \right\|_h^2 + \frac{c^2}{2} \|D_h U^0\|_{h,e}^2 - \frac{c^4 k^2}{8} \|A_h U^0\|_h^2. \quad (4.46)$$

Thus

$$\sqrt{E^{1/2}} \leq \left(\frac{\|P^0\|_h^2 + c^2 \|D_h U^0\|_{h,e}^2}{2} \right)^{1/2} + \frac{k}{2\sqrt{2}} \|S^0\|_h. \quad (4.47)$$

No separate uniform bound on $A_h U^0$ is needed. If the total source instead satisfies $\|S^m\|_h \leq F_*$, summing (4.44) gives

$$E^{m+1/2} \leq \left(\sqrt{E^{1/2}} + \frac{F_* t_m}{\sqrt{2\theta}} \right)^2. \quad (4.48)$$

For two grid solutions, their difference satisfies the same linear difference operator with source $G(U^m) - G(\tilde{U}^m) + \mathcal{R}^m - \tilde{\mathcal{R}}^m$. The Lipschitz inequality bounds the nonlinear difference by $L_G \|U^m - \tilde{U}^m\|_h$. The proof of (4.45) therefore applies to the difference energy, with initial and forcing differences on its right-hand side. Together with (4.42) and (4.47), this proves stability in the discrete displacement, gradient, and velocity norms.

Proposition 4.1 (Convergence for a smooth wave solution). *Assume G is globally Lipschitz and that the Dirichlet wave problem (4.27) has a solution $v \in C^4([0, \ell] \times [0, T])$. Use exact sampled initial data*

$$U_j^0 = v(x_j, 0), \quad P_j^0 = v_t(x_j, 0),$$

the starting step (4.39), and $\mathcal{R}^m = 0$. Let $V_j^m = v(x_j, t_m)$ and $\varepsilon^m = U^m - V^m$. Under the fixed CFL condition (4.41),

$$\max_{0 \leq m \leq N} \|\varepsilon^m\|_h + \max_{0 \leq m \leq N} c \|D_h \varepsilon^m\|_{h,e} + \max_{0 \leq m \leq N-1} \left\| \frac{\varepsilon^{m+1} - \varepsilon^m}{k} \right\|_h \leq C_T (h^2 + k^2), \quad (4.49)$$

where C_T is independent of h and k .

Proof. Taylor's formula for the centred time and spatial differences gives the residual

$$\tau^m = \frac{V^{m+1} - 2V^m + V^{m-1}}{k^2} + c^2 A_h V^m - G(V^m), \quad \|\tau^m\|_h \leq C(k^2 + h^2).$$

The constant depends on bounds for v_{tttt} and v_{xxxx} and on ℓ . The error consequently satisfies

$$\frac{\varepsilon^{m+1} - 2\varepsilon^m + \varepsilon^{m-1}}{k^2} + c^2 A_h \varepsilon^m = G(U^m) - G(V^m) - \tau^m.$$

Let $E_\varepsilon^{m+1/2}$ be the quadratic energy (4.40) evaluated on the error. The difference stability argument proves

$$\sqrt{E_\varepsilon^{m+1/2}} \leq e^{Kt_m} \left(\sqrt{E_\varepsilon^{1/2}} + \frac{k}{\sqrt{2\theta}} \sum_{j=1}^m \|\tau^j\|_h \right).$$

For the starting step, Taylor expansion at $t = 0$ gives $\varepsilon^0 = 0$ and $\|\varepsilon^1\|_h \leq C(k^3 + k^2 h^2)$. The spatial cross term in $E_\varepsilon^{1/2}$ therefore vanishes, and

$$\sqrt{E_\varepsilon^{1/2}} = \frac{\|\varepsilon^1\|_h}{\sqrt{2k}} \leq C(k^2 + kh^2) \leq C_T(k^2 + h^2).$$

Since $mk \leq T$, the accumulated residual is also bounded by $C_T(k^2 + h^2)$. Coercivity at both endpoints of each time interval controls the error gradients and time differences. The discrete Poincaré inequality controls the displacement error and completes (4.49). $\square$

The convergence estimate concerns the difference between the numerical solution and exact nodal samples, together with their discrete derivatives. Its constants depend on the fixed CFL margin and the stated regularity of the exact solution.

5. Fractional Dirichlet energies and nonlinear graph spaces

Fractional energy estimates depend on the realization of the spatial operator and on the regularity of the nonlinear composition. We therefore specify the operator before introducing the nonlinear interaction. The resulting form estimates lead to a complete metric space that controls both a function and its nonlinear image. For power nonlinearities, this construction yields a precise integrability threshold, compactness below that threshold, and a concentration sequence proving its optimality. The final applications connect these properties to diffusion dissipation and wave energy.

