

# NONLINEAR ANALYSIS OF PLASMA DYNAMICAL CHAOTIC SYSTEM: HIDDEN MULTISTABILITY WITH CIRCUIT IMPLEMENTATION

Xinwang Liu<sup>1</sup>, Javeria Ayub<sup>2</sup>, Muhammad Aqeel<sup>2,†</sup>  
and Umer Farooq<sup>1</sup>

**Abstract** In this article, we introduce hidden multistability in plasma dynamical chaotic system with coexisting hidden and self-excited attractors. Plasma is the branch of physics that is the result of nonlinear oscillation, instability, and turbulence of particle interaction and scattering waves. The multistability in the plasma dynamical chaotic system is due to the radio-frequency oscillation of an electron gas and the excitation of a radioactive oscillation of the ion-acoustic wave. The plasma dynamical chaotic system is constructed by the amplitude of these oscillations, which exhibits chaos for particular values of parameters. Hidden multistability is investigated and confirmed in the plasma dynamical chaotic system with five major analytical tools, such as parameter constraints, basin of attraction, bifurcation diagram, Lyapunov exponents, and Poincaré section. Moreover, by utilizing the Multisim we have done a circuit implementation and subsequently validated the mathematical model. The findings of this study will provide helpful insight into the plasma dynamical chaotic system in targeting multi-stability and chaos.

**Keywords** Hidden multistability, chaos, bifurcation, Poincaré map, basin of attraction, Lyapunov exponents, chaotic circuit.

**MSC(2010)** 34C28, 37G35, 34C23, 37D45.

## 1. Introduction

In 1978, Pikovsky and colleagues embarked on pioneering work in the study of plasma chaotic systems [34], sparking a wave of exploration by researchers into the dynamics of plasma dynamical chaotic system, exemplified by studies such as [11, 19, 39]. Despite these efforts, many questions remained unanswered. Key aspects, such as parametric constraints for multistability and regions of chaos, have largely escaped thorough investigation in prior studies [6, 23, 43]. This research aims to delve into these uncharted territories, revealing the concealed behaviors within plasma dynamical chaotic system.

Multistability, a characteristic inherent in chaotic dynamical systems [10, 20, 22, 25, 27], is an intriguing topic. The significance of multistability transcends the realm of physics and extends into numerous fields, including the control of complex systems, secure communication in networked systems, and modeling biological and ecological systems [2–5, 28, 31, 35]. For instance,

<sup>†</sup>The corresponding author.

<sup>1</sup>College of Mathematical Sciences, Harbin Engineering University, 150001, China

<sup>2</sup>Applied Mathematics and Statistics, Institute of Space Technology, Islamabad 44000, Pakistan  
Email: liuxinwang@hrbeu.edu.cn(X. Liu), hchodary@gmail.com(J. Ayub),  
medraqeel@gmail.com(M. Aqeel), umer\_farooq@hrbeu.edu.cn(U. Farooq)

in the domain of control theory, the analysis of multistable behavior can provide insights into designing strategies for stabilizing chaotic systems [2–5, 8, 26, 35, 38]. In secure communication, understanding the multistability of chaotic systems can lead to the development of more robust communication protocols. Similarly, in biological and ecological systems, multistability can offer valuable insights into the dynamics of interconnected systems.

A multistable phenomenon has been recently reported for a few continuous monostable chaotic systems after discretization, including the use of numerical integration methods with variable symmetry. This means that the nature of the dynamics of the original system can be changed by the way the system is discretized.

Therefore reliable numerical solutions should be stable towards time step refinement, consistent among different numerical solutions, have a basin of attraction structure and (where possible) be supported by analytical predictions and/or continuation and/or experimental observations [1, 12, 13, 16, 17, 24, 33].

Numerical simulations were performed in MATLAB using the classical fourth-order Runge Kutta (RK4) method with adaptive step size. Unless otherwise stated, the system was integrated over the time interval ( $t \in [0, T]$ ). To investigate the multistable behavior of the proposed system, simulations were carried out for different initial conditions while keeping the system parameters fixed at ( $\alpha = 7$ ), ( $\beta = 1$ ), and ( $\gamma = 4$ ), unless otherwise stated.

The application of chaotic systems in electronic circuits presents its own set of challenges and opportunities. This research demonstrates how the implementation of chaotic systems in electronic circuits can open new avenues in secure communication, random number generation, nonlinear dynamics research, and analog computing [7, 9, 18, 40].

Multisim, through its SPICE simulation engine, typically employs the trapezoidal integration method or the Gear (Backward Differentiation Formula, BDF) method for transient circuit analysis. Although both methods are designed to provide stable numerical solutions for stiff circuit equations, they can still influence the qualitative behavior of nonlinear circuits, including the appearance of multistability, hidden attractors, and chaotic dynamics.

These numerical effects have been extensively discussed in the SPICE literature and should be considered carefully when interpreting the dynamics obtained from nonlinear circuit simulations [29, 41, 42].

The Multisim simulations were performed using *Multisim*14.2, which is based on the SPICE simulation engine. Transient analysis was carried out using both the Trapezoidal and Gear integration methods, with the second-order Gear method employed to enhance numerical stability. The maximum simulation time step was set to  $0.5\mu s$ , as this parameter plays a crucial role in accurately capturing the chaotic dynamics of the system. Each simulation was run for a total duration of  $0.5s$ , and the transient analysis was initiated using user-defined initial capacitor voltages without operating-point initialization. The default SPICE convergence tolerances were maintained throughout the simulations, with  $REL TOL = 0.001$ ,  $ABSTOL = 1pA$ , and  $VNTOL = 1\mu V$ . To investigate the existence of coexisting attractors, the initial capacitor voltages were systematically varied while keeping all circuit parameters unchanged. The numerical and circuit simulations both used adaptive time-step algorithms. The MATLAB simulation was performed over a fixed interval of 50 s using *ode45*, while the NI Multisim simulation was continued until a stable chaotic attractor was observed on the oscilloscope.

The main contribution of this work is not only to identify additional attractors. It lies in the systematic discovery and comprehensive characterization of previously unreported hidden multistability in the plasma dynamical system. Although this plasma dynamical chaotic system

has been widely studied for its chaotic dynamics, bifurcation characteristics, and equilibrium behavior, earlier investigations have mainly concentrated on self-excited attractors that can be obtained by initiating trajectories in the neighborhood of unstable equilibrium points [14, 15, 21, 32, 36, 37]. In contrast, the present study reveals the existence of hidden attractors whose basins of attraction are not connected to the vicinity of any equilibrium point.

In this article, we comprehensively analyze parameter constraints, basin of attraction, bifurcation diagrams, Lyapunov exponents, and Poincaré sections of chaotic systems to diagnose multistability. The examination of these elements serves as a foundation for understanding the behavior of plasma dynamical chaotic system, its stability, and its hidden dynamics.

The following sections provide an in-depth exploration of these concepts. In Section 2, we delve into the stability analysis of plasma chaotic systems through parametric constraints. Sections 3 uncover the localization of hidden multistability and the investigation of hidden and self-excited attractors, respectively. Finally, Section 5 and 6 presents discussion and our concluding remarks.

## 2. Plasma chaotic model

The nonlinear dynamical physical model presented is derived from the nonlinear interaction of three resonantly coupled waves in a non-isothermal magnetoactive plasma.

### 2.1. Mathematical modeling

In this system [30], a whistler wave acts as the external pump and transfers energy to a plasma wave and an ion-sound wave through parametric interaction. Efficient energy transfer occurs only when the resonance conditions,

$$Q = K + X, \quad \omega Q = \omega K + \Omega X$$

are satisfied, ensuring the conservation of momentum and energy. To describe the confinement of the parametric instability, the authors [30], introduce a second plasma wave satisfying  $K_1 = K - X$ , and  $\omega_{K_1} = \omega_K - \Omega X$ , which receives energy through decay interaction. This resonantly coupled wave triplet forms the physical basis of the proposed dynamical model.

Parametric excitation is a nonlinear process in which the external pump wave supplies energy to two resonantly coupled daughter waves rather than exciting a single wave directly. In this model, the whistler wave continuously feeds energy into the plasma and ion-sound waves once the instability threshold is exceeded. Meanwhile, part of this energy is transferred to another plasma wave through decay interaction, preventing unlimited growth of the instability. Therefore, the system dynamics are governed by the interplay between pump-induced excitation, linear damping, and nonlinear energy exchange, which determines whether the plasma dynamical chaotic system reaches a stable state, periodic oscillations, or chaotic behavior.

To obtain the mathematical model, the authors [30], begin with the governing hydrodynamic equations for electron oscillations and the kinetic equations for ion-sound waves. These equations are then reduced to a system of coupled nonlinear amplitude equations that incorporate linear damping, pump-driven excitation, and quadratic nonlinear wave interactions. By introducing suitable amplitude, phase, and time normalizations, the system is transformed into a dimensionless form characterized by the three parameters

$$\alpha = \frac{|W_K E_Q|}{\rho_{K_1}}, \quad \beta = \frac{\rho_K}{\rho_{K_1}}, \quad \gamma = \frac{\rho_K}{\rho_{K_1}},$$

where  $\alpha$  is the normalized pump strength, while  $\beta$  and  $\gamma$  represent the normalized damping rates of the plasma and ion-sound waves, respectively. The parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  are dimensionless quantities that describe the relationship between energy input and energy dissipation in the plasma dynamical chaotic system. The parameter  $\alpha$  represents the ratio of the effective pump strength to the damping of the secondary plasma wave.

A small value of  $\alpha$  indicates that energy dissipation is more dominant than energy input, causing the system to remain stable. As  $\alpha$  increases, the external pump wave injects more energy into the plasma dynamical chaotic system, intensifying the nonlinear interactions among the coupled waves. This increased energy input may drive the system through bifurcations and eventually into chaotic dynamics. The parameter  $\beta$  represents the relative damping of the primary plasma wave. Lower values of  $\beta$  correspond to weaker damping, enabling the plasma wave to retain energy for a longer period and participate more actively in the nonlinear energy exchange. In contrast, higher values of  $\beta$  increase energy dissipation, thereby suppressing nonlinear oscillations and promoting system stability. Likewise,  $\gamma$  characterizes the relative damping of the ion-sound wave. When  $\gamma$  is small, the ion-sound wave persists for a longer duration, strengthening the nonlinear coupling between the interacting waves. As  $\gamma$  increases, the ion-sound wave loses energy more rapidly, reducing its contribution to the overall dynamics. Therefore,  $\alpha$  primarily controls the energy supplied to the plasma dynamical chaotic system, whereas  $\beta$  and  $\gamma$  regulate how efficiently that energy is dissipated. Together, these parameters determine the stability, bifurcation behavior, and long-term nonlinear evolution of the plasma dynamical chaotic system.

Finally, complete phase synchronization allows the complex amplitudes to be expressed as real variables  $(x_1, x_2, x_3)$ , resulting in the three-dimensional autonomous model presented in equation (2.1). Despite this simplification, the reduced system preserves the essential physics of nonlinear wave interaction and provides an effective framework for studying stability, bifurcations, and chaotic dynamics.

The following system of differential equations (2.1) describes the dynamics of the wave amplitudes with  $t \rightarrow \infty$  in [34].

$$\begin{aligned} \dot{x}_1 &= -x_2x_3 - \beta x_1 + \alpha x_2, \\ \dot{x}_2 &= x_1x_3 - \gamma x_2 + \alpha x_1, \\ \dot{x}_3 &= x_1x_2 - x_3, \end{aligned} \tag{2.1}$$

where  $x_1, x_2, x_3$  are real wave amplitudes and  $\alpha$  is proportional to the amplitude of pump, which is the electric field,  $\beta$  and  $\gamma$  are the damping decrements of the excited hybrid and acoustic waves. System is dissipative for all values of  $\alpha, \beta, \gamma \geq 0$ , whereas the divergences is  $\nabla x_{i's} = (-\beta - \gamma - 1) < 0$ , for all  $i = 1, 2, 3$ .

Plasma chaotic attractor is presented in Figure 1, where  $\alpha, \beta$  and  $\gamma$  are real chaotic parameters. Sensitivity dependencies on initial condition is the root factor of the existence of chaos. Graphs in Figure 2 show the sensitive dependence of the Plasma chaotic model through two nearby trajectories. To investigate the hidden multistability parametric constraints are defined by stability analysis to deeply understand the dynamics.

