

BIFURCATION ANALYSIS AND CHAOS CONTROL FOR A DISCRETE SPACE–TIME LOKTA–VOLTERRA DIFFUSION EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract This paper investigates a discrete Lotka–Volterra competition diffusion system subject to Neumann boundary conditions. By employing normal form theory and center manifold reduction, we show that the system exhibits a flip bifurcation at the unique positive equilibrium, whereas neither a Neimark–Sacker nor a fold bifurcation is observed, moreover, a Turing bifurcation is identified. To regulate the bifurcation behavior, both direct and indirect state feedback control strategies are implemented. Under appropriate parameter conditions, these control schemes effectively stabilize the system by suppressing chaos and bifurcation-induced instabilities. Extensive numerical simulations are conducted to validate the theoretical results. The bifurcation analysis framework developed herein is generalizable to other discrete diffusion systems with different boundary configurations.

Keywords Bifurcation, chaos control, stability, center manifold reduction.

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1. Introduction

The Lotka–Volterra model, formulated by Alfred J. Lotka and V. Volterra [12, 21], provides a foundational framework in mathematical ecology for studying interspecies interactions, including predator–prey dynamics and competitive exclusion. Its continuous time formulation has been extensively studied for capturing key ecological processes such as population oscillations, coexistence, and extinction [1, 6, 14, 18, 19]. However, increasing recognition that many ecosystems evolve in discrete time steps, driven by seasonal reproduction, environmental fluctuations, or sampling protocols, has positioned the discrete time Lotka–Volterra model as an essential analytical tool. As a result, this discrete–time version has become indispensable for analyzing population dynamics in systems with non–overlapping generations or pulsed reproductive events.

A discrete time Lotka–Volterra competition model, which incorporates density–dependent growth to bound population dynamics between two competing species, takes the following form [11]

$$\begin{cases} x(t+1) = x(t) \exp \left[r_1 - a_{11}x(t) - a_{12}y(t) \right], \\ y(t+1) = y(t) \exp \left[r_2 - a_{21}x(t) - a_{22}y(t) \right], \end{cases} \quad (1.1)$$

where $x(t)$ and $y(t)$ denotes, respectively, the size or density of individuals in species x and

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y at time t , r_i represents intrinsic growth rates (ensuring population growth at low densities). a_{ii} represents intraspecific competition coefficients (self-regulation), and a_{ij} represents Inter-specific competition coefficients (between-species competition) for $1 \leq i, j \leq 2$.

Let $u(t) = a_{11}x_t$, $v(t) = a_{22}y_t$, and $r_1 = r_2$, $a_{11} = a_{22}$, $a_{12} = a_{21}$, $\frac{a_{12}}{a_{11}} = \frac{a_{21}}{a_{22}} = a$, we have the following symmetrically formulated form:

$$\begin{cases} u(t+1) = u(t) \exp \left[r - u(t) - av(t) \right], \\ v(t+1) = v(t) \exp \left[r - au(t) - v(t) \right]. \end{cases} \quad (1.2)$$

The pioneering theoretical contributions of Skellam [17] and Turing [20] established the foundational premise that spatial structure is essential for population stability—a view strongly supported by subsequent studies [5, 8, 9, 22, 25]. Spatial models are therefore not merely an alternative but often a necessity, as they uniquely capture the localized dynamics of inherently spatial processes, such as disease transmission and resource competition. A discrete spatiotemporal Lotka–Volterra diffusion model subject to Neumann boundary conditions can be formulated as follow.

$$\begin{cases} u_i^{t+1} = u_i^t \exp \left[r - u_i^t - av_i^t \right] + d_1 \nabla^2 u_i^t, \\ v_i^{t+1} = v_i^t \exp \left[r - au_i^t - v_i^t \right] + d_2 \nabla^2 v_i^t, \\ \Delta u_0^t = \Delta u_m^t = \Delta v_0^t = \Delta v_m^t = 0, i = 1, 2, \dots, m, t \in Z^+, \end{cases} \quad (1.3)$$

where d_1 and d_2 are diffusion coefficients, and $\nabla^2 u_i^t = u_{i+1}^t - 2u_i^t + u_{i-1}^t$, $\nabla^2 v_i^t = v_{i+1}^t - 2v_i^t + v_{i-1}^t$. If $d_1 = d_2 = d$, the system (1.3) can be written as

$$\begin{cases} u_i^{t+1} = u_i^t \exp \left[r - u_i^t - av_i^t \right] + d \nabla^2 u_i^t, \\ v_i^{t+1} = v_i^t \exp \left[r - au_i^t - v_i^t \right] + d \nabla^2 v_i^t, \\ \Delta u_0^t = \Delta u_m^t = \Delta v_0^t = \Delta v_m^t = 0, i = 1, 2, \dots, m, t \in Z^+. \end{cases} \quad (1.4)$$