5.1. Dirichlet realizations and nonlinear energy inequalities

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $H = L^2(\Omega)$, and $0 < \alpha < 2$. Set $s = \alpha/2$. We distinguish two Dirichlet realizations [12, 15]. For the spectral realization, the positive Dirichlet Laplacian A from Section 2 gives

$$B_{\text{sp}} = A^s, \quad D(B_{\text{sp}}) = \left\{ u \in H : \sum_{j=1}^{\infty} \lambda_j^{2s} |(u, \phi_j)_H|^2 < \infty \right\}. \quad (5.1)$$

Its action is defined by the eigenfunction expansion $B_{\text{sp}} u = \sum_j \lambda_j^s (u, \phi_j)_H \phi_j$. The homogeneous boundary condition is encoded by the underlying Dirichlet operator.

For the restricted integral realization, let $\tilde{u}$ denote the zero extension of u to $\mathbb{R}^n$ and define

$$\mathcal{V}_{\text{int}} = \{u \in H : \tilde{u} \in H^s(\mathbb{R}^n)\}. \quad (5.2)$$

The closed form is

$$\mathfrak{e}_{\text{int}}(u, v) = \frac{C_{n,\alpha}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))}{|x - y|^{n+\alpha}} dx dy, \quad (5.3)$$

where

$$C_{n,\alpha} = \frac{2^\alpha \Gamma((n + \alpha)/2)}{\pi^{n/2} |\Gamma(-\alpha/2)|}.$$

The associated positive self-adjoint operator is denoted by B_{int} . On smooth compactly supported functions,

$$B_{\text{int}} u(x) = C_{n,\alpha} \text{P. V.} \int_{\mathbb{R}^n} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x - y|^{n+\alpha}} dy, \quad x \in \Omega. \quad (5.4)$$

Here the Dirichlet condition is prescribed on the entire exterior $\mathbb{R}^n \setminus \Omega$. With the unitary Fourier transform,

$$\mathfrak{e}_{\text{int}}(u, u) = \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{u}(\xi)|^2 d\xi. \quad (5.5)$$

Thus the nonlocal integral contains the factor $C_{n,\alpha}/2$ when it represents the squared energy seminorm [18].

In the statements common to both realizations, fix either operator and write

$$\mathcal{V}_B = D(B^{1/2}), \quad \mathfrak{e}_B(u, v) = (B^{1/2}u, B^{1/2}v)_H, \quad \mathfrak{e}_B(u) = \mathfrak{e}_B(u, u). \quad (5.6)$$

We equip $\mathcal{V}_B$ with $\|u\|_{\mathcal{V}_B}^2 = \|u\|_2^2 + \mathfrak{e}_B(u)$. Both operators have compact resolvent and a positive first eigenvalue μ_1 . In the spectral case, $\mu_1 = \lambda_1^{\alpha/2}$; in the integral case, μ_1 is the first eigenvalue of B_{int} . Consequently,

$$\|u\|_2 \leq \mu_1^{-1/2} \mathfrak{e}_B(u)^{1/2}, \quad u \in \mathcal{V}_B. \quad (5.7)$$

The constant $C_{\alpha,\Omega} = \mu_1^{-1/2}$ is understood relative to the chosen realization. Spectral expansion also gives

$$\mathfrak{e}_B(u) \leq \mu_1^{-1} \|Bu\|_2^2, \quad \|u\|_2^2 \leq \mu_1^{-2} \|Bu\|_2^2, \quad u \in D(B). \quad (5.8)$$

Both forms admit a representation

$$\begin{aligned} \mathfrak{e}_B(u, v) &= \frac{1}{2} \iint_{\Omega \times \Omega} (u(x) - u(y))(v(x) - v(y)) J_B(x, y) dx dy \\ &\quad + \int_{\Omega} \kappa_B(x) u(x) v(x) dx, \end{aligned} \quad (5.9)$$

where J_B is symmetric and nonnegative and $\kappa_B \geq 0$. For B_{int} ,

$$J_B(x, y) = C_{n,\alpha} |x - y|^{-n-\alpha}, \quad \kappa_B(x) = C_{n,\alpha} \int_{\Omega^c} |x - y|^{-n-\alpha} dy.$$

For B_{sp} , let $p_t(x, y)$ be the Dirichlet heat kernel of A and $c_s = s/\Gamma(1-s)$. Then

$$\begin{aligned} J_B(x, y) &= c_s \int_0^\infty p_t(x, y) \frac{dt}{t^{1+s}}, \\ \kappa_B(x) &= c_s \int_0^\infty \left(1 - \int_{\Omega} p_t(x, y) dy \right) \frac{dt}{t^{1+s}}. \end{aligned} \quad (5.10)$$

These formulas follow by expanding the semigroup representation

$$\mathfrak{e}_{A^s}(u) = c_s \int_0^\infty (u - e^{-tA}u)_H \frac{dt}{t^{1+s}}.$$

The nonnegativity of the heat kernel and its mass bound $\int_{\Omega} p_t(x, y) dy \leq 1$ justify the quadratic formula by Tonelli's theorem; polarization gives the bilinear identity. This representation is part of the established fractional Dirichlet theory [9, 22].

