

CLASSICAL AND BAYESIAN ESTIMATION OF NADARAJAH–HAGHIGHI RELIABILITY MEASURES UNDER ADAPTIVE PROGRESSIVE FIRST-FAILURE CENSORING WITH APPLICATIONS*

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Abstract This paper investigates statistical inference for the Nadarajah–Haghighi distribution under the adaptive progressive first-failure censoring scheme, a flexible life-testing design that integrates adaptive time control with guaranteed failure termination. The censoring mechanism generalizes several classical structures and provides enhanced flexibility in managing experimental duration and resource allocation. Maximum likelihood estimation procedures are developed for the unknown model parameters and the associated reliability measures. Two types of approximate confidence intervals are constructed using the observed Fisher information matrix together with the delta method for reliability characteristics. From a Bayesian perspective, independent gamma prior distributions are assumed for the positive model parameters. Bayes estimators are obtained under the squared error loss function using the Markov chain Monte Carlo simulation implemented through the Metropolis–Hastings algorithm. The simulation-based approach also enables the construction of two types of credible intervals for the parameters and reliability measures. An extensive Monte Carlo study is conducted to assess the finite-sample performance of the proposed classical and Bayesian estimators under various censoring configurations. The proposed methods are further illustrated through the analysis of two real-data applications arising from distinct engineering and biomedical domains. These case studies are deliberately chosen to highlight the flexibility and robustness of the proposed methodology in capturing diverse data characteristics, including skewness, heterogeneity, and complex lifetime behavior. The findings demonstrate that the developed inferential framework provides reliable and efficient estimation for the Nadarajah–Haghighi model under complex adaptive censoring schemes.

Keywords Adaptive progressive first–failure censoring, reliability measures, Nadarajah–Haghighi distribution, likelihood function, Bayesian estimation.

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List of abbreviations

Abbreviation	Definition
ACIs	Approximate Confidence Intervals
ACIs-NA	ACIs based on the Normal Approximation
ACIs-NT	ACIs based on Log-Transformation
AIL	Average Interval Length
APFFC	Adaptive Progressive First-Failure Censoring
BCIs	Bayes Credible Intervals
CDF	Cumulative Distribution Function
CP	Coverage Probability
HPD	Highest Posterior Density
HRF	Hazard Rate Function
KS	Kolmogorov–Smirnov
MRAB	Mean Absolute Bias
MCMC	Markov Chain Monte Carlo
M–H	Metropolis–Hastings
MLEs	Maximum Likelihood Estimators
NH	Nadarajah–Haghighi
P–P	Probability–Probability Plot
PDF	Probability Density Function
PFFC	Progressive First-Failure Censoring
PTIIC	Progressive Type-II Censoring
Q–Q	Quantile–Quantile Plot
RF	Reliability Function
RMSE	Root Mean Squared Error
St.D	Standard Deviation
Std.Ers	Standard Errors
TTT	Total Time on Test

1. Introduction

In life-testing experiments and survival investigations, the idealized assumption that every unit placed on test can be observed until failure is rarely attainable in practice. Real experimental studies are commonly constrained by limited testing horizons, substantial operational expenditures, and logistical or administrative limitations. Consequently, experiments are often terminated before all units fail, leading to incomplete lifetime observations. In such circumstances, censoring mechanisms constitute a fundamental component of reliability analysis, as they provide a rigorous probabilistic framework for incorporating partially observed lifetimes into the inferential procedure. By properly accounting for censored data, one can obtain sta-

tistically valid and efficient estimators without requiring excessively long or prohibitively costly experimental runs.

One of the most widely employed censoring mechanisms in reliability analysis is the progressive Type-II censoring (PTIIC) scheme, which represents a flexible generalization of the classical Type-II censoring plan. Under this design, an initial sample of n units is placed on test, and the experiment proceeds until a preassigned number m of failures is observed. Let $X_{i:m:n}$ denote the i -th ordered observed failure time under the scheme. At the occurrence of the i -th failure, a predetermined number S_i of the remaining surviving units is randomly withdrawn from the experiment according to a pre-specified removal vector $(S_1, S_2, \dots, S_m)$ satisfying the constraint $\sum_{i=1}^m S_i = n - m$. This structured removal mechanism enables the experimenter to effectively regulate the duration of the test, optimize the allocation of available resources, and reduce operational costs, all while preserving inferential information derived from progressively censored order statistics. The conceptual foundation of progressive censoring can be traced back to the pioneering contribution of Cohen [11]. Since then, the theoretical properties, likelihood construction, and inferential procedures under the PTIIC scheme have been systematically developed and extensively investigated in the reliability literature; see, among others, Balakrishnan and Aggarwala [7], Guo and Gui [18], and Brito et al. [10].

Wu and Kuş [41] proposed the progressive first-failure censoring (PFFC) scheme as an organized life-testing design that integrates the fundamental concepts of first-failure censoring and progressive censoring into a unified framework. In this setting, n independent and homogeneous groups, each containing k identical units, are simultaneously placed under observation. Prior to initiating the experiment, the total number of first failures to be recorded, denoted by m , together with the progressive removal vector $\mathbf{S} = (S_1, S_2, \dots, S_m)$, are specified in advance. The experiment evolves by sequentially observing the earliest failure among all groups currently under test. When the first-failure is detected, the entire group in which the failure occurs is withdrawn from the experiment, along with an additional S_1 surviving groups randomly selected in accordance with the predetermined censoring plan. The same procedure is repeated at subsequent stages: At the i -th observed first-failure, the corresponding failed group and S_i other surviving groups are removed from the test for $i = 2, \dots, m$. The life-testing experiment is terminated upon observing the m -th first-failure ($m < n$), at which time the last failed group and the remaining S_m designated groups are withdrawn.

The PFFC scheme has received substantial attention in the reliability literature due to its practical relevance and analytical tractability. This interest has led to an expanding body of research devoted to both classical and Bayesian inference for various lifetime models under this censoring structure. Statistical estimation problems within the PFFC framework have been examined for several commonly used lifetime distributions, including Weibull distribution by Ahmadi et al. [2], generalized inverted exponential distribution by Dube et al. [14], inverse power Lomax distribution by Shi and Shi [38], inverted generalized linear exponential distribution by Mahmoud al. [23], and extended Rayleigh distribution by Abu-Moussa et al. [27], among others.

More recently, Fathi [17] introduced the adaptive PFFC (APFFC) scheme as a refinement of the classical PFFC structure by embedding the adaptive progressive censoring principle originally developed by Ng et al. [33]. The essential motivation behind this modification is to combine the guaranteed-failure feature of failure-based termination with the practical need for flexible time control in modern life-testing experiments. Under the APFFC framework, the total number of first failures to be observed, m , is specified in advance, thereby preserving the fundamental property of failure-censored experiments that ensures the availability of sufficient failure information for inference. However, unlike the conventional PFFC scheme where the

removal vector is rigidly followed throughout the experiment, the adaptive version allows the censoring mechanism to be dynamically adjusted in response to the actual experimental progression. This flexibility provides improved control over test duration and resource allocation without compromising inferential reliability.

The APFFC scheme can be described as follows: Consider a life-testing experiment conducted on n independent and homogeneous groups, each composed of k identical units. Before the experiment begins, the experimenter determines three design components: The target number of first failures m , a preassigned ideal termination time τ , and a progressive removal vector $\mathbf{S} = (S_1, \dots, S_m)$. The test proceeds sequentially by observing the ordered first-failure times $X_{1:m:n}, \dots, X_{m:m:n}$. At the occurrence of the i -th observed first-failure, denoted by $X_{i:m:n}$, the entire failed group is withdrawn from the experiment. Additionally, S_i surviving groups are randomly selected and removed according to the specified removal plan. The experiment terminates according to one of the following two mutually exclusive scenarios:

1. **Case I:** If the m -th observed first-failure time $X_{m:m:n}$ occurs before the predetermined time τ , that is, $X_{m:m:n} < \tau$, the experiment is terminated at $X_{m:m:n}$. In this situation, the APFFC scheme coincides with the classical PFFC scheme with a fixed removal pattern.
2. **Case II:** If the preassigned time threshold τ is reached before observing the m -th first-failure, that is, if $X_{r:m:n} < \tau < X_{r+1:m:n}$, where r denotes the number of first failures recorded prior to time τ with $r < m$, then the censoring mechanism is adaptively modified. In this case, the progressive removal plan is suspended from that stage onward, and no additional surviving groups are withdrawn. The experiment then proceeds without further removals until the m -th first-failure is observed.

This adaptive adjustment ensures that the predetermined number of failures is ultimately obtained for statistical inference, while preventing unnecessary extension of the experiment beyond the planned time horizon. In the adaptive setting where the preassigned time threshold is reached before the occurrence of the m -th first-failure, that is, when $\tau < X_{m:m:n}$, the progressive removal mechanism is suspended from time τ onward. This operationally corresponds to setting $(S_{r+1}, \dots, S_{m-1}) = 0$.

Let $\mathbf{x} = (x_1, \dots, x_m)$, denote an observed APFFC sample, where $x_i = x_{i:m:n}$, $i = 1, \dots, m$, arising from a continuous lifetime distribution with probability density function (PDF) $f(x; \boldsymbol{\varphi})$ and cumulative distribution function (CDF) $F(x; \boldsymbol{\varphi})$, where $\boldsymbol{\varphi}$ is the unknown parameter vector. Under the APFFC design, the corresponding joint likelihood function can be formulated, up to a multiplicative constant independent of $\boldsymbol{\varphi}$, as follows

$$L(\boldsymbol{\varphi}; \mathbf{x}) = \prod_{i=1}^m f(x_i; \boldsymbol{\varphi}) \prod_{i=1}^r [1 - F(x_i; \boldsymbol{\varphi})]^{R_i} \prod_{i=r+1}^{m-1} [1 - F(x_i; \boldsymbol{\varphi})]^{k-1} [1 - F(x_m; \boldsymbol{\varphi})]^{R^*}, \quad (1.1)$$

where $R_i = k(S_i + 1) - 1$ and $R^* = k(n - m + 1 - \sum_{i=1}^r S_i) - 1$.

The survival terms in (1.1) represent the contributions of all censored and withdrawn units associated with the observed ordered first-failure times. For convenience, the likelihood function can be expressed in the following compact form:

$$L(\boldsymbol{\varphi}; \mathbf{x}) = \prod_{i=1}^m f(x_i; \boldsymbol{\varphi}) [1 - F(x_i; \boldsymbol{\varphi})]^{w_i}, \quad (1.2)$$

where the weights w_i characterize the APFFC removal mechanism and are defined as

$$w_i = \begin{cases} R_i, & i = 1, \dots, r, \\ k - 1, & i = r + 1, \dots, m - 1, \\ R^*, & i = m. \end{cases}$$

An appropriate lifetime distribution is essential for reliable statistical inference, particularly under complex censoring mechanisms where both model flexibility and computational tractability are important. Although numerous lifetime models have been proposed in the literature, many classical distributions are unable to adequately capture the diverse hazard rate patterns observed in practical reliability and survival data. Among the available alternatives, the two-parameter Nadarajah–Haghighi (NH) distribution, introduced by Nadarajah and Haghighi [29], has attracted considerable attention due to its flexibility and relatively simple structure. Let X be a non-negative random variable following the NH distribution with parameter vector $\varphi = (a, b)^\top$, where $a > 0$ and $b > 0$ denote the shape and scale parameters, respectively. The corresponding PDF and CDF are given by

$$f(x; \varphi) = ab(1 + bx)^{a-1} e^{1-(1+bx)^a}, \quad x > 0, a, b > 0 \quad (1.3)$$

and

$$F(x; \varphi) = 1 - e^{1-(1+bx)^a}. \quad (1.4)$$

The reliability function (RF), and hazard rate function (HRF) are given, respectively, by

$$R(x; \varphi) = e^{1-(1+bx)^a} \quad (1.5)$$

and

$$H(x; \varphi) = ab(1 + bx)^{a-1}. \quad (1.6)$$

Nadarajah and Haghighi [29] demonstrated that the NH density function is capable of exhibiting both monotone decreasing and unimodal shapes, depending on the values of the shape parameter. Moreover, the corresponding HRF may assume increasing, decreasing, or constant forms, thereby providing flexibility comparable to that of the Weibull, gamma, and exponentiated exponential distributions. Owing to these attractive structural properties, the NH model has received considerable attention in the statistical literature, and various authors have examined inferential procedures for its unknown parameters under different sampling schemes. For example, Sana and Faizan [37], Wu and Gui [40], Dey et al. [13], Elshahhat et al. [16], Almarashi et al. [3], and Abushal [1].

The motivation of the present study stems from the practical need to develop efficient inferential procedures for flexible lifetime models under realistic and highly structured censoring mechanisms. Among the available censoring schemes, the APFFC scheme provides an attractive framework by combining guaranteed-failure termination with adaptive time control, thereby improving experimental efficiency while encompassing several widely used censoring schemes, including PTIIC, PFFC, and adaptive progressive Type-II censoring, as special cases. On the other hand, the NH distribution provides a flexible two-parameter lifetime model capable of accommodating various hazard rate shapes. Despite these attractive features, comprehensive classical and Bayesian inference for the NH distribution under the APFFC scheme has not been adequately investigated. Accordingly, this study develops inferential procedures for the NH model parameters and their associated reliability measures under the APFFC scheme.

The main contributions of this work are summarized as follows:

- Develop maximum likelihood estimation procedures for the model parameters and associated reliability measures under the APFFC scheme.
- Construct two types of approximate confidence intervals (ACIs) based on the normal approximation and logarithmic transformation.
- Establish a Bayesian inferential framework using independent gamma priors and the Metropolis–Hastings (M–H) algorithm to obtain Bayes estimators together with Bayesian credible intervals (BCIs) and highest posterior density (HPD) intervals.
- Evaluate the finite-sample performance of the proposed classical and Bayesian procedures through extensive Monte Carlo simulation studies under various censoring configurations.
- Illustrate the practical applicability of the proposed methodology using two real lifetime datasets from engineering and biomedical applications.

The remainder of this paper is structured as follows. In Section 2, likelihood-based inference for the NH distribution under the APFFC scheme is developed. Section 3 presents the Bayesian inferential framework, where posterior distributions and Bayesian estimators are obtained under suitable prior assumptions. In Section 4, an extensive simulation study is carried out to evaluate the performance of the proposed classical and Bayesian procedures. Section 5 demonstrates the practical usefulness of the developed methodology through the analysis of real lifetime datasets. Finally, Section 6 concludes the paper by summarizing the principal findings and outlining potential avenues for further research.

2. Likelihood estimation

In this section, likelihood-based estimation procedures for the unknown parameter vector φ of the NH distribution, together with the associated reliability measures, are developed under the APFFC scheme. The section is organized into two parts. We first consider point estimation by deriving the MLEs through maximization of the corresponding log-likelihood function, which generally requires numerical optimization methods due to the nonlinear structure of the likelihood equations. We then establish interval estimation procedures by invoking the large-sample properties of the MLEs to construct ACIs for both the model parameters and the reliability characteristics.

2.1. Point estimation

Let $\mathbf{x} = (x_1, \dots, x_m)$ denote an observed sample obtained under the APFFC scheme from the NH distribution, whose PDF and CDF are defined in (1.3) and (1.4), respectively. Based on the APFFC structure, the corresponding joint likelihood function given in (1.2) can be written as follows

$$L(\varphi; \mathbf{x}) = (ab)^m \exp \left[(a-1) \sum_{i=1}^m \log(1 + bx_i) - \sum_{i=1}^m (1 + w_i)(1 + bx_i)^a \right]. \quad (2.1)$$

The log-likelihood function corresponding to (2.1) follows

$$\mathcal{L}(\varphi; \mathbf{x}) = m \log(a) + m \log(b) + (a-1) \sum_{i=1}^m \log(1 + bx_i) - \sum_{i=1}^m (1 + w_i)(1 + bx_i)^a. \quad (2.2)$$

The likelihood equations for the unknown parameters are obtained by differentiating the log-likelihood function $\mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})$ with respect to a and b and equating the resulting first-order partial derivatives to zero. Accordingly, the normal equations corresponding to a and b are given by

$$\frac{\partial \mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})}{\partial a} = \frac{m}{a} + \sum_{i=1}^m \log(1 + bx_i) - \sum_{i=1}^m (1 + w_i)(1 + bx_i)^a \log(1 + bx_i) = 0 \quad (2.3)$$

and

$$\frac{\partial \mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})}{\partial b} = \frac{m}{b} + (a - 1) \sum_{i=1}^m \frac{x_i}{1 + bx_i} - a \sum_{i=1}^m x_i(1 + w_i)(1 + bx_i)^{a-1} = 0. \quad (2.4)$$

The MLEs of a and b , denoted by $\hat{a}$ and $\hat{b}$, are obtained by simultaneously solving the nonlinear likelihood equations in (2.3) and (2.4). It is evident that both score equations are intrinsically coupled through the unknown parameters a and b , where each equation contains nonlinear combinations of the terms, such as $(1 + bx_i)^a$, $(1 + bx_i)^{a-1}$, and $\log(1 + bx_i)$. Consequently, neither parameter can be isolated analytically from the corresponding score equation, and the resulting system does not admit explicit closed-form solutions. Therefore, the MLEs must be obtained through numerical optimization of the log-likelihood function in (2.2). In the present study, the numerical optimization is performed using the Newton–Raphson (NR) algorithm implemented in the `maxLik` package in R (version 4.2.2) proposed by Henningsen and Toomet [20]. The iterative optimization procedure starts from suitable positive initial values and proceeds until convergence is achieved according to a predefined numerical tolerance. Let

$$\boldsymbol{\varphi} = (a, b)^\top,$$

and denote the score vector by

$$\mathbf{U}(\boldsymbol{\varphi}) = \left(\frac{\partial \mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})}{\partial a}, \frac{\partial \mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})}{\partial b} \right)^\top,$$

where $\mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})$ is the log-likelihood function given in (2.2). Furthermore, let

$$\mathbf{H}(\boldsymbol{\varphi}) = \left[\frac{\partial^2 \mathcal{L}(\boldsymbol{\varphi}; \mathbf{x})}{\partial \varphi_i \partial \varphi_j} \right]$$

denote the Hessian matrix.

Starting from a suitable initial value $\boldsymbol{\varphi}^{(0)} = (a^{(0)}, b^{(0)})^\top$, the NR algorithm updates the parameter vector according to

$$\boldsymbol{\varphi}^{(s+1)} = \boldsymbol{\varphi}^{(s)} - \mathbf{H}^{-1}(\boldsymbol{\varphi}^{(s)}) \mathbf{U}(\boldsymbol{\varphi}^{(s)}), \quad s = 0, 1, \dots,$$

until convergence is achieved. The iterative procedure is terminated when the relative change between two successive parameter estimates satisfies

$$\left\| \boldsymbol{\varphi}^{(s+1)} - \boldsymbol{\varphi}^{(s)} \right\| < 10^{-6},$$

or, equivalently, when the absolute difference between two successive log-likelihood values satisfies

$$\left| \mathcal{L}(\boldsymbol{\varphi}^{(s+1)}; \mathbf{x}) - \mathcal{L}(\boldsymbol{\varphi}^{(s)}; \mathbf{x}) \right| < 10^{-6}.$$

The converged solution, $\hat{\varphi} = (\hat{a}, \hat{b})^\top$, is then taken as the MLE. In the proceeding numerical computations, the likelihood optimization is performed using the NR algorithm implemented in the `maxLik` package in R (version 4.2.2) proposed by Henningsen and Toomet [20].

Moreover, by virtue of the invariance property of MLE method, the MLEs of the RF $R(t)$ and the HRF $H(t)$ at mission time t , are obtained by substituting $\hat{a}$ and $\hat{b}$ into their respective functional forms in (1.5) and (1.6), respectively. Accordingly, these estimates are given by

$$\hat{R}(t) = e^{1-(1+\hat{b}t)^{\hat{a}}}$$

and

$$\hat{H}(t) = \hat{a}\hat{b}(1 + \hat{b}t)^{\hat{a}-1}.$$

2.2. Interval estimation

The ACIs for the unknown parameters a and b as well as the reliability measures can be constructed by using the large-sample properties of the MLEs. Under standard regularity conditions, the MLE $\hat{\varphi} = (\hat{a}, \hat{b})^\top$ is asymptotically normally distributed; see Meeker et al. [25] for more details. Specifically, $\hat{\varphi} \xrightarrow{d} \mathcal{N}(\varphi, \mathbf{I}^{-1}(\varphi))$, where $\mathbf{I}(\varphi)$ denotes the Fisher information matrix associated with the NH model under the APFFC scheme. In regular situations, the asymptotic covariance matrix of $\hat{\varphi}$ is obtained from the inverse of the Fisher information matrix, which is defined through the second-order partial derivatives of the log-likelihood function. However, under the APFFC design, the log-likelihood function possesses a complicated structure due to the adaptive removal mechanism, rendering the analytical evaluation of the expected Fisher information matrix analytically cumbersome.

Accordingly, the asymptotic variance-covariance matrix of the MLE is approximated using the observed Fisher information matrix, obtained from the negative Hessian of the log-likelihood function in (2.2) evaluated at $\hat{\varphi}$. Thus, the estimated variance-covariance can be written as

$$\mathbf{I}^{-1}(\hat{\varphi}) = \left(\begin{array}{cc} -\frac{\mathcal{L}(\varphi; \mathbf{x})}{\partial a^2} & -\frac{\mathcal{L}(\varphi; \mathbf{x})}{\partial a \partial b} \\ -\frac{\mathcal{L}(\varphi; \mathbf{x})}{\partial b \partial a} & -\frac{\mathcal{L}(\varphi; \mathbf{x})}{\partial b^2} \end{array} \right)_{\varphi=\hat{\varphi}}^{-1}, \quad (2.5)$$

where

$$\begin{aligned} \frac{\partial^2 \mathcal{L}(\varphi; \mathbf{x})}{\partial a^2} &= -\frac{m}{a} - \sum_{i=1}^m (1 + w_i)(1 + bx_i)^a \log^2(1 + bx_i), \\ \frac{\partial^2 \mathcal{L}(\varphi; \mathbf{x})}{\partial b^2} &= -\frac{m}{b^2} - (a-1) \sum_{i=1}^m \frac{x_i^2}{(1 + bx_i)^2} - a(a-1) \sum_{i=1}^m x_i^2 (1 + w_i)(1 + bx_i)^{a-2} \end{aligned}$$

and

$$\frac{\partial^2 \mathcal{L}(\varphi; \mathbf{x})}{\partial a \partial b} = \sum_{i=1}^m \frac{x_i}{1 + bx_i} - \sum_{i=1}^m (1 + w_i)(1 + bx_i)^{a-1} [1 + a \log(1 + bx_i)].$$

Accordingly, for a prescribed significance level α ($0 < \alpha < 1$), the two-sided ACIs based on the normal approximation of the MLEs (denoted by ACIs-NA) for the parameters a and b with confidence level $100(1 - \alpha)\%$ can be constructed, respectively, as

$$\hat{a} \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{a})}, \quad \hat{b} \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{b})},$$

where $\widehat{\text{Var}}(\hat{a})$ and $\widehat{\text{Var}}(\hat{b})$ denote the diagonal elements of the estimated variance-covariance matrix $\mathbf{I}^{-1}(\hat{\boldsymbol{\varphi}})$ provided by (2.5), and $z_{\alpha/2}$ represents the upper $\alpha/2$ quantile of the standard normal distribution.

An additional component of interval estimation concerns the construction of confidence bounds for the reliability measures, such as $R(t)$ and $H(t)$. Since these quantities are nonlinear functions of the parameter vector $\boldsymbol{\varphi}$, deriving exact expressions for their sampling variances is analytically intractable. To overcome this difficulty, we employ the delta method, which provides an asymptotic approximation to the variance of a smooth function of the MLEs. In particular, the reliability measures are treated as differentiable functions of (a, b) , and their variances are approximated through a first-order Taylor expansion evaluated at $(\hat{a}, \hat{b})$, combined with the estimated variance-covariance matrix of the parameters, see for more details Raykov and Marcoulides [35], and Beutner [9]. Hence, the asymptotic variances of $\hat{R}(t)$ and $\hat{H}(t)$ can be approximated, respectively, by

$$\widehat{\text{Var}}(\hat{R}(t)) \approx \nabla_R \mathbf{I}^{-1}(\hat{\boldsymbol{\varphi}}) \nabla_R^\top \Big|_{\boldsymbol{\varphi}=\hat{\boldsymbol{\varphi}}},$$

and

$$\widehat{\text{Var}}(\hat{H}(t)) \approx \nabla_H \mathbf{I}^{-1}(\hat{\boldsymbol{\varphi}}) \nabla_H^\top \Big|_{\boldsymbol{\varphi}=\hat{\boldsymbol{\varphi}}},$$

where $\nabla_R = \left(\frac{\partial R(t; \boldsymbol{\varphi})}{\partial a}, \frac{\partial R(t; \boldsymbol{\varphi})}{\partial b} \right)$ and $\nabla_H = \left(\frac{\partial H(t; \boldsymbol{\varphi})}{\partial a}, \frac{\partial H(t; \boldsymbol{\varphi})}{\partial b} \right)$, where

$$\begin{aligned} \frac{\partial R(t; \boldsymbol{\varphi})}{\partial a} &= R(t; \boldsymbol{\varphi})(1 + bt)^a \log(1 + bt), \\ \frac{\partial R(t; \boldsymbol{\varphi})}{\partial b} &= atR(t; \boldsymbol{\varphi})(1 + bt)^{a-1}, \\ \frac{\partial H(t; \boldsymbol{\varphi})}{\partial a} &= b(1 + bt)^{a-1} [1 + a \log(1 + bt)] \end{aligned}$$

and

$$\frac{\partial H(t; \boldsymbol{\varphi})}{\partial b} = a(1 + bt)^{a-1} \left[1 + \frac{bt(a-1)}{1 + bt} \right].$$

Consequently, the ACIs-NA for $R(t)$ and $h(t)$, with confidence level $100(1 - \alpha)\%$, can be constructed using the normal approximation as

$$\hat{R}(t) \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{R}(t))} \quad \text{and} \quad \hat{h}(t) \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{h}(t))}.$$

It should be emphasized that confidence intervals constructed using this normal approximation are based on asymptotic theory. Therefore, when the effective sample size is small or the censoring level is heavy, the finite-sample distribution of the estimators may deviate from normality, potentially affecting coverage accuracy.

The ACIs-NA for the model parameters and associated reliability measures may exhibit certain practical limitations. In particular, these intervals may produce negative lower bounds for inherently positive parameters or generate upper bounds exceeding one when applied to the RF. Such outcomes violate the natural parameter constraints and may reduce interpretability. To address these issues, we construct alternative ACIs based on the normal approximation of the logarithmically transformed estimators (denoted by ACIs-NT). The logarithmic transformation

stabilizes variability and ensures that the resulting interval bounds respect the inherent positivity constraints of the parameters and reliability measures. For the unknown parameter a , the $100(1 - \alpha)\%$ ACI-NT can be expressed as

$$\hat{a} \exp \left(\pm z_{\alpha/2} \frac{\sqrt{\widehat{\text{Var}}(\hat{a})}}{\hat{a}} \right).$$

The $100(1 - \alpha)\%$ ACIs-NT for the parameter b , as well as for the RF and HRF, can be constructed analogously.

3. Bayesian estimation

Bayesian inference has become an important methodological tool in the analysis of lifetime and reliability data, offering a probabilistic framework that complements classical likelihood-based approaches. In reliability studies, Bayesian methods are particularly appealing because they allow the incorporation of prior information about the model parameters together with the information contained in the observed sample, into a single unified posterior distribution. This integration is especially valuable in life-testing experiments conducted under complex censoring mechanisms, where the effective sample size may be limited. Within this framework, statistical inference is carried out through the posterior distribution of the parameter vector, which reflects both prior beliefs and the likelihood contribution under the APFFC scheme. In this section, we develop Bayesian estimation procedures for the NH distribution based on APFFC data. The Bayes estimates of the model parameters and reliability measures are obtained using the simulation-based methods implemented via Markov chain Monte Carlo (MCMC) techniques, which allow for full posterior exploration and interval estimation.

3.1. Prior and posterior distributions

By inspecting the likelihood function given in (2.1), it becomes evident that no natural conjugate prior distributions exist for the two unknown parameters a and b of the NH model under the APFFC scheme. Consequently, appropriate prior distributions must be selected in a manner that respects the support of the parameters ($a > 0, b > 0$), offers sufficient flexibility to represent various prior beliefs, and at the same time does not introduce unnecessary analytical or computational complexity into the posterior structure. In light of these considerations, we adopt gamma prior distributions for both parameters. The gamma family is supported on the positive real line and is sufficiently flexible to accommodate both informative and weakly informative prior specifications through suitable choices of its hyper-parameters. Moreover, it leads to a well-defined posterior density that can be efficiently handled using numerical optimization or simulation-based techniques.

Assuming that a and b are independent and each follows a gamma distribution, the joint prior density, up to a proportionality constant, can be written as

$$\pi(\varphi) \propto a^{\eta_1-1} b^{\eta_2-1} e^{-\kappa_1 a - \kappa_2 b}, a > 0, b > 0, \quad (3.1)$$

where $\eta_1, \kappa_1 > 0$ and $\eta_2, \kappa_2 > 0$ are the corresponding hyper-parameters. The gamma distribution has been widely employed in the literature as a prior specification for the model parameters

of the NH distribution by several researchers; see, for example, Newer et al. [32], and Azimi and Esmailian [6].

By combining the likelihood function given in (2.1) with the joint prior distribution specified in (3.1), the resulting joint posterior density of φ can be expressed as follows:

$$\mathcal{P}(\varphi|\mathbf{x}) = Aa^{m+\eta_1-1}b^{m+\eta_2-1} \exp \left[(a-1) \sum_{i=1}^m \log(1+bx_i) - \sum_{i=1}^m (1+w_i)(1+bx_i)^a - \kappa_1 a - \kappa_2 b \right], \quad (3.2)$$

where

$$A = \int_0^\infty \int_0^\infty \pi(\varphi) L(\varphi; \mathbf{x}) da db.$$

It is clear from the expression of the joint posterior density that it possesses a complicated analytical form, which makes the derivation of closed-form marginal posterior distributions for a and b intractable. Consequently, direct analytical evaluation of posterior summaries is not feasible. Although approximation techniques, such as Lindley's method, may be used to obtain approximate Bayes estimators when closed-form solutions are unavailable, their implementation requires the evaluation of complicated third-order derivatives of the log-posterior distribution under the APFFC scheme. Furthermore, these methods are primarily intended for point estimation and do not directly provide Bayesian credible intervals. For these reasons, MCMC methods are adopted in this study, as they offer a flexible framework for obtaining both Bayes estimators and posterior interval estimates.

3.2. MCMC-based Bayesian inference

To derive Bayes estimators from the posterior distribution given in (3.2), it is essential to adopt a suitable loss function that formalizes the cost associated with estimation errors. The choice of loss function determines the form of the resulting Bayes estimator. In the present study, we employ the symmetric squared error loss function, under which the Bayes estimator coincides with the posterior mean of the parameter of interest. Accordingly, for any parametric function of the parameter vector, say $\psi(\varphi)$, the corresponding Bayes estimator is obtained by taking its posterior expectation with respect to the joint posterior distribution in (3.2), that is

$$\tilde{\psi}(\varphi) = \int_0^\infty \int_0^\infty \psi(\varphi) \mathcal{P}(\varphi|\mathbf{x}) da db. \quad (3.3)$$

It follows from (3.3) that the Bayes estimator of $\psi(\varphi)$ under the squared error loss function does not admit a closed-form expression due to the analytical complexity of the joint posterior density. To circumvent this challenge, we resort to simulation-based inference using the MCMC techniques. The MCMC approach allows direct sampling from the posterior distribution specified in (3.2), thereby enabling the simultaneous computation of Bayes point estimates, posterior summaries, and constructing BCIs and HPD intervals in a unified framework. For the implementation of the MCMC algorithm, we first derive the full conditional posterior distributions of the parameters a and b , respectively, which form the basis for the subsequent sampling scheme. From (3.2), the full conditional distributions of a and b are given, respectively, by

$$\mathcal{P}_1(a|b, \mathbf{x}) \propto a^{m+\eta_1-1} \exp \left\{ a \left[\sum_{i=1}^m \log(1+bx_i) - \kappa_1 \right] - \sum_{i=1}^m (1+w_i)(1+bx_i)^a \right\} \quad (3.4)$$

and

$$\mathcal{P}_2(b|a, \mathbf{x}) \propto b^{m+\eta_2-1} \exp \left[(a-1) \sum_{i=1}^m \log(1+bx_i) - \sum_{i=1}^m (1+w_i)(1+bx_i)^a - \kappa_2 b \right]. \quad (3.5)$$

It is evident that the full conditional posterior distributions of a and b in (3.4) and (3.5) do not belong to any standard parametric family from which direct sampling can be performed using conventional techniques. As a result, generating independent samples from these conditional distributions in closed form is not feasible. To address this difficulty, we employ the M–H algorithm within the MCMC framework. In implementing the M–H procedure, normal proposal distributions are adopted to generate candidate values for a and b . Since all model parameters are strictly positive, the posterior distribution is supported only on the positive parameter space. Although normal random-walk proposal distributions are adopted to generate candidate values, any proposed value outside the admissible domain has zero posterior density and is therefore automatically rejected through the M–H acceptance step. Consequently, all retained posterior samples satisfy the required positivity constraints. This simulation-based strategy enables us to approximate the posterior distribution and, consequently, to compute the Bayes estimates as well as to construct the HPD credible intervals for the model parameters and the associated reliability measures.

To generate posterior samples from the full conditional distributions of a and b , the following steps of the M–H algorithm are implemented:

1. Initialize the Markov chain by selecting starting values $a^{(0)}$ and $b^{(0)}$. Here, we take the MLEs as starting values.
2. Set the iteration index $l = 1$.
3. Generate a new value of $a^{(l)}$ using the M–H step:
 - (a) Propose a candidate value a^* from a normal proposal distribution centered at the previous iterate $a^* \sim \mathcal{N}\left(a^{(l-1)}, \widehat{\text{Var}}(\hat{a})\right)$, where $\widehat{\text{Var}}(\hat{a})$ denotes the estimated variance of the MLE $\hat{a}$, obtained from the estimated variance–covariance matrix.
 - (b) Compute the acceptance probability

$$AP_a = \min \left\{ 1, \frac{\mathcal{P}_1(a^*|b^{(l-1)}, \mathbf{x})}{\mathcal{P}_1(a^{(l-1)}|b^{(l-1)}, \mathbf{x})} \right\}.$$

- (c) Set

$$a^{(l)} = \begin{cases} a^*, & \text{with probability } AP_a, \\ a^{(l-1)}, & \text{otherwise.} \end{cases}$$

4. Generate $b^{(l)}$ similarly using another M–H step:
 - (a) Propose a candidate value b^* from $b^* \sim \mathcal{N}\left(b^{(l-1)}, \widehat{\text{Var}}(\hat{b})\right)$.
 - (b) Compute the acceptance probability

$$AP_b = \min \left\{ 1, \frac{\mathcal{P}_2(b^*|a^{(l-1)}, \mathbf{x})}{\mathcal{P}_2(b^{(l-1)}|a^{(l-1)}, \mathbf{x})} \right\}.$$

(c) Set

$$b^{(l)} = \begin{cases} b^*, & \text{with probability } AP_b, \\ b^{(l-1)}, & \text{otherwise.} \end{cases}$$

5. At iteration l , compute the RF and HRF using the updated parameter values:

$$R^{(l)}(t) = e^{1-(1+b^{(l)}t)a^{(l)}} \quad \text{and} \quad H^{(l)}(t) = a^{(l)}b^{(l)}(1+b^{(l)}t)^{a^{(l)}-1}.$$

6. Set $l = l + 1$ and repeat Steps 3–6 until a total of $\mathcal{M}$ iterations is reached.

7. After discarding an appropriate burn-in period, retain the generated posterior samples

$$\left\{ a^{(l)}, b^{(l)}, R^{(l)}(t), H^{(l)}(t) \right\}, \quad l = 1, \dots, \mathcal{M},$$

which are then used to compute Bayes point estimates, BCIs and HPD intervals for the model parameters and the associated reliability measures.

For any model parameter, say a , the Bayes estimate under the squared error loss function can be readily obtained from the generated MCMC samples. Let $\{a^{(l)} : l = 1, \dots, \mathcal{M}\}$ denote the retained posterior draws of a after discarding the burn-in period. Then, the Bayes estimate of a is given by the posterior mean as

$$\tilde{a} = \frac{1}{\mathcal{M}} \sum_{l=1}^{\mathcal{M}} a^{(l)}.$$

Two types of posterior interval estimates are considered in this study. The BCIs are constructed using the equal-tailed posterior quantiles, whereas the HPD intervals are obtained as the shortest intervals containing the specified posterior probability. While both approaches provide valid Bayesian interval estimates, the HPD intervals are generally more informative for asymmetric posterior distributions. Therefore, reporting both interval types offers a more comprehensive assessment of posterior uncertainty. To construct the $100(1 - \alpha)\%$ BCI and HPD credible interval for a , the retained samples are first sorted in ascending order,

$$a^{[1]} \leq a^{[2]} \leq \dots \leq a^{[\mathcal{M}]}$$

For a given credibility level $100(1 - \alpha)\%$, the BCI of the unknown parameter a , can be obtained as follows

$$\left\{ a^{[\alpha\mathcal{M}/2]}, a^{[(1-\alpha/2)\mathcal{M}]} \right\}.$$

To compute the HPD of the unknown parameter a , define the integer $\varrho = \lfloor \alpha\mathcal{M} \rfloor$. All candidate intervals of the form

$$\left(a^{[i]}, a^{[i+\mathcal{M}-\varrho]} \right), \quad i = 1, 2, \dots, \varrho,$$

are then constructed, and the interval with the minimum length is selected. The resulting shortest interval constitutes the $100(1 - \alpha)\%$ HPD credible interval for a . The Bayes estimates, BCIs and HPD credible intervals for the remaining parameter b , as well as for the reliability measures $R(t)$ and $H(t)$, can be obtained in exactly the same manner using their corresponding posterior MCMC samples.

4. Monte Carlo comparisons

This section presents an extensive evaluation of the proposed theoretical results, focusing primarily on both point estimation and interval estimation of the parameters of the $NH(a, b)$ distribution. Moreover, the corresponding reliability measures, including the reliability function $R(t)$ and the hazard rate function $H(t)$, are thoroughly investigated. To assess the finite-sample behavior of the proposed estimators, a Monte Carlo simulation study is conducted based on random samples generated from the $NH(0.5, 1.5)$ model. It is worth noting here that the NH parameter values were selected as representative values that produce a practically relevant life-time model and were used consistently in all simulation experiments. The primary objective of the simulation study is to evaluate the finite-sample performance of the proposed classical and Bayesian estimators under different APFFC sampling configurations. Consequently, only the experimental design factors, including the sample size n , group size k , termination time T , and censoring schemes $\mathbf{S}$, were varied across the simulation scenarios, whereas the underlying model parameters remained fixed. The APFFC scheme is applied over 1,000 independent simulation runs. For the fixed time point $t = 0.1$, the true reliability and hazard rate values are approximately $R(t) = 0.9302$ and $H(t) = 0.6994$, respectively, and these quantities serve as reference values for evaluating estimator accuracy. The estimation performance for a , b , $R(t)$, and $H(t)$ is examined across a range of experimental settings by varying the threshold time $T \in \{1, 2\}$, group size $k \in \{2, 5\}$, and total sample size $n \in \{20, 50, 80\}$, together with several censoring patterns characterized by the removal vector $\mathbf{S}$. A detailed summary of all simulation scenarios and design configurations is reported in Table 1. To further specify, the proposed progressive censoring schemes $\mathcal{S}[1]$, $\mathcal{S}[2]$ and $\mathcal{S}[3]$ function as left, center and right censoring charts respectively.

Table 1. Various PTIIC designs in Monte Carlo simulations.

Test	n	m	$\mathbf{S}$	Test	n	m	$\mathbf{S}$
$\mathcal{S}[1]$	20	10	$(5^2, 0^8)$	$\mathcal{S}[1]$	20	15	$(5^1, 0^{14})$
$\mathcal{S}[2]$			$(0^4, 5^2, 0^4)$	$\mathcal{S}[2]$			$(0^7, 5^1, 0^7)$
$\mathcal{S}[3]$			$(0^8, 5^2)$	$\mathcal{S}[3]$			$(0^{14}, 5^1)$
$\mathcal{S}[1]$	50	20	$(5^6, 0^{14})$	$\mathcal{S}[1]$	50	40	$(5^2, 0^{38})$
$\mathcal{S}[2]$			$(0^7, 5^6, 0^7)$	$\mathcal{S}[2]$			$(0^{19}, 5^2, 0^{19})$
$\mathcal{S}[3]$			$(0^{14}, 5^6)$	$\mathcal{S}[3]$			$(0^{38}, 5^2)$
$\mathcal{S}[1]$	80	30	$(5^{10}, 0^{20})$	$\mathcal{S}[1]$	80	60	$(5^4, 0^{56})$
$\mathcal{S}[2]$			$(0^{10}, 5^{10}, 0^{10})$	$\mathcal{S}[2]$			$(0^{28}, 5^4, 0^{28})$
$\mathcal{S}[3]$			$(0^{20}, 5^{10})$	$\mathcal{S}[3]$			$(0^{56}, 5^4)$

For each simulation scenario, 1,000 APFFC samples were generated, and all numerical analyses were carried out in the R statistical computing environment (version 4.2.2) by R Core Team [39] using two widely adopted packages: (i) `maxLik` (version 1.5-2.2), developed by Henningsen and Toomet [20], for maximum likelihood estimation; and (ii) `coda` (version 0.19-4.1), introduced by Plummer et al. [34], for Bayesian posterior analysis. Specifically, the `maxLik` package was employed to maximize the log-likelihood function via the NR algorithm, yielding

the MLEs of a and b . Subsequently, the MLEs of the RF $R(t)$ and the HRF $H(t)$ were obtained using the invariance property of $\hat{a}$ and $\hat{b}$. The observed Fisher information matrix was then used to compute the standard errors and construct the corresponding ACIs. Posterior inference was conducted using the `coda` package, which provides a comprehensive framework for analyzing MCMC output.

Moreover, the maximum likelihood estimation via the NR algorithm was initialized at $(0.1, 0.1)$, which lies within the admissible parameter space. To assess the robustness of the optimization procedure, several alternative positive starting values, including $(0.1, 0.1)$, $(0.5, 0.5)$, $(1, 1)$, and $(2, 2)$, were also considered. In all cases, the algorithm converged to the same optimum of the log-likelihood function, demonstrating the numerical stability of the estimation procedure. In contrast, the Bayesian MCMC algorithm was initialized at the corresponding true parameter values, namely $(a, b) = (0.5, 1.5)$, to evaluate the performance of the proposed Bayesian estimators under controlled simulation settings. Additional numerical experiments using different positive initial values produced virtually identical posterior summaries, indicating that the MCMC algorithm is insensitive to reasonable choices of initial values.

Bayesian inference is conducted via an MCMC procedure comprising a total of $\mathcal{M} = 12,000$ iterations, of which the first $\mathcal{B} = 2,000$ iterations are discarded to mitigate the effect of initial transients. The resulting posterior samples are then used to compute approximate Bayesian point estimates, along with the corresponding 95% Bayesian credible and highest posterior density (BCI/HPD) intervals for the parameters a and b , as well as for the reliability function $R(t)$ and hazard rate function $H(t)$. The frequentist estimates of a and b , denoted by $\hat{a}$ and $\hat{b}$, are obtained by numerically solving the associated score equations using the Newton–Raphson algorithm as implemented in the `maxLik` package. The iterative optimization routine is deemed to have converged when the absolute difference between successive log-likelihood values falls below the tolerance level of 10^{-6} . All simulations and inferential procedures are executed entirely within the R environment.