Bifurcation theory occupies a central position in the study of nonlinear dynamical systems, particularly in ecological modeling where it provides critical insights into regime shifts and stability transitions. In recent years, significant achievements have been made in the bifurcation analysis of discrete dynamical systems describing interspecies interactions (see [2, 3, 10, 16, 27]). Research demonstrates that the center manifold theorem and normal form theory constitute the core methodological framework for analyzing bifurcation behaviors in such systems. The center manifold theorem reduces high-dimensional systems to low-dimensional center manifolds, while normal form theory transforms differential (difference) equations into topologically equivalent systems with simplest forms in the neighborhood of critical parameters, thereby effectively revealing the bifurcation directions and stability characteristics of bifurcation solutions. However, the primary bottleneck in practical applications lies in the precise computation of center manifolds, a challenge that becomes particularly pronounced in multivariate scenarios. Current research predominantly focuses on bifurcation analysis of two-variable or four-variable difference equations, while a systematic theoretical framework for general n -variable cases or partial difference equations (such as discrete diffusion equations) remains notably absent. The reference [24] investigated flip bifurcation in a logistic discrete diffusion system with Neumann boundary conditions, determining the bifurcation equations, proposing computation methods

for center manifolds, and identifying certain dynamic characteristics. A coupled map lattices are considered and the system can own a complex dynamics including fluctuation, periodicity and even chaotic behavior in [13]. To the best of the authors' knowledge, there is no result on the bifurcations such as fold, flip and Neimark–Sacker bifurcation of the discrete Lotka–Volterra competition diffusion system (1.4).

When a nonlinear dynamical system undergoes bifurcation, the original equilibrium state may be disrupted, potentially leading to chaotic behavior that renders the system unpredictable in the long term. In ecological systems, to ensure favorable development of biological populations and maintain dynamic equilibrium of the ecosystem, artificial regulatory measures are often implemented. Mathematically, this corresponds to adding feedback control terms to the model [4, 15]. The introduction of feedback control can prevent drastic fluctuations in population sizes.

For dynamical systems, a key focus in feedback control research involves designing appropriate feedback mechanisms that simultaneously ensure the existence of positive equilibrium points in the closed-loop system and global stability of these equilibrium states. This control-theoretic approach is particularly crucial for ecological management, where maintaining population stability while avoiding extinction or explosion scenarios represents a fundamental challenge. The feedback control strategy essentially modifies the system's bifurcation structure, potentially suppressing undesirable chaotic regimes and stabilizing population dynamics within sustainable bounds.

So, in this paper, drawing inspiration from the previously cited works (and references therein) related with bifurcation analysis on discrete system, we focus on the following main findings:

- The flip bifurcation is analyzed when the bifurcation parameter d passes through a critical value for the system (1.4) by means of the theory of the normal form and center manifolds. And the bifurcation analysis method developed here can be extended to other discrete diffusion systems with different boundary conditions. The work presented in this paper is an extension of the work in [13] to two-dimensional discrete diffusion systems, and to some extent addresses the open problem raised in [29].
- The conditions for the occurrence of Turing bifurcation is also obtained.
- Two different control techniques are primarily designed to manage period-doubling bifurcation.
- Phase portraits, maximum Lyapunov exponents, and bifurcation diagrams for the discrete systems are illustrated. Numerical results are provided to validate the theoretical results.

The remainder of this paper is organized as follows. Section 2 applies the center manifold method and normal form theory to the discrete time competition systems with linear diffusion under Neumann boundary conditions (1.4). It establishes conditions for flip bifurcation and further derives criteria for the occurrence of Turing bifurcation. In Section 3, a direct state feedback controller and an indirect state feedback controller are designed for the system (1.4). The influence of different control parameters on the bifurcation characteristics is analyzed, demonstrating that adjusting these parameters can alter the properties of the bifurcation point. This enables the delay or advancement of both flip bifurcation and Turing bifurcation, thereby achieving the intended control objectives. Section 4 presents numerical simulations that support the theoretical findings and illustrate the system's complex dynamical behaviors. Finally, Section 5 summarizes the results and provides interpretations of the obtained outcomes.

2. Stability and bifurcation analysis

2.1. Equilibrium point and its stability

In the present section, we investigate the stability at equilibrium points for the system (1.4). If $\Pi = (u^*, v^*)$ is an equilibrium point for the system (1.4), then we get

$$\begin{cases} u^* = u^* \exp[r - u^* - av^*], \\ v^* = v^* \exp[r - au^* - v^*]. \end{cases} \quad (2.1)$$

Obviously, system (1.4) has four equilibrium points, namely, $\Pi_0 = (0, 0)$, $\Pi_1 = (0, r)$, $\Pi_2 = (r, 0)$, $\Pi_3 = (\frac{r}{1+a}, \frac{r}{1+a})$, in which Π_3 is the unique positive equilibrium point.

To explore the stability conditions of the system (1.4), the following lemma needs to be presented first.