## 2.2. Stability analysis of plasma dynamical chaotic system

Plasma chaotic model in (2.1), possess three real equilibrium points,  $E_0 = (0, 0, 0)$  and  $E_{1,2} = (\pm\sqrt{mn}, \pm\sqrt{n/m}, n)$  where as,  $m = \frac{\alpha-n}{\beta}$ ,  $n = \sqrt{\sigma_0}$  and  $\sigma_0 = \alpha^2 - \beta\gamma$ . When  $\alpha^2 \leq \beta\gamma$  only

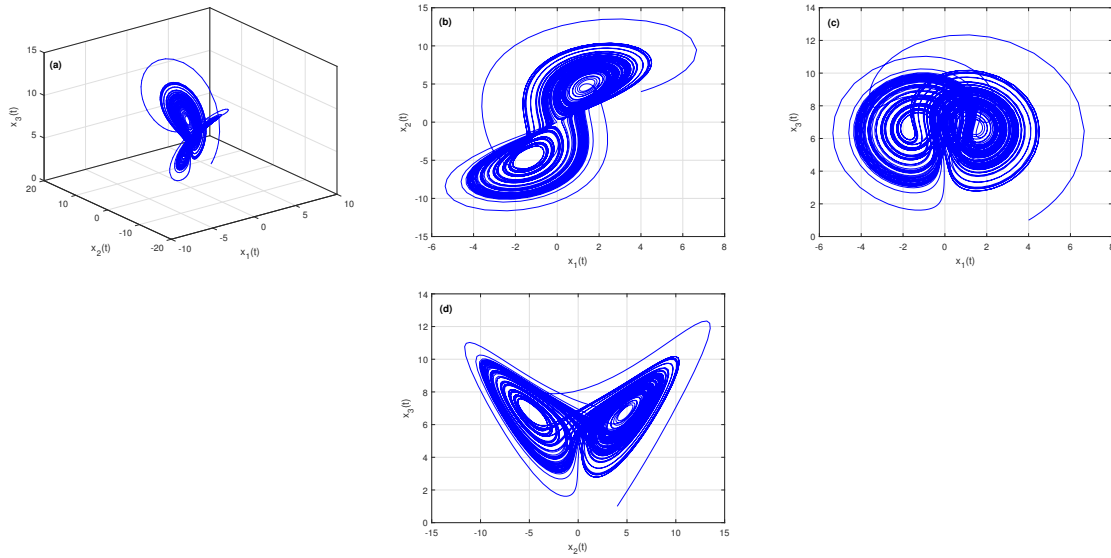


**Figure 1.** Plasma chaotic attractor with the initial condition  $(x_1, x_2, x_3) = (4, 4, 1)$  and parameters values are  $\alpha = 7$ ,  $\beta = 1$  and  $\gamma = 4$ , (a)  $x_1x_2x_3$ -Phase Space (b)  $x_1x_2$ -Phase Plane (c)  $x_1x_3$ -Phase Plane & (d)  $x_2x_3$ -Phase Plane.

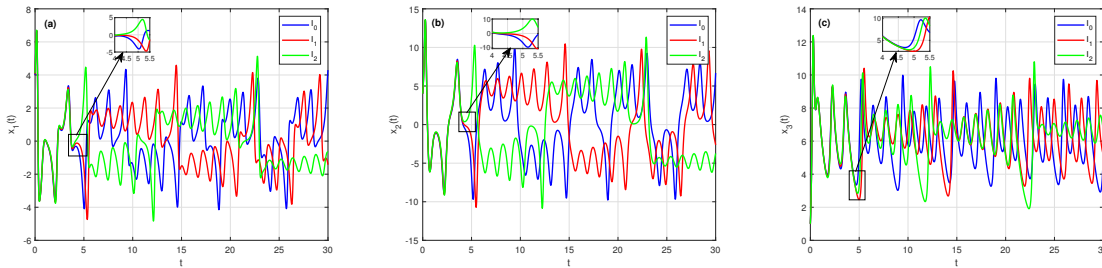


**Figure 2.** Plasma chaotic attractor with the initial conditions  $I_0(x_1, x_2, x_3) = (4, 4, 1)$ ,  $I_1(x_1, x_2, x_3) = (4.05, 4, 1)$  and  $I_2(x_1, x_2, x_3) = (4.10, 4, 1)$ , parameters values are  $\alpha = 7$ ,  $\beta = 1$  and  $\gamma = 4$ , (a)  $x_1$  (b)  $x_2$  (c)  $x_3$  trajectories.

$E_0$  exists and for  $\alpha^2 > \beta\gamma$  all equilibrium points  $E_{0,1,2}$  exist. So, the following is the stability analysis according to the Routh Hurwitz criteria.

**Theorem 2.1.** *If  $\alpha, \beta, \gamma > 0$ ,  $\alpha^2 \leq \beta\gamma$  then  $E_0$  is stable. Otherwise, for  $\alpha^2 > \beta\gamma$ ,  $E_0$  is unstable.*

**Theorem 2.2.** *If  $\alpha, \beta, \gamma > 0$ ,  $\alpha(\gamma(\alpha - \sqrt{\sigma_0}) + \beta(\alpha + \sqrt{\sigma_0})) > 0$  and  $\alpha(1 + \beta + \gamma)(\gamma(\alpha - \sqrt{\sigma_0}) + \beta(\alpha + \sqrt{\sigma_0})) > 4\beta\gamma\sigma_0$  then  $E_{1,2}$  symmetric equilibrium points are stable. Otherwise,  $E_{1,2}$  are unstable.*

Parametric constraints represents three regions  $R_a$ ,  $R_b$  and  $R_c$ , whereas in  $R_a$  only  $E_0$  exists and in remaining regions all equilibrium points exist. Also,  $R_a$  is a region where  $E_0$  is stable,  $R_b$  is the region where  $E_{1,2}$  are stable, and  $R_c$  is the region where all equilibrium points  $E_{0,1,2}$  are unstable. Numerically computed points in Figure 3, have been considered in Table 1 presents the stability analysis on these points.

Calculation of the basin of attraction is based on discrete points in  $x_1x_2$ -plane around equilibrium points. In Figure 4 the cross-section of  $x_1x_2$ -plane at  $x_3 = 3$  represents the basin

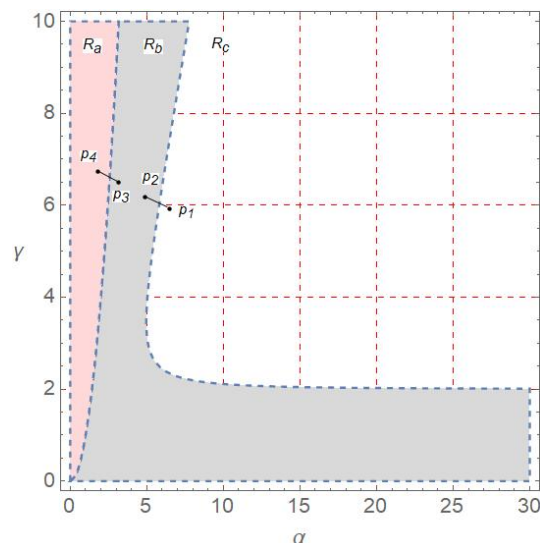


**Figure 3.** Parametric constraints representation of Plasma chaotic system (2.1),  $(\alpha, \gamma)$  for  $\beta = 1$ , whereas four different points has been taken  $p_1, p_2, p_3$  and  $p_4$  through numerical computation.