For the Bayesian framework, two distinct and mutually independent prior configurations are specified for the hyperparameters (η_i, κ_i) , $i = 1, 2$, corresponding to the NH model parameters a and b . By considering multiple prior settings, the sensitivity of the posterior inference to prior information can be evaluated, allowing an assessment of the stability and robustness of the proposed Bayesian methodology. These prior settings are selected to represent different levels of prior information and are defined as follows:

- Prior-1: $(\eta_1, \eta_2) = (2.5, 7.5)$ and $\kappa_i = 5$, $i = 1, 2$.
- Prior-2: $(\eta_1, \eta_2) = (5, 15)$ and $\kappa_i = 10$, $i = 1, 2$.

It is worth noticing here that the hyperparameter values corresponding to (η_i, κ_i) (for $i = 1, 2$) were selected to facilitate a sensitivity analysis of the proposed Bayesian procedure under different degrees of prior information. Specifically, the gamma priors were chosen such that both prior specifications possess identical prior means while differing in their prior variances. Since a gamma distribution with shape parameter η and rate parameter κ has mean η/κ and variance η/κ^2 , Prior-1 yields

$$E(a) = 0.5, \quad \text{Var}(a) = 0.10, \quad E(b) = 1.5, \quad \text{Var}(b) = 0.30,$$

whereas Prior-2 gives

$$E(a) = 0.5, \quad \text{Var}(a) = 0.05, \quad E(b) = 1.5, \quad \text{Var}(b) = 0.15.$$

Hence, both prior specifications are centered at the same parameter values, while Prior-2 is more concentrated than Prior-1 and therefore reflects stronger prior information. Subsequently, these prior settings enable us to assess the sensitivity and robustness of the Bayesian inference with respect to the precision of the prior distributions.

It is important to emphasize that the quantities (η_i, κ_i) (for $i = 1, 2$) are fixed hyperparameters specifying the gamma prior distributions and are not unknown model parameters subject to numerical estimation. Consequently, no initial values are required for the hyperparameters in the proposed inferential procedures. The numerical optimization is performed only with respect to the NH model parameters $\phi = (a, b)^\top$, for which suitable positive starting values are required when solving the nonlinear likelihood equations.

To assess the convergence and mixing behavior of the proposed MCMC algorithm, four complementary diagnostic tools are employed, namely the autocorrelation function (ACF), trace plot, and ergodic mean plot convergence diagnostic. Figure 1 presents these diagnostic plots for the model parameters a and b , together with the RF $R(t)$ and HRF $H(t)$. The ACF plots exhibit a rapid decay toward zero as the lag increases, indicating weak serial dependence among successive posterior samples and demonstrating good mixing of the Markov chains. The trace plots fluctuate randomly around stable posterior means without exhibiting noticeable trends or systematic drifts, suggesting that the generated chains have reached their stationary distributions and adequately explored the posterior space. The ergodic mean plots stabilize quickly after the initial iterations and remain nearly constant thereafter, confirming the numerical stability of the posterior estimates and providing additional evidence of convergence.

In addition to the trace plots, autocorrelation functions, and ergodic mean plots, the Brooks–Gelman–Rubin (BGR) diagnostic was employed as a generalized form of the classical Gelman–Rubin convergence diagnostic; see Figure 2. This diagnostic aims to assess the convergence and mixing of the MCMC algorithm, though two parallel Markov chains with dispersed initial values were generated. The BGR procedure evaluates the potential scale reduction factor by comparing the within-chain and between-chain variability and, in the present implementation, monitors both the posterior median and the 97.5% posterior quantile. A potential scale reduction factor approaching one indicates that the parallel chains have reached a common stationary distribution and that their between-chain and within-chain variability is in close agreement. The convergence assessment was conducted for all quantities monitored in the Bayesian analysis, namely the NH model parameters a and b and the associated reliability measures $R(t)$ and $H(t)$. The BGR resulting diagnostics consistently indicated satisfactory convergence for all monitored quantities. The MCMC algorithm was run for a total of $\mathcal{M} = 12,000$ iterations, with the first $\mathcal{B} = 2,000$ iterations discarded as burn-in. Therefore, 10,000 posterior iterations were retained for subsequent Bayesian estimation and interval construction. The convergence plots shown in Figure 2 are presented using these post-burn-in iterations so that the horizontal axis ranges over the 10,000 retained iterations. After the burn-in period, the BGR potential scale reduction factors for the median and 97.5% posterior quantile rapidly approached one and remained close to one thereafter for a , b , $R(t)$, and $H(t)$. This behavior, together with the corresponding trace, autocorrelation, and ergodic mean diagnostics, provides evidence that the MCMC chains have satisfactorily converged and that the resulting posterior summaries are numerically stable.

The estimation findings of the model parameters a , b , $R(t)$, and $H(t)$, collectively denoted by ζ_j , $j = 1, 2, 3, 4$, are evaluated using several standard performance criteria, defined as follows.

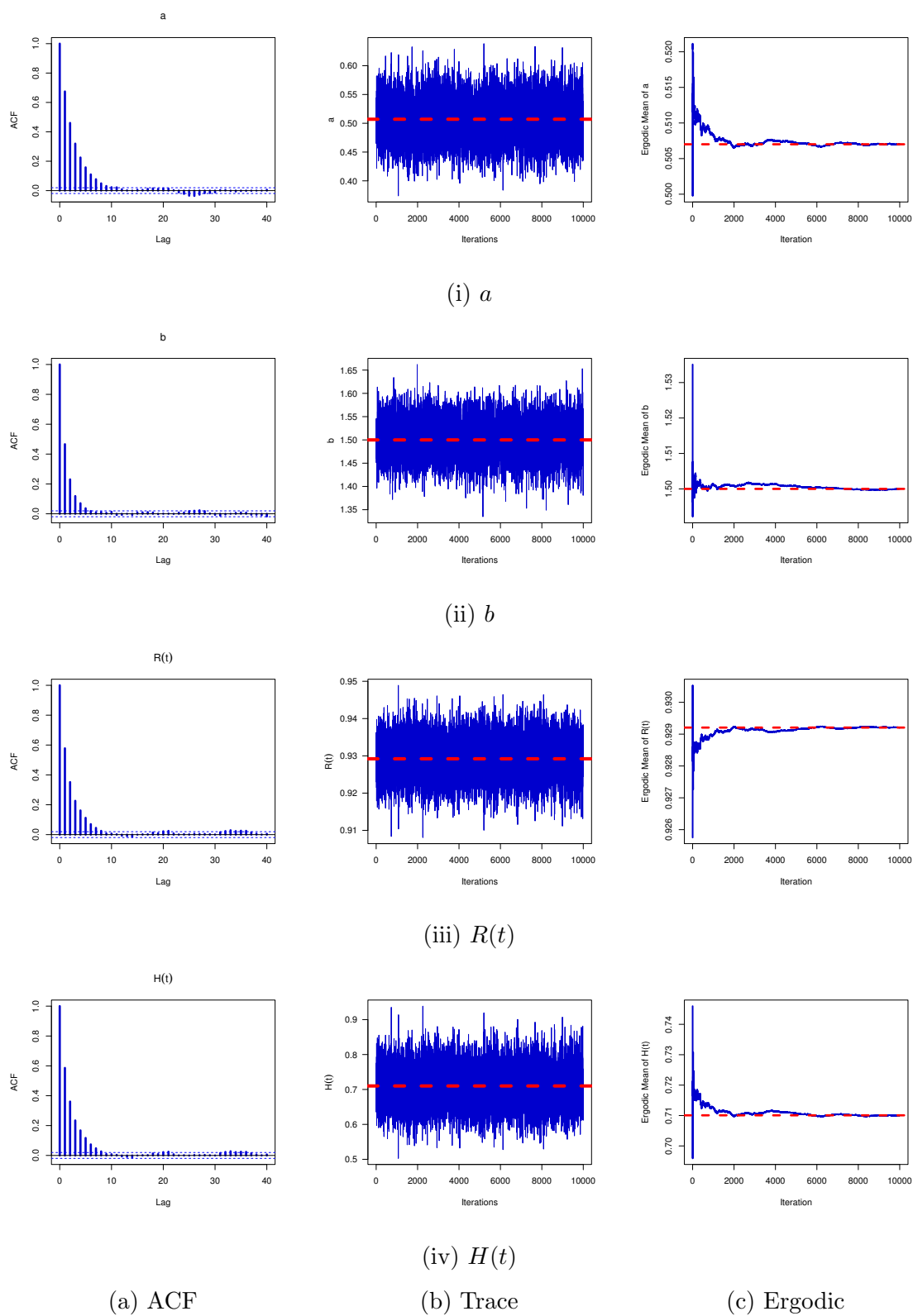


Figure 1. Convergence diagnostics for the MCMC outputs of a , b , $R(t)$, and $H(t)$.

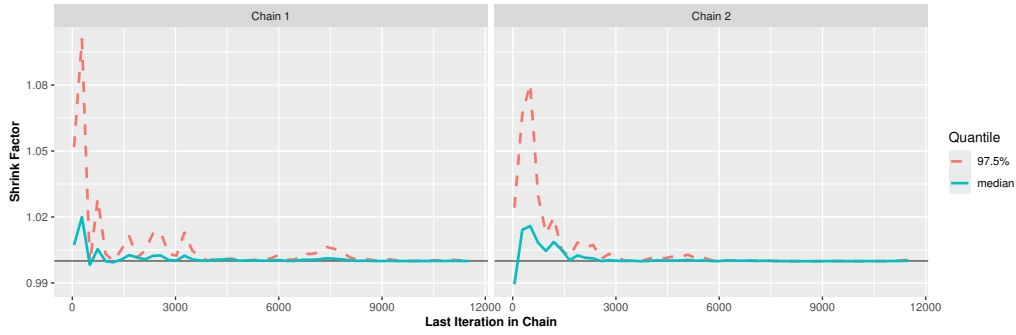


Figure 2. The BGR diagnostic of MCMC outputs.

- Root Mean Squared Error (RMSE):

$$\text{RMSE}(\ddot{\zeta}_j) \left[\frac{1}{1000} \sum_{i=1}^{1000} (\ddot{\zeta}_j^{[i]} - \zeta_j)^2 \right]^{1/2},$$

which measures the overall estimation accuracy by combining bias and variability.

- Mean Relative Absolute Bias (MRAB):

$$\text{MRAB}(\ddot{\zeta}_j) = \frac{1}{1000} \sum_{i=1}^{1000} \frac{|\ddot{\zeta}_j^{[i]} - \zeta_j|}{\zeta_j},$$

quantifying the average relative deviation of the estimator from its true value.

- Average Interval Length (AIL):

$$\text{AIL}_{95\%}(\zeta_j) \frac{1}{1000} \sum_{i=1}^{1000} \left(\mathcal{U}_{\zeta_j^{[i]}} \mathcal{L}_{\zeta_j^{[i]}} \right),$$

where $\mathcal{L}_{\zeta_j^{[i]}}$ and $\mathcal{U}_{\zeta_j^{[i]}}$ denote the lower and upper bounds of the corresponding 95% confidence (or credible) interval.

- Coverage Probability (CP):

$$\text{CP}_{95\%}(\zeta_j) \frac{1}{1000} \sum_{i=1}^{1000} \mathbb{I} \left(\mathcal{L}_{\zeta_j^{[i]}} \leq \zeta_j \leq \mathcal{U}_{\zeta_j^{[i]}} \right),$$

where $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (a, b, R(t), H(t))$, and $\mathbb{I}(\cdot)$ denotes the indicator function, which equals one if the true parameter value is contained within the interval and zero otherwise.

An examination of Tables 2–9 focuses on estimation procedures that simultaneously attain smaller RMSE, MRAB, and AIL values, while CPs are close to, or exceeding, their nominal levels. Based on these numerical results, the principal findings of the simulation study may be summarized as follows:

- Both point and interval estimators for the model parameters a and b , together with the associated reliability quantities $R(t)$ and $H(t)$, exhibit stable and satisfactory performance across all considered experimental settings.
- Increasing the total sample size n leads to a clear enhancement in estimation accuracy for all parameters. This improvement is reflected through uniformly decreasing RMSE, MRAB, and AIL measures, thereby demonstrating higher statistical efficiency and supporting the consistency of the proposed estimators.
- The effect of increasing the group size k reveals several systematic behaviors:
 - The RMSE and MRAB values for a , b , $R(t)$, and $H(t)$ generally decrease, indicating improved estimation precision.
 - The AILs for 95% ACI-NA/ACI-NT estimates of a and b tend to increase, while those for $R(t)$ and $H(t)$ decrease.
 - The AILs for 95% BCI/HPD estimates of a , b , $R(t)$, and $H(t)$ go up.
 - The CPs associated with 95% ACI-NA/ACI-NT estimates of a and b display a marginal decrease, while those corresponding to $R(t)$ and $H(t)$ show a noticeable improvement.
 - The CPs associated with 95% BCI/HPD estimates of a , b , $R(t)$, and $H(t)$ decreased.
- A comparative analysis of the censoring schemes $\mathcal{S}[i]$, $i = 1, 2, 3$, defined in Table 1, indicates that the estimators of a and $H(t)$ perform most favorably under $\mathcal{S}[3]$ (right censoring), the estimators of b achieve superior accuracy under $\mathcal{S}[2]$ (middle censoring), and the estimators of $R(t)$ attain their best performance under $\mathcal{S}[1]$ (left censoring).
- Bayesian estimators constructed under both informative prior settings consistently outperform their frequentist counterparts. This advantage is evident through smaller RMSE, MRAB, and AIL values, as well as improved CPs, underscoring the effectiveness of incorporating prior information.
- Due to the use of informative priors, both BCI and HPD credible intervals systematically yield shorter AILs and higher CPs when compared with the corresponding ACI-NA/ACI-NT estimates.
- As the termination time T of the APFFC scheme increases, substantial reductions in RMSE, MRAB, and AIL are observed for a , b , $R(t)$, and $H(t)$, accompanied by higher CP values. This improvement is attributable to the extended observation window, which captures additional failure information and enhances inference on the underlying lifetime distribution.
- For most simulation scenarios, the empirical CPs for all parameters remain close to the nominal 95% level, confirming the reliability and adequacy of the proposed interval estimation procedures.
- Overall, the Bayesian approach based on informative priors is strongly recommended for efficient inference on the lifetime characteristics of the model when APFFC data are available.

Table 2. RMSE (first column) and MRAB (second column) results for a .

(n, m)	Test	MLE		Bayesian				MLE		Bayesian			
Prior→				1		2				1		2	
$(k, T) \rightarrow$		(2, 1)						(2, 2)					
(20,10)	$\mathcal{S}[1]$	1.3731	0.8141	0.3642	0.3622	0.2865	0.2767	1.3283	0.7890	0.3634	0.3614	0.2848	0.2743
	$\mathcal{S}[2]$	1.4539	0.8905	0.3669	0.3642	0.2972	0.2825	1.3530	0.8024	0.3662	0.3635	0.2958	0.2804
	$\mathcal{S}[3]$	1.3209	0.7688	0.3609	0.3572	0.2862	0.2755	1.2427	0.7359	0.3601	0.3564	0.2846	0.2732
(20,15)	$\mathcal{S}[1]$	1.1232	0.4240	0.3447	0.3440	0.2430	0.2317	0.9556	0.4236	0.3446	0.3438	0.2383	0.2269
	$\mathcal{S}[2]$	1.2980	0.7254	0.3539	0.3532	0.2477	0.2352	1.1254	0.6462	0.3533	0.3525	0.2433	0.2308
	$\mathcal{S}[3]$	0.9297	0.4144	0.3332	0.3313	0.2420	0.2292	0.8907	0.4132	0.3324	0.3302	0.2363	0.2239
(50,20)	$\mathcal{S}[1]$	0.8513	0.4084	0.2724	0.2712	0.2413	0.2288	0.7645	0.4079	0.2715	0.2700	0.2356	0.2216
	$\mathcal{S}[2]$	0.7475	0.3943	0.2685	0.2640	0.2250	0.2039	0.5830	0.3936	0.2640	0.2592	0.2173	0.1956
	$\mathcal{S}[3]$	0.4530	0.3868	0.2676	0.2628	0.2186	0.2027	0.3993	0.3857	0.2639	0.2590	0.2104	0.1941
(50,40)	$\mathcal{S}[1]$	0.3871	0.3675	0.2558	0.2528	0.1859	0.1786	0.3731	0.3549	0.2542	0.2473	0.1834	0.1755
	$\mathcal{S}[2]$	0.4523	0.3757	0.2595	0.2542	0.2076	0.1982	0.3871	0.3593	0.2545	0.2529	0.2058	0.1929
	$\mathcal{S}[3]$	0.3737	0.3539	0.2550	0.2420	0.1856	0.1736	0.3611	0.3333	0.2540	0.2400	0.1833	0.1694
(80,30)	$\mathcal{S}[1]$	0.3595	0.3465	0.2441	0.2396	0.1481	0.1183	0.3536	0.3179	0.2429	0.2380	0.1339	0.1006
	$\mathcal{S}[2]$	0.3550	0.3092	0.2071	0.1924	0.1271	0.1099	0.3466	0.3056	0.2039	0.1888	0.1169	0.0911
	$\mathcal{S}[3]$	0.3474	0.2988	0.2008	0.1866	0.1178	0.0979	0.3284	0.2637	0.1971	0.1828	0.1003	0.0797
(80,60)	$\mathcal{S}[1]$	0.2927	0.2632	0.1879	0.1684	0.0749	0.0605	0.2906	0.2481	0.1795	0.1584	0.0689	0.0573
	$\mathcal{S}[2]$	0.3244	0.2738	0.1939	0.1711	0.0892	0.0724	0.3215	0.2548	0.1861	0.1606	0.0850	0.0705
	$\mathcal{S}[3]$	0.2741	0.1889	0.1420	0.1120	0.0741	0.0601	0.2664	0.1728	0.1263	0.0930	0.0680	0.0552
$(k, T) \rightarrow$		(5, 1)						(5, 2)					
(20,10)	$\mathcal{S}[1]$	0.3491	0.3412	0.3108	0.2967	0.2844	0.2558	0.3419	0.3385	0.3080	0.2944	0.2792	0.2524
	$\mathcal{S}[2]$	0.3760	0.3574	0.3180	0.3108	0.3029	0.2646	0.3604	0.3498	0.3156	0.3080	0.3008	0.2605
	$\mathcal{S}[3]$	0.3389	0.3175	0.3081	0.2761	0.2646	0.2406	0.3256	0.3149	0.3055	0.2719	0.2605	0.2315
(20,15)	$\mathcal{S}[1]$	0.3039	0.2936	0.2763	0.2557	0.2406	0.2062	0.2941	0.2863	0.2721	0.2492	0.2349	0.1970
	$\mathcal{S}[2]$	0.3155	0.3046	0.2844	0.2649	0.2558	0.2288	0.3129	0.3044	0.2792	0.2605	0.2524	0.2227
	$\mathcal{S}[3]$	0.2894	0.2755	0.2631	0.2370	0.2315	0.2023	0.2841	0.2726	0.2587	0.2363	0.2273	0.1955
(50,20)	$\mathcal{S}[1]$	0.2672	0.2634	0.2429	0.2289	0.2229	0.2022	0.2652	0.2633	0.2394	0.2244	0.2107	0.1882
	$\mathcal{S}[2]$	0.2608	0.2368	0.2268	0.2232	0.1841	0.1706	0.2605	0.2331	0.2268	0.2191	0.1815	0.1705
	$\mathcal{S}[3]$	0.2539	0.2266	0.2208	0.2109	0.1732	0.1629	0.2493	0.2239	0.2182	0.2061	0.1687	0.1581
(50,40)	$\mathcal{S}[1]$	0.2448	0.2117	0.1902	0.1841	0.1599	0.1411	0.2225	0.2107	0.1849	0.1738	0.1479	0.1332
	$\mathcal{S}[2]$	0.2452	0.2229	0.2093	0.2027	0.1703	0.1525	0.2358	0.2168	0.2050	0.1985	0.1674	0.1487
	$\mathcal{S}[3]$	0.2007	0.1824	0.1751	0.1706	0.1472	0.1287	0.1962	0.1815	0.1715	0.1705	0.1372	0.1226
(80,30)	$\mathcal{S}[1]$	0.1593	0.1498	0.1448	0.1368	0.1259	0.1174	0.1574	0.1497	0.1404	0.1320	0.1197	0.1148
	$\mathcal{S}[2]$	0.1843	0.1676	0.1609	0.1472	0.1365	0.1259	0.1693	0.1625	0.1490	0.1448	0.1287	0.1181
	$\mathcal{S}[3]$	0.1486	0.1368	0.1351	0.1186	0.1149	0.1060	0.1427	0.1364	0.1295	0.1186	0.1142	0.1060
(80,60)	$\mathcal{S}[1]$	0.1247	0.1176	0.1134	0.1053	0.0981	0.0925	0.1211	0.1167	0.1084	0.1053	0.0941	0.0925
	$\mathcal{S}[2]$	0.1241	0.1128	0.1025	0.0981	0.0861	0.0803	0.1129	0.1046	0.1021	0.0931	0.0817	0.0663
	$\mathcal{S}[3]$	0.1122	0.1011	0.0877	0.0808	0.0679	0.0543	0.1020	0.1009	0.0836	0.0790	0.0632	0.0503

5. Real data applications

To illustrate the practical performance and broad applicability of the proposed estimation procedures, this section presents two real-data applications arising from distinct engineering and biomedical domains. The first application analyzes diamond size data from a large mining area in south-west Africa, representing an engineering and geostatistical context with nonstandard distributional features and direct economic relevance. The second application focuses on lifetime data from gastric cancer patients receiving combined chemotherapy and radiation therapy, re-

Table 3. RMSE (first column) and MRAB (second column) results for b .