Lemma 2.1 ([24]). *The eigenvalue problem*

$$\begin{cases} -\nabla^2 \phi_{j-1} = \mu \phi_j, \\ \Delta \phi_0 = \Delta \phi_m = 0, \end{cases} \quad (2.2)$$

admits a solution

$$\mu_j = 4 \sin^2 \frac{(j-1)\pi}{2m}, 0 = \mu_1 < \mu_2 < \cdots < \mu_m,$$

with its corresponding eigenvector

$$\varphi_i^{(j)} = \cos \frac{(j-1)(2i-1)\pi}{2m}$$

for $i \in [1, m]$.

The linearization operator of (1.4) about the equilibrium point Π is:

$$L = \begin{pmatrix} d\nabla^2 + A & B \\ C & d\nabla^2 + D \end{pmatrix} = \begin{pmatrix} d\nabla^2 + 1 - u^* & -au^* \\ -av^* & d\nabla^2 + 1 - v^* \end{pmatrix}.$$

Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{j=1}^m \begin{pmatrix} a_j \\ b_j \end{pmatrix} \varphi^{(j)}, j = 1, 2, \dots, m,$$

be an eigenfunction for L with eigenvalue β , that is $L(\phi, \psi)^T = \beta(\phi, \psi)^T$. Then from a straightforward analysis, we can obtain

$$L_j \begin{pmatrix} a_j \\ b_j \end{pmatrix} = \beta_j \begin{pmatrix} a_j \\ b_j \end{pmatrix}, j = 1, 2, \dots, m,$$

where

$$L_j = \begin{pmatrix} A - d\mu_j & B \\ C & D - d\mu_j \end{pmatrix} = \begin{pmatrix} 1 - u^* - d\mu_j & -au^* \\ -av^* & 1 - v^* - d\mu_j \end{pmatrix}.$$

It is a fact that the eigenvalues of L are given by the eigenvalues of L_j for $j = 1, 2, \dots, m$, and the characteristic equation of L_j is

$$(\beta_j - 1 + u^* + d\mu_j)(\beta_j - 1 + v^* + d\mu_j) - a^2 u^* v^* = 0.$$

Case 1 (the equilibrium point Π_0). At the equilibrium point, the the characteristic equation of L_j is

$$(\beta_j - 1 + d\mu_j)(\beta_j - 1 + d\mu_j) = 0,$$

which has eigenvalues

$$\beta_{1j} = \beta_{2j} = 1 - d\mu_j.$$

After some computations, the following result can be derived.

Theorem 2.1. *If $|\beta_{1j}| < 1$ and $|\beta_{2j}| < 1$ namely, $0 < d\mu_j < 2$, the system (1.4) at the equilibrium point Π_0 is locally asymptotically stable.*

Case 2 (the equilibrium point Π_1). At the equilibrium point, the the characteristic equation of L_j is

$$(\beta_j - 1 + d\mu_j)(\beta_j - 1 + r + d\mu_j) = 0,$$

which has eigenvalues

$$\beta_{1j} = 1 - d\mu_j, \beta_{2j} = 1 - r - d\mu_j.$$

The analysis yields the following result.

Theorem 2.2. *If $|\beta_{1j}| < 1$ and $|\beta_{2j}| < 1$ namely, $0 < d\mu_j < 2 - r$, the system (1.4) at the equilibrium point Π_1 is locally asymptotically stable.*

Case 3 (the equilibrium point Π_2). At the equilibrium point, the the characteristic equation of L_j is

$$(\beta_j - 1 + r + d\mu_j)(\beta_j - 1 + d\mu_j) = 0,$$

which has eigenvalues

$$\beta_{1j} = 1 - r - d\mu_j, \beta_{2j} = 1 - d\mu_j.$$

Upon computation, we obtain the following result.

Theorem 2.3. *If $|\beta_{1j}| < 1$ and $|\beta_{2j}| < 1$ namely, $0 < d\mu_j < 2 - r$, the system (1.4) at the equilibrium point Π_2 is locally asymptotically stable.*

Case 4 (the equilibrium point Π_3). At the equilibrium point, the the characteristic equation of L_j is

$$(\beta_j - 1 + \frac{r}{1+a} + d\mu_j)(\beta_j - 1 + \frac{r}{1+a} + d\mu_j) - a^2 \frac{r^2}{(1+a)^2} = 0,$$

which has eigenvalues

$$\beta_{1j} = 1 - r - d\mu_j, \beta_{2j} = 1 + \frac{(a-1)r}{a+1} - d\mu_j.$$

Following the necessary computations, the following result is derived.

Theorem 2.4. *Assume that $|\beta_{1j}| < 1$ and $|\beta_{2j}| < 1$ namely, $\max\{0, \frac{(a-1)r}{a+1}\} < d\mu_j < \min\{2 - r, 2 + \frac{(a-1)r}{a+1}\}$, the system (1.4) at the equilibrium point Π_3 is locally asymptotically stable, otherwise, the system may undergo a bifurcation.*

2.2. Bifurcation analysis

In this part, we mainly study bifurcation in system (1.4) at its positive equilibrium point Π_3 . From the above analysis, the following facts can be obtained.