Theorem 5.1 (Nonlinear interaction and secant bounds). *Suppose $v, g(v) \in \mathcal{V}_B$. Define*

$$\mathcal{I}_B(v, g) = \mathfrak{e}_B(g(v), v). \quad (5.11)$$

Then

$$|\mathcal{I}_B(v, g)| \leq \mathfrak{e}_B(g(v))^{1/2} \mathfrak{e}_B(v)^{1/2}. \quad (5.12)$$

If $v \in D(B)$, this interaction equals $(g(v), Bv)_H$, and

$$|(g(v), Bv)_H| \leq C_{\alpha, \Omega} \mathfrak{e}_B(g(v))^{1/2} \|Bv\|_2. \quad (5.13)$$

If $g(0) = 0$ and g is globally Lipschitz with constant L , then $g(v) \in \mathcal{V}_B$ for every $v \in \mathcal{V}_B$ and

$$\mathfrak{e}_B(g(v)) \leq L^2 \mathfrak{e}_B(v), \quad |\mathcal{I}_B(v, g)| \leq L \mathfrak{e}_B(v). \quad (5.14)$$

More precisely, assume that $M > 0$, $0 \leq m \leq M$, and

$$m \leq \frac{g(a) - g(b)}{a - b} \leq M \quad \text{for all } a \neq b, \quad g(0) = 0. \quad (5.15)$$

Then

$$\mathfrak{e}_B(g(v)) + mM \mathfrak{e}_B(v) \leq (m + M) \mathcal{I}_B(v, g), \quad m \mathfrak{e}_B(v) \leq \mathcal{I}_B(v, g) \leq M \mathfrak{e}_B(v). \quad (5.16)$$

Proof. Cauchy–Schwarz in H , applied to $B^{1/2}g(v)$ and $B^{1/2}v$, proves (5.12). For $v \in D(B)$, self-adjointness identifies the form pairing with $(g(v), Bv)_H$. Estimate (5.8) then proves (5.13). In particular, the form estimate remains meaningful without requiring $Bv \in H$.

For a Lipschitz g fixing zero,

$$|g(a) - g(b)| \leq L|a - b|, \quad |g(a)| \leq L|a|.$$

These inequalities bound, respectively, the jump and killing terms in (5.9). They also give $g(v) \in H$. The finite-energy characterization of the form domain therefore yields $g(v) \in \mathcal{V}_B$ and the first inequality in (5.14). The second follows from (5.12).

Under (5.15), put $d = a - b$ and $z = g(a) - g(b)$. The secant condition implies

$$(z - md)(Md - z) \geq 0,$$

or, equivalently,

$$z^2 + mMd^2 \leq (m + M)zd.$$

Integrating this inequality against the jump kernel proves the corresponding part of (5.16). Taking $b = 0$ and using $g(0) = 0$ proves the killing contribution. Finally, $md^2 \leq zd \leq Md^2$, together with its counterpart at $b = 0$, gives the remaining two-sided bound. $\square$

For a monotone M -Lipschitz map fixing zero, the choice $m = 0$ gives the useful dissipation estimate

$$\mathfrak{e}_B(g(v)) \leq M \mathcal{I}_B(v, g). \quad (5.17)$$

The constant in (5.13) is optimal over this class: Take $g(t) = Mt$ and let v be a first eigenfunction of B . The interaction bounds are consequences of the Dirichlet form structure, rather than a separate fractional chain rule.

For unbounded power nonlinearities, the relevant dissipation controls a different nonlinear image. The following Stroock–Varopoulos estimate is stated in a form that does not assume fractional regularity of the reaction term [3].