(n, m)	Test	MLE		Bayesian				MLE		Bayesian			
Prior $\rightarrow$				1		2				1		2	
$(k, T) \rightarrow$		(2, 1)						(2, 2)					
(20,10)	$\mathcal{S}[1]$	1.4255	1.1951	0.6743	0.6431	0.5934	0.4485	1.2402	1.0271	0.6597	0.6383	0.5122	0.3420
	$\mathcal{S}[2]$	1.1395	1.0666	0.6396	0.6267	0.4379	0.3547	1.1191	0.7912	0.6293	0.6166	0.3586	0.2912
	$\mathcal{S}[3]$	1.1628	1.0673	0.6426	0.6309	0.4562	0.3635	1.1575	0.9711	0.6396	0.6283	0.4128	0.2993
(20,15)	$\mathcal{S}[1]$	1.0941	0.7787	0.6190	0.5878	0.3652	0.3252	1.0699	0.6719	0.6073	0.5738	0.3425	0.2671
	$\mathcal{S}[2]$	0.9045	0.7396	0.5674	0.5526	0.3155	0.2600	0.7402	0.5724	0.5567	0.5429	0.3134	0.2536
	$\mathcal{S}[3]$	1.0508	0.7536	0.5747	0.5590	0.3352	0.2884	1.0060	0.5890	0.5719	0.5587	0.3170	0.2669
(50,20)	$\mathcal{S}[1]$	0.7341	0.6824	0.5319	0.5073	0.3088	0.2597	0.7256	0.5671	0.5120	0.4965	0.2991	0.2489
	$\mathcal{S}[2]$	0.7091	0.5247	0.4321	0.3121	0.2932	0.2264	0.6760	0.4479	0.4124	0.3070	0.2461	0.2063
	$\mathcal{S}[3]$	0.7226	0.5264	0.4825	0.4091	0.3072	0.2573	0.7207	0.4957	0.4747	0.3948	0.2877	0.2390
(50,40)	$\mathcal{S}[1]$	0.6678	0.5036	0.3939	0.3066	0.2450	0.2225	0.6429	0.4027	0.3902	0.2770	0.2378	0.1950
	$\mathcal{S}[2]$	0.6425	0.3855	0.3646	0.2773	0.2370	0.2083	0.5834	0.3681	0.3477	0.2489	0.2205	0.1904
	$\mathcal{S}[3]$	0.6599	0.4200	0.3858	0.3049	0.2437	0.2225	0.6355	0.3894	0.3638	0.2707	0.2270	0.1937
(80,30)	$\mathcal{S}[1]$	0.6243	0.3628	0.3466	0.2757	0.2302	0.2056	0.5519	0.3604	0.3179	0.2428	0.2156	0.1855
	$\mathcal{S}[2]$	0.4705	0.3533	0.2804	0.2379	0.2056	0.1830	0.4540	0.3309	0.2572	0.2159	0.1904	0.1729
	$\mathcal{S}[3]$	0.5710	0.3551	0.3300	0.2570	0.2181	0.1930	0.5239	0.3440	0.3122	0.2405	0.2008	0.1842
(80,60)	$\mathcal{S}[1]$	0.3590	0.3372	0.2797	0.2265	0.2051	0.1760	0.3531	0.2831	0.2514	0.2103	0.1807	0.1686
	$\mathcal{S}[2]$	0.3506	0.2803	0.1968	0.1869	0.1643	0.1611	0.3113	0.2610	0.1919	0.1783	0.1620	0.1522
	$\mathcal{S}[3]$	0.3526	0.2818	0.2675	0.2144	0.1991	0.1671	0.3186	0.2735	0.2264	0.2003	0.1705	0.1648
$(k, T) \rightarrow$		(5, 1)						(5, 2)					
(20,10)	$\mathcal{S}[1]$	0.8945	0.6381	0.5685	0.5579	0.3666	0.3407	0.7677	0.6381	0.5579	0.4216	0.3632	0.3241
	$\mathcal{S}[2]$	0.4815	0.4276	0.4060	0.3849	0.3347	0.3084	0.4543	0.4113	0.3918	0.3727	0.3347	0.3084
	$\mathcal{S}[3]$	0.7542	0.5378	0.4704	0.4177	0.3604	0.3311	0.6038	0.5378	0.4704	0.4144	0.3577	0.3211
(20,15)	$\mathcal{S}[1]$	0.4497	0.4015	0.3849	0.3760	0.3269	0.2957	0.4365	0.3928	0.3808	0.3692	0.3235	0.2948
	$\mathcal{S}[2]$	0.3948	0.3618	0.3401	0.3161	0.3146	0.2896	0.3713	0.3564	0.3390	0.3152	0.3125	0.2888
	$\mathcal{S}[3]$	0.4206	0.3720	0.3546	0.3343	0.3235	0.2948	0.3720	0.3640	0.3546	0.3252	0.3229	0.2898
(50,20)	$\mathcal{S}[1]$	0.3690	0.3515	0.3390	0.3149	0.3035	0.2687	0.3594	0.3490	0.3333	0.3038	0.2858	0.2439
	$\mathcal{S}[2]$	0.3267	0.3242	0.3087	0.2980	0.2805	0.2376	0.3267	0.3090	0.2980	0.2807	0.2417	0.1999
	$\mathcal{S}[3]$	0.3438	0.3331	0.3287	0.3035	0.2934	0.2643	0.3374	0.3321	0.3286	0.2996	0.2684	0.2374
(50,40)	$\mathcal{S}[1]$	0.3259	0.3228	0.2913	0.2779	0.2416	0.2072	0.3259	0.3040	0.2913	0.2730	0.2257	0.1996
	$\mathcal{S}[2]$	0.3137	0.2858	0.2686	0.2383	0.2021	0.1744	0.2986	0.2736	0.2595	0.2296	0.1899	0.1568
	$\mathcal{S}[3]$	0.3214	0.2893	0.2741	0.2601	0.2299	0.2021	0.3126	0.2778	0.2732	0.2379	0.2026	0.1744
(80,30)	$\mathcal{S}[1]$	0.2977	0.2364	0.2324	0.2006	0.1810	0.1620	0.2371	0.2330	0.2051	0.1940	0.1685	0.1563
	$\mathcal{S}[2]$	0.2082	0.1952	0.1862	0.1797	0.1650	0.1568	0.1970	0.1938	0.1810	0.1698	0.1650	0.1463
	$\mathcal{S}[3]$	0.2897	0.2324	0.2006	0.1875	0.1786	0.1595	0.2344	0.2184	0.1970	0.1803	0.1661	0.1467
(80,60)	$\mathcal{S}[1]$	0.2054	0.1910	0.1834	0.1687	0.1648	0.1463	0.1964	0.1895	0.1786	0.1685	0.1638	0.1462
	$\mathcal{S}[2]$	0.1895	0.1881	0.1683	0.1605	0.1507	0.1126	0.1884	0.1873	0.1681	0.1580	0.1301	0.0918
	$\mathcal{S}[3]$	0.1898	0.1895	0.1803	0.1683	0.1615	0.1462	0.1898	0.1894	0.1698	0.1668	0.1615	0.1428

flecting a biomedical survival-analysis setting of high clinical importance. A concise description of each data set is provided below (see Table 10), demonstrating the effectiveness of the proposed models across markedly different applied scenarios.

- Application 1: Diamond size data collected from a large mining area in south-west Africa provide a scientifically rich and practically important setting for advanced statistical modeling. Such data reflect complex geological formation processes, operational extraction mechanisms, and natural heterogeneity inherent in mineral resources, making them ideal for studying nonstandard distributional shapes, skewness, and tail behavior. From a prac-

Table 4. RMSE (first column) and MRAB (second column) results for $R(t)$.

(n, m)	Test	MLE		Bayesian				MLE		Bayesian			
Prior→				1		2				1		2	
$(k, T) \rightarrow$		(2, 1)						(2, 2)					
(20,10)	$\mathcal{S}[1]$	0.4824	0.4623	0.4248	0.3945	0.3841	0.3635	0.4807	0.4610	0.4102	0.3909	0.3806	0.3603
	$\mathcal{S}[2]$	0.4841	0.4639	0.4284	0.4006	0.3918	0.3652	0.4825	0.4622	0.4263	0.3951	0.3885	0.3620
	$\mathcal{S}[3]$	0.4870	0.4688	0.4397	0.4247	0.3937	0.3744	0.4854	0.4671	0.4364	0.4088	0.3904	0.3713
(20,15)	$\mathcal{S}[1]$	0.4356	0.4214	0.3344	0.3094	0.3012	0.2867	0.4333	0.4194	0.3106	0.3017	0.2950	0.2541
	$\mathcal{S}[2]$	0.4728	0.4613	0.3694	0.3278	0.3196	0.3011	0.4712	0.4597	0.3609	0.3267	0.3086	0.2972
	$\mathcal{S}[3]$	0.4745	0.4614	0.3832	0.3339	0.3276	0.3089	0.4729	0.4600	0.3794	0.3279	0.3207	0.3015
(50,20)	$\mathcal{S}[1]$	0.2925	0.2760	0.2710	0.2411	0.2330	0.1951	0.2874	0.2743	0.2660	0.2361	0.2133	0.1886
	$\mathcal{S}[2]$	0.3021	0.2868	0.2808	0.2423	0.2395	0.2165	0.2975	0.2814	0.2728	0.2417	0.2325	0.1912
	$\mathcal{S}[3]$	0.3150	0.3000	0.2896	0.2844	0.2463	0.2210	0.3031	0.2910	0.2874	0.2587	0.2399	0.2157
(50,40)	$\mathcal{S}[1]$	0.2857	0.2600	0.2454	0.2174	0.1613	0.1392	0.2772	0.2476	0.2422	0.2117	0.1481	0.1232
	$\mathcal{S}[2]$	0.2903	0.2602	0.2541	0.2233	0.2120	0.1921	0.2800	0.2544	0.2510	0.2214	0.2038	0.1636
	$\mathcal{S}[3]$	0.2916	0.2747	0.2666	0.2300	0.2267	0.1922	0.2805	0.2728	0.2654	0.2269	0.2104	0.1636
(80,30)	$\mathcal{S}[1]$	0.2384	0.2157	0.2092	0.1921	0.1259	0.0996	0.2304	0.2138	0.2052	0.1837	0.1234	0.0994
	$\mathcal{S}[2]$	0.2576	0.2334	0.2249	0.1939	0.1356	0.1120	0.2508	0.2323	0.2248	0.1891	0.1271	0.1002
	$\mathcal{S}[3]$	0.2852	0.2550	0.2381	0.1978	0.1379	0.1128	0.2757	0.2398	0.2367	0.1895	0.1280	0.1019
(80,60)	$\mathcal{S}[1]$	0.1305	0.1263	0.1053	0.1022	0.0891	0.0692	0.1291	0.1103	0.1041	0.0901	0.0825	0.0652
	$\mathcal{S}[2]$	0.1894	0.1644	0.1433	0.1254	0.0922	0.0726	0.1772	0.1549	0.1398	0.1123	0.0836	0.0658
	$\mathcal{S}[3]$	0.2023	0.1806	0.1661	0.1445	0.1008	0.0804	0.1960	0.1686	0.1574	0.1174	0.0840	0.0668
$(k, T) \rightarrow$		(5, 1)						(5, 2)					
(20,10)	$\mathcal{S}[1]$	0.4666	0.4629	0.4453	0.4422	0.3712	0.3675	0.4645	0.4615	0.4428	0.4390	0.3711	0.3672
	$\mathcal{S}[2]$	0.4812	0.4730	0.4492	0.4466	0.4035	0.3991	0.4800	0.4719	0.4468	0.4436	0.4026	0.3986
	$\mathcal{S}[3]$	0.4536	0.4035	0.3991	0.3891	0.3694	0.3643	0.4520	0.4026	0.3986	0.3795	0.3669	0.3606
(20,15)	$\mathcal{S}[1]$	0.4054	0.3606	0.3341	0.3290	0.3281	0.3225	0.3965	0.3538	0.3332	0.3282	0.3280	0.3224
	$\mathcal{S}[2]$	0.4330	0.3749	0.3712	0.3675	0.3662	0.3610	0.4297	0.3715	0.3711	0.3672	0.3643	0.3582
	$\mathcal{S}[3]$	0.3334	0.3313	0.3275	0.3260	0.3217	0.2753	0.3328	0.3281	0.3274	0.3218	0.3101	0.2523
(50,20)	$\mathcal{S}[1]$	0.3027	0.2984	0.2523	0.2449	0.2282	0.2198	0.3008	0.2940	0.2483	0.2383	0.2238	0.2170
	$\mathcal{S}[2]$	0.3083	0.3009	0.2940	0.2770	0.2580	0.2424	0.3043	0.2967	0.2791	0.2768	0.2575	0.2180
	$\mathcal{S}[3]$	0.2967	0.2768	0.2424	0.2264	0.2177	0.2100	0.2770	0.2580	0.2294	0.2236	0.2166	0.1969
(50,40)	$\mathcal{S}[1]$	0.2618	0.2432	0.2283	0.2120	0.2041	0.1926	0.2475	0.2327	0.2170	0.2051	0.2008	0.1838
	$\mathcal{S}[2]$	0.2654	0.2490	0.2348	0.2247	0.2090	0.1971	0.2513	0.2348	0.2276	0.2223	0.2065	0.1952
	$\mathcal{S}[3]$	0.2294	0.2077	0.2041	0.1996	0.1876	0.1813	0.2101	0.2068	0.2019	0.1926	0.1838	0.1600
(80,30)	$\mathcal{S}[1]$	0.2262	0.1915	0.1799	0.1617	0.1522	0.1349	0.1982	0.1827	0.1777	0.1607	0.1387	0.1234
	$\mathcal{S}[2]$	0.2274	0.2028	0.1982	0.1813	0.1752	0.1600	0.2051	0.2008	0.1876	0.1769	0.1630	0.1468
	$\mathcal{S}[3]$	0.1902	0.1718	0.1457	0.1386	0.1234	0.1115	0.1735	0.1566	0.1395	0.1297	0.1217	0.1010
(80,60)	$\mathcal{S}[1]$	0.1231	0.1035	0.0894	0.0851	0.0789	0.0690	0.1056	0.0964	0.0876	0.0817	0.0692	0.0639
	$\mathcal{S}[2]$	0.1732	0.1387	0.1264	0.1172	0.1100	0.1084	0.1480	0.1348	0.1264	0.1106	0.1100	0.0933
	$\mathcal{S}[3]$	0.0952	0.0842	0.0807	0.0784	0.0682	0.0646	0.0866	0.0839	0.0797	0.0683	0.0677	0.0624

tical perspective, accurate modeling of diamond sizes supports improved resource valuation, production planning, and risk assessment in mining operations, directly influencing economic efficiency and decision-making; see Mohammed et al. [26]. This application examines 24 diamond size distribution data collecting from a large mining area in south-west Africa; see Maurya and Goyal [24] for additional details.

- Application 2: Gastric cancer is a malignant disease characterized by the uncontrolled growth of abnormal cells in the lining of the stomach, often developing gradually and potentially spreading to other parts of the body if not detected early; see Hartgrink et al. [19].

Table 5. RMSE (first column) and MRAB (second column) results for $H(t)$.

(n, m)	Test	MLE		Bayesian				MLE		Bayesian			
				1		2				1		2	
Prior→													
$(k, T) \rightarrow$		(2, 1)								(2, 2)			
(20,10)	$\mathcal{S}[1]$	0.5150	0.4792	0.4410	0.4291	0.4045	0.3815	0.4995	0.4777	0.4395	0.4224	0.4015	0.3785
	$\mathcal{S}[2]$	0.5499	0.4832	0.4473	0.4323	0.4057	0.3887	0.5250	0.4817	0.4446	0.4272	0.4027	0.3859
	$\mathcal{S}[3]$	0.4946	0.4762	0.4264	0.4179	0.3967	0.3787	0.4931	0.4750	0.4257	0.4121	0.3935	0.3757
(20,15)	$\mathcal{S}[1]$	0.4848	0.4742	0.3884	0.3619	0.3319	0.3231	0.4834	0.4730	0.3833	0.3564	0.3223	0.3116
	$\mathcal{S}[2]$	0.4871	0.4756	0.4047	0.3901	0.3393	0.3234	0.4858	0.4746	0.4025	0.3879	0.3328	0.3162
	$\mathcal{S}[3]$	0.4496	0.4369	0.3438	0.3393	0.3224	0.3099	0.4476	0.4351	0.3400	0.3364	0.3146	0.3020
(50,20)	$\mathcal{S}[1]$	0.3393	0.3197	0.3036	0.3008	0.2739	0.2360	0.3308	0.3097	0.3029	0.2815	0.2560	0.2295
	$\mathcal{S}[2]$	0.3167	0.3003	0.2888	0.2747	0.2512	0.2344	0.3128	0.2925	0.2863	0.2730	0.2449	0.2122
	$\mathcal{S}[3]$	0.3090	0.2957	0.2870	0.2701	0.2466	0.2126	0.3033	0.2915	0.2713	0.2482	0.2289	0.2066
(50,40)	$\mathcal{S}[1]$	0.3064	0.2806	0.2389	0.2184	0.2178	0.1987	0.2989	0.2745	0.2360	0.2182	0.2095	0.1782
	$\mathcal{S}[2]$	0.3079	0.2944	0.2680	0.2556	0.2407	0.2103	0.3008	0.2587	0.2526	0.2454	0.2247	0.1825
	$\mathcal{S}[3]$	0.3051	0.2796	0.2072	0.1752	0.1685	0.1530	0.2956	0.2641	0.2063	0.1718	0.1633	0.1416
(80,30)	$\mathcal{S}[1]$	0.3032	0.2746	0.2060	0.1712	0.1416	0.1142	0.2928	0.2577	0.1980	0.1671	0.1358	0.1086
	$\mathcal{S}[2]$	0.2692	0.2466	0.1967	0.1562	0.1409	0.1138	0.2628	0.2396	0.1963	0.1559	0.1334	0.1052
	$\mathcal{S}[3]$	0.2515	0.2299	0.1457	0.1287	0.1125	0.1035	0.2440	0.2219	0.1323	0.1280	0.1047	0.1033
(80,60)	$\mathcal{S}[1]$	0.1896	0.1556	0.1190	0.0982	0.0923	0.0751	0.1675	0.1276	0.1130	0.0953	0.0818	0.0712
	$\mathcal{S}[2]$	0.1931	0.1597	0.1324	0.1097	0.0974	0.0806	0.1701	0.1365	0.1209	0.1015	0.0884	0.0794
	$\mathcal{S}[3]$	0.1365	0.1083	0.1035	0.0870	0.0855	0.0727	0.1345	0.1081	0.0933	0.0867	0.0730	0.0687
$(k, T) \rightarrow$		(5, 1)								(5, 2)			
(20,10)	$\mathcal{S}[1]$	0.4727	0.4602	0.4556	0.4533	0.3890	0.3827	0.4703	0.4586	0.4543	0.4515	0.3854	0.3793
	$\mathcal{S}[2]$	0.4889	0.4705	0.4588	0.4566	0.4156	0.4115	0.4875	0.4693	0.4575	0.4548	0.4128	0.4089
	$\mathcal{S}[3]$	0.4540	0.4198	0.4156	0.4115	0.3773	0.3723	0.4527	0.4187	0.4128	0.4089	0.3772	0.3712
(20,15)	$\mathcal{S}[1]$	0.4278	0.4166	0.3452	0.3430	0.3399	0.3377	0.4274	0.4160	0.3446	0.3407	0.3394	0.3315
	$\mathcal{S}[2]$	0.4351	0.4179	0.3890	0.3827	0.3764	0.3707	0.4327	0.4175	0.3854	0.3793	0.3758	0.3671
	$\mathcal{S}[3]$	0.4195	0.4021	0.3434	0.3421	0.3381	0.3370	0.4189	0.3999	0.3430	0.3406	0.3377	0.3305
(50,20)	$\mathcal{S}[1]$	0.4007	0.3907	0.3211	0.3159	0.3088	0.2895	0.3928	0.3821	0.3173	0.3100	0.2922	0.2741
	$\mathcal{S}[2]$	0.3638	0.3581	0.3169	0.3128	0.2575	0.2330	0.3638	0.3581	0.3153	0.3093	0.2384	0.2285
	$\mathcal{S}[3]$	0.3400	0.3088	0.2950	0.2895	0.2369	0.2215	0.3258	0.3067	0.2922	0.2741	0.2256	0.2095
(50,40)	$\mathcal{S}[1]$	0.2763	0.2594	0.2454	0.2381	0.2140	0.1996	0.2628	0.2456	0.2395	0.2261	0.2114	0.1938
	$\mathcal{S}[2]$	0.2799	0.2665	0.2526	0.2484	0.2270	0.2111	0.2741	0.2652	0.2517	0.2344	0.2196	0.2069
	$\mathcal{S}[3]$	0.2395	0.2194	0.2149	0.2106	0.1996	0.1900	0.2344	0.2194	0.2114	0.2012	0.1938	0.1735
(80,30)	$\mathcal{S}[1]$	0.2153	0.1944	0.1876	0.1673	0.1634	0.1407	0.2078	0.1917	0.1830	0.1662	0.1486	0.1307
	$\mathcal{S}[2]$	0.2216	0.2078	0.2059	0.1914	0.1891	0.1735	0.2140	0.2072	0.2012	0.1900	0.1835	0.1584
	$\mathcal{S}[3]$	0.1932	0.1805	0.1629	0.1530	0.1463	0.1307	0.1812	0.1639	0.1555	0.1466	0.1359	0.1206
(80,60)	$\mathcal{S}[1]$	0.1741	0.1493	0.1446	0.1428	0.1273	0.1259	0.1526	0.1463	0.1428	0.1347	0.1273	0.1172
	$\mathcal{S}[2]$	0.1550	0.1407	0.0980	0.0874	0.0805	0.0772	0.1489	0.1333	0.0952	0.0873	0.0761	0.0688
	$\mathcal{S}[3]$	0.1369	0.1245	0.0976	0.0870	0.0795	0.0724	0.1282	0.1155	0.0944	0.0848	0.0755	0.0667

It remains one of the leading causes of cancer-related mortality worldwide, and optimizing therapeutic strategies is a major priority in both clinical oncology and biostatistical research. The analysis of data from a group of gastric cancer patients undergoing combined chemotherapy and radiation therapy provides a valuable framework for understanding treatment efficacy, toxicity profiles, and patient survival patterns. The benefits of this analysis include improved estimation of survival and response measures, thereby supporting personalized treatment planning, enhancing prognostic assessment, and contributing to more reliable clinical decision-making, thereby strengthening the role of quantitative methods in advancing cancer therapy research. In this context, this application evaluates

Table 6. AIL (first column) and CP (second column) of the 95% interval estimates for a .

(k, T)	(n, m)	Test	ACI-NA		ACI-NT		BCI				HPD			
							1		2		1		2	
Prior→														
(2,1)	(20,10)	$S[1]$	0.339	0.945	0.370	0.940	0.296	0.951	0.230	0.961	0.271	0.955	0.220	0.963
		$S[2]$	0.363	0.941	0.392	0.937	0.314	0.949	0.288	0.953	0.290	0.952	0.261	0.957
		$S[3]$	0.332	0.946	0.356	0.942	0.285	0.953	0.224	0.962	0.266	0.956	0.212	0.964
	(20,15)	$S[1]$	0.313	0.949	0.316	0.948	0.253	0.958	0.198	0.966	0.250	0.958	0.197	0.966
		$S[2]$	0.325	0.947	0.342	0.945	0.273	0.955	0.201	0.966	0.260	0.957	0.198	0.966
		$S[3]$	0.287	0.953	0.302	0.950	0.242	0.960	0.192	0.967	0.229	0.961	0.191	0.967
	(50,20)	$S[1]$	0.286	0.953	0.300	0.951	0.240	0.960	0.150	0.973	0.229	0.961	0.146	0.974
		$S[2]$	0.247	0.959	0.282	0.954	0.225	0.962	0.144	0.974	0.198	0.966	0.143	0.974
		$S[3]$	0.234	0.961	0.270	0.955	0.214	0.964	0.120	0.978	0.187	0.968	0.116	0.978
	(50,40)	$S[1]$	0.181	0.969	0.227	0.962	0.165	0.971	0.103	0.980	0.132	0.976	0.094	0.982
		$S[2]$	0.189	0.967	0.228	0.962	0.183	0.968	0.106	0.980	0.152	0.973	0.094	0.982
		$S[3]$	0.177	0.969	0.221	0.963	0.164	0.971	0.101	0.981	0.132	0.976	0.091	0.982
	(80,30)	$S[1]$	0.143	0.974	0.150	0.973	0.120	0.978	0.099	0.981	0.114	0.979	0.089	0.982
		$S[2]$	0.143	0.974	0.149	0.973	0.119	0.978	0.094	0.982	0.114	0.979	0.088	0.982
		$S[3]$	0.124	0.977	0.147	0.974	0.118	0.978	0.092	0.982	0.099	0.981	0.087	0.983
	(80,60)	$S[1]$	0.115	0.978	0.144	0.974	0.111	0.979	0.072	0.985	0.088	0.982	0.065	0.986
		$S[2]$	0.123	0.977	0.144	0.974	0.115	0.978	0.091	0.982	0.099	0.981	0.086	0.983
		$S[3]$	0.110	0.979	0.136	0.975	0.108	0.979	0.049	0.988	0.088	0.983	0.046	0.989
(2,2)	(20,10)	$S[1]$	0.335	0.945	0.367	0.941	0.294	0.952	0.229	0.961	0.268	0.956	0.219	0.963
		$S[2]$	0.360	0.942	0.391	0.937	0.313	0.949	0.282	0.953	0.288	0.953	0.260	0.957
		$S[3]$	0.327	0.947	0.354	0.943	0.283	0.953	0.223	0.962	0.261	0.957	0.210	0.964
	(20,15)	$S[1]$	0.310	0.949	0.315	0.949	0.252	0.958	0.198	0.966	0.248	0.959	0.197	0.966
		$S[2]$	0.324	0.947	0.338	0.945	0.270	0.955	0.200	0.966	0.259	0.957	0.198	0.966
		$S[3]$	0.286	0.953	0.301	0.951	0.241	0.960	0.192	0.967	0.228	0.961	0.190	0.967
	(50,20)	$S[1]$	0.285	0.953	0.299	0.951	0.239	0.960	0.149	0.973	0.228	0.962	0.145	0.974
		$S[2]$	0.240	0.960	0.281	0.954	0.225	0.962	0.144	0.974	0.192	0.967	0.143	0.974
		$S[3]$	0.216	0.963	0.267	0.956	0.209	0.964	0.120	0.978	0.167	0.971	0.116	0.978
	(50,40)	$S[1]$	0.181	0.969	0.226	0.962	0.162	0.971	0.102	0.980	0.129	0.976	0.093	0.982
		$S[2]$	0.184	0.968	0.228	0.962	0.182	0.968	0.106	0.980	0.147	0.974	0.094	0.982
		$S[3]$	0.176	0.969	0.221	0.963	0.150	0.973	0.100	0.981	0.120	0.978	0.090	0.982
	(80,30)	$S[1]$	0.139	0.975	0.150	0.973	0.120	0.978	0.098	0.981	0.111	0.979	0.089	0.982
		$S[2]$	0.136	0.975	0.149	0.973	0.119	0.978	0.093	0.982	0.109	0.979	0.088	0.983
		$S[3]$	0.122	0.977	0.147	0.974	0.117	0.978	0.091	0.982	0.098	0.981	0.087	0.983
	(80,60)	$S[1]$	0.114	0.979	0.143	0.974	0.110	0.979	0.071	0.985	0.088	0.982	0.064	0.986
		$S[2]$	0.118	0.978	0.144	0.974	0.115	0.978	0.091	0.982	0.094	0.982	0.086	0.983
		$S[3]$	0.109	0.979	0.134	0.976	0.107	0.980	0.048	0.989	0.087	0.983	0.045	0.989
(5,1)	(20,10)	$S[1]$	0.445	0.943	0.467	0.940	0.373	0.952	0.314	0.960	0.356	0.954	0.308	0.961
		$S[2]$	0.450	0.942	0.511	0.934	0.409	0.947	0.346	0.956	0.360	0.954	0.316	0.960
		$S[3]$	0.387	0.950	0.439	0.943	0.351	0.955	0.307	0.961	0.310	0.960	0.293	0.963
	(20,15)	$S[1]$	0.353	0.955	0.387	0.950	0.310	0.960	0.278	0.965	0.283	0.964	0.261	0.967
		$S[2]$	0.362	0.954	0.416	0.946	0.333	0.957	0.283	0.964	0.290	0.963	0.278	0.965
		$S[3]$	0.349	0.955	0.353	0.955	0.283	0.964	0.276	0.965	0.279	0.964	0.257	0.967
	(50,20)	$S[1]$	0.346	0.956	0.349	0.955	0.279	0.964	0.257	0.967	0.277	0.965	0.226	0.971
		$S[2]$	0.310	0.960	0.345	0.956	0.276	0.965	0.237	0.970	0.248	0.968	0.219	0.972
		$S[3]$	0.296	0.962	0.320	0.959	0.256	0.967	0.218	0.972	0.236	0.970	0.202	0.975
	(50,40)	$S[1]$	0.282	0.964	0.297	0.962	0.237	0.970	0.201	0.975	0.226	0.971	0.197	0.975
		$S[2]$	0.295	0.962	0.303	0.961	0.243	0.969	0.204	0.974	0.236	0.970	0.200	0.975
		$S[3]$	0.273	0.965	0.284	0.964	0.227	0.971	0.198	0.975	0.218	0.972	0.188	0.976
	(80,30)	$S[1]$	0.258	0.967	0.271	0.965	0.217	0.973	0.166	0.979	0.207	0.974	0.161	0.980
		$S[2]$	0.260	0.967	0.275	0.965	0.220	0.972	0.186	0.977	0.208	0.974	0.184	0.977
		$S[3]$	0.248	0.969	0.252	0.968	0.202	0.975	0.159	0.980	0.198	0.975	0.145	0.982
	(80,60)	$S[1]$	0.195	0.975	0.205	0.974	0.164	0.979	0.154	0.981	0.156	0.981	0.140	0.983
		$S[2]$	0.161	0.980	0.190	0.976	0.152	0.981	0.126	0.984	0.129	0.984	0.122	0.985
		$S[3]$	0.155	0.981	0.186	0.977	0.149	0.981	0.122	0.985	0.124	0.985	0.116	0.986
(5,2)	(20,10)	$S[1]$	0.438	0.944	0.455	0.941	0.364	0.953	0.312	0.960	0.350	0.955	0.306	0.961
		$S[2]$	0.439	0.943	0.506	0.935	0.405	0.948	0.345	0.956	0.352	0.955	0.311	0.960
		$S[3]$	0.385	0.950	0.435	0.944	0.348	0.955	0.305	0.961	0.308	0.961	0.292	0.963
	(20,15)	$S[1]$	0.351	0.955	0.385	0.950	0.308	0.961	0.276	0.965	0.281	0.964	0.260	0.967
		$S[2]$	0.360	0.954	0.409	0.947	0.327	0.958	0.282	0.964	0.288	0.963	0.276	0.965
		$S[3]$	0.347	0.955	0.351	0.955	0.281	0.964	0.276	0.965	0.278	0.965	0.257	0.967
	(50,20)	$S[1]$	0.346	0.956	0.347	0.955	0.278	0.965	0.257	0.967	0.277	0.965	0.223	0.972
		$S[2]$	0.309	0.960	0.344	0.956	0.275	0.965	0.237	0.970	0.247	0.969	0.217	0.973
		$S[3]$	0.293	0.963	0.319	0.959	0.255	0.968	0.216	0.973	0.234	0.970	0.201	0.975
	(50,40)	$S[1]$	0.279	0.964	0.296	0.962	0.237	0.970	0.200	0.975	0.223	0.972	0.197	0.975
		$S[2]$	0.293	0.963	0.302	0.961	0.241	0.969	0.202	0.975	0.234	0.970	0.198	0.975
		$S[3]$	0.271	0.965	0.282	0.964	0.226	0.971	0.198	0.975	0.217	0.973	0.186	0.977
	(80,30)	$S[1]$	0.254	0.968	0.271	0.966	0.216	0.973	0.165	0.979	0.203	0.974	0.160	0.980
		$S[2]$	0.259	0.967	0.275	0.965	0.220	0.972	0.185	0.977	0.207	0.974	0.183	0.977
		$S[3]$	0.246	0.969	0.251	0.968	0.201	0.975	0.159	0.980	0.197	0.975	0.144	0.982
	(80,60)	$S[1]$	0.194	0.976	0.197	0.975	0.157	0.980	0.147	0.982	0.156	0.981	0.140	0.983
		$S[2]$	0.161	0.980	0.189	0.976	0.151	0.981	0.126	0.984	0.129	0.984	0.116	0.986
		$S[3]$	0.155	0.981	0.186	0.977	0.149	0.982	0.122	0.985	0.124	0.985	0.114	0.986

Table 7. AIL (first column) and CP (second column) of the 95% interval estimates for b .

(k, T)	(n, m)	Test	ACI-NA		ACI-NT		BCI				HPD			
							1		2		1		2	
Prior→														
(2,1)	(20,10)	$S[1]$	1.129	0.917	1.163	0.914	1.011	0.926	0.704	0.952	0.982	0.929	0.610	0.959
		$S[2]$	0.960	0.931	1.003	0.927	0.873	0.938	0.626	0.958	0.835	0.941	0.499	0.968
		$S[3]$	1.028	0.925	1.084	0.920	0.942	0.932	0.629	0.958	0.894	0.936	0.547	0.964
	(20,15)	$S[1]$	0.956	0.931	0.989	0.928	0.860	0.939	0.606	0.960	0.832	0.941	0.496	0.969
		$S[2]$	0.886	0.937	0.957	0.931	0.832	0.941	0.549	0.964	0.770	0.946	0.487	0.969
		$S[3]$	0.902	0.935	0.961	0.931	0.836	0.941	0.567	0.963	0.784	0.945	0.492	0.969
	(50,20)	$S[1]$	0.776	0.946	0.834	0.941	0.726	0.950	0.522	0.967	0.675	0.954	0.487	0.969
		$S[2]$	0.728	0.950	0.796	0.944	0.692	0.953	0.473	0.971	0.633	0.957	0.464	0.971
		$S[3]$	0.774	0.946	0.811	0.943	0.705	0.952	0.486	0.969	0.673	0.954	0.467	0.971
	(50,40)	$S[1]$	0.718	0.950	0.762	0.947	0.663	0.955	0.463	0.971	0.624	0.958	0.457	0.972
		$S[2]$	0.669	0.954	0.754	0.947	0.656	0.956	0.458	0.972	0.581	0.962	0.445	0.973
		$S[3]$	0.674	0.954	0.757	0.947	0.659	0.955	0.459	0.972	0.584	0.961	0.452	0.972
	(80,30)	$S[1]$	0.664	0.955	0.754	0.948	0.656	0.956	0.456	0.972	0.572	0.962	0.431	0.974
		$S[2]$	0.655	0.956	0.665	0.955	0.578	0.962	0.436	0.974	0.569	0.963	0.412	0.975
		$S[3]$	0.657	0.955	0.682	0.953	0.593	0.961	0.454	0.972	0.572	0.962	0.419	0.975
	(80,60)	$S[1]$	0.646	0.956	0.657	0.955	0.572	0.962	0.429	0.974	0.562	0.963	0.397	0.977
		$S[2]$	0.566	0.963	0.637	0.957	0.554	0.964	0.399	0.977	0.492	0.969	0.379	0.978
		$S[3]$	0.631	0.958	0.641	0.957	0.557	0.964	0.412	0.975	0.549	0.964	0.390	0.977
(2,2)	(20,10)	$S[1]$	1.017	0.926	1.145	0.915	0.995	0.928	0.701	0.952	0.810	0.943	0.608	0.959
		$S[2]$	0.908	0.935	1.001	0.927	0.871	0.938	0.624	0.958	0.720	0.950	0.494	0.969
		$S[3]$	0.981	0.929	1.070	0.922	0.930	0.933	0.625	0.958	0.723	0.950	0.536	0.965
	(20,15)	$S[1]$	0.898	0.936	0.976	0.929	0.849	0.940	0.604	0.960	0.697	0.952	0.491	0.969
		$S[2]$	0.868	0.938	0.916	0.934	0.797	0.944	0.547	0.964	0.632	0.958	0.468	0.971
		$S[3]$	0.873	0.938	0.923	0.934	0.802	0.944	0.567	0.963	0.652	0.956	0.471	0.971
	(50,20)	$S[1]$	0.737	0.949	0.789	0.945	0.686	0.953	0.503	0.968	0.600	0.960	0.468	0.971
		$S[2]$	0.713	0.951	0.747	0.948	0.649	0.956	0.471	0.971	0.544	0.965	0.462	0.971
		$S[3]$	0.717	0.951	0.784	0.945	0.682	0.953	0.481	0.970	0.559	0.963	0.467	0.971
	(50,40)	$S[1]$	0.678	0.954	0.739	0.949	0.643	0.957	0.462	0.971	0.533	0.966	0.444	0.973
		$S[2]$	0.668	0.955	0.669	0.954	0.582	0.962	0.451	0.972	0.527	0.966	0.422	0.975
		$S[3]$	0.671	0.954	0.682	0.953	0.593	0.961	0.453	0.972	0.528	0.966	0.436	0.974
	(80,30)	$S[1]$	0.657	0.955	0.665	0.955	0.578	0.962	0.442	0.973	0.524	0.966	0.413	0.975
		$S[2]$	0.578	0.962	0.658	0.955	0.572	0.962	0.435	0.974	0.501	0.968	0.391	0.977
		$S[3]$	0.596	0.960	0.659	0.955	0.573	0.962	0.441	0.973	0.522	0.967	0.411	0.976
	(80,60)	$S[1]$	0.573	0.962	0.650	0.956	0.565	0.963	0.405	0.976	0.493	0.969	0.391	0.977
		$S[2]$	0.556	0.964	0.567	0.963	0.493	0.969	0.390	0.977	0.459	0.972	0.370	0.979
		$S[3]$	0.559	0.963	0.633	0.957	0.550	0.964	0.401	0.976	0.474	0.970	0.376	0.978
(5,1)	(20,10)	$S[1]$	1.181	0.921	1.330	0.911	1.157	0.923	0.636	0.959	1.006	0.933	0.619	0.960
		$S[2]$	1.116	0.926	1.160	0.923	1.009	0.933	0.572	0.963	0.971	0.936	0.553	0.965
		$S[3]$	1.121	0.925	1.197	0.920	1.040	0.931	0.581	0.963	0.975	0.936	0.554	0.965
	(20,15)	$S[1]$	1.084	0.928	1.085	0.928	0.943	0.938	0.556	0.965	0.942	0.938	0.529	0.966
		$S[2]$	1.060	0.930	1.073	0.929	0.933	0.938	0.521	0.967	0.922	0.939	0.510	0.968
		$S[3]$	1.065	0.929	1.084	0.928	0.943	0.938	0.529	0.966	0.926	0.939	0.514	0.967
	(50,20)	$S[1]$	1.046	0.932	1.037	0.931	0.901	0.941	0.519	0.967	0.892	0.941	0.498	0.969
		$S[2]$	1.012	0.933	1.022	0.932	0.889	0.942	0.475	0.970	0.880	0.942	0.470	0.970
		$S[3]$	1.026	0.932	1.029	0.932	0.895	0.941	0.509	0.968	0.892	0.941	0.486	0.969
	(50,40)	$S[1]$	0.944	0.938	0.988	0.935	0.860	0.944	0.470	0.970	0.821	0.946	0.469	0.971
		$S[2]$	0.851	0.944	0.867	0.943	0.753	0.951	0.464	0.971	0.740	0.952	0.459	0.971
		$S[3]$	0.942	0.938	0.945	0.938	0.822	0.946	0.466	0.971	0.819	0.946	0.464	0.971
	(80,30)	$S[1]$	0.761	0.950	0.837	0.945	0.728	0.953	0.463	0.971	0.662	0.957	0.449	0.972
		$S[2]$	0.758	0.951	0.769	0.950	0.669	0.957	0.453	0.972	0.659	0.957	0.443	0.972
		$S[3]$	0.761	0.950	0.818	0.946	0.712	0.954	0.462	0.971	0.661	0.957	0.445	0.972
	(80,60)	$S[1]$	0.746	0.951	0.764	0.950	0.665	0.957	0.445	0.972	0.648	0.958	0.441	0.973
		$S[2]$	0.634	0.959	0.723	0.953	0.616	0.960	0.428	0.973	0.492	0.969	0.413	0.974
		$S[3]$	0.652	0.958	0.739	0.952	0.643	0.959	0.441	0.973	0.559	0.964	0.428	0.973
(5,2)	(20,10)	$S[1]$	1.157	0.923	1.261	0.916	1.097	0.927	0.636	0.959	0.731	0.952	0.619	0.960
		$S[2]$	1.012	0.933	1.132	0.925	0.985	0.935	0.557	0.964	0.658	0.957	0.543	0.965
		$S[3]$	1.044	0.931	1.134	0.925	0.986	0.935	0.581	0.963	0.668	0.957	0.553	0.965
	(20,15)	$S[1]$	0.943	0.938	1.084	0.928	0.942	0.938	0.541	0.966	0.639	0.959	0.518	0.967
		$S[2]$	0.933	0.938	1.060	0.930	0.922	0.939	0.511	0.968	0.599	0.962	0.509	0.968
		$S[3]$	0.943	0.938	1.065	0.929	0.926	0.939	0.525	0.967	0.609	0.961	0.513	0.967
	(50,20)	$S[1]$	0.904	0.940	1.031	0.932	0.896	0.941	0.508	0.968	0.596	0.962	0.497	0.969
		$S[2]$	0.892	0.941	1.015	0.933	0.883	0.942	0.473	0.970	0.546	0.965	0.469	0.971
		$S[3]$	0.896	0.941	1.026	0.932	0.892	0.941	0.495	0.969	0.586	0.962	0.485	0.969
	(50,40)	$S[1]$	0.866	0.943	0.950	0.937	0.826	0.946	0.470	0.971	0.540	0.966	0.468	0.971
		$S[2]$	0.763	0.950	0.863	0.943	0.751	0.951	0.463	0.971	0.534	0.966	0.448	0.972
		$S[3]$	0.825	0.946	0.943	0.938	0.820	0.946	0.465	0.971	0.535	0.966	0.463	0.971
	(80,30)	$S[1]$	0.752	0.951	0.783	0.949	0.681	0.956	0.461	0.971	0.532	0.966	0.441	0.972
		$S[2]$	0.681	0.956	0.758	0.951	0.659	0.957	0.453	0.972	0.521	0.967	0.438	0.973
		$S[3]$	0.734	0.952	0.775	0.949	0.674	0.956	0.453	0.972	0.531	0.966	0.439	0.973
	(80,60)	$S[1]$	0.665	0.957	0.746	0.951	0.648	0.958	0.441	0.972	0.511	0.968	0.407	0.975
		$S[2]$	0.629	0.959	0.708	0.954	0.510	0.968	0.422	0.974	0.444	0.972	0.390	0.976
		$S[3]$	0.643	0.959	0.736	0.952	0.640	0.959	0.439	0.973	0.507	0.968	0.395	0.976

Table 8. AIL (first column) and CP (second column) of the 95% interval estimates for $R(t)$.