Theorem 2.5. *A Neimark–Sacker bifurcation is rigorously excluded at the positive equilibrium point Π_3 of system (1.4).*

Proof. Because all β_j are real, it is analytically shown that system (1.4) precludes the occurrence of a Neimark–Sacker bifurcation. $\square$

Theorem 2.6. *Assume that $0 < a < 1$, then the system (1.4) does not undergo a fold bifurcation.*

Proof. Because $\beta_{1j} = 1 - r - d\mu_j = 1$ or $\beta_{2j} = 1 + \frac{(a-1)r}{a+1} - d\mu_j = 1$ can not hold for all $r, d > 0$ and $0 < a < 1$, then the system (1.4) does not undergo a fold bifurcation. The proof is over. $\square$

Next, treating d as the bifurcation parameter, we apply the center manifold theory and normal form analysis to derive the conditions for a flip bifurcation.

Because $\beta_{1j} < \beta_{2j}$, and

$$\begin{aligned}\beta_{1,m} &< \beta_{1,m-1} < \cdots < \beta_{1,1}, \\ \beta_{2,m} &< \beta_{2,m-1} < \cdots < \beta_{2,1},\end{aligned}$$

we can obtain

$$1 - r - d\mu_m = \beta_{1,m} \leq \beta_j \leq \beta_{2,1} = 1 + \frac{(a-1)r}{a+1}.$$

If $0 < a < 1$, then $0 < \beta_{2,1} < 1$, and it does not change with the parameter d . Consider $1 - r - d\mu_m = -1$, we can obtain the critical parameter $d^* = \frac{2-r}{\mu_m}$.

The system (1.4) is qualitatively equivalent to the system that follows.

$$\begin{pmatrix} u_1^{t+1} \\ u_2^{t+1} \\ u_3^{t+1} \\ \vdots \\ u_{m-1}^{t+1} \\ u_m^{t+1} \\ v_1^{t+1} \\ v_2^{t+1} \\ v_3^{t+1} \\ \vdots \\ v_{m-1}^{t+1} \\ v_m^{t+1} \end{pmatrix} = \begin{pmatrix} u_1^t \exp \left[r - u_1^t - av_1^t \right] + d(-u_1^t + u_2^t) \\ u_2^t \exp \left[r - u_2^t - av_2^t \right] + d(u_1^t - 2u_2^t + u_3^t) \\ u_3^t \exp \left[r - u_3^t - av_3^t \right] + d(u_2^t - 2u_3^t + u_4^t) \\ \vdots \\ u_{m-1}^t \exp \left[r - u_{m-1}^t - av_{m-1}^t \right] + d(u_{m-2}^t - 2u_{m-1}^t + u_m^t) \\ u_m^t \exp \left[r - u_m^t - av_m^t \right] + d(u_{m-1}^t - u_m^t) \\ v_1^t \exp \left[r - au_1^t - v_1^t \right] + d(-v_1^t + v_2^t) \\ v_2^t \exp \left[r - au_2^t - v_2^t \right] + d(v_1^t - 2v_2^t + v_3^t) \\ v_3^t \exp \left[r - au_3^t - v_3^t \right] + d(v_2^t - 2v_3^t + v_4^t) \\ \vdots \\ v_{m-1}^t \exp \left[r - au_{m-1}^t - v_{m-1}^t \right] + d(v_{m-2}^t - 2v_{m-1}^t + v_m^t) \\ v_m^t \exp \left[r - au_m^t - v_m^t \right] + d(v_{m-1}^t - v_m^t) \end{pmatrix}. \quad (2.3)$$

Let $\xi_i^t = u_i^t - u^*$, $\eta_i^t = v_i^t - v^*$, $\delta^t = d - d^*$, $i = 1, 2, \dots$, for convenience, ξ_i^t is still written as u_i^t , η_i^t is still written as v_i^t , then

$$U^{t+1} = AU^t + F, \quad (2.4)$$

in which

$$U^t = \left(u_1^t \ u_2^t \ u_3^t \ \cdots \ u_{m-1}^t \ u_m^t \ v_1^t \ v_2^t \ v_3^t \ \cdots \ v_{m-1}^t \ v_m^t \right)^T, \quad A = \begin{pmatrix} B & D \\ D & B \end{pmatrix},$$

$$B = \begin{pmatrix} 1 - \frac{r}{1+a} - d^* & d^* & 0 & \cdots & 0 & 0 \\ d^* & 1 - \frac{r}{1+a} - 2d^* & d^* & \cdots & 0 & 0 \\ 0 & d^* & 1 - \frac{r}{1+a} - 2d^* & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 - \frac{r}{1+a} - 2d^* & d^* \\ 0 & 0 & 0 & \cdots & d^* & 1 - \frac{r}{1+a} - d^* \end{pmatrix}_{m \times m},$$