**Table 1.** Analysis of numerically selected points from parametric constraints in Figure 3 of the plasma dynamical chaotic system (2.1). Here, “FP” stands for fixed points and “EV” stands for eigenvalues.

| $(\alpha, \gamma)$     | FP             | EV                                                                                                         | Analysis                                   |
|------------------------|----------------|------------------------------------------------------------------------------------------------------------|--------------------------------------------|
| $p_1 = (6.161, 6.108)$ | $E_0, E_{1,2}$ | $E_0 : Re(\lambda_{1,2}) < 0, Re(\lambda_3) > 0$ &<br>$E_{1,2} : Re(\lambda_1) < 0, Re(\lambda_{2,3}) > 0$ | $E_{0,1,2}$ are unstable                   |
| $p_2 = (5.713, 6.204)$ | $E_0, E_{1,2}$ | $E_0 : Re(\lambda_{1,2}) < 0, Re(\lambda_3) > 0$ &<br>$E_{1,2} : Re(\lambda_{1,2,3}) < 0$                  | $E_0$ is unstable and $E_{1,2}$ are stable |
| $p_3 = (2.75, 7.293)$  | $E_0, E_{1,2}$ | $E_0 : Re(\lambda_{1,2}) < 0, Re(\lambda_3) > 0$ &<br>$E_{1,2} : Re(\lambda_{1,2,3}) < 0$                  | $E_0$ is unstable and $E_{1,2}$ are stable |
| $p_4 = (2.381, 7.421)$ | $E_0$          | $E_0 : Re(\lambda_{1,2,3}) < 0$                                                                            | $E_0$ is stable                            |

of attraction for different values of  $\alpha$  and for  $\beta = 1, \gamma = 4$ . This technique is based on discrete points connected with four different colors, orange, light blue, dark blue, and yellow. As the plasma dynamical chaotic system consists of three equilibrium points, the dark/light blue color is the basin of attraction for the symmetric equilibrium, the orange color is the basin of attraction for equilibrium at origin, whereas the yellow color approaches infinity. For simplicity, our system also proposed infinity symmetric fractals which show self-excited chaotic attractor of the plasma dynamical chaotic system for  $\alpha = 5.5$ .

### 3. Localization of hidden multi-stability

Although the Pikovsky et al. [30], investigates equation (2.1) by varying the normalized pump strength  $\alpha$ , any of the three parameters,  $\alpha, \beta$ , or  $\gamma$ , can be selected as the bifurcation parameter because each controls a different physical aspect of the plasma dynamical chaotic system. The parameter  $\alpha$  governs the energy injected by the external pump wave, whereas  $\beta$  and  $\gamma$  repre-

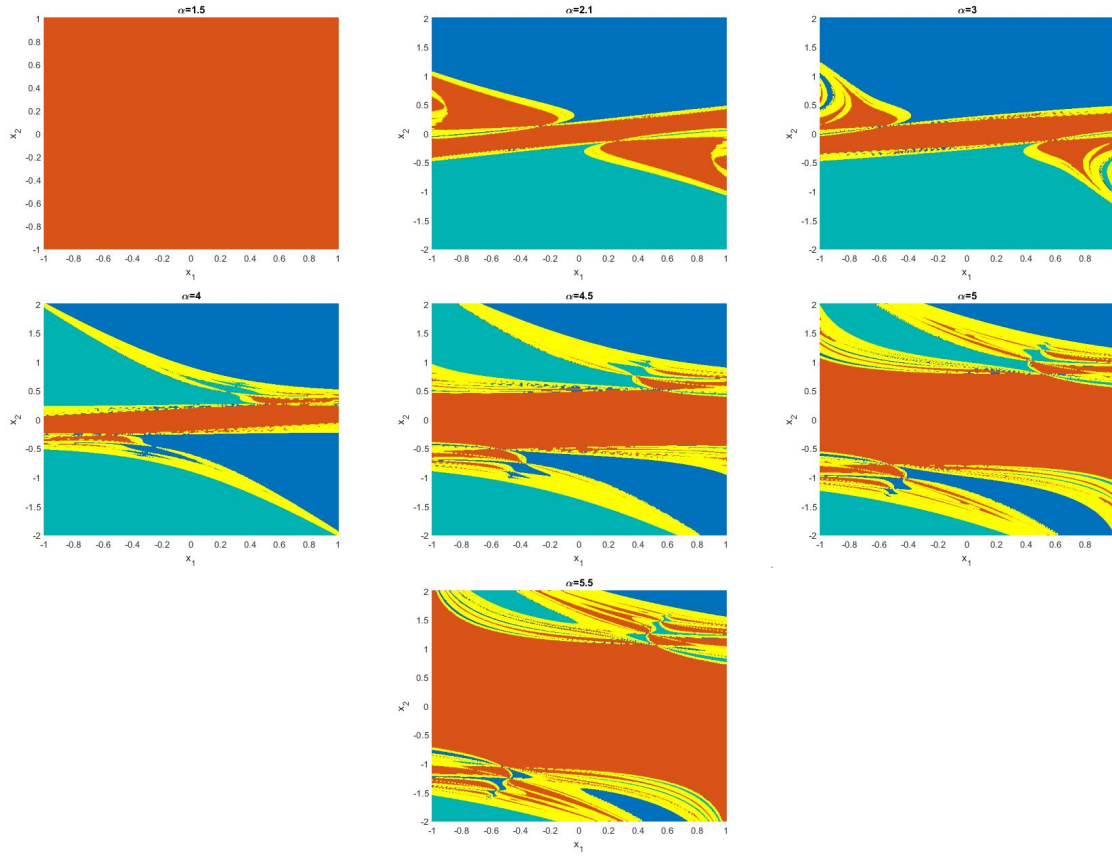


**Figure 4.** Basin of attraction in two-dimensional cross-section  $x_1x_2$ -plane at  $x_3 = 3$  for different values of  $\alpha$  and for  $\beta = 1$ , and  $\gamma = 4$ .

sent the normalized damping rates of the plasma and ion-sound waves, respectively. Therefore, varying  $\alpha$  changes the energy supplied to the system, while varying  $\beta$  or  $\gamma$  modifies the rate of energy dissipation. As a result, changing any of these parameters can alter the stability of the system and trigger transitions between stable equilibrium, periodic oscillations, chaotic dynamics, or multistable states. The existence of multi-stability is confirmed through the bifurcation diagram with the variation of bifurcation parameters. Furthermore, these regions can be investigated for hidden multi-stability. We have investigated multi-stability through a bifurcation diagram, phase portraits, and a Poincaré section, for two different initial conditions,  $I_0(x_1, x_2, x_3) = (0.1, 0.1, 0.1)$  in blue colour,  $I_1(x_1, x_2, x_3) = (-1, 0, -1)$  in red colour. Initially we fix  $\beta = 1$ ,  $\gamma = 4$ , and investigate hidden multistability for  $\alpha > 2$  when system possesses three equilibrium points, whereas  $\alpha \leq 2$  system has only one equilibrium point  $E_0$ . The dynamics of the plasma chaotic system is investigated through Lyapunov exponents with the variation of  $\alpha$ . Lyapunov exponents in Figure 5 presented the existences of chaotic attractor in regions  $\alpha \in (4.9, 13.5)$  and  $\alpha \in (14., 16)$ , for one positive Lyapunov exponent. Bifurcation diagrams are generated for two different initial conditions with the variation of bifurcation parameter  $\alpha$  in Figure 6.