Proposition 5.1 (Power dissipation). *For $r > 1$, write*

$$f_r(t) = |t|^{r-1}t, \quad p = \frac{r+1}{2}, \quad c_r = \frac{4r}{(r+1)^2}.$$

If $u \in D(B)$ and $f_r(u) \in H$, then $f_p(u) \in \mathcal{V}_B$ and

$$(f_r(u), Bu)_H \geq c_r \mathfrak{e}_B(f_p(u)). \quad (5.18)$$

Proof. First,

$$\|f_p(u)\|_2^2 = \int_{\Omega} u f_r(u) dx \leq \|u\|_2 \|f_r(u)\|_2,$$

so $f_p(u) \in H$. For real numbers $a > b$, the fundamental theorem of calculus and Cauchy–Schwarz give

$$\begin{aligned} |f_p(a) - f_p(b)|^2 &= p^2 \left(\int_b^a |t|^{(r-1)/2} dt \right)^2 \\ &\leq p^2 (a - b) \int_b^a |t|^{r-1} dt \\ &= \frac{p^2}{r} (a - b) (f_r(a) - f_r(b)). \end{aligned}$$

By symmetry the resulting inequality holds for all a, b . Moreover, $a f_r(a) = f_p(a)^2$ and $0 < c_r \leq 1$.

Both realizations generate a symmetric sub-Markov semigroup $T_\tau = e^{-\tau B}$. This follows by subordination in the spectral case and from the Markov property of the integral form in the restricted case. Its kernel $q_\tau(x, y)$ is nonnegative, symmetric, and has mass at most one. For $a, b \in H$, define

$$\mathfrak{e}_\tau(a, b) = \tau^{-1} (a, (I - T_\tau)b)_H.$$

Expanding this expression gives

$$\begin{aligned} \mathfrak{e}_\tau(a, b) &= \frac{1}{2\tau} \iint_{\Omega \times \Omega} (a(x) - a(y))(b(x) - b(y)) q_\tau(x, y) dx dy \\ &\quad + \frac{1}{\tau} \int_{\Omega} a(x) b(x) \left(1 - \int_{\Omega} q_\tau(x, y) dy \right) dx. \end{aligned}$$

Applying the scalar inequalities to both terms yields

$$\mathfrak{e}_\tau(f_r(u), u) \geq c_r \mathfrak{e}_\tau(f_p(u), f_p(u)).$$

Since $u \in D(B)$, $\tau^{-1}(I - T_\tau)u \rightarrow Bu$ in H . On the other hand, the spectral theorem and $(1 - e^{-\tau\mu})/\tau \uparrow \mu$ as $\tau \downarrow 0$ show that the expression on the right increases to $c_r \mathfrak{e}_B(f_p(u))$, with value $+\infty$ if $f_p(u) \notin \mathcal{V}_B$. The finite limit of the left-hand side proves both membership in the form domain and (5.18). $\square$

5.2. Complete nonlinear graph spaces and sharp embeddings

The regularity of $f(u)$ can now be incorporated into a well-defined metric structure. This construction concerns the graph of a superposition operator and is related to the classical theory of nonlinear composition [2, 5].

Definition 5.1 (Nonlinear graph domain). Fix B , and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous with $f(0) = 0$. Define

$$\mathcal{A}_{f,B}(\Omega) = \{u \in \mathcal{V}_B : f(u) \in \mathcal{V}_B\}. \quad (5.19)$$

For $u \in \mathcal{A}_{f,B}(\Omega)$, set

$$\mathcal{J}_{f,B}(u) = (\|u\|_2^2 + \mathfrak{e}_B(u) + \mathfrak{e}_B(f(u)))^{1/2}. \quad (5.20)$$

For $u, v \in \mathcal{A}_{f,B}(\Omega)$, define

$$d_{f,B}(u, v) = (\|u - v\|_2^2 + \mathfrak{e}_B(u - v) + \mathfrak{e}_B(f(u) - f(v)))^{1/2}. \quad (5.21)$$

The nonlinear difference in (5.21) is $f(u) - f(v)$. The functional satisfies $\mathcal{J}_{f,B}(u) = d_{f,B}(u, 0)$, but it is generally not a norm on a vector space. This distinction gives the construction a precise meaning without imposing linearity on the graph domain.

Theorem 5.2 (Completeness, compactness, and lower semicontinuity). *The pair $(\mathcal{A}_{f,B}(\Omega), d_{f,B})$ is a complete metric space. Every sequence (u_j) with $\sup_j \mathcal{J}_{f,B}(u_j) < \infty$ has a subsequence and a limit $u \in \mathcal{A}_{f,B}(\Omega)$ such that*

$$u_j \rightarrow u \quad \text{in } H, \quad f(u_j) \rightarrow f(u) \quad \text{in } H. \quad (5.22)$$

The extended functional

$$\mathcal{F}_{f,B}(u) = \begin{cases} \mathcal{J}_{f,B}(u)^2, & u \in \mathcal{A}_{f,B}(\Omega), \\ +\infty, & u \in H \setminus \mathcal{A}_{f,B}(\Omega) \end{cases} \quad (5.23)$$

is lower semicontinuous in H and has compact sublevel sets in H .

Proof. Consider the injective map

$$\Gamma_f(u) = (u, B^{1/2}u, B^{1/2}f(u)) \in H \times H \times H.$$

The distance (5.21) is exactly the distance between $\Gamma_f(u)$ and $\Gamma_f(v)$ in this Hilbert product. It therefore satisfies symmetry and the triangle inequality; its first component ensures that zero distance implies equality almost everywhere.

Let (u_j) be Cauchy in this distance. Then it converges to some u in $\mathcal{V}_B$. The sequence $B^{1/2}f(u_j)$ is Cauchy in H , and the fractional Poincaré inequality gives

$$\|f(u_j) - f(u_k)\|_2 \leq \mu_1^{-1/2} \|B^{1/2}(f(u_j) - f(u_k))\|_2.$$

Thus $(f(u_j))$ is Cauchy in $\mathcal{V}_B$ and converges there to some z . Taking a common almost-everywhere subsequence of the two H convergences and using scalar continuity of f gives $z = f(u)$ almost everywhere. Consequently $u \in \mathcal{A}_{f,B}(\Omega)$ and $d_{f,B}(u_j, u) \rightarrow 0$. This proves completeness.

Now suppose only that $\mathcal{J}_{f,B}(u_j)$ is bounded. The fractional Poincaré inequality bounds both u_j and $f(u_j)$ in $\mathcal{V}_B$. Compactness of $\mathcal{V}_B \hookrightarrow H$ and weak compactness in $\mathcal{V}_B$ give, along a common subsequence,

$$u_j \rightharpoonup u, \quad f(u_j) \rightharpoonup z \quad \text{in } \mathcal{V}_B,$$

with both convergences strong in H . Scalar continuity again identifies $z = f(u)$. Weak lower semicontinuity yields

$$\mathcal{J}_{f,B}(u)^2 \leq \liminf_{j \rightarrow \infty} \mathcal{J}_{f,B}(u_j)^2.$$

For an arbitrary sequence converging in H , apply this argument to a subsequence realizing a finite lower limit of $\mathcal{J}_{f,B}$; an infinite lower limit needs no further argument. This proves lower semicontinuity. The same compactness argument shows that every sublevel set is sequentially compact in H , and hence compact. No compactness in the graph metric is asserted. $\square$

The graph metric also controls the nonlinear interaction quantitatively. For $u, v \in \mathcal{A}_{f,B}(\Omega)$,

$$|\mathcal{I}_B(u, f) - \mathcal{I}_B(v, f)| \leq \frac{\mathcal{J}_{f,B}(u) + \mathcal{J}_{f,B}(v)}{2} d_{f,B}(u, v). \quad (5.24)$$

Indeed, put $a_u = B^{1/2}u$ and $b_u = B^{1/2}f(u)$. The identity

$$2((a_u, b_u)_H - (a_v, b_v)_H) = (a_u + a_v, b_u - b_v)_H + (b_u + b_v, a_u - a_v)_H$$

and Cauchy–Schwarz in $H \times H$ prove (5.24). In particular, $|\mathcal{I}_B(u, f)| \leq \mathcal{J}_{f,B}(u)^2/2$.

If f is globally L -Lipschitz and fixes zero, then

$$\mathcal{A}_{f,B}(\Omega) = \mathcal{V}_B, \quad \|u\|_{\mathcal{V}_B} \leq \mathcal{J}_{f,B}(u) \leq \sqrt{1 + L^2} \|u\|_{\mathcal{V}_B}. \quad (5.25)$$

Thus nonlinear dependence does not by itself produce a different underlying set. For $f(t) = \lambda t$, one has

$$\mathcal{J}_{f,B}(u)^2 = \|u\|_2^2 + (1 + \lambda^2) \mathfrak{e}_B(u), \quad (5.26)$$

which is a genuine equivalent Hilbert norm.

Example 5.1 (Failure of the norm axioms for a power map). Let $f_p(t) = |t|^{p-1}t$, where $p > 1$, and choose a nonzero $u \in \mathcal{A}_{f_p,B}(\Omega)$. Write $a = \|u\|_2^2 + \mathfrak{e}_B(u)$ and $b = \mathfrak{e}_B(f_p(u))$. The spectral gap implies $b > 0$. For $t > 0$,

$$\mathcal{J}_{f_p,B}(tu)^2 = t^2 a + t^{2p} b.$$

Therefore $\mathcal{J}_{f_p,B}(2u) > 2\mathcal{J}_{f_p,B}(u)$. Both absolute homogeneity and the norm triangle inequality fail. The metric (5.21) nevertheless remains valid and complete.

The next result makes the additional information carried by a power nonlinearity explicit. We use the restricted realization here so that both the embedding and its sharpness can be expressed through the whole-space fractional form.

Theorem 5.3 (Sharp integrability and compactness for power graphs). *Assume $n > \alpha$, $p > 1$, $B = B_{\text{int}}$, and $f_p(t) = |t|^{p-1}t$. Set*

$$r_* = \frac{2n}{n - \alpha}, \quad q_* = pr_* = \frac{2np}{n - \alpha}. \quad (5.27)$$

If C_S is the fractional Sobolev constant, then

$$\|u\|_{L^{q_*}(\Omega)}^p \leq C_S \mathfrak{e}_B(f_p(u))^{1/2}, \quad u \in \mathcal{A}_{f_p,B}(\Omega). \quad (5.28)$$

Moreover,

$$\|u - v\|_{L^{q_*}(\Omega)}^p \leq 2^{p-1} C_S d_{f_p, B}(u, v). \quad (5.29)$$

Every $\mathcal{J}_{f_p, B}$ -bounded sequence has a subsequence converging strongly in $L^q(\Omega)$ for each fixed $1 \leq q < q_*$. The critical exponent is optimal: Bounded graph sublevels need not be relatively compact in $L^{q_*}(\Omega)$ and need not be bounded in $L^q(\Omega)$ when $q > q_*$.

Proof. The fractional Sobolev inequality gives

$$\|z\|_{L^{r_*}(\Omega)} \leq C_S \mathfrak{e}_B(z)^{1/2}, \quad z \in \mathcal{V}_B$$

[18]. Applying it to $z = f_p(u)$ and using $\|f_p(u)\|_{L^{r_*}} = \|u\|_{L^{pr_*}}^p$ proves (5.28).

For real a, b ,

$$|a - b|^p \leq 2^{p-1} |f_p(a) - f_p(b)|. \quad (5.30)$$

For arguments of the same sign, this follows from $(x + y)^p \geq x^p + y^p$ for $x, y \geq 0$; for opposite signs, it follows from $(x + y)^p \leq 2^{p-1}(x^p + y^p)$. Taking the L^{r_*} norm in (5.30) and applying the Sobolev inequality to $f_p(u) - f_p(v)$ proves (5.29).

For a bounded graph sequence, Theorem 5.2 gives a limit u in the graph domain and strong H convergence of the nonlinear images. Their uniform L^{r_*} bound and interpolation then give strong convergence of $f_p(u_j)$ in every $L^r(\Omega)$ with $2 \leq r < r_*$ along the same subsequence. Inequality (5.30) then gives strong convergence of u_j in $L^{pr}(\Omega)$. The remaining exponents $1 \leq q < 2p$ follow from strong L^{2p} convergence and the finite measure of Ω .

To prove optimality, choose $x_0 \in \Omega$ and a nonzero, nonnegative $\eta \in C_c^\infty(B_1(0))$. For sufficiently small $\varepsilon > 0$, define

$$\beta = \frac{n - \alpha}{2p}, \quad u_\varepsilon(x) = \varepsilon^{-\beta} \eta\left(\frac{x - x_0}{\varepsilon}\right).$$

All these functions are supported inside Ω . The function η^p is Lipschitz and compactly supported, so it has finite fractional energy. Write $[\zeta]_\alpha^2$ for the whole-space quadratic form in (5.3). A change of variables gives

$$\begin{aligned} \mathfrak{e}_B(f_p(u_\varepsilon)) &= [\eta^p]_\alpha^2, \\ \mathfrak{e}_B(u_\varepsilon) &= \varepsilon^{(n-\alpha)(1-1/p)} [\eta]_\alpha^2, \\ \|u_\varepsilon\|_2^2 &= \varepsilon^{n-(n-\alpha)/p} \|\eta\|_2^2. \end{aligned} \quad (5.31)$$

Thus the graph functional is bounded, while $u_\varepsilon \rightarrow 0$ in $\mathcal{V}_B$. For every finite $q \geq 1$,

$$\|u_\varepsilon\|_{L^q(\Omega)} = \varepsilon^{n/q-\beta} \|\eta\|_{L^q(\mathbb{R}^n)}. \quad (5.32)$$

At $q = q_*$ the norm is a fixed positive number, whereas $u_\varepsilon \rightarrow 0$ almost everywhere. Hence no subsequence can converge strongly in $L^{q_*}(\Omega)$. For $q > q_*$, the exponent in (5.32) is negative and the norm diverges. The L^∞ norm also diverges. This proves both optimality statements. $\square$

The same concentration sequence shows that the graph metric can distinguish sequences converging in the underlying fractional energy norm: Its nonlinear energy stays strictly positive in (5.31). This is a quantitative distinction between the two convergence requirements.