$$D = \begin{pmatrix} -\frac{ar}{1+a} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{ar}{1+a} & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{ar}{1+a} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{ar}{1+a} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{ar}{1+a} \end{pmatrix}_{m \times m},$$

$$F = \begin{pmatrix} f_1(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ f_2(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ f_3(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ \vdots \\ f_{m-1}(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ f_m(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ g_1(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ g_2(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ g_3(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ \vdots \\ g_{m-1}(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\ g_m(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \end{pmatrix},$$

here

$$\begin{aligned}
 & f_i(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\
 = & \begin{cases} \frac{1}{2}(\frac{r}{1+a} - 2)(u_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(v_i^t)^2 - u_i^t \delta^t + u_{i+1}^t \delta^t \\ + \frac{1}{3}(3 - \frac{r}{1+a})(u_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})u_i^t (v_i^t)^2 + \frac{a}{2}(2 - \frac{r}{1+a})(u_i^t)^2 v_i^t \\ - \frac{a^3 r}{6(1+a)}(v_i^t)^3 + O(|u_i^t| + |u_{i+1}^t| + |v_i^t| + |\delta^t|)^4, & i = 1, \\ \\ \frac{1}{2}(\frac{r}{1+a} - 2)(u_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(v_i^t)^2 + u_{i-1}^t \delta^t \\ - 2u_i^t \delta^t + u_{i+1}^t \delta^t + \frac{1}{3}(3 - \frac{r}{1+a})(u_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})u_i^t (v_i^t)^2 \\ + \frac{a}{2}(2 - \frac{r}{1+a})(u_i^t)^2 v_i^t - \frac{a^3 r}{6(1+a)}(v_i^t)^3 \\ + O(|u_{i-1}^t| + |u_i^t| + |u_{i+1}^t| + |v_i^t| + |\delta^t|)^4, & 2 \leq i \leq m-1, \\ \\ \frac{1}{2}(\frac{r}{1+a} - 2)(u_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(v_i^t)^2 + u_{i-1}^t \delta^t \\ - u_i^t \delta^t + \frac{1}{3}(3 - \frac{r}{1+a})(u_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})u_i^t (v_i^t)^2 \\ + \frac{a}{2}(2 - \frac{r}{1+a})(u_i^t)^2 v_i^t - \frac{a^3 r}{6(1+a)}(v_i^t)^3 \\ + O(|u_{i-1}^t| + |u_i^t| + |v_i^t| + |\delta^t|)^4, & i = m, \end{cases} \\
 & g_i(u_1^t, u_2^t, u_3^t, \dots, u_{m-1}^t, u_m^t, v_1^t, v_2^t, v_3^t, \dots, v_{m-1}^t, v_m^t, \delta^t) \\
 = & \begin{cases} \frac{1}{2}(\frac{r}{1+a} - 2)(v_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(u_i^t)^2 - v_i^t \delta^t \\ + v_{i+1}^t \delta^t + \frac{1}{3}(3 - \frac{r}{1+a})(v_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})v_i^t (u_i^t)^2 \\ + \frac{a}{2}(2 - \frac{r}{1+a})(v_i^t)^2 u_i^t - \frac{a^3 r}{6(1+a)}(u_i^t)^3 \\ + O(|v_i^t| + |v_{i+1}^t| + |u_i^t| + |\delta^t|)^4, & i = 1, \\ \\ \frac{1}{2}(\frac{r}{1+a} - 2)(v_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(u_i^t)^2 + v_{i-1}^t \delta^t \\ - 2v_i^t \delta^t + v_{i+1}^t \delta^t + \frac{1}{3}(3 - \frac{r}{1+a})(v_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})v_i^t (u_i^t)^2 \\ + \frac{a}{2}(2 - \frac{r}{1+a})(v_i^t)^2 u_i^t - \frac{a^3 r}{6(1+a)}(u_i^t)^3 \\ + O(|v_{i-1}^t| + |v_i^t| + |v_{i+1}^t| + |u_i^t| + |\delta^t|)^4, & 2 \leq i \leq m-1, \\ \\ \frac{1}{2}(\frac{r}{1+a} - 2)(v_i^t)^2 + a(\frac{r}{1+a} - 1)u_i^t v_i^t + \frac{a^2 r}{2(1+a)}(u_i^t)^2 + v_{i-1}^t \delta^t \\ - v_i^t \delta^t + \frac{1}{3}(3 - \frac{r}{1+a})(v_i^t)^3 + \frac{a^2}{2}(1 - \frac{r}{1+a})v_i^t (u_i^t)^2 \\ + \frac{a}{2}(2 - \frac{r}{1+a})(v_i^t)^2 u_i^t - \frac{a^3 r}{6(1+a)}(u_i^t)^3 \\ + O(|v_{i-1}^t| + |v_i^t| + |u_i^t| + |\delta^t|)^4, & i = m. \end{cases}
 \end{aligned}$$

Then construct the translation

$$\begin{pmatrix} u_1^t \\ u_2^t \\ u_3^t \\ \vdots \\ u_{m-1}^t \\ u_m^t \\ v_1^t \\ v_2^t \\ v_3^t \\ \vdots \\ v_{m-1}^t \\ v_m^t \end{pmatrix} = T \begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \\ \vdots \\ x_{m-1}^t \\ x_m^t \\ y_1^t \\ y_2^t \\ y_3^t \\ \vdots \\ y_{m-1}^t \\ y_m^t \end{pmatrix},$$

in which

$$T = \begin{pmatrix} T_1 & E \\ E & T_1 \end{pmatrix}_{2m \times 2m}, \quad T_1 = \begin{pmatrix} e_{11} & e_{21} & e_{31} & \cdots & e_{m-1,1} & e_{m1} \\ e_{12} & e_{22} & e_{32} & \cdots & e_{m-1,2} & e_{m2} \\ e_{13} & e_{23} & e_{33} & \cdots & e_{m-1,3} & e_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{1,m-1} & e_{2,m-1} & e_{3,m-1} & \cdots & e_{m-1,m-1} & e_{m,m-1} \\ e_{1m} & e_{2m} & e_{3m} & \cdots & e_{m-1,m} & e_{mm} \end{pmatrix}_{m \times m}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{m}}\varphi_1^{(1)} & \sqrt{\frac{2}{m}}\varphi_1^{(2)} & \sqrt{\frac{2}{m}}\varphi_1^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_1^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_1^{(m)} \\ \frac{1}{\sqrt{m}}\varphi_2^{(1)} & \sqrt{\frac{2}{m}}\varphi_2^{(2)} & \sqrt{\frac{2}{m}}\varphi_2^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_2^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_2^{(m)} \\ \frac{1}{\sqrt{m}}\varphi_3^{(1)} & \sqrt{\frac{2}{m}}\varphi_3^{(2)} & \sqrt{\frac{2}{m}}\varphi_3^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_3^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_3^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{m}}\varphi_{m-1}^{(1)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(2)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(m)} \\ \frac{1}{\sqrt{m}}\varphi_m^{(1)} & \sqrt{\frac{2}{m}}\varphi_m^{(2)} & \sqrt{\frac{2}{m}}\varphi_m^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_m^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_m^{(m)} \end{pmatrix}_{m \times m}.$$

E is a $m \times m$ unit matrix, and

$$T^{-1} = T',$$

then the system (2.4) can be reformulated as follows

$$X^{t+1} = MX^t + G,$$

where

$$X^t = \left(x_1^t x_2^t x_3^t \cdots x_{m-1}^t x_m^t y_1^t y_2^t y_3^t \cdots y_{m-1}^t y_m^t \delta^t \right)^T, \quad M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(2m+1) \times (2m+1)},$$

$$M_1 = \begin{pmatrix} 1 + \frac{(a-1)r}{a+1} - d^* \mu_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 + \frac{(a-1)r}{a+1} - d^* \mu_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 + \frac{(a-1)r}{a+1} - d^* \mu_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 + \frac{(a-1)r}{a+1} - d^* \mu_m \end{pmatrix}_{m \times m},$$

$$M_2 = \begin{pmatrix} 1 - r - d^* \mu_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 - r - d^* \mu_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 - r - d^* \mu_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 - r - d^* \mu_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}_{m \times m},$$

$$G = \begin{pmatrix} F_1(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ F_2(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ F_3(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ \vdots \\ F_{m-1}(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ F_m(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ G_1(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ G_2(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ G_3(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ \vdots \\ G_{m-1}(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ G_m(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \cdots, y_{m-1}^t, y_m^t, \delta^t) \\ 0 \end{pmatrix},$$

here

$$\begin{aligned}
& F_j(x_1^t, x_2^t, x_3^t, \dots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \dots, y_{m-1}^t, y_m^t, \delta^t) \\
&= \frac{1}{2} \left(\frac{r}{1+a} - 2 \right) \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 + \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right)^2 \right] \\
&+ a \left(\frac{r}{1+a} - 1 \right) \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \right. \\
&+ \left. \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right) \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right) \right] + \frac{a^2 r}{2(1+a)} \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 \right. \\
&+ \left. \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right)^2 \right] - \sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \delta^t - \sum_{i=2}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \delta^t \\
&+ \sum_{i=1}^{m-1} e_{ji} \left(y_i^t + \sum_{k=1}^m e_{k,i+1} x_k^t \right) \delta^t + \sum_{i=2}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{k,i-1} x_k^t \right) \delta^t \\
&+ \frac{1}{3} \left(3 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^3 + \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right)^3 \right] \\
&+ \frac{a^2}{2} \left(1 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 \right. \\
&+ \left. \left(y_i^t + \sum_{k=1}^m e_{kj} x_k^t \right)^2 \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right) \right] \\
&+ \frac{a}{2} \left(2 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \right. \\
&+ \left. \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right) \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right)^2 \right] - \frac{a^3 r}{6(1+a)} \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^3 \right. \\
&+ \left. \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right)^3 \right] + O(|x_i^t| + |y_i^t| + |\delta^t|)^4, \\
& G_j(x_1^t, x_2^t, x_3^t, \dots, x_{m-1}^t, x_m^t, y_1^t, y_2^t, y_3^t, \dots, y_{m-1}^t, y_m^t, \delta^t) \\
&= \frac{1}{2} \left(\frac{r}{1+a} - 2 \right) \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 + \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right)^2 \right] \\
&+ a \left(\frac{r}{1+a} - 1 \right) \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \right. \\
&+ \left. \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right) \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right) \right] \\
&+ \frac{a^2 r}{2(1+a)} \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 + \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right)^2 \right] \\
&- \sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \delta^t - \sum_{i=2}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \delta^t
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m-1} e_{ji} \left(x_i^t + \sum_{k=1}^m e_{k,i+1} y_k^t \right) \delta^t + \sum_{i=2}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{k,i-1} y_k^t \right) \delta^t \\
& + \frac{1}{3} \left(3 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^3 + \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right)^3 \right] \\
& + \frac{a^2}{2} \left(1 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 \right. \\
& \left. + \left(x_i^t + \sum_{k=1}^m e_{kj} y_k^t \right)^2 \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right) \right] + \frac{a}{2} \left(2 - \frac{r}{1+a} \right) \\
& \left[\sum_{i=1}^m e_{ji} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) + \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right) \left(y_j^t + \sum_{k=1}^m e_{kj} x_k^t \right)^2 \right] \\
& - \frac{a^3 r}{6(1+a)} \left[\sum_{i=1}^m e_{ji} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^3 + \left(x_j^t + \sum_{k=1}^m e_{kj} y_k^t \right)^3 \right] + O\left((|x_i^t| + |y_i^t| + |\delta^t|)^4\right).
\end{aligned}$$

By the center manifold theorem, the dynamical system admits a local center manifold $W^c(0, 0, \dots, 0)$, which can be locally approximated as:

$$\begin{aligned}
& W^c(\underbrace{0, 0, \dots, 0}_{2m+1}) \\
& = \left\{ (x_1^t, x_2^t, \dots, x_m^t, y_1^t, y_2^t, \dots, y_m^t, \delta^t) \left| \begin{cases} x_j^t = h_{1j}(y_m^t, \delta^t), j = 1, 2, \dots, m, \\ y_i^t = h_{2i}(y_m^t, \delta^t), i = 1, 2, \dots, m-1, \end{cases} \right. \right\},
\end{aligned}$$

where

$$\begin{cases} x_j^t = h_{1j}(y_m^t, \delta^t) = a_{j1}(y_m^t)^2 + a_{j2}y_m^t\delta^t + a_{j3}(\delta^t)^2 + O((|y_m^t| + |\delta^t|)^3), j = 1, 2, \dots, m, \\ y_i^t = h_{2i}(y_m^t, \delta^t) = b_{i1}(y_m^t)^2 + b_{i2}y_m^t\delta^t + b_{i3}(\delta^t)^2 + O((|y_m^t| + |\delta^t|)^3), i = 1, 2, \dots, m-1, \end{cases}$$

and $O((|y_m^t| + |\delta^t|)^3)$ is a function with order at least 3 in the variables.

By computation, we have

$$\begin{aligned}
a_{j1} &= \left[\left(\frac{r}{1+a} - 2 \right) e_{jm} + \frac{a^2 r}{2(1+a)} \left(\sum_{i=1}^m e_{ji} e_{mi}^2 + 1 \right) \right] \frac{a+1}{d^* \mu_j (a+1) - (a-1)r}, \\
a_{j2} &= - \sum_{k=1}^m e_{jk} e_{mk} - \sum_{k=2}^m e_{jk} e_{mk} + \sum_{k=1}^{m-1} e_{jk} e_{m,k+1} + \sum_{k=2}^m e_{jk} e_{m,k-1}, \\
a_{j3} &= 0, j = 1, 2, \dots, m, \\
b_{i1} &= \left[\left(\frac{r}{1+a} - 2 \right) e_{im} + \frac{a^2 r}{2(1+a)} \left(\sum_{k=1}^m e_{ik} e_{mk}^2 + 1 \right) \right] \frac{1}{r + d^* \mu_i}, \\
b_{i2} &= - \sum_{k=1}^m e_{ik} e_{mk} - \sum_{k=2}^m e_{ik} e_{mk} + \sum_{k=1}^{m-1} e_{ik} e_{m,k+1} + \sum_{k=2}^m e_{ik} e_{m,k-1}, \\
b_{i3} &= 0, j = 1, 2, \dots, m-1.
\end{aligned}$$

Therefore, the following map, which is (2.4) restricted to the center manifold W^c , is considered

$$\begin{aligned}
 H : y_m^{t+1} = & -y_m^t + \frac{1}{2} \left(\frac{r}{1+a} - 2 \right) \left[\sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 + \left(y_m^t + \sum_{k=1}^m e_{km} x_k^t \right)^2 \right] \\
 & + a \left(\frac{r}{1+a} - 1 \right) \left[\sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \right. \\
 & + \left. \left(x_m^t + \sum_{k=1}^m e_{km} y_k^t \right) \left(y_m^t + \sum_{k=1}^m e_{km} x_k^t \right) \right] + \frac{a^2 r}{2(1+a)} \left[\sum_{i=1}^m e_{mi} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 \right. \\
 & + \left. \left(x_m^t + \sum_{k=1}^m e_{km} y_k^t \right)^2 \right] - \sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \delta^t - \sum_{i=2}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \delta^t \\
 & + \sum_{i=1}^{m-1} e_{mi} \left(x_i^t + \sum_{k=1}^m e_{k,i+1} y_k^t \right) \delta^t + \sum_{i=2}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{k,i-1} y_k^t \right) \delta^t \\
 & + \frac{1}{3} \left(3 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^3 + \left(y_m^t + \sum_{k=1}^m e_{km} x_k^t \right)^3 \right] \\
 & + \frac{a^2}{2} \left(1 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right) \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^2 + \left(x_i^t + \sum_{k=1}^m e_{km} y_k^t \right)^2 \right. \\
 & \times \left. \left(y_m^t + \sum_{k=1}^m e_{km} x_k^t \right) \right] + \frac{a}{2} \left(2 - \frac{r}{1+a} \right) \left[\sum_{i=1}^m e_{mi} \left(x_i^t + \sum_{k=1}^m e_{ki} y_k^t \right)^2 \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right) \right. \\
 & + \left. \left(x_m^t + \sum_{k=1}^m e_{km} y_k^t \right) \left(y_j^t + \sum_{k=1}^m e_{km} x_k^t \right)^2 \right] \\
 & - \frac{a^3 r}{6(1+a)} \left[\sum_{i=1}^m e_{mi} \left(y_i^t + \sum_{k=1}^m e_{ki} x_k^t \right)^3 + \left(x_m^t + \sum_{k=1}^m e_{km} y_k^t \right)^3 \right] + O(|y_m^t| + |\delta^t|)^4.
 \end{aligned}$$

Notice that

$$\left\{ \begin{aligned}
 & H(0,0) = 0, \\
 & \frac{\partial H}{\partial \delta} \Big|_{(0,0)} = 0, \\
 & \frac{\partial H}{\partial y_m} \Big|_{(0,0)} = -1, \\
 & \frac{\partial^2 H}{\partial y_m \partial \delta} \Big|_{(0,0)} = - \sum_{k=1}^m e_{mk}^2 - \sum_{k=2}^m e_{mk}^2 + \sum_{k=1}^{m-1} e_{mk} e_{m,k+1} + \sum_{k=2}^m e_{mk} e_{m,k-1}, \\
 & \frac{\partial^2 H}{\partial y_m^2} \Big|_{(0,0)} = \left(\frac{r}{1+a} - 2 \right) \left(\sum_{k=1}^m e_{mk}^2 + 2 \right) + 2a \left(\frac{r}{1+a} - 1 \right) e_{mm}^2, \\
 & \frac{\partial^2 H}{\partial \delta^2} \Big|_{(0,0)} = 0, \\
 & \frac{\partial^3 H}{\partial y_m^3} \Big|_{(0,0)} = - \frac{2a^3 r}{1+a} e_{mm}^3.
 \end{aligned} \right.$$

In order for map (1.4) to undergo a flip bifurcation, we require that two discriminatory quantities $H1$ and $H2$ are not zero [23], where

$$\begin{aligned} H1 &: \left(\frac{\partial H}{\partial \delta} \frac{\partial^2 H}{\partial y_m^2} + 2 \frac{\partial^2 H}{\partial y_m \partial \delta} \right) \Big|_{(0,0)} \\ &= 2 \left(- \sum_{k=1}^m e_{mk}^2 - \sum_{k=2}^m e_{mk}^2 + \sum_{k=1}^{m-1} e_{mk} e_{m,k+1} + \sum_{k=2}^m e_{ik} e_{m,k-1} \right), \\ H2 &: \left(\frac{1}{2} \left(\frac{\partial^2 H}{\partial y_m^2} \right)^2 + \frac{1}{3} \frac{\partial^3 H}{\partial y_m^3} \right) \Big|_{(0,0)} \\ &= \frac{1}{2} \left[\left(\frac{r}{1+a} - 2 \right) \left(\sum_{k=1}^m e_{mk}^2 + 2 \right) + 2a \left(\frac{r}{1+a} - 1 \right) e_{mm}^2 \right]^2 - \frac{2a^3 r}{3(1+a)} e_{mm}^3. \end{aligned}$$

Based on the above analysis, the following Theorem is presented to illustrate the occurrence of flip bifurcation about the positