

DEFINITION, EXISTENCE, NEGATIVITY, AND INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR MTH-CONVEX FUNCTIONS

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Abstract In the paper, the authors introduce a concept of MTH-convex functions, provide an illustrative example of the MTH-convex functions, prove that every MTH-convex function is negative, and establish several new integral inequalities of the Hermite–Hadamard type for this class of functions.

Keywords MTH-convex function, property, example, convex function, integral inequality, Hermite–Hadamard type.

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1. Introduction

Throughout this paper, we adopt the notations

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_- = (-\infty, 0), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

The following definition is the classical concept of convex functions.

Definition 1.1 ([13, Definition 1.1.1]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on an interval I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Theorem 1.2 ([13, p. 37]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I for $a, b \in I$ with $a < b$. Then the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

is valid.

The inequality (1.1) is well known in the literature as the Hermite–Hadamard integral inequality. It has been refined and generalized by a number of mathematicians. For more information, please refer to [4, 13], for example.

In [18], Tunç and Yildırım defined MT-convexity as follows.

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Definition 1.3 ([18, Definition 2]). Let $I \subseteq \mathbb{R}$ be an interval. A nonnegative function $f : I \rightarrow \mathbb{R}_0$ is said to be MT-convex if the inequality

$$f(tx + (1-t)y) \leq \frac{1}{2} \left[\frac{\sqrt{t}}{\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{\sqrt{t}} f(y) \right]$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

In the paper [18], the following integral inequalities for MT-convex functions were established.

Theorem 1.4 ([16, Theorem 1] and [18, Theorem 2]). Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \rightarrow \mathbb{R}_0$ is MT-convex such that $a, b \in I$ with $a < b$ and $f \in L_1([a, b])$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx$$

and

$$\frac{2}{b-a} \int_a^b \tau(x) f(x) dx \leq \frac{f(a) + f(b)}{2},$$

where

$$\tau(x) = \frac{\sqrt{(b-x)(x-a)}}{b-a}, \quad x \in [a, b].$$

In [15], the concept of MT- h -convex functions was introduced and some Hermite–Hadamard type inequalities of MT- h -convex functions were established via classical and generalized fractional integrals. For other recent results concerning the Hermite–Hadamard, Ostrowski, and Simpson type inequalities of MT-convex functions, see [3, 5–12, 14–17] and the references cited therein.

In [20], the concept of GT-convex functions was introduced.

Definition 1.5 ([20, p. 3954]). Let $I \subseteq \mathbb{R}_+$ be an interval. A function $f : I \rightarrow \mathbb{R}_0$ is said to be GT-convex if the inequality

$$f(x^t y^{1-t}) \leq \frac{1}{2} \left[\frac{\sqrt{t}}{\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{\sqrt{t}} f(y) \right]$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

In [20], Wang et al. established the following integral inequalities.

Theorem 1.6 ([20, Theorem 3.1]). Suppose that $I \subseteq \mathbb{R}_+$ is a real interval. If a function $f : I \rightarrow \mathbb{R}_0$ is GT-convex on I such that $a, b \in I$ with $a < b$ and $f \in L_1([a, b])$, then

$$f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{\pi}{4} [f(a) + f(b)].$$

In [1], Bai et al. introduced the concept “HT-convex functions” and established some Hermite–Hadamard type integral inequalities of HT-convex functions.

Definition 1.7 ([1, p. 4]). Suppose that $I \subseteq \mathbb{R}_+$ is an interval and $f : I \rightarrow \mathbb{R}_0$ is a nonnegative function. If for all $x, y \in I$ and $t \in (0, 1)$,

$$f\left(\frac{xy}{ty + (1-t)x}\right) \leq \frac{1}{2} \left[\frac{\sqrt{t}}{\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{\sqrt{t}} f(y) \right],$$

then f is said to be an HT-convex function.

Theorem 1.8 ([1, Theorem 4.1]). Suppose that $I \subseteq \mathbb{R}_- \cup \mathbb{R}_+$ is a real interval. If $f : I \rightarrow \mathbb{R}_0$ is HT-convex such that $a, b \in I$ with $a < b$ and $f \in L_1([a, b])$, then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{\pi}{4}[f(a) + f(b)].$$

In this paper, we will introduce a new concept of MTH-convex functions, give an example of MTH-convex functions, study their basic properties, and establish several integral inequalities of the Hermite–Hadamard type for MTH-convex functions.

2. MTH-convex functions and their properties

For $x, y \in \mathbb{R}$ and $t \in [0, 1]$ with $xy > 0$, denote

$$A_t(x, y) = tx + (1-t)y \quad \text{and} \quad H_t(x, y) = \left(\frac{t}{x} + \frac{1-t}{y}\right)^{-1}. \quad (2.1)$$

In particular, the quantities

$$A(x, y) = A_{1/2}(x, y) \quad \text{and} \quad H(x, y) = H_{1/2}(x, y)$$

are called the arithmetic mean and harmonic mean, respectively.

For $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$, it is well known that

$$H_t(x, y) \leq A_t(x, y). \quad (2.2)$$

If $x, y \in \mathbb{R}_-$ and $t \in [0, 1]$, the reversed inequality of (2.2) holds.

We now introduce the concept of MTH-convex functions.

Definition 2.1. Let $I \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}_- \cup \mathbb{R}_+$ be intervals. A function $f : I \rightarrow D$ is said to be MTH-convex on I if the inequality

$$f(tx + (1-t)y) \leq \frac{2}{\frac{\sqrt{t}}{\sqrt{1-t}} \frac{1}{f(x)} + \frac{\sqrt{1-t}}{\sqrt{t}} \frac{1}{f(y)}} \quad (2.3)$$

holds for all $x, y \in I$ and $t \in (0, 1)$. If the inequality (2.3) is reversed, then we say that f is an MTH-concave function.

Remark 2.2. Under the conditions of Definition 2.1, the inequality (2.3) is equivalent to

$$f(A_t(x, y)) \leq 2\sqrt{t(1-t)} H_t(f(x), f(y)) \quad (2.4)$$

for $x, y \in I$ and $t \in (0, 1)$.

Proposition 2.3. Let $I \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}_- \cup \mathbb{R}_+$ be intervals, and $f : I \rightarrow D$ be a function. If f is MTH-convex (or MTH-concave, respectively), then it is negative (or positive, respectively).

Proof. For all $x = y \in I$ and $t \in (0, 1) \setminus \{\frac{1}{2}\}$, by the MTH-convexity of the function f , we have

$$f(x) = f(tx + (1-t)y) \leq \frac{2}{\frac{\sqrt{t}}{\sqrt{1-t}} \frac{1}{f(x)} + \frac{\sqrt{1-t}}{\sqrt{t}} \frac{1}{f(y)}} = 2\sqrt{t(1-t)} f(x). \quad (2.5)$$

Since $f(x) \neq 0$ and $0 < \sqrt{t(1-t)} < \frac{1}{2}$ for any $x \in I$ and $t \in (0, 1)$ with $t \neq \frac{1}{2}$, we derive that $f(x) < 0$ for all $x \in I$.

Similarly, using the MTH-concavity of the function f , we can easily see that the reversed inequality of (2.5) holds. Hence, it follows that $f(x) > 0$ for all $x \in I$. Proposition 2.3 is thus proved. $\square$

Example 2.4. Let $f(x) = x^3$ for $x \in \mathbb{R}_-$. Then f is MTH-convex on $\mathbb{R}_-$.

Proof. According to Remark 2.2, we only need to prove the inequality

$$[tx + (1-t)y]^3 \leq \frac{2\sqrt{t(1-t)}(xy)^3}{ty^3 + (1-t)x^3} \quad (2.6)$$

for $x, y \in \mathbb{R}_-$ and $t \in (0, 1)$.

For all $x, y \in \mathbb{R}_-$ and $t \in (0, 1)$, without loss of generality, we assume that $y \leq x$. Putting $u = \frac{x}{y} \in (0, 1]$ in the inequality (2.6), dividing both sides by y^3 , and noticing that $y^3 < 0$ reverses the inequality sign and $t + (1-t)u^3 > 0$, we obtain

$$[tu + (1-t)]^3 [t + (1-t)u^3] \geq 2\sqrt{t(1-t)}u^3, \quad t \in (0, 1), \quad u \in (0, 1].$$

For all $t \in (0, 1)$, define

$$g(u) = [tu + (1-t)]^3 \left[\frac{t}{u^3} + (1-t) \right], \quad t \in (0, 1), \quad u \in (0, 1].$$

Direct differentiation yields

$$g'(u) = 3t(1-t)[tu + (1-t)]^2 \frac{u^4 - 1}{u^4} \leq 0, \quad t \in (0, 1), \quad u \in (0, 1].$$

Therefore, the function $g(u)$ is decreasing on $(0, 1]$. Hence, we acquire

$$[tu + (1-t)]^3 \left[\frac{t}{u^3} + (1-t) \right] \geq 1 \geq 2\sqrt{t(1-t)}, \quad t \in (0, 1), \quad u \in (0, 1].$$

Accordingly, the function $f(x) = x^3$ is MTH-convex on $\mathbb{R}_-$. The proof of Example 2.4 is thus complete. $\square$

Remark 2.5. In [19, Remark 2.3 and Proposition 2.8], the MTH-convex functions were mentioned, but not studied in detail.

In [2, 21, 22], several mathematicians investigated other similar convexities of functions.

3. Two lemmas

In order to establish several integral inequalities of the Hermite–Hadamard type for MTH-convex functions, we need the following two lemmas.

Lemma 3.1. Let $x, y \in \mathbb{R}_-$. Then

$$P(x, y) = \int_0^1 \sqrt{t(1-t)} H_t(x, y) dt = \begin{cases} \frac{\pi xy(x+y+2\sqrt{xy})}{2(y-x)^2}, & x \neq y, \\ \frac{\pi}{8}x, & x = y \end{cases}$$

and

$$Q(x, y) = \int_0^1 H_t(x, y) dt = \begin{cases} \frac{xy \ln \frac{y}{x}}{y - x}, & x \neq y, \\ x, & x = y, \end{cases}$$

where $H_t(x, y)$ is defined by (2.1).

Proof. If $x = y$, then $H_t(x, x) = x$ for $t \in (0, 1)$, so

$$P(x, x) = x \int_0^1 \sqrt{t(1-t)} dt = \frac{\pi}{8}x \quad \text{and} \quad Q(x, x) = x.$$

Let $x \neq y$. Since $H_t(x, y) = H_{1-t}(y, x)$, the substitution $t \rightarrow 1 - t$ shows that P and Q are symmetric in x and y , as are the right-hand sides in the lemma. Hence we may assume $y < x < 0$. Put $w = \frac{y-x}{x} > 0$. Then $1 + w = \frac{y}{x}$ and

$$H_t(x, y) = \frac{xy}{ty + (1-t)x} = \frac{y}{1+tw}.$$

Direct integration gives

$$Q(x, y) = y \int_0^1 \frac{dt}{1+tw} = \frac{y \ln(1+w)}{w} = \frac{xy \ln \frac{y}{x}}{y-x}.$$

Set

$$I(w) = \int_0^1 \frac{\sqrt{t(1-t)}}{1+tw} dt,$$

so that $P(x, y) = yI(w)$. The substitution $t = \sin^2 \theta$ yields

$$I(w) = 2 \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1+w \sin^2 \theta} d\theta.$$

With $s = \sin^2 \theta$, we have the partial-fraction identity

$$\frac{s(1-s)}{1+ws} = -\frac{s}{w} + \frac{w+1}{w^2} - \frac{w+1}{w^2} \frac{1}{1+ws}.$$

Together with

$$\int_0^{\pi/2} \sin^2 \theta d\theta = \frac{\pi}{4}, \quad \int_0^{\pi/2} d\theta = \frac{\pi}{2}, \quad \int_0^{\pi/2} \frac{d\theta}{1+w \sin^2 \theta} = \frac{\pi}{2\sqrt{1+w}},$$

the last of which follows from $u = \tan \theta$, this gives

$$I(w) = -\frac{\pi}{2w} + \frac{(w+1)\pi}{w^2} - \frac{\pi\sqrt{1+w}}{w^2} = \frac{\pi(w+2-2\sqrt{1+w})}{2w^2}.$$

Finally, using

$$w+2 = \frac{x+y}{x}, \quad w^2 = \frac{(y-x)^2}{x^2}, \quad \sqrt{1+w} = \sqrt{\frac{y}{x}}, \quad \text{and} \quad x\sqrt{\frac{y}{x}} = -\sqrt{xy}$$

(because $x < 0$), we obtain

$$\begin{aligned} P(x, y) &= \frac{\pi y(w + 2 - 2\sqrt{1 + w})}{2w^2} \\ &= \frac{\pi xy\left(x + y - 2x\sqrt{\frac{y}{x}}\right)}{2(y - x)^2} \\ &= \frac{\pi xy(x + y + 2\sqrt{xy})}{2(y - x)^2}. \end{aligned}$$

The proof of Lemma 3.1 is thus complete. $\square$

Lemma 3.2. *Let $x, y \in \mathbb{R}_-$. Then*

$$\begin{aligned} E(x, y) &= \int_0^1 t(1 - t) H_t(x, y) dt \\ &= \begin{cases} \frac{xy(y^2 - x^2 - 2xy \ln \frac{y}{x})}{2(y - x)^3}, & x \neq y, \\ \frac{1}{6}x, & x = y, \end{cases} \\ D(x, y) &= \int_0^1 t(1 - t) H_t^2(x, y) dt \\ &= \begin{cases} \frac{(xy)^2[(x + y) \ln \frac{y}{x} - 2(y - x)]}{(y - x)^3}, & x \neq y, \\ \frac{1}{6}x^2, & x = y, \end{cases} \end{aligned}$$

and

$$\begin{aligned} C(x, y) &= \int_0^1 \sqrt{t(1 - t)} H_t^2(x, y) dt \\ &= \begin{cases} \frac{\pi(xy)^2(|x + y| - 2\sqrt{xy})}{2(y - x)^2\sqrt{xy}}, & x \neq y, \\ \frac{\pi}{8}x^2, & x = y, \end{cases} \end{aligned}$$

where $H_t(x, y)$ is defined by (2.1).

Proof. If $x \neq y$, assume that $y < x$. Putting $w = \frac{y-x}{x}$ and direct computation give

$$\begin{aligned} \int_0^1 t(1 - t) H_t(x, y) dt &= y \frac{w(w + 2) - 2(w + 1) \ln(w + 1)}{2w^3} \\ &= \frac{xy(y^2 - x^2 - 2xy \ln \frac{y}{x})}{2(y - x)^3}, \\ \int_0^1 t(1 - t) H_t^2(x, y) dt &= y^2 \frac{(w + 2) \ln(w + 1) - 2w}{w^3} \\ &= \frac{(xy)^2[(x + y) \ln \frac{y}{x} - 2(y - x)]}{(y - x)^3}, \end{aligned}$$

and

$$\begin{aligned}
 \int_0^1 \sqrt{t(1-t)} H_t^2(x, y) dt &= \frac{\pi y^2 (\sqrt{1+w} - 1)^2}{2w^2 \sqrt{1+w}} \\
 &= \frac{\pi x y^2 \left((x+y) - 2x \sqrt{\frac{y}{x}} \right)}{2(y-x)^2 \sqrt{\frac{y}{x}}} \\
 &= \frac{\pi x y^2 [(x+y) + 2\sqrt{xy}]}{2(y-x)^2 \sqrt{\frac{y}{x}}} \\
 &= \frac{\pi (xy)^2 (|x+y| - 2\sqrt{xy})}{2(y-x)^2 \sqrt{xy}},
 \end{aligned}$$

where $xy > 0$, and the last step uses that $x+y = -|x+y|$ and

$$\sqrt{\frac{y}{x}} = \frac{\sqrt{xy}}{|x|} = -\frac{\sqrt{xy}}{x}$$

since $x, y < 0$. The proof of Lemma 3.2 is thus completed. $\square$

4. Integral inequalities for MTH-convex functions

Now we start to establish several new integral inequalities of the Hermite–Hadamard type for MTH-convex functions.

Theorem 4.1. *Let $I \subseteq \mathbb{R}$ be an interval and $a, b \in I$ with $a < b$. If the function $f : I \rightarrow \mathbb{R}_-$ is MTH-convex on I and $f \in L_1([a, b])$, then*

$$\frac{1}{b-a} \int_a^b f(x) dx \leq 2P(f(a), f(b)),$$

where $P(x, y)$ is defined in Lemma 3.1.

Proof. Making the variable change $x = A_t(a, b)$ for $t \in (0, 1)$, using the MTH-convexity of f , employing the inequality (2.4), and utilizing Lemma 3.1, we acquire

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b f(x) dx &= \int_0^1 f(A_t(a, b)) dt \\
 &\leq 2 \int_0^1 \sqrt{t(1-t)} H_t(f(a), f(b)) dt \\
 &= 2P(f(a), f(b)).
 \end{aligned}$$

Theorem 4.1 is thus proved. $\square$

Theorem 4.2. *Let $I \subseteq \mathbb{R}$ be a real interval and $a, b \in I$ with $a < b$. If the function $f : I \rightarrow \mathbb{R}_-$ is MTH-convex on I and $f \in L_1([a, b])$, then*

$$\frac{1}{b-a} \int_a^b \frac{f(x)}{\tau(x)} dx \leq 2Q(f(a), f(b)), \quad (4.1)$$

where $Q(x, y)$ is defined in Lemma 3.1 and $\tau(x)$ is defined by

$$\tau(x) = \frac{\sqrt{(x-a)(b-x)}}{b-a}, \quad x \in [a, b]. \quad (4.2)$$

Proof. Making the variable change $x = A_t(a, b)$ for $t \in (0, 1)$ leads to $\tau(x) = \sqrt{t(1-t)}$. From the MTH-convexity of f , we deduce

$$\frac{1}{b-a} \int_a^b \frac{f(x)}{\tau(x)} dx = \int_0^1 \frac{f(A_t(a, b))}{\sqrt{t(1-t)}} dt \leq 2 \int_0^1 H_t(f(a), f(b)) dt.$$

By Lemma 3.1, the inequality (4.1) holds. The proof of Theorem 4.2 is complete. $\square$

Theorem 4.3. Let $I \subseteq \mathbb{R}$ be a real interval and $a, b \in I$ with $a < b$. If the function $f : I \rightarrow \mathbb{R}_-$ is MTH-convex on I and $f \in L_1([a, b])$, then

$$\frac{1}{b-a} \int_a^b \tau(x) f(x) dx \leq 2E(f(a), f(b)),$$

where $E(x, y)$ is defined in Lemma 3.2 and $\tau(x)$ is defined in Theorem 4.2.

Proof. Taking the variable change $x = A_t(a, b)$ for $t \in (0, 1)$ and using Lemma 3.2, we derive

$$\begin{aligned} \frac{1}{b-a} \int_a^b \tau(x) f(x) dx &= \int_0^1 \sqrt{t(1-t)} f(A_t(a, b)) dt \\ &\leq 2 \int_0^1 t(1-t) H_t(f(a), f(b)) dt \\ &= 2E(f(a), f(b)). \end{aligned}$$

Theorem 4.3 is proved. $\square$

Theorem 4.4. Let $I \subseteq \mathbb{R}$ be an interval and $a, b \in I$ with $a < b$. If the functions $f, g : I \rightarrow \mathbb{R}_-$ are MTH-convex on I and $fg \in L_1([a, b])$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \geq \left[\frac{H(f(a), g(a))H(f(b), g(b))}{G(f(a)g(a), f(b)g(b))} \right]^2 D(f(a) + g(a), f(b) + g(b)),$$

where $H(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}}$ is the harmonic mean, $G(x, y) = \sqrt{xy}$ is the geometric mean, and $D(x, y)$ is defined in Lemma 3.2.

Proof. Since f and g are MTH-convex functions, using the inequality (2.4), we obtain

$$f(A_t(a, b)) \leq 2\sqrt{t(1-t)} H_t(f(a), f(b)) \quad (4.3)$$

and

$$g(A_t(a, b)) \leq 2\sqrt{t(1-t)} H_t(g(a), g(b))$$

for all $t \in (0, 1)$. Utilizing the negativity $f(x) < 0$ and $g(x) < 0$ for $x \in I$ and the geometric-arithmetic mean inequality, we have

$$[tf(b) + (1-t)f(a)][tg(b) + (1-t)g(a)] \leq \frac{1}{4}[t(f(b) + g(b)) + (1-t)(f(a) + g(a))]^2. \quad (4.4)$$

Thus, from the inequalities (4.3) to (4.4), it follows that

$$\begin{aligned} f(A_t(a, b))g(A_t(a, b)) &\geq 4t(1-t)H_t(f(a), f(b))H_t(g(a), g(b)) \\ &\geq \frac{4t(1-t)f(a)f(b)g(a)g(b)}{[tf(b) + (1-t)f(a)][tg(b) + (1-t)g(a)]} \\ &\geq \frac{16t(1-t)f(a)f(b)g(a)g(b)}{[t(f(b) + g(b)) + (1-t)(f(a) + g(a))]^2} \end{aligned} \quad (4.5)$$

for all $t \in (0, 1)$.