5.3. Diffusion dissipation and fractional wave energy

The following applications use the chosen operator B throughout. In particular, the spectral and exterior Dirichlet conditions retain the meanings specified above.

Example 5.2 (Fractional diffusion with a Lipschitz reaction). Consider

$$v_t + Bv = f(v), \quad v(0) = v_0 \in \mathcal{V}_B, \quad (5.33)$$

where $f(0) = 0$ and f is globally Lipschitz with constant L . Let v be a solution satisfying

$$v \in H^1(0, T; H) \cap L^2(0, T; D(B)) \cap C([0, T]; \mathcal{V}_B). \quad (5.34)$$

For an energy solution, this regularity follows by applying the spectral proof of Theorem 4.1 to B : the evaluated source belongs to $L^2(0, T; H)$ because $|f(v)| \leq L|v|$.

Testing the equation by v yields

$$\frac{1}{2} \frac{d}{dt} \|v\|_2^2 + \mathfrak{e}_B(v) = (f(v), v)_H. \quad (5.35)$$

Testing instead by Bv gives the identity involving the nonlinear fractional interaction:

$$\frac{1}{2} \frac{d}{dt} \mathfrak{e}_B(v) + \|Bv\|_2^2 = \mathcal{I}_B(v, f). \quad (5.36)$$

The form chain rule is justified by the regularity in (5.34) and spectral approximation. Using (5.14), (5.7), and (5.8) gives

$$\|v(t)\|_{\mathcal{V}_B}^2 \leq e^{-2(\mu_1 - L)t} \|v_0\|_{\mathcal{V}_B}^2. \quad (5.37)$$

Thus $L < \mu_1$ gives exponential decay. The rate is attained when $f(t) = Lt$ and the initial value is a first eigenfunction. This example separates the basic L^2 balance from the form balance in which $\mathcal{I}_B(v, f)$ actually occurs.

Example 5.3 (Power absorption and nonlinear graph regularity). Let $r > 1$ and consider

$$u_t + Bu + f_r(u) = h, \quad u(0) = u_0 \in \mathcal{V}_B, \quad h \in L^2(0, T; H). \quad (5.38)$$

Assume that u has the regularity (5.34) and $f_r(u) \in L^2(0, T; H)$, and that the equation holds in H almost everywhere. These hypotheses concern an existing strong solution. Set $p = (r + 1)/2$ and $c_r = 4r/(r + 1)^2$.

Pairing the equation with Bu gives

$$\frac{1}{2} \frac{d}{dt} \mathfrak{e}_B(u) + \|Bu\|_2^2 + (f_r(u), Bu)_H = (h, Bu)_H.$$

Proposition 5.1 and Young's inequality imply

$$\frac{d}{dt} \mathfrak{e}_B(u) + \|Bu\|_2^2 + 2c_r \mathfrak{e}_B(f_p(u)) \leq \|h\|_2^2.$$

In particular, the form regularity of $f_p(u(t))$ is a conclusion for almost every time. Defining $R_t = \mathfrak{e}_B(u_0) + \int_0^t \|h(s)\|_2^2 ds$, integration yields

$$\mathfrak{e}_B(u(t)) + \int_0^t \|Bu(s)\|_2^2 ds + 2c_r \int_0^t \mathfrak{e}_B(f_p(u(s))) ds \leq R_t. \quad (5.39)$$

By (5.8),

$$\|u\|_2^2 + \mathfrak{e}_B(u) \leq (\mu_1^{-2} + \mu_1^{-1}) \|Bu\|_2^2.$$

Therefore

$$\int_0^T \mathcal{J}_{f_p, B}(u(t))^2 dt \leq \max \left\{ \mu_1^{-2} + \mu_1^{-1}, \frac{1}{2c_r} \right\} R_T. \quad (5.40)$$

This identifies a nonlinear graph functional controlled by the equation's dissipation.

If $B = B_{\text{int}}$ and $n > \alpha$, Theorem 5.3 further gives

$$\int_0^T \|u(t)\|_{L^{n(r+1)/(n-\alpha)}(\Omega)}^{r+1} dt \leq \frac{C_S^2}{2c_r} R_T. \quad (5.41)$$

The nonlinear exponent is determined by the absorption: the reaction is $f_r(u)$, while the graph regularity concerns $f_{(r+1)/2}(u)$. Estimate (5.40) is an integrated bound; it does not assert that the graph functional is pointwise decreasing in time.