In the bifurcation analysis,  $\alpha$  is selected as the bifurcation parameter because it directly controls the external energy supplied to the plasma dynamical chaotic system. Physically, increasing  $\alpha$  corresponds to increasing the amplitude of the pump wave, thereby enhancing the parametric excitation of the plasma and ion-sound waves. When  $\alpha$  is below the instability threshold,

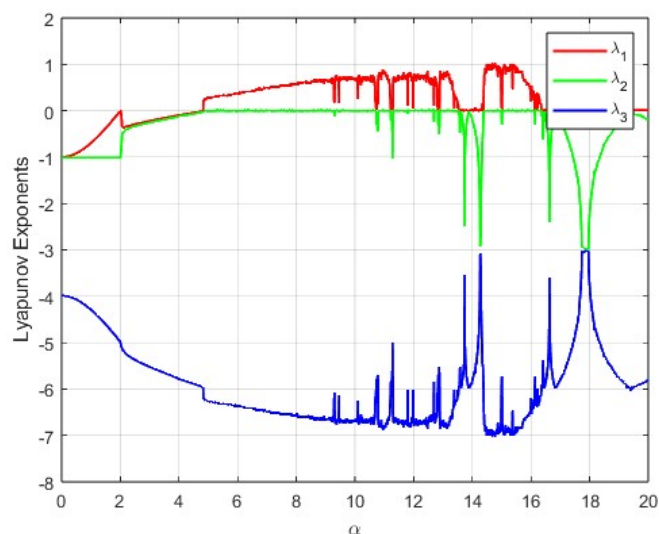


**Figure 5.** Lyapunov exponents with the initial condition  $(x_1, x_2, x_3) = (4, 4, 1)$  and the variation of  $\alpha$  for  $\beta = 1$ ,  $\gamma = 4$ .

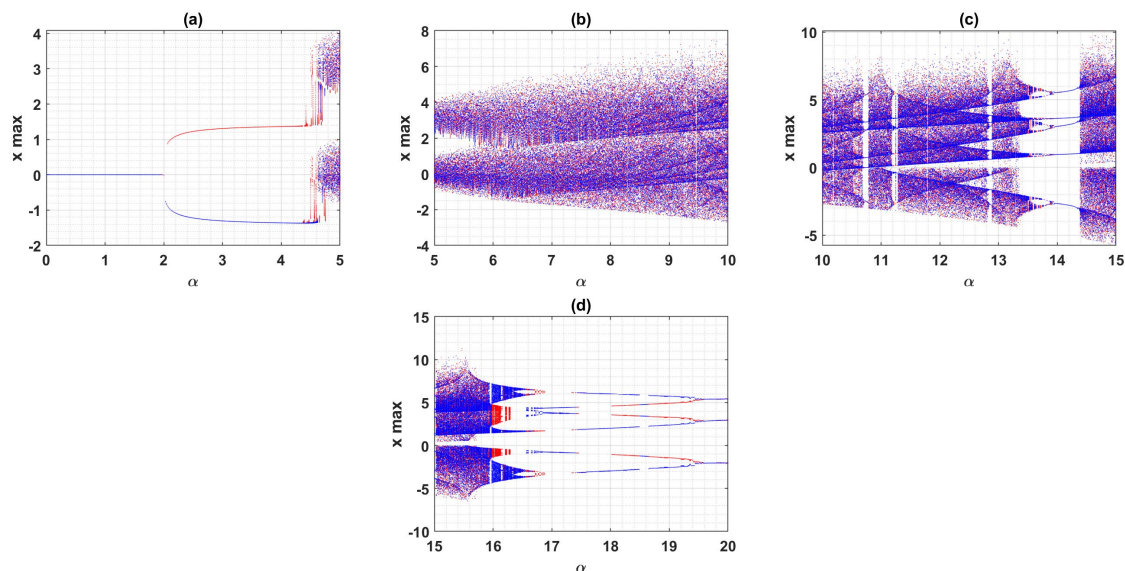


**Figure 6.** Multi-stability in subfigures of bifurcation diagram with the variation of  $\alpha$  for  $\beta = 1$ ,  $\gamma = 4$ . A comparison of bifurcation diagram is presented for two different initial conditions,  $I_0(x_1, x_2, x_3) = (0.1, 0.1, 0.1)$  in blue colour,  $I_1(x_1, x_2, x_3) = (-1, 0, -1)$  in red colour.

damping dominates and the system converges to a stable equilibrium. As  $\alpha$  increases, nonlinear interactions become stronger, leading to bifurcations, periodic oscillations, and eventually chaotic dynamics.

Multi-stability exists for small intervals in Plasma chaotic system (2.1), hence hidden multi-stability can be investigated in these regions only.

Two different initial conditions exhibit distinct stable branches over the same range of  $\alpha$ , the system exhibits multistability. Physically, this means that even with identical plasma dynami-

cal chaotic system parameters and pump strength, the nonlinear wave interaction can support multiple stable long-term states. Although the pump injects the same amount of energy into the plasma dynamical chaotic system, different initial wave amplitudes lead to different pathways of energy redistribution among the interacting waves for small intervals. Consequently, the plasma dynamical chaotic system may evolve toward different periodic attractors under identical operating conditions. This demonstrates that the long-term behavior of equation (2.1) depends not only on the pump strength but also on the initial state of the system, reflecting the coexistence of multiple stable energy-distribution mechanisms in the nonlinear plasma dynamical chaotic system.

Multi-stability exists in the above bifurcation diagrams for  $2 \leq \alpha \leq 4.5$ ,  $16 \leq \alpha \leq 17$ , and  $18 \leq \alpha \leq 19$ . Now we only discuss multi-stability in above mentioned regions. Furthermore, multi-stability in bifurcation diagrams is confirmed through Poincaré sections and phase portraits in Figures 7, 8 and 9. Here we have utilized parametric constraints from Figure 3 and

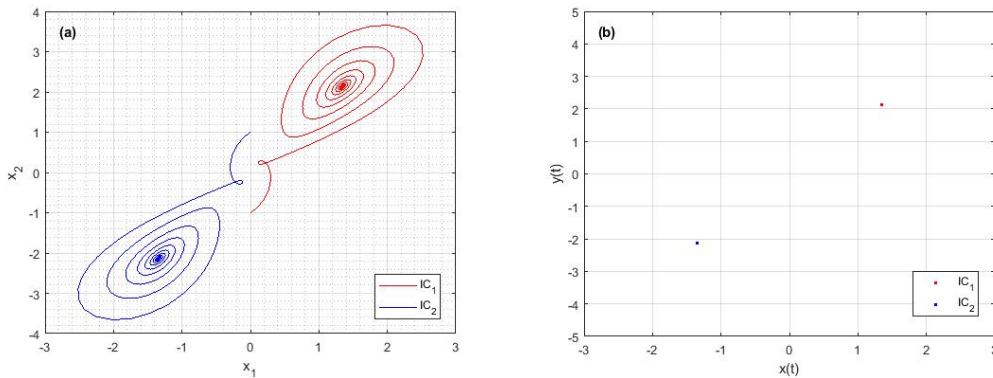


**Figure 7.** (Case01:  $2 \leq \alpha \leq 4.5$ ) Multi-stability:  $\alpha = 3$  for  $\beta = 1$ ,  $\gamma = 4$ . Comparison of phase portraits and Poincaré section are presented for two different initial conditions: (a) Phase plane  $x_1x_2$ -plane and (b) Poincaré section.

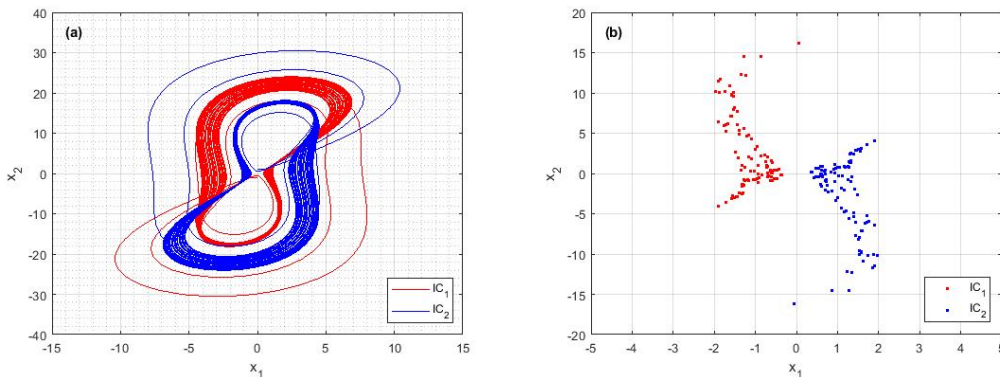


**Figure 8.** (Case02:  $16 \leq \alpha \leq 17$ ) Multi-stability:  $\alpha = 16.2$  for  $\beta = 1$ ,  $\gamma = 4$ . Comparison of phase portraits and Poincaré section are presented for two different initial conditions: (a) Phase plane  $x_1x_2$ -plane and (b) Poincaré section.

confirm the hidden multi-stability attractors, through random initial conditions in Figure 4.

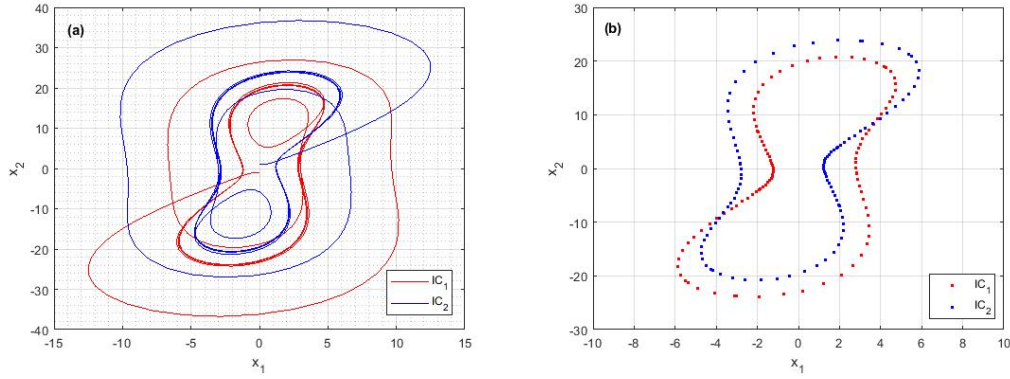


**Figure 9.** (Case03:  $18 \leq \alpha \leq 19$ ) Multi-stability:  $\alpha = 18.5$  for  $\beta = 1$ ,  $\gamma = 4$ . Comparison of phase portraits and Poincaré section are presented for two different initial conditions: (a) Phase plane  $x_1x_2$ -plane and (b) Poincaré section.

#### 4. Circuit implementation of plasma dynamical chaotic system

Chaotic circuits are chaotic systems, that represent chaotic signals, and also indicate the physical existence of chaotic systems through electronic circuits. Circuit implementation is the most important way of studying the transition from theory to the application of chaos to engineering practice. The mathematical representation of Plasma chaotic circuits starts from the system of integral equations which is also the solution set of the Plasma chaotic system.

$$\begin{aligned} x_1 &= - \int (x_2x_3 + \beta x_1 + \alpha(-x_2))dt, \\ x_2 &= - \int ((-x_1)x_3 + \gamma x_2 + \alpha(-x_1))dt, \\ x_3 &= - \int (x_1(-x_2) + x_3)dt, \end{aligned} \quad (4.1)$$

now we can replace coefficients with suitable resistances and capacitors.

$$\begin{aligned} x_1 &= \frac{1}{C_1} \int \left[ \left( \frac{1}{R_1} \right) x_2x_3 + \left( \frac{1}{R_2} \right) x_1 + \left( \frac{R_{10}}{R_3R_9} \right) (-x_2) \right] dt, \\ x_2 &= \frac{1}{C_2} \int \left[ \left( \frac{R_5}{R_4R_6} \right) (-x_1)x_3 + \left( \frac{1}{R_7} \right) x_2 + \left[ \left( \frac{R_5}{R_4R_8} \right) (-x_1) \right] \right] dt, \\ x_3 &= \frac{1}{C_3} \int \left[ \left( \frac{R_{10}}{R_9R_{11}} \right) x_1(-x_2) + \left( \frac{1}{R_{12}} \right) x_3 \right] dt. \end{aligned} \quad (4.2)$$