(k, T)	(n, m)	Test	ACI-NA		ACI-NT		BCI				HPD			
							1		2		1		2	
Prior→														
(2,1)	(20,10)	$S[1]$	1.228	0.911	1.120	0.918	0.482	0.958	0.476	0.959	0.407	0.963	0.378	0.965
		$S[2]$	1.316	0.906	1.198	0.913	0.499	0.957	0.491	0.958	0.407	0.963	0.391	0.964
		$S[3]$	1.373	0.902	1.294	0.907	0.505	0.957	0.501	0.957	0.497	0.957	0.493	0.957
	(20,15)	$S[1]$	0.868	0.934	0.766	0.940	0.412	0.963	0.389	0.964	0.353	0.966	0.336	0.967
		$S[2]$	0.893	0.932	0.830	0.936	0.427	0.962	0.408	0.963	0.369	0.965	0.354	0.966
		$S[3]$	1.174	0.915	0.914	0.931	0.430	0.961	0.415	0.962	0.373	0.965	0.357	0.966
	(50,20)	$S[1]$	0.708	0.944	0.670	0.946	0.345	0.967	0.331	0.968	0.222	0.975	0.210	0.975
		$S[2]$	0.807	0.938	0.740	0.942	0.345	0.967	0.334	0.968	0.239	0.973	0.229	0.974
		$S[3]$	0.828	0.936	0.748	0.941	0.350	0.966	0.334	0.967	0.254	0.973	0.235	0.974
	(50,40)	$S[1]$	0.630	0.949	0.514	0.956	0.277	0.971	0.269	0.972	0.208	0.975	0.164	0.978
		$S[2]$	0.647	0.948	0.545	0.954	0.278	0.971	0.270	0.972	0.214	0.975	0.165	0.978
		$S[3]$	0.658	0.947	0.578	0.952	0.284	0.971	0.277	0.971	0.217	0.975	0.177	0.977
	(80,30)	$S[1]$	0.486	0.958	0.388	0.964	0.231	0.974	0.190	0.977	0.160	0.978	0.155	0.979
		$S[2]$	0.554	0.954	0.499	0.957	0.242	0.973	0.200	0.976	0.162	0.978	0.158	0.979
		$S[3]$	0.590	0.951	0.512	0.956	0.271	0.971	0.226	0.974	0.177	0.977	0.161	0.978
	(80,60)	$S[1]$	0.399	0.963	0.320	0.968	0.203	0.976	0.184	0.977	0.157	0.979	0.134	0.980
		$S[2]$	0.406	0.963	0.361	0.966	0.204	0.976	0.189	0.977	0.158	0.979	0.138	0.980
		$S[3]$	0.483	0.958	0.376	0.965	0.209	0.975	0.190	0.977	0.160	0.978	0.152	0.979
(2,2)	(20,10)	$S[1]$	1.183	0.914	1.119	0.918	0.478	0.958	0.470	0.959	0.397	0.964	0.377	0.965
		$S[2]$	1.236	0.911	1.162	0.915	0.496	0.957	0.486	0.958	0.399	0.963	0.390	0.964
		$S[3]$	1.314	0.906	1.193	0.913	0.503	0.957	0.498	0.957	0.496	0.957	0.492	0.958
	(20,15)	$S[1]$	0.868	0.934	0.766	0.940	0.406	0.963	0.388	0.964	0.350	0.966	0.336	0.967
		$S[2]$	0.892	0.932	0.829	0.936	0.423	0.962	0.407	0.963	0.368	0.965	0.354	0.966
		$S[3]$	1.140	0.917	0.913	0.931	0.425	0.962	0.414	0.962	0.373	0.965	0.356	0.966
	(50,20)	$S[1]$	0.707	0.944	0.670	0.946	0.344	0.967	0.325	0.968	0.220	0.975	0.210	0.975
		$S[2]$	0.807	0.938	0.740	0.942	0.345	0.967	0.329	0.968	0.237	0.974	0.229	0.974
		$S[3]$	0.828	0.936	0.747	0.941	0.349	0.967	0.330	0.968	0.251	0.973	0.235	0.974
	(50,40)	$S[1]$	0.630	0.949	0.514	0.956	0.274	0.971	0.267	0.972	0.207	0.976	0.164	0.978
		$S[2]$	0.647	0.948	0.545	0.954	0.276	0.971	0.268	0.972	0.213	0.975	0.165	0.978
		$S[3]$	0.658	0.947	0.577	0.952	0.282	0.971	0.276	0.971	0.214	0.975	0.176	0.977
	(80,30)	$S[1]$	0.486	0.958	0.388	0.964	0.230	0.974	0.190	0.977	0.159	0.979	0.155	0.979
		$S[2]$	0.554	0.954	0.499	0.957	0.239	0.973	0.200	0.976	0.160	0.978	0.158	0.979
		$S[3]$	0.590	0.951	0.512	0.956	0.269	0.972	0.225	0.974	0.173	0.978	0.160	0.978
	(80,60)	$S[1]$	0.399	0.963	0.320	0.968	0.200	0.976	0.184	0.977	0.155	0.979	0.131	0.980
		$S[2]$	0.406	0.963	0.361	0.966	0.203	0.976	0.188	0.977	0.156	0.979	0.137	0.980
		$S[3]$	0.483	0.958	0.376	0.965	0.208	0.975	0.189	0.977	0.157	0.979	0.152	0.979
(5,1)	(20,10)	$S[1]$	0.787	0.922	0.704	0.929	0.567	0.941	0.552	0.943	0.474	0.950	0.470	0.950
		$S[2]$	0.822	0.919	0.781	0.923	0.643	0.935	0.571	0.941	0.534	0.944	0.486	0.949
		$S[3]$	0.676	0.932	0.676	0.932	0.537	0.944	0.483	0.949	0.465	0.950	0.421	0.954
	(20,15)	$S[1]$	0.586	0.940	0.518	0.946	0.483	0.949	0.421	0.954	0.351	0.960	0.329	0.962
		$S[2]$	0.636	0.935	0.636	0.935	0.505	0.947	0.450	0.952	0.442	0.952	0.395	0.956
		$S[3]$	0.530	0.945	0.517	0.946	0.450	0.952	0.395	0.956	0.345	0.961	0.329	0.962
	(50,20)	$S[1]$	0.474	0.950	0.457	0.951	0.310	0.964	0.304	0.964	0.288	0.966	0.277	0.967
		$S[2]$	0.524	0.945	0.484	0.949	0.314	0.964	0.307	0.964	0.291	0.966	0.277	0.967
		$S[3]$	0.462	0.951	0.437	0.953	0.302	0.965	0.300	0.965	0.271	0.967	0.270	0.967
	(50,40)	$S[1]$	0.437	0.953	0.368	0.959	0.295	0.965	0.254	0.969	0.248	0.969	0.231	0.971
		$S[2]$	0.449	0.952	0.370	0.959	0.300	0.965	0.278	0.967	0.261	0.968	0.256	0.969
		$S[3]$	0.424	0.954	0.367	0.959	0.278	0.967	0.253	0.969	0.247	0.969	0.224	0.971
	(80,30)	$S[1]$	0.382	0.958	0.293	0.965	0.262	0.968	0.243	0.970	0.240	0.970	0.202	0.973
		$S[2]$	0.423	0.954	0.363	0.959	0.264	0.968	0.246	0.970	0.245	0.970	0.223	0.972
		$S[3]$	0.346	0.961	0.266	0.968	0.245	0.970	0.235	0.970	0.229	0.971	0.201	0.973
	(80,60)	$S[1]$	0.258	0.969	0.235	0.970	0.224	0.971	0.212	0.972	0.201	0.973	0.183	0.975
		$S[2]$	0.320	0.963	0.242	0.970	0.224	0.971	0.214	0.972	0.207	0.973	0.198	0.974
		$S[3]$	0.214	0.972	0.207	0.973	0.202	0.973	0.200	0.974	0.179	0.975	0.161	0.977
(5,2)	(20,10)	$S[1]$	0.787	0.922	0.704	0.929	0.565	0.942	0.547	0.943	0.471	0.950	0.461	0.951
		$S[2]$	0.822	0.919	0.781	0.923	0.643	0.935	0.565	0.942	0.532	0.944	0.481	0.949
		$S[3]$	0.676	0.932	0.676	0.932	0.532	0.944	0.475	0.949	0.457	0.951	0.417	0.954
	(20,15)	$S[1]$	0.585	0.940	0.518	0.946	0.475	0.949	0.417	0.954	0.348	0.961	0.328	0.962
		$S[2]$	0.636	0.935	0.636	0.935	0.500	0.947	0.444	0.952	0.436	0.953	0.391	0.957
		$S[3]$	0.530	0.945	0.517	0.946	0.444	0.952	0.391	0.957	0.344	0.961	0.326	0.963
	(50,20)	$S[1]$	0.474	0.950	0.457	0.951	0.308	0.964	0.303	0.965	0.287	0.966	0.275	0.967
		$S[2]$	0.524	0.945	0.483	0.949	0.312	0.964	0.304	0.964	0.288	0.966	0.276	0.967
		$S[3]$	0.462	0.951	0.437	0.953	0.300	0.965	0.300	0.965	0.271	0.967	0.270	0.967
	(50,40)	$S[1]$	0.437	0.953	0.368	0.959	0.295	0.965	0.252	0.969	0.247	0.969	0.229	0.971
		$S[2]$	0.449	0.952	0.370	0.959	0.300	0.965	0.277	0.967	0.261	0.968	0.256	0.969
		$S[3]$	0.424	0.954	0.367	0.959	0.277	0.967	0.252	0.969	0.246	0.970	0.224	0.971
	(80,30)	$S[1]$	0.382	0.958	0.293	0.965	0.262	0.968	0.243	0.970	0.240	0.970	0.202	0.973
		$S[2]$	0.423	0.954	0.363	0.959	0.264	0.968	0.245	0.970	0.245	0.970	0.221	0.972
		$S[3]$	0.346	0.961	0.266	0.968	0.244	0.970	0.234	0.971	0.227	0.971	0.200	0.974
	(80,60)	$S[1]$	0.257	0.969	0.234	0.971	0.223	0.972	0.207	0.973	0.201	0.973	0.182	0.975
		$S[2]$	0.320	0.963	0.242	0.970	0.224	0.971	0.213	0.972	0.207	0.973	0.198	0.974
		$S[3]$	0.213	0.972	0.205	0.973	0.202	0.973	0.200	0.974	0.178	0.975	0.161	0.977

Table 9. AIL (first column) and CP (second column) of the 95% interval estimates for $H(t)$.

(k, T)	(n, m)	Test	ACI-NA		ACI-NT		BCI				HPD			
							1		2		1		2	
Prior→														
(2,1)	(20,10)	$S[1]$	1.017	0.926	0.922	0.932	0.527	0.958	0.515	0.959	0.415	0.965	0.399	0.967
		$S[2]$	1.019	0.926	0.942	0.931	0.532	0.958	0.521	0.959	0.520	0.959	0.502	0.960
		$S[3]$	0.963	0.930	0.821	0.939	0.501	0.960	0.496	0.960	0.415	0.965	0.387	0.967
	(20,15)	$S[1]$	0.842	0.938	0.752	0.943	0.432	0.964	0.413	0.966	0.368	0.969	0.352	0.970
		$S[2]$	0.891	0.934	0.796	0.941	0.433	0.964	0.421	0.965	0.372	0.968	0.355	0.969
		$S[3]$	0.683	0.948	0.615	0.952	0.414	0.966	0.391	0.967	0.349	0.970	0.335	0.971
	(50,20)	$S[1]$	0.632	0.951	0.596	0.954	0.365	0.969	0.351	0.970	0.251	0.976	0.237	0.977
		$S[2]$	0.618	0.952	0.549	0.957	0.360	0.969	0.351	0.970	0.237	0.977	0.229	0.978
		$S[3]$	0.527	0.958	0.488	0.961	0.359	0.969	0.347	0.970	0.217	0.978	0.208	0.979
	(50,40)	$S[1]$	0.476	0.961	0.458	0.963	0.273	0.975	0.268	0.975	0.202	0.979	0.162	0.982
		$S[2]$	0.483	0.961	0.472	0.962	0.281	0.974	0.278	0.974	0.205	0.979	0.174	0.981
		$S[3]$	0.473	0.962	0.453	0.963	0.270	0.975	0.266	0.975	0.198	0.980	0.162	0.982
	(80,30)	$S[1]$	0.459	0.963	0.421	0.965	0.268	0.975	0.228	0.978	0.167	0.982	0.156	0.982
		$S[2]$	0.420	0.965	0.382	0.968	0.233	0.977	0.201	0.979	0.153	0.983	0.151	0.983
		$S[3]$	0.371	0.968	0.345	0.970	0.225	0.978	0.189	0.980	0.151	0.983	0.150	0.983
	(80,60)	$S[1]$	0.335	0.971	0.322	0.971	0.197	0.980	0.186	0.980	0.149	0.983	0.139	0.983
		$S[2]$	0.342	0.970	0.334	0.971	0.201	0.979	0.186	0.980	0.150	0.983	0.146	0.983
		$S[3]$	0.315	0.972	0.295	0.973	0.195	0.980	0.181	0.981	0.148	0.983	0.131	0.984
(2,2)	(20,10)	$S[1]$	0.968	0.929	0.899	0.934	0.520	0.959	0.512	0.959	0.408	0.966	0.398	0.967
		$S[2]$	1.012	0.926	0.922	0.932	0.530	0.958	0.521	0.959	0.520	0.959	0.500	0.960
		$S[3]$	0.913	0.933	0.807	0.940	0.498	0.960	0.493	0.960	0.407	0.966	0.385	0.967
	(20,15)	$S[1]$	0.769	0.942	0.699	0.947	0.428	0.965	0.412	0.966	0.368	0.969	0.352	0.970
		$S[2]$	0.871	0.936	0.766	0.943	0.430	0.964	0.419	0.965	0.372	0.968	0.355	0.969
		$S[3]$	0.632	0.951	0.577	0.955	0.409	0.966	0.391	0.967	0.347	0.970	0.334	0.971
	(50,20)	$S[1]$	0.597	0.954	0.568	0.955	0.364	0.969	0.346	0.970	0.250	0.976	0.236	0.977
		$S[2]$	0.563	0.956	0.524	0.958	0.358	0.969	0.345	0.970	0.236	0.977	0.228	0.978
		$S[3]$	0.496	0.960	0.469	0.962	0.358	0.969	0.342	0.970	0.215	0.978	0.208	0.979
	(50,40)	$S[1]$	0.465	0.962	0.440	0.964	0.272	0.975	0.267	0.975	0.200	0.979	0.162	0.982
		$S[2]$	0.475	0.962	0.444	0.964	0.280	0.974	0.276	0.974	0.201	0.979	0.172	0.981
		$S[3]$	0.457	0.963	0.437	0.964	0.266	0.975	0.266	0.975	0.197	0.980	0.161	0.982
	(80,30)	$S[1]$	0.426	0.965	0.414	0.966	0.266	0.975	0.227	0.978	0.164	0.982	0.154	0.982
		$S[2]$	0.388	0.967	0.365	0.969	0.231	0.977	0.200	0.980	0.152	0.983	0.151	0.983
		$S[3]$	0.355	0.969	0.343	0.970	0.225	0.978	0.189	0.980	0.151	0.983	0.149	0.983
	(80,60)	$S[1]$	0.335	0.971	0.319	0.972	0.197	0.980	0.185	0.980	0.146	0.983	0.135	0.984
		$S[2]$	0.337	0.971	0.324	0.971	0.200	0.979	0.185	0.980	0.149	0.983	0.145	0.983
		$S[3]$	0.299	0.973	0.281	0.974	0.192	0.980	0.178	0.981	0.146	0.983	0.131	0.984
(5,1)	(20,10)	$S[1]$	0.624	0.938	0.592	0.941	0.582	0.942	0.569	0.943	0.491	0.951	0.489	0.951
		$S[2]$	0.682	0.932	0.680	0.933	0.640	0.937	0.594	0.941	0.561	0.944	0.519	0.948
		$S[3]$	0.568	0.944	0.560	0.944	0.538	0.946	0.501	0.950	0.486	0.951	0.433	0.956
	(20,15)	$S[1]$	0.501	0.950	0.499	0.950	0.462	0.953	0.433	0.956	0.348	0.964	0.331	0.966
		$S[2]$	0.552	0.945	0.526	0.947	0.517	0.948	0.454	0.954	0.446	0.955	0.397	0.959
		$S[3]$	0.495	0.950	0.452	0.954	0.423	0.957	0.397	0.959	0.345	0.964	0.330	0.966
	(50,20)	$S[1]$	0.405	0.959	0.391	0.960	0.329	0.966	0.317	0.967	0.307	0.968	0.292	0.969
		$S[2]$	0.375	0.961	0.368	0.962	0.325	0.966	0.317	0.967	0.305	0.968	0.292	0.969
		$S[3]$	0.370	0.962	0.356	0.963	0.303	0.968	0.302	0.968	0.279	0.970	0.278	0.970
	(50,40)	$S[1]$	0.310	0.968	0.295	0.969	0.292	0.969	0.253	0.973	0.246	0.973	0.233	0.975
		$S[2]$	0.327	0.966	0.318	0.967	0.297	0.969	0.280	0.970	0.268	0.971	0.265	0.972
		$S[3]$	0.300	0.968	0.280	0.970	0.279	0.970	0.252	0.973	0.242	0.974	0.226	0.975
	(80,30)	$S[1]$	0.282	0.970	0.266	0.972	0.257	0.972	0.242	0.974	0.241	0.974	0.201	0.978
		$S[2]$	0.282	0.970	0.278	0.970	0.263	0.972	0.245	0.973	0.241	0.974	0.223	0.976
		$S[3]$	0.252	0.973	0.240	0.974	0.231	0.975	0.229	0.975	0.221	0.976	0.200	0.978
	(80,60)	$S[1]$	0.239	0.974	0.238	0.974	0.227	0.975	0.219	0.976	0.206	0.977	0.197	0.978
		$S[2]$	0.231	0.975	0.228	0.975	0.222	0.976	0.202	0.977	0.197	0.978	0.180	0.980
		$S[3]$	0.206	0.977	0.201	0.978	0.199	0.978	0.189	0.979	0.176	0.980	0.168	0.981
(5,2)	(20,10)	$S[1]$	0.620	0.939	0.584	0.942	0.573	0.943	0.560	0.944	0.490	0.951	0.485	0.951
		$S[2]$	0.681	0.933	0.679	0.934	0.639	0.937	0.586	0.942	0.560	0.944	0.517	0.948
		$S[3]$	0.566	0.944	0.552	0.945	0.526	0.947	0.490	0.951	0.479	0.952	0.430	0.956
	(20,15)	$S[1]$	0.499	0.950	0.490	0.951	0.462	0.953	0.430	0.956	0.346	0.964	0.331	0.966
		$S[2]$	0.545	0.946	0.519	0.948	0.508	0.949	0.450	0.954	0.445	0.955	0.395	0.960
		$S[3]$	0.454	0.954	0.445	0.955	0.413	0.958	0.395	0.960	0.342	0.965	0.328	0.966
	(50,20)	$S[1]$	0.399	0.959	0.379	0.961	0.325	0.966	0.313	0.967	0.305	0.968	0.291	0.969
		$S[2]$	0.372	0.962	0.366	0.962	0.321	0.967	0.313	0.967	0.303	0.968	0.290	0.969
		$S[3]$	0.362	0.963	0.348	0.964	0.303	0.968	0.302	0.968	0.279	0.970	0.278	0.970
	(50,40)	$S[1]$	0.304	0.968	0.293	0.969	0.290	0.969	0.252	0.973	0.242	0.974	0.231	0.975
		$S[2]$	0.322	0.966	0.306	0.968	0.296	0.969	0.279	0.970	0.268	0.971	0.265	0.972
		$S[3]$	0.297	0.969	0.279	0.970	0.269	0.971	0.251	0.973	0.242	0.974	0.225	0.975
	(80,30)	$S[1]$	0.271	0.971	0.262	0.972	0.257	0.972	0.241	0.974	0.240	0.974	0.200	0.978
		$S[2]$	0.279	0.970	0.267	0.972	0.262	0.972	0.245	0.974	0.240	0.974	0.222	0.976
		$S[3]$	0.241	0.974	0.240	0.974	0.230	0.975	0.229	0.975	0.215	0.976	0.199	0.978
	(80,60)	$S[1]$	0.238	0.974	0.231	0.975	0.227	0.975	0.212	0.977	0.205	0.977	0.196	0.978
		$S[2]$	0.229	0.975	0.223	0.976	0.222	0.976	0.198	0.978	0.192	0.978	0.179	0.980
		$S[3]$	0.205	0.977	0.200	0.978	0.196	0.978	0.183	0.979	0.175	0.980	0.164	0.981

the lifetime of 36 cancer patients by a group of gastric given chemotherapy and radiation treatment. This dataset was first provided by Bekker et al. [8] and later reanalyzed by El-Sagheer et al. [36].

Table 10. Data points of diamond sizes and gastric cancers.

Diamond Size Data											
1	1.5	2	2	2	2	2.5	3	3	4.5	5	7
7	7.5	9	16.5	20.5	28	39	40.5	69.5	137	257.5	358
Gastric Cancer Data											
0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282	0.296	0.334	0.395
0.458	0.466	0.501	0.507	0.529	0.534	0.540	0.570	0.641	0.644	0.696	0.841
0.863	1.099	1.219	1.271	1.326	1.447	1.485	1.533	1.581	1.589	2.178	2.343

To evaluate the adequacy of the NH distribution, the MLEs of the model parameters a and b , along with their corresponding standard errors (Std.Ers), were obtained using the data summarized in Table 10. Furthermore, the Kolmogorov–Smirnov ($\mathcal{KS}$) goodness-of-fit statistic and its associated P -value were computed to formally assess model suitability. The numerical results indicate that the NH model provides a satisfactory fit for both datasets under consideration. For each dataset, Figure 3 illustrates the fitted and empirical reliability curves, as well as the probability–probability (P–P) and quantile–quantile (Q–Q) plots. The strong agreement observed between the empirical patterns and the corresponding model-based estimates across these graphical diagnostics corroborates the goodness-of-fit results reported in Table 11. In particular, the panels displayed in Figure 3(a)–(c) reveal a high degree of consistency between the theoretical and empirical representations. In addition, scaled total time on test (TTT) plots, likelihood contour maps, and violin plots are presented in Figure 4 to further explore the data characteristics. The estimated hazard rate functions shown in Figure 4(a) exhibit increasing and upside-down bathtub-shaped behaviors for the diamond and cancer datasets, respectively, which are fully consistent with the theoretical hazard rate properties of the NH distribution. Moreover, the contour plots in Figure 4(b) confirm the existence and uniqueness of the MLEs of a and b . Consequently, the resulting estimates $\hat{a}$ and $\hat{b}$ reported in Table 11 were adopted as initial values in all subsequent numerical and computational analyses. Finally, Figure 4(c) reveals that the distribution of the given diamond dataset is extremely right-skewed, with most observations concentrated near the lower end and a few very large outliers stretching the upper tail substantially. In contrast, for the gastric cancer dataset as shown in Figure 4(c), the data distribution appears unimodal and moderately right-skewed, with values fairly concentrated around the median and no evidence of extreme outliers.

To facilitate a deeper interpretation of the estimation outcomes derived from the complete diamond and gastric cancer datasets, a sequence of first-failure censored samples was constructed under various grouping structures. Specifically, each dataset was randomly divided into independent groups of fixed sizes, with $k = 2$ and 3 considered for the diamond data and $k = 2$ and 4 for the gastric cancer data. The resulting first-failure samples corresponding to these grouping schemes are reported in Table 12. Building on these samples, multiple APFFC samples were subsequently generated under different combinations of the termination time T , effective sample

Table 11. Summary fit of the NH distribution from diamond and gastric cancer datasets.

Data	MLE(Std.Er)		$\mathcal{KS}(P\text{-value})$
	a	b	
Diamond	0.3028(0.0586)	0.5388(0.3145)	0.1329(0.7688)
Cancer	0.8593(0.0871)	1.7963(0.1533)	0.0712(0.5345)

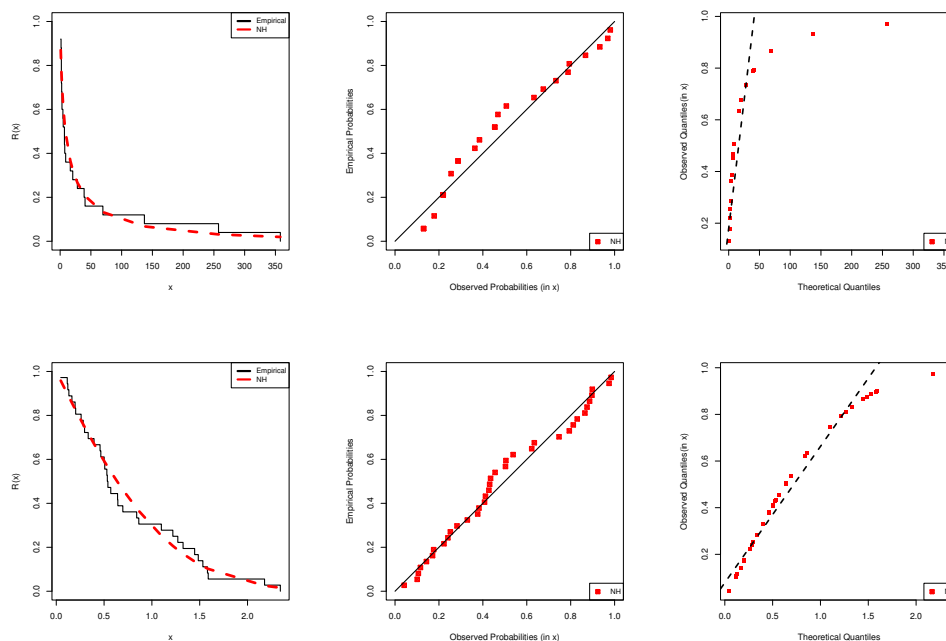


Figure 3. The RF (left), PP (middle), and QQ (right) fitting diagrams of the NH distribution from diamond (top) and gastric cancer (bottom) datasets.

size m , and censoring scheme $\mathbf{S}$, as summarized in Table 13.

For each APFFC configuration, Table 13 presents the MLEs of the parameters a , b , $R(t)$, and $H(t)$, evaluated at $t = 5$ for the diamond data and $t = 0.5$ for the gastric cancer data. The corresponding 95% asymptotic intervals (including ACI-NA and ACI-NT) and their widths were also computed. From a Bayesian perspective, posterior inference was performed using the MCMC algorithm described in Section 3, with a total of $\mathcal{M} = 30,000$ iterations, of which the first 5,000 iterations were discarded as burn-in. Under the SEL, posterior means and the associated 95% BCI and HPD intervals were obtained for all unknown quantities.

For both applications, the NH model parameters were assigned weak gamma prior distributions with moderate scale parameters to provide diffuse yet proper prior support. Specifically, the hyperparameters were set as $\eta_i = \kappa_i = 0.001$ for $i = 1, 2$, ensuring minimal prior influence on posterior inference while maintaining numerical stability in the Bayesian computations. Tables 14 and 15 summarize the resulting point estimates (along with their standard errors) and 95% interval bounds (together with their widths) for a , b , $R(t)$, and $H(t)$.

For the real data analysis, the maximum likelihood estimation was initialized at the positive

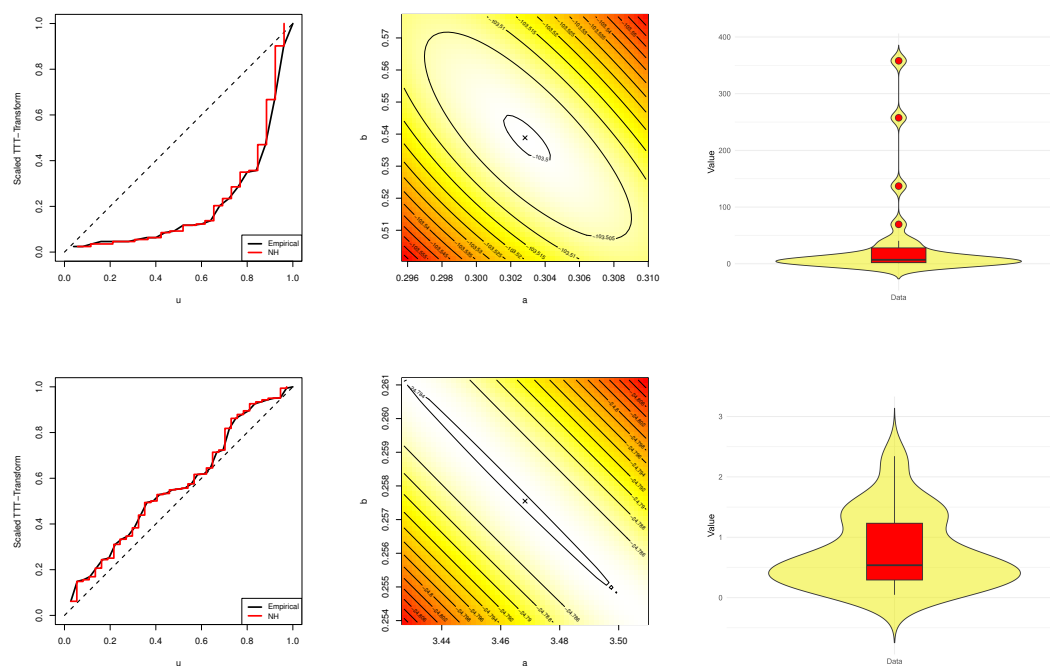


Figure 4. The TTT (left), Contour (middle), and Violin (right) diagrams of the NH distribution from diamond (top) and gastric cancer (bottom) datasets.

starting values $(0.1, 0.1)$ and subsequently optimized using the NR algorithm. The resulting MLEs were then employed as the initial values for the MCMC algorithm in the Bayesian analysis. Using the corresponding MLEs as starting values places the Markov chains in a high-posterior-density region, thereby shortening the burn-in period, improving computational efficiency, and facilitating rapid convergence to the target posterior distribution.

A comparison between the likelihood-based and Bayesian estimators reported in Tables 14 and 15 clearly demonstrates the superiority of the Bayesian approach. Both symmetric and asymmetric Bayesian estimators consistently deliver more accurate point estimates and tighter interval estimates for all parameters, as reflected by smaller standard errors and shorter interval lengths. These findings underscore the efficiency gains achieved by incorporating prior information within the Bayesian framework, particularly under adaptive censoring schemes.

To illustrate the existence and practical uniqueness of the estimators through the geometry of the objective surface, Figure 5 displays contour plots of the objective function $(S[ij])$, for $i = 1, 2$ and $j = 1, 2, \dots, 6$ over the (a, b) parameter space. These contour plots serve as diagnostic tools for the associated optimization problem. In each case, the presence of a single, clearly defined minimum provides strong empirical evidence supporting both the existence and uniqueness of the resulting estimates.

To further assess the convergence properties of the Bayesian estimation procedure, Figures 6 and 7 present MCMC posterior histograms with Gaussian kernel density estimates, along with corresponding trace plots for the parameters a , b , $R(t)$, and $H(t)$, using the APFFC samples $S[i1]$ ($i = 1, 2$) as representative examples. In these figures, the solid line denotes the posterior mean, while the dashed lines correspond to the lower and upper bounds of the HPD credible interval. The trace plots reveal stable fluctuations around constant posterior means, with no visible

Table 12. Different first-failure censored samples from diamond and gastric cancer datasets.

k	1	2	3	4	5	6	7	8	9	10	11	12
Diamond Data												
2	1	2	2.5	3	4.5	5	7.5	16.5	20.5	39	69.5	137
3	1	3	5	9	16.5	20.5	28	40.5				
Cancer Data												
2	0.115	0.164	0.197	0.203	0.260	0.282	0.296	0.395	0.458	0.466	0.501	0.529
	0.540	0.641	0.696	0.841	1.271	1.326	1.485	1.581				
4	0.047	0.197	0.282	0.458	0.501	0.534	0.641	0.863	1.219	1.326		

Table 13. Different APFFC samples from diamond and gastric cancer datasets.

Sample	(n, m, k)	$\mathbf{S}$	$\tau(r)$	R^*	Times										
Diamond Data															
S[11]	(24,10,2)	$(1^2, 0^8)$	70(10)	0	1	2	2.5	3	4.5	7.5	16.5	20.5	39	69.5	
S[12]		$(1, 0^8, 1)$	45(9)	2	1	2	2.5	3	4.5	5	16.5	20.5	39	69.5	
S[13]		$(0^8, 1^2)$	20(8)	4	1	2	2.5	3	4.5	5	7.5	16.5	20.5	39	
S[14]	(24,6,3)	$(1^2, 0^4)$	30(6)	0	1	3	9	16.5	20.5	28					
S[15]		$(1, 0^4, 1)$	20(5)	3	1	3	5	9	16.5	28					
S[16]		$(0^4, 1^2)$	10(4)	6	1	3	5	9	16.5	20.5					
Cancer Data															
S[21]	(40,15,2)	$(1^5, 0^{10})$	1.5(15)	0	0.115	0.164	0.197	0.203	0.282	0.296	0.395	0.466	0.501	0.529	
					0.540	0.696	0.841	1.326	1.485						
S[22]		$(0^5, 1^5, 0^5)$	0.51(9)	2	0.115	0.164	0.197	0.203	0.260	0.282	0.296	0.458	0.501	0.529	
					0.540	0.641	0.696	0.841	1.271						
S[23]		$(0^{10}, 1^5)$	0.75(13)	4	0.115	0.164	0.197	0.203	0.260	0.282	0.296	0.395	0.458	0.466	
					0.501	0.540	0.641	0.841	1.326						
S[24]	(40,8,4)	$(1^2, 0^6)$	1.25(8)	0	0.047	0.197	0.282	0.458	0.534	0.641	0.863	1.219			
S[25]		$(0^3, 1^2, 0^3)$	0.47(4)	4	0.047	0.197	0.282	0.458	0.501	0.534	0.863	1.326			
S[26]		$(0^6, 1^2)$	0.55(6)	8	0.047	0.197	0.282	0.458	0.501	0.534	0.641	0.863			

trends, drifts, or regime shifts, indicating that the Markov chains have successfully converged to their stationary distributions. Moreover, the bounded variability and sustained overlap with the credible bands suggest efficient mixing and support the asymptotic convergence of the MCMC algorithm.

Posterior summary statistics were computed based on the retained 25,000 post-burn-in MCMC samples for each parameter, including the posterior mean, mode, first quartile (Q_1), median, third quartile (Q_3), standard deviation (St.D), and skewness; see Table 16. The numerical summaries reported therein are fully consistent with the point and interval estimates presented in Table 14 and align well with the graphical diagnostics shown in Figures 6 and 7.

Additional insight into the posterior behavior is provided by the skewness measures. For both diamond and gastric cancer datasets, the posterior distributions of a , b , $R(t)$, and $H(t)$ appear well balanced and approximately symmetric, further confirming the stability and robustness of

Table 14. Estimates of a , b , $R(t)$, and $H(t)$ from diamond data.

Sample	Par.	MLE		95% ACI-NA			95% BCI		
		MCMC		95% ACI-NT			95% HPD		
		Est.	Std.Er	Low.	Upp.	Width	Low.	Upp.	Width
S[11]	a	0.4100	0.2453	0.0071	0.8908	0.8837	0.4079	0.4117	0.0039
		0.4098	0.0010	0.1269	1.3245	1.1976	0.4079	0.4117	0.0038
	b	0.1283	0.1441	0.0054	0.4108	0.4053	0.1260	0.1299	0.0039
		0.1279	0.0011	0.0142	1.1594	1.1452	0.1260	0.1299	0.0039
	$R(t)$	0.7982	0.0806	0.6403	0.9561	0.3158	0.7962	0.8014	0.0052
		0.7988	0.0014	0.6550	0.9729	0.3179	0.7961	0.8013	0.0052
	$H(t)$	0.0393	0.0134	0.0130	0.0656	0.0526	0.0386	0.0397	0.0010
		0.0392	0.0003	0.0201	0.0767	0.0566	0.0387	0.0397	0.0010
	a	0.3021	0.1511	0.0060	0.5982	0.5922	0.2998	0.3037	0.0039
		0.3018	0.0011	0.1134	0.8050	0.6916	0.2998	0.3037	0.0039
S[12]	b	0.2034	0.2108	0.0097	0.6166	0.6068	0.2012	0.2051	0.0039
		0.2031	0.0010	0.0267	1.5505	1.5238	0.2011	0.2050	0.0039
	$R(t)$	0.7897	0.0795	0.6339	0.9455	0.3116	0.7882	0.7921	0.0040
		0.7901	0.0011	0.6483	0.9619	0.3136	0.7881	0.7920	0.0040
	$H(t)$	0.0377	0.0119	0.0143	0.0610	0.0467	0.0372	0.0380	0.0008
		0.0376	0.0002	0.0203	0.0700	0.0498	0.0372	0.0379	0.0008
	a	0.3369	0.1861	0.0278	0.7016	0.6739	0.3348	0.3386	0.0039
		0.3367	0.0010	0.1141	0.9946	0.8805	0.3347	0.3386	0.0039
	b	0.1948	0.2047	0.0643	0.5961	0.5318	0.1927	0.1966	0.0039
		0.1946	0.0010	0.0248	1.5279	1.5031	0.1926	0.1966	0.0039
S[13]	$R(t)$	0.7729	0.0797	0.6167	0.9291	0.3124	0.7712	0.7754	0.0042
		0.7733	0.0011	0.6315	0.9460	0.3146	0.7712	0.7754	0.0042
	$H(t)$	0.0418	0.0130	0.0164	0.0673	0.0509	0.0413	0.0422	0.0008
		0.0417	0.0002	0.0228	0.0769	0.0541	0.0413	0.0422	0.0008
	a	0.6432	0.3759	0.0936	1.3800	1.2864	0.6413	0.6452	0.0039
		0.6432	0.0010	0.2046	2.0224	1.8178	0.6413	0.6452	0.0039
	b	0.7042	1.3347	0.0000	3.3201	3.3201	0.7022	0.7062	0.0040
		0.7042	0.0010	0.0172	28.909	28.892	0.7022	0.7062	0.0040
	$R(t)$	0.1942	0.2265	0.0000	0.6382	0.6382	0.1925	0.1958	0.0033
		0.1942	0.0009	0.0197	1.9110	1.8912	0.1925	0.1959	0.0033
S[14]	$H(t)$	0.2644	0.1157	0.0376	0.4911	0.4535	0.2627	0.2661	0.0033
		0.2644	0.0009	0.1121	0.6233	0.5112	0.2627	0.2660	0.0033
	a	0.6508	1.4579	0.0000	3.5082	3.5082	0.6487	0.6526	0.0039
		0.6505	0.0010	0.0081	52.524	52.516	0.6486	0.6524	0.0038
	b	0.0374	0.1100	0.0075	0.2530	0.2455	0.0338	0.0375	0.0037
		0.0355	0.0021	0.0001	11.889	11.889	0.0338	0.0372	0.0034
	$R(t)$	0.8886	0.0658	0.7596	1.0176	0.2580	0.8885	0.8986	0.0101
		0.8939	0.0059	0.7686	1.0275	0.2589	0.8893	0.8986	0.0093
	$H(t)$	0.0229	0.0113	0.0008	0.0451	0.0443	0.0208	0.0230	0.0021
		0.0218	0.0013	0.0087	0.0602	0.0515	0.0208	0.0228	0.0020
S[15]	a	0.6321	1.2745	0.0000	3.1300	3.1300	0.6311	0.6330	0.0020
		0.6320	0.0005	0.0121	32.895	32.883	0.6311	0.6330	0.0020
	b	0.0624	0.1804	0.0000	0.4159	0.4159	0.0612	0.0632	0.0020
		0.0622	0.0005	0.0002	18.089	18.089	0.0612	0.0632	0.0020
	$R(t)$	0.8293	0.1066	0.6204	1.0383	0.4179	0.8274	0.8321	0.0047
		0.8298	0.0013	0.6446	1.0670	0.4223	0.8274	0.8321	0.0047
	$H(t)$	0.0357	0.0174	0.0016	0.0697	0.0681	0.0351	0.0361	0.0010
		0.0356	0.0003	0.0137	0.0926	0.0789	0.0351	0.0361	0.0010

Table 15. Estimates of a , b , $R(t)$, and $H(t)$ from diamond data.

Sample	Par.	MLE		95% ACI-NA			95% BCI		
		MCMC		95% ACI-NT			95% HPD		
		Est.	Std.Er	Low.	Upp.	Width	Low.	Upp.	Width
S[11]	a	0.9752	1.4556	0.0000	3.8281	3.8281	0.9655	0.9850	0.0195
		0.9751	0.0050	0.0523	18.181	18.128	0.9655	0.9850	0.0195
	b	0.4708	0.8607	0.0000	2.1577	2.1577	0.4608	0.4806	0.0198
		0.4706	0.0050	0.0131	16.941	16.928	0.4607	0.4804	0.0198
	$R(t)$	0.7954	0.0548	0.6880	0.9027	0.2147	0.7911	0.7998	0.0087
		0.7955	0.0022	0.6950	0.9103	0.2153	0.7910	0.7997	0.0087
	$H(t)$	0.4567	0.1179	0.2256	0.6878	0.4622	0.4455	0.4677	0.0221
		0.4565	0.0057	0.2754	0.7575	0.4821	0.4456	0.4677	0.0221
S[12]	a	1.4393	2.6781	0.0000	6.6883	6.6883	1.4295	1.4492	0.0197
		1.4393	0.0050	0.0375	55.204	55.167	1.4294	1.4490	0.0197
	b	0.3856	0.8384	0.0000	2.0288	2.0288	0.3758	0.3955	0.0197
		0.3855	0.0050	0.0054	27.335	27.329	0.3756	0.3953	0.0197
	$R(t)$	0.7491	0.0620	0.6276	0.8706	0.2430	0.7432	0.7551	0.0119
		0.7492	0.0030	0.6369	0.8810	0.2441	0.7434	0.7552	0.0118
	$H(t)$	0.5998	0.1565	0.2931	0.9064	0.6133	0.5825	0.6168	0.0343
		0.5996	0.0087	0.3597	1.0001	0.6404	0.5823	0.6164	0.0341
S[13]	a	0.9067	1.2175	0.0000	3.2929	3.2929	0.8970	0.9165	0.0195
		0.9066	0.0050	0.0652	12.602	12.537	0.8970	0.9165	0.0195
	b	0.6073	1.0118	0.0000	2.5904	2.5904	0.5973	0.6172	0.0198
		0.6072	0.0050	0.0232	15.906	15.883	0.5972	0.6170	0.0198
	$R(t)$	0.7620	0.0589	0.6466	0.8774	0.2307	0.7578	0.7662	0.0084
		0.7621	0.0021	0.6550	0.8866	0.2316	0.7578	0.7662	0.0084
	$H(t)$	0.5372	0.1414	0.2600	0.8144	0.5544	0.5259	0.5484	0.0225
		0.5370	0.0057	0.3207	0.9000	0.5793	0.5258	0.5483	0.0224
S[14]	a	0.7760	1.5651	0.0000	3.8436	3.8436	0.7662	0.7858	0.0195
		0.7758	0.0050	0.0149	40.429	40.414	0.7663	0.7858	0.0195
	b	0.6595	1.7254	0.0000	4.0413	4.0413	0.6495	0.6693	0.0198
		0.6594	0.0050	0.0039	111.21	111.20	0.6494	0.6692	0.0198
	$R(t)$	0.7807	0.0865	0.6112	0.9503	0.3390	0.7768	0.7847	0.0079
		0.7808	0.0020	0.6284	0.9701	0.3417	0.7769	0.7848	0.0079
	$H(t)$	0.4801	0.1694	0.1481	0.8122	0.6641	0.4698	0.4901	0.0203
		0.4799	0.0052	0.2404	0.9588	0.7183	0.4699	0.4902	0.0202
S[15]	a	0.9487	1.4718	0.0000	3.8333	3.8333	0.9390	0.9585	0.0195
		0.9486	0.0050	0.0453	19.848	19.802	0.9390	0.9585	0.0195
	b	0.8612	1.8277	0.0000	4.4435	4.4435	0.8512	0.8711	0.0198
		0.8611	0.0051	0.0134	55.163	55.150	0.8511	0.8709	0.0198
	$R(t)$	0.6673	0.1164	0.4392	0.8954	0.4562	0.6628	0.6718	0.0091
		0.6673	0.0023	0.4741	0.9393	0.4652	0.6628	0.6719	0.0091
	$H(t)$	0.8021	0.2836	0.2462	1.3579	1.1117	0.7877	0.8165	0.0288
		0.8020	0.0073	0.4011	1.6040	1.2029	0.7878	0.8165	0.0288
S[16]	a	1.1456	3.4646	0.0000	7.9361	7.9361	1.1359	1.1555	0.0195
		1.1456	0.0050	0.0031	429.77	429.77	1.1359	1.1555	0.0195

the Bayesian inference.

6. Conclusion

In this study, statistical inference for the Nadarajah–Haghighi distribution was developed under the adaptive progressive first-failure censoring scheme, a flexible life-testing design that inte-

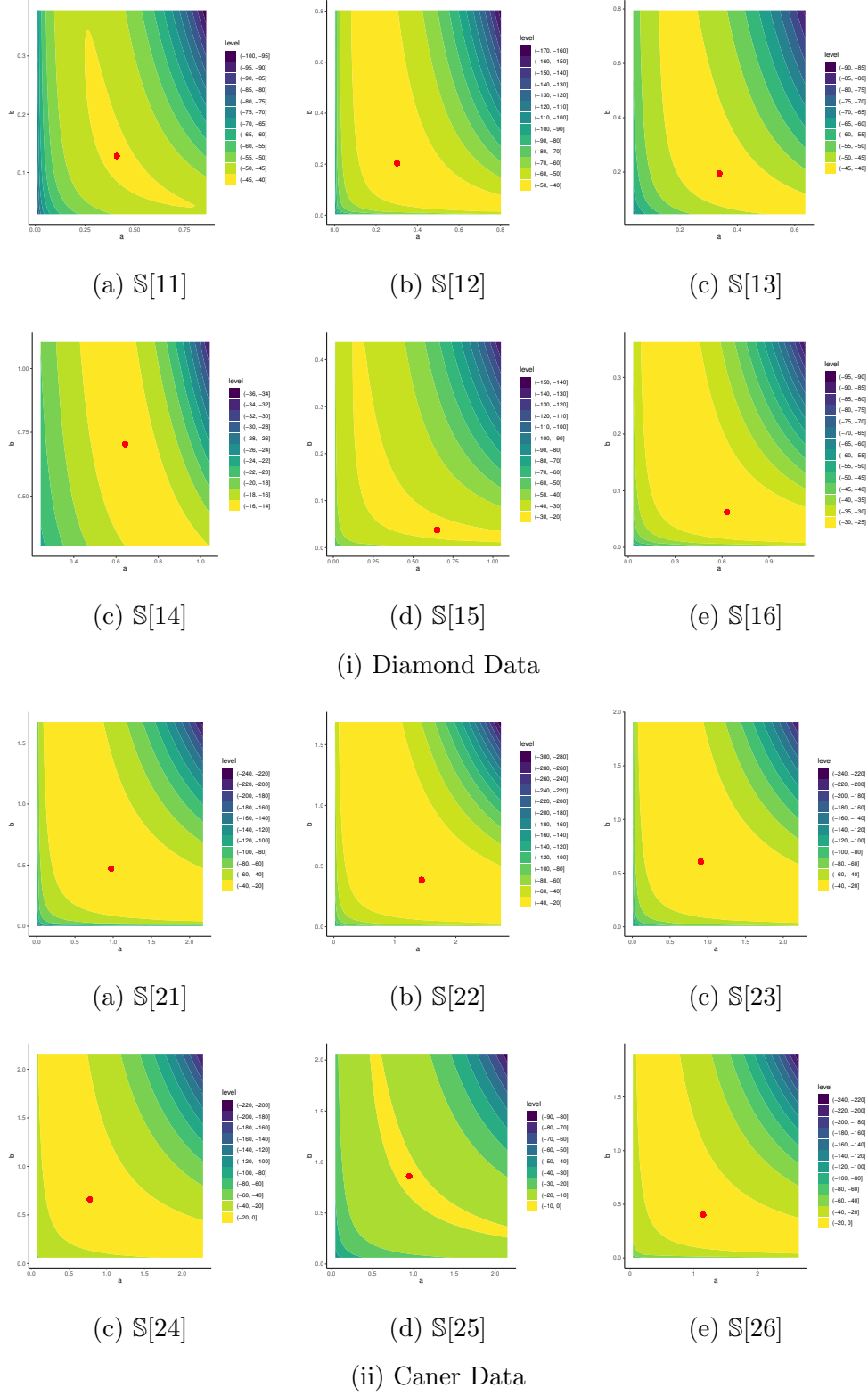


Figure 5. Contour maps of a and b from diamond and gastric cancer datasets.

Table 16. Statistics for 30,000 posterior iterations of a , b , $R(t)$, and $H(t)$ from diamond and gastric cancer datasets.

Sample	Par.	Mean	Mode	Q_1	Median	Q_3	St.D	Skew.
Diamond Data								
S[11]	a	0.410	0.409	0.409	0.410	0.410	0.0010	0.0021
	b	0.1279	0.1244	0.1273	0.1279	0.1286	0.0010	-0.0158
	$R(t)$	0.7988	0.8037	0.7979	0.7988	0.7997	0.0013	0.0380
	$H(t)$	0.0392	0.0382	0.0390	0.0392	0.0393	0.0003	-0.0348
S[12]	a	0.3018	0.2994	0.3011	0.3018	0.3025	0.0010	0.0351
	b	0.2031	0.2015	0.2024	0.2031	0.2038	0.0010	0.0286
	$R(t)$	0.7901	0.7929	0.7895	0.7901	0.7908	0.0010	-0.0148
	$H(t)$	0.0376	0.0370	0.0374	0.0376	0.0377	0.0002	0.0215
S[13]	a	0.3367	0.3343	0.3360	0.3367	0.3374	0.0010	0.0379
	b	0.1946	0.1930	0.1939	0.1946	0.1953	0.0010	0.0312
	$R(t)$	0.7733	0.7763	0.7726	0.7733	0.7740	0.0011	-0.0016
	$H(t)$	0.0417	0.0411	0.0416	0.0417	0.0419	0.0002	0.0060
S[14]	a	0.6432	0.6411	0.6425	0.6432	0.6439	0.0010	0.0415
	b	0.7042	0.7029	0.7035	0.7042	0.7049	0.0010	0.0210
	$R(t)$	0.1942	0.1946	0.1936	0.1942	0.1948	0.0009	-0.0173
	$H(t)$	0.2644	0.2628	0.2638	0.2644	0.2650	0.0009	0.0374
S[15]	a	0.6505	0.6495	0.6498	0.6505	0.6512	0.0010	0.1890
	b	0.0355	0.0342	0.0348	0.0354	0.0362	0.0010	0.3446
	$R(t)$	0.8939	0.8977	0.8921	0.8940	0.8959	0.0026	-0.3306
	$H(t)$	0.0218	0.0210	0.0214	0.0218	0.0222	0.0006	0.3342
S[16]	a	0.6320	0.6319	0.6317	0.6320	0.6324	0.0005	0.0205
	b	0.0622	0.0608	0.0618	0.0622	0.0625	0.0005	0.0464
	$R(t)$	0.8298	0.8331	0.8289	0.8298	0.8306	0.0012	-0.0368
	$H(t)$	0.0356	0.0348	0.0354	0.0356	0.0358	0.0003	0.0389
Cancer Data								
S[11]	a	0.975	0.964	0.972	0.975	0.978	0.0050	0.0414
	b	0.4706	0.4601	0.4672	0.4706	0.4740	0.0050	0.0263
	$R(t)$	0.7955	0.7948	0.7940	0.7955	0.7970	0.0022	-0.0267
	$H(t)$	0.4565	0.4418	0.4527	0.4565	0.4603	0.0056	0.0373
S[12]	a	1.4393	1.4273	1.4359	1.4392	1.4426	0.0050	0.0161
	b	0.3855	0.3755	0.3822	0.3855	0.3889	0.0050	0.0148
	$R(t)$	0.7492	0.7480	0.7471	0.7492	0.7512	0.0030	-0.0115
	$H(t)$	0.5996	0.5836	0.5937	0.5995	0.6054	0.0087	0.0267
S[13]	a	0.9066	0.8958	0.9032	0.9066	0.9100	0.0050	0.0425
	b	0.6072	0.5966	0.6037	0.6072	0.6106	0.0050	0.0254
	$R(t)$	0.7621	0.7620	0.7606	0.7621	0.7635	0.0021	-0.0266
	$H(t)$	0.5370	0.5220	0.5332	0.5370	0.5409	0.0057	0.0382
S[14]	a	0.7758	0.7651	0.7724	0.7758	0.7792	0.0050	0.0419
	b	0.6594	0.6486	0.6559	0.6594	0.6628	0.0050	0.0239
	$R(t)$	0.7808	0.7811	0.7795	0.7808	0.7822	0.0020	-0.0243
	$H(t)$	0.4799	0.4666	0.4764	0.4799	0.4834	0.0051	0.0357
S[15]	a	0.9486	0.9403	0.9452	0.9486	0.9520	0.0050	0.0417
	b	0.8611	0.8505	0.8577	0.8611	0.8645	0.0051	0.0208
	$R(t)$	0.6673	0.6684	0.6658	0.6674	0.6689	0.0023	-0.0223
	$H(t)$	0.8020	0.7830	0.7970	0.8019	0.8069	0.0073	0.0377
S[16]	a	1.1456	1.1347	1.1421	1.1455	1.1489	0.0050	0.0387
	b	0.4018	0.3913	0.3984	0.4018	0.4052	0.0050	0.0253
	$R(t)$	0.7919	0.7907	0.7902	0.7919	0.7936	0.0025	-0.0237
	$H(t)$	0.4727	0.4561	0.4683	0.4727	0.4772	0.0066	0.0355

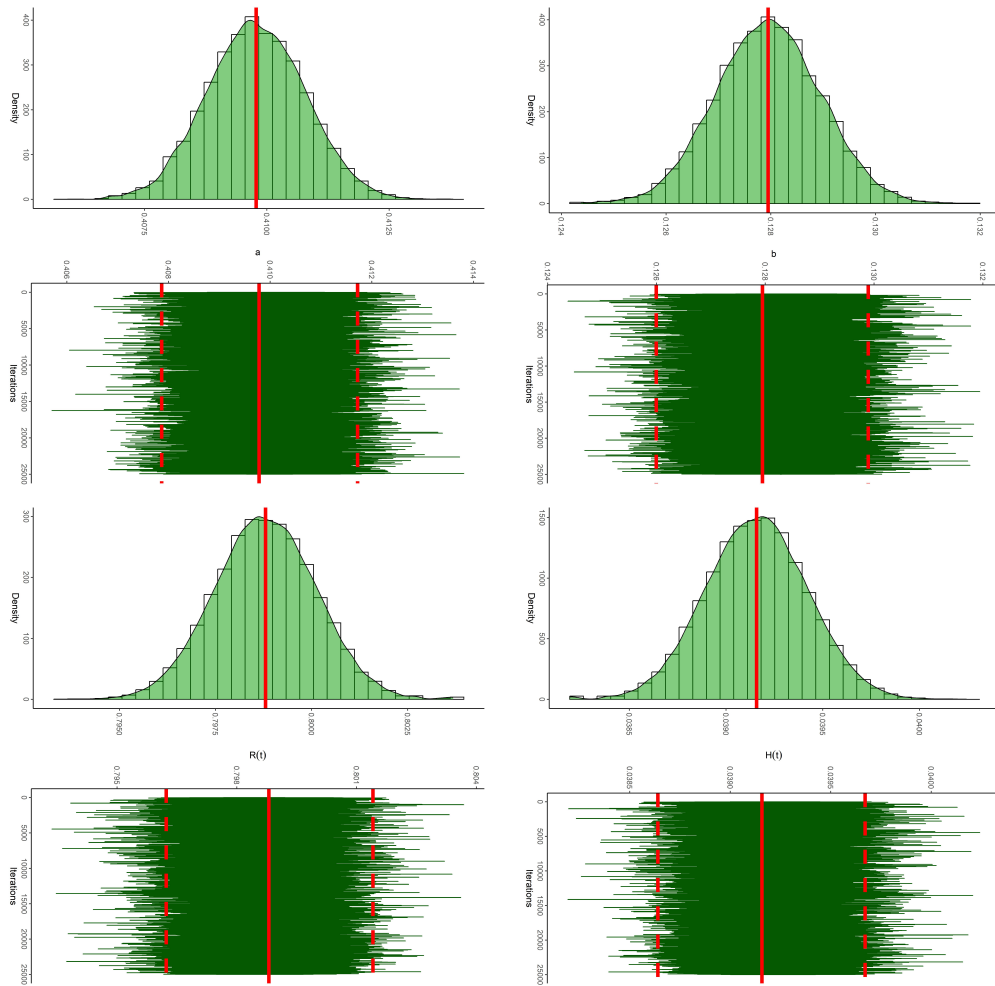


Figure 6. Convergence diagnostics of a , b , $R(t)$, and $H(t)$ from diamond data.

grates adaptive time control with guaranteed failure termination. The maximum likelihood estimation procedures were established for the model parameters and the associated reliability measures. Approximate confidence intervals were constructed using the observed Fisher information matrix, while the delta method was employed to obtain interval estimation for the reliability and hazard rate functions. From a Bayesian perspective, independent gamma prior distributions were assumed for the positive parameters, and posterior inference was carried out using the Markov chain Monte Carlo simulation implemented through the Metropolis–Hastings algorithm. The simulation-based approach enabled the computation of Bayes point estimates as well as highest posterior density credible intervals for the parameters and reliability characteristics. The finite-sample performance of the proposed estimators was examined through an extensive Monte Carlo simulation study under various censoring configurations. The results indicated that both classical and Bayesian procedures provided reliable inference, with Bayesian methods demonstrating competitive performance, particularly under moderate and heavy censoring levels. The applicability of the proposed methodology was further confirmed through the analysis of real lifetime data. Overall, the developed inferential framework offers a comprehensive and practical approach for analyzing lifetime data arising from adaptive progressive

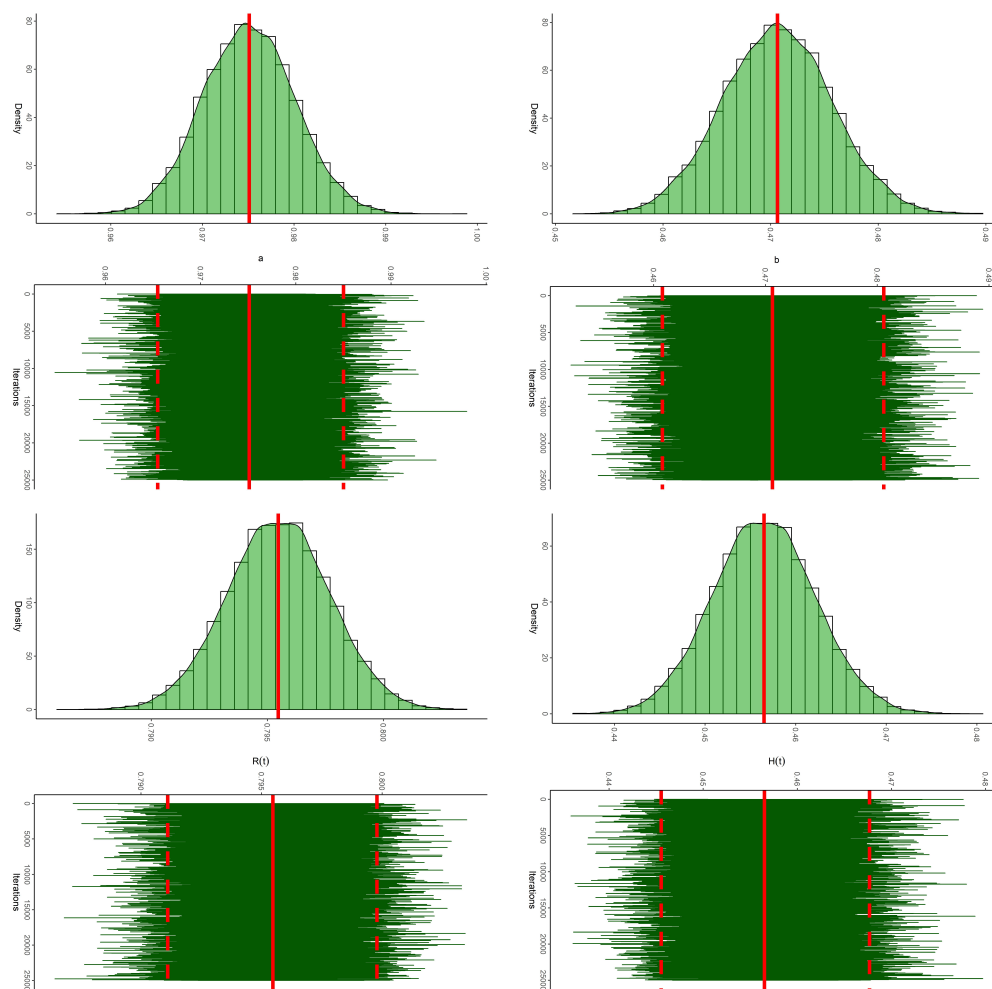


Figure 7. Convergence diagnostics of a , b , $R(t)$, and $H(t)$ from gastric cancer data.

first-failure censoring experiments.

The present study has some limitations that also provide opportunities for future research. The proposed inferential framework is developed specifically for the Nadarajah–Haghighi distribution under the adaptive progressive first-failure censoring scheme, and its extension to other lifetime distributions and censoring schemes deserves further investigation. Moreover, while the maximum likelihood approach relies on asymptotic theory, the Bayesian procedure requires Markov chain Monte Carlo sampling, which may be computationally intensive for more complex models or large-scale simulation studies. Although the proposed interval estimation methods demonstrated satisfactory performance in the simulation study, bootstrap-based confidence intervals may offer an attractive alternative, particularly for finite samples. However, their implementation requires repeated numerical optimization, resulting in substantial computational effort. Therefore, the development of efficient bootstrap-based inference under adaptive progressive first-failure censoring constitutes another promising direction for future research.

Author contribution. Conceptualization: Refah Alotaibi and Mazen Nassar; Methodology: Refah Alotaibi and Mazen Nassar; Investigation: Mazen Nassar and Ahmed Elshahhat; Funding Acquisition: Refah Alotaibi; Formal Analysis: Mazen Nassar and Ahmed Elshahhat; Software

/ Coding: Ahmed Elshahhat; Visualization: Ahmed Elshahhat; Writing – Review & Editing: Refah Alotaibi, Mazen Nassar, and Ahmed Elshahhat.

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Data availability. The authors confirm that the data supporting the findings of this study are available within the article.

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Conflicts of interest. The authors declare no conflict of interest.

Use of generative-AI tools declaration. The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Appendix 1: R scripts

This part presents the main functions of the proposed NH model, implemented in the R environment.

- **The PDF and CDF:**

```
pdf_NH <- function(par,x){
  a  = par[1]
  b  = par[2]
  res = a*b*(1+b*x)^(a-1)*exp(1-(1+b*x)^(a))
  return(res)
}
```

```
cdf_NH <- function(par,x){
  a  = par[1]
  b  = par[2]
  res = 1-exp(1-(1+b*x)^(a))
  return(res)
}
```

- **The KS Test:**

```
res_NH <- goodness.fit(pdf = pdf_NH, cdf = cdf_NH,
                        starts = c(0.1,0.1),
                        data = x, method = "BFGS",
                        domain = c(0, Inf), mle = NULL)

print(res_NH)
```

- **The MLE NR-based:**

```

loglik <- function(param){
  a = param[1]
  b = param[2]
  ll = sum(log(PDF_1(a,b,x[1:m]))) +
        sum((k*(R[1:r]+1)-1)*log(1-CDF_1(a,b,x[1:r]))) +
        sum((n-m-sum(k*(R_r[1:r]+1)-1))*log(1-CDF_1(a,b,x[m])))
}
out = maxLik(loglik,startvalue)

```

- The Bayes MCMC-based:

```

LL <- function(param){
  a = param[1]
  b = param[2]
  ll = sum(log(PDF_1(a,b,x[1:m]))) +
        sum((k*(R[1:r]+1)-1)*log(1-CDF_1(a,b,x[1:r]))) +
        sum((n-m-sum(k*(R_r[1:r]+1)-1))*log(1-CDF_1(a,b,x[m])))
  return(ll)
}

Pri <- function(param){
  a = param[1]
  b = param[2]
  a_prior = dgamma(a, shape=eta_1, rate=kappa_1, log = T)
  b_prior = dgamma(b, shape=eta_2, rate=kappa_2, log = T)
  return(a_prior+b_prior)
}

posterior <- function(param){
  return (LL(param)+ Pri(param))
}

proposalfunction <- function(param){
  return(rnorm(no.par, mean=param, sd=c(sqrt(V_11),sqrt(V_22))))
}

run_metropolis_MCMC <- function(startvalue, iterations){
  chain = array(dim = c(iterations+1,no.par))
  chain[1,] = startvalue
  for (i in 1:iterations){
    proposal = proposalfunction(chain[i,])
    probab = exp(posterior(proposal) - posterior(chain[i,]))
    if (runif(1) < probab){
      chain[i+1,] = proposal
    }else{
      chain[i+1,] = chain[i,]
    }
  }
  return(mcmc(chain))
}

startvalue = c(coef(out)[1],coef(out)[2])

```

```
chain = run_metropolis_MCMC(startvalue, 12000)
```

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