Example 5.4 (Fractional wave energy and the potential threshold). Consider the fractional wave equation

$$w_{tt} + Bw + f(w) = F, \quad w(0) = w_0, \quad w_t(0) = w_1. \quad (5.42)$$

Assume that f is globally Lipschitz, that $F \in C([0, T]; H)$, and that

$$w \in C([0, T]; D(B)) \cap C^1([0, T]; \mathcal{V}_B) \cap C^2([0, T]; H).$$

In particular, $w_0 \in D(B)$ and $w_1 \in \mathcal{V}_B$. Define the potential

$$\Pi(t) = \int_0^t f(a) da. \quad (5.43)$$

Global Lipschitz continuity implies $|\Pi(t)| \leq |f(0)||t| + L|t|^2/2$, so its integral is finite on H . The corresponding energy is

$$\mathcal{E}_B(t) = \frac{1}{2} \|w_t(t)\|_2^2 + \frac{1}{2} \mathfrak{e}_B(w(t)) + \int_{\Omega} \Pi(w(x, t)) dx. \quad (5.44)$$

The potential functional is continuously differentiable on H , with derivative $f(w)$. Pairing (5.42) with w_t consequently gives

$$\frac{d}{dt} \mathcal{E}_B(t) = (F(t), w_t(t))_H. \quad (5.45)$$

When $F = 0$, this proves

$$\mathcal{E}_B(t) = \mathcal{E}_B(0). \quad (5.46)$$

To obtain control of the energy norm, assume additionally that, for some $C_0 \geq 0$ and $0 \leq \eta < \mu_1$,

$$\Pi(a) \geq -\frac{\eta}{2} a^2 - C_0 \quad \text{for every } a \in \mathbb{R}. \quad (5.47)$$

The shifted energy $\widehat{\mathcal{E}}_B = \mathcal{E}_B + C_0|\Omega|$ then satisfies

$$\widehat{\mathcal{E}}_B(t) \geq \frac{1}{2} \|w_t(t)\|_2^2 + \frac{1}{2} \left(1 - \frac{\eta}{\mu_1} \right) \mathfrak{e}_B(w(t)) \geq 0. \quad (5.48)$$

Using (5.45), dividing by $2\sqrt{\widehat{\mathcal{E}}_B + \varepsilon}$, and then letting $\varepsilon \downarrow 0$ gives

$$\sqrt{\widehat{\mathcal{E}}_B(t)} \leq \sqrt{\widehat{\mathcal{E}}_B(0)} + \frac{1}{\sqrt{2}} \int_0^t \|F(s)\|_2 ds. \quad (5.49)$$

This argument includes zero-energy initial data. For monotone f with $f(0) = 0$, the potential is nonnegative, so one can take $\eta = C_0 = 0$.

The strict potential threshold has a direct spectral interpretation. For $f(a) = -\eta a$, $F = 0$, and $w(t) = z(t)\psi_1$, where $B\psi_1 = \mu_1\psi_1$, the equation becomes

$$z'' + (\mu_1 - \eta)z = 0.$$

At $\eta = \mu_1$, nonzero initial velocity permits linear growth of the displacement; for $\eta > \mu_1$, an exponentially growing mode exists. Thus conservation alone does not guarantee bounded displacement when the potential destroys coercivity.

The nonlinear graph domain supplies a complete metric setting, an optimal integrability scale for power maps, and quantitative continuity of the form interaction. Its role in evolution equations is determined by the proved energy balance: Power absorption controls graph regularity through integrated dissipation, whereas the wave equation is governed by its kinetic, fractional, and potential energies.

6. Conclusion

In this paper, we introduced a nonlinear functional framework for the analysis of fractional nonlinear partial differential equations. The main objective was to extend the classical Sobolev setting by taking into account not only the regularity of the solution but also the regularity associated with its nonlinear component.

The proposed construction provides a natural functional setting in which the solution and its nonlinear image can be considered simultaneously. It reduces to the classical fractional Sobolev framework in the linear case, while allowing additional information to be incorporated when the governing equation contains a nonlinear mapping.

The analytical results developed in this work show how the proposed construction can be used to derive energy and regularity estimates for fractional nonlinear equations. The resulting estimates provide control of the solution and its nonlinear contribution and establish connections with the classical Sobolev regularity theory.

The examples considered in the paper illustrate the applicability of the proposed approach to fractional nonlinear problems. They also demonstrate how the introduced functional setting can be used to investigate important properties of solutions, including regularity and stability-related behavior.

The present work provides a foundation for further developments in the analysis of fractional nonlinear partial differential equations. Future investigations may consider more general nonlinear operators, different types of fractional operators, higher-dimensional problems, and the development of analytical and numerical methods within the proposed nonlinear functional setting.

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