The chaotic circuit of the plasma dynamical chaotic system designed on Multisim 14.2 is represented below:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C_1} \left[ \left( \frac{1}{R_1} \right) x_2x_3 + \left( \frac{1}{R_2} \right) x_1 + \left( \frac{R_{10}}{R_3R_9} \right) (-x_2) \right], \\ \dot{x}_2 &= \frac{1}{C_2} \left[ \left( \frac{R_5}{R_4R_6} \right) (-x_1)x_3 + \left( \frac{1}{R_7} \right) x_2 + \left[ \left( \frac{R_5}{R_4R_8} \right) (-x_1) \right] \right], \\ \dot{x}_3 &= \frac{1}{C_3} \left[ \left( \frac{R_{10}}{R_9R_{11}} \right) x_1(-x_2) + \left( \frac{1}{R_{12}} \right) x_3 \right], \end{aligned} \quad (4.3)$$

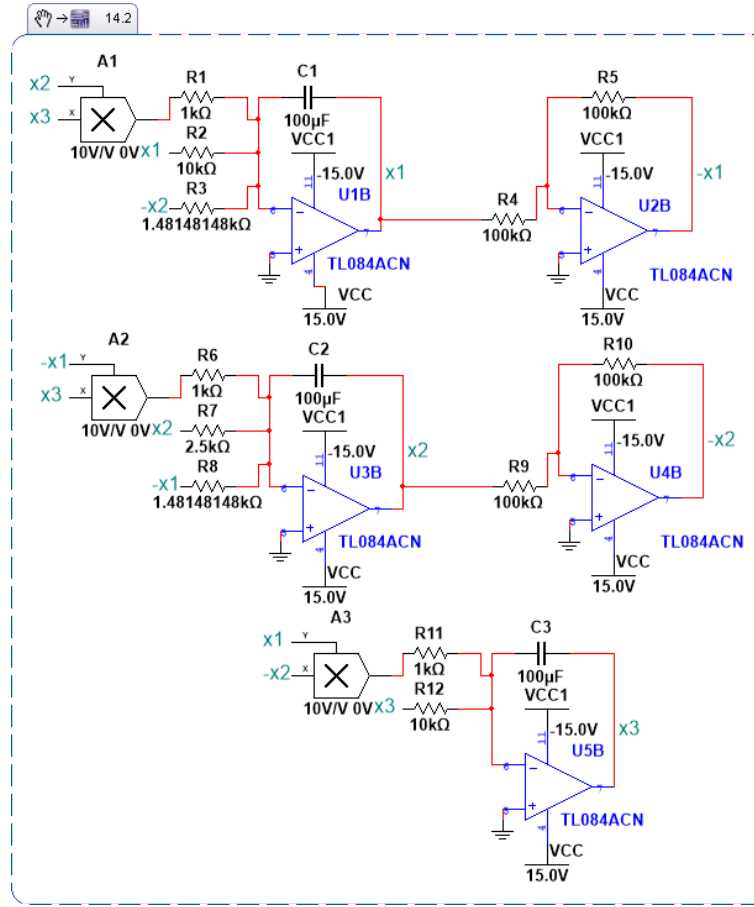


**Figure 10.** Circuit design of plasma dynamical chaotic system on Multisim 14.2, for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 6.75$ .

whereas,  $C_1 = C_2 = C_3 = 100\mu F$ ,  $R_1 = R_6 = R_{11} = 1k\Omega$ ,  $R_2 = R_{12} = 10k\Omega$ ,  $R_3 = R_8 = 1.48148148k\Omega$ ,  $R_4 = R_5 = R_9 = R_{10} = 100k\Omega$ , and  $R_7 = 2.5k\Omega$ . The electronic chaotic circuit of the plasma dynamical chaotic system is represented in Figure 10, whereas chaotic signals are presented in Figures 11, 12, and 13. In the following Figures 14, 15, we see the change in trajectories for different values of  $\alpha$ . In Figure 14  $\alpha = 16.2$  and  $R_3 = R_8 = 0.617283951k\Omega$  simulation shows the chaos. Also Figure 15  $\alpha = 18.5$  and  $R_3 = R_8 = 0.540540541k\Omega$  simulation shows the period doubling.

## 5. Discussion

The dynamical investigations carried out in this study demonstrate that the proposed three-dimensional plasma dynamical chaotic system 2.1 exhibits a rich spectrum of nonlinear behavior. The stability results clearly distinguish the transition from regular motion to chaotic regimes as system parameters vary. The presence of sensitivity to initial conditions in Figure 2, further supported by positive Lyapunov exponents in Figure 5, confirms the existence of deterministic chaos and highlights the system's unpredictability over time. The bifurcation diagrams shown in Figure 6 illustrate several qualitative changes in the system dynamics, indicating parameter-dependent transitions between stable, periodic and chaotic solutions, and multiple attractors.

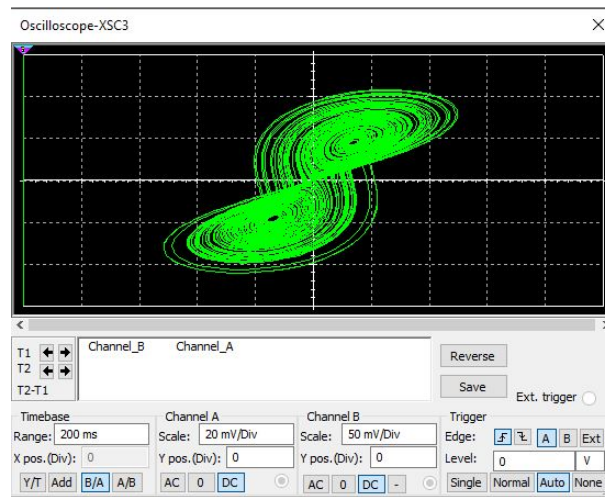


**Figure 11.** Simulation results of circuit design in Figure 10, in  $x_1x_2$ – plane: Plasma dynamical chaotic system for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 6.75$ .

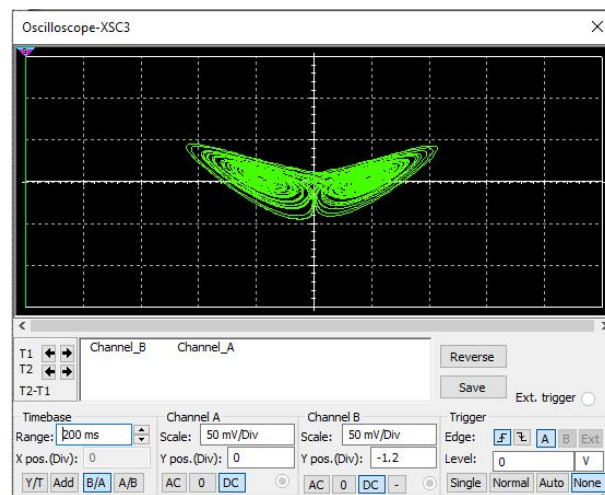


**Figure 12.** Simulation results of circuit design in Figure 10, in  $x_2x_3$ – plane: Plasma dynamical chaotic system for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 6.75$ .

The Poincaré maps shown in the Figures 7, 8 and 9 show the evolution of the system from closed invariant curves to more intricate structures resembling fractals as the time series progresses. Furthermore, the basin of attraction analysis in Figure 4 demonstrates a strong sensitivity to initial conditions, revealing that with different initial conditions, the system can approach to different long-time attractors, thus verifying the multistable behavior of the model. Lastly, the electronic circuit realization of Figures 11, 12 and 13 proves that the chaotic dynamics predicted are not only theoretical or numerical but can also be attained physically. In conclusion, all these results validate the reliability of the proposed chaotic plasma dynamical chaotic system and highlight its practical application in secure communication, random signal generation, nonlinear information processing, and beyond.

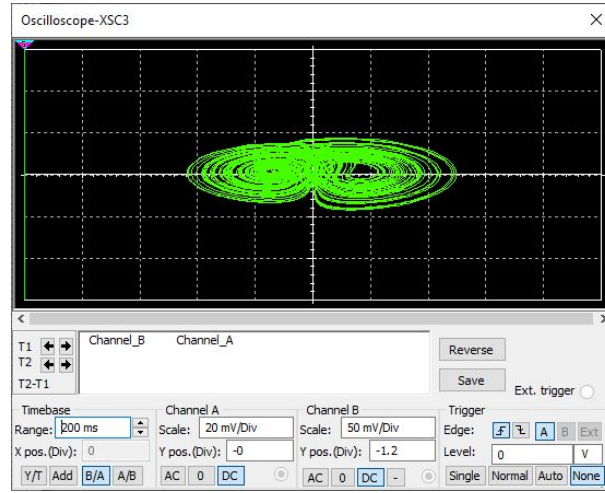


**Figure 13.** Simulation results of circuit design in Figure 10, in  $x_1x_3$ – plane: Plasma dynamical chaotic system for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 6.75$ .

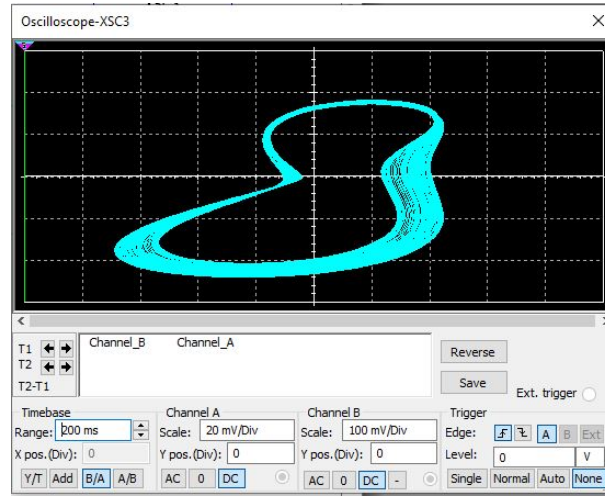


**Figure 14.** Simulation results of circuit design in Figure 10, in  $x_1x_2$ – plane: Plasma system for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 16.2$ .

## 6. Conclusion

In summary, a novel multi-stable hidden plasma dynamical chaotic system has been introduced in this article with different analytical tools, such as basin of bifurcation diagram, Poincaré map and Lyapunov exponents. The chaotic study of this new multistable plasma dynamical chaotic system reveals the existence of a self-excited attractor for the selecting appropriate value of system parameters where all equilibrium points are unstable. Moreover, a systematic procedure is proposed to reveal the multistability of a chaotic system. In this study, it is proposed that for diagnosing the multistability in a chaotic system, there are five important parameters (Parameter constraints, basin of attraction, bifurcation diagram, Lyapunov exponents, Poincaré section) that can confirm the multistability in a chaotic system. Finally, the implementation of plasma dynamical chaotic system in electronic circuits has presented the potential to open up

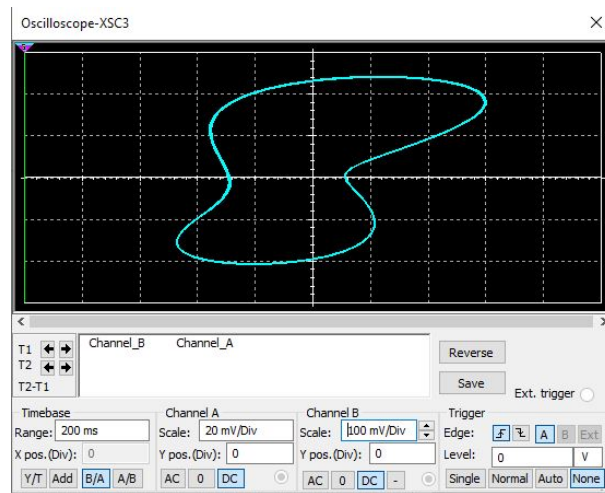


**Figure 15.** Simulation results of circuit design in Figure 10, in  $x_1x_2$ - plane: Plasma dynamical chaotic system for  $\beta = 1$ ,  $\gamma = 4$ , and  $\alpha = 18.5$ .

new avenues for research and innovation in a wide range of fields.

**Conflict of interest/Competing interests.** Conflict of interest: The authors declare that they have no conflict of interest.

**Ethics approval and consent to participate.** The research presented in this paper is entirely theoretical and computational. No human subjects, personal data, or animal testing were included in this study. As a result, institutional ethical approval and consent to participate were not applicable. The work complies with all relevant guidelines for computational and mathematical research.

**Consent for publication.** All authors confirm that this manuscript has been prepared with full accuracy and integrity. We agree to its submission to the journal and provide our consent for publication upon acceptance. We also acknowledge that the final published version may undergo editorial adjustments in accordance with the journal's policies. No part of this work is currently under publication consideration elsewhere, and we accept responsibility for the content presented in this article.

**Data availability & Materials availability & Code availability.** Data supporting the findings of this study are available from the corresponding author on a reasonable request.

**Author contribution.**

- Xinwang Liu: Methodology, data curation.
- Javeria Ayub: Conceptualization, methodology, data curation, writing - original draft.
- Muhammad Aqeel: Formal analysis, investigation, visualization, writing - review & editing.
- Umer Farooq: Review & editing.

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