Integrating on both sides of the inequality (4.5) with respect to $t \in (0, 1)$ and applying the integral transform $x = ta + (1-t)b$ for $t \in (0, 1)$, we gain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x) \, dx &= \int_0^1 f(A_t(a, b))g(A_t(a, b)) \, dt \\ &\geq \int_0^1 \frac{16t(1-t)f(a)f(b)g(a)g(b)}{[t(f(b) + g(b)) + (1-t)(f(a) + g(a))]^2} \, dt. \end{aligned}$$

Using Lemma 3.2, we acquire

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x) \, dx &\geq \frac{16f(a)f(b)g(a)g(b)}{[(f(a) + g(a))(f(b) + g(b))]^2} \\ &\quad \times \int_0^1 t(1-t)H_t^2(f(a) + g(a), f(b) + g(b)) \, dt \\ &= \frac{16f(a)f(b)g(a)g(b)}{[(f(a) + g(a))(f(b) + g(b))]^2} D(f(a) + g(a), f(b) + g(b)) \\ &= \left[\frac{H(f(a), g(a))H(f(b), g(b))}{G(f(a)g(a), f(b)g(b))} \right]^2 D(f(a) + g(a), f(b) + g(b)). \end{aligned} \quad (4.6)$$

Theorem 4.4 is thus proved. $\square$

Theorem 4.5. Let $I \subseteq \mathbb{R}$ be a real interval and $a, b \in I$ with $a < b$. If the functions $f, g : I \rightarrow \mathbb{R}_+$ are MTH-convex on I and $fg \in L_1([a, b])$, then

$$\frac{1}{b-a} \int_a^b \frac{f(x)g(x)}{\tau(x)} \, dx \geq \left[\frac{H(f(a), g(a))H(f(b), g(b))}{G(f(a)g(a), f(b)g(b))} \right]^2 C(f(a) + g(a), f(b) + g(b)),$$

where $H(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}}$ is the harmonic mean, $G(x, y) = \sqrt{xy}$ is the geometric mean, $C(x, y)$ is defined in Lemma 3.2, and $\tau(x)$ is defined by (4.2).

Proof. By virtue of the method used in the proof of Theorem 4.4 and in view of the inequalities (4.5) and (4.6), we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b \frac{f(x)g(x)}{\tau(x)} \, dx &= \int_0^1 \frac{f(A_t(a, b))g(A_t(a, b))}{\sqrt{t(1-t)}} \, dt \\ &\geq \int_0^1 \frac{4t(1-t)H_t(f(a), f(b))H_t(g(a), g(b))}{\sqrt{t(1-t)}} \, dt \\ &= \int_0^1 4\sqrt{t(1-t)} H_t(f(a), f(b))H_t(g(a), g(b)) \, dt \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 \frac{16\sqrt{t(1-t)} f(a)f(b)g(a)g(b)}{[t(f(b)+g(b))+(1-t)(f(a)+g(a))]^2} dt \\
&= \frac{16f(a)f(b)g(a)g(b)}{[(f(a)+g(a))(f(b)+g(b))]^2} \\
&\quad \times \int_0^1 \sqrt{t(1-t)} H_t^2(f(a)+g(a), f(b)+g(b)) dt \\
&= \frac{16f(a)f(b)g(a)g(b)}{[(f(a)+g(a))(f(b)+g(b))]^2} C(f(a)+g(a), f(b)+g(b)).
\end{aligned}$$

Thus, the proof of Theorem 4.5 is concluded. $\square$

5. Conclusions

In Section 1, we reviewed recent developments of several generalized convex functions, including the classical convex functions, MT-convex functions, GT-convex functions, and HT-convex functions.

In Section 2, we introduced the new concept of MTH-convex functions, proved that every MTH-convex (respectively, MTH-concave) function is negative (respectively, positive), and provided an illustrative example of MTH-convex functions.

In Section 3, two simple lemmas were prepared for Section 4.

In Section 4, we presented five integral inequalities of the Hermite–Hadamard type for MTH-convex functions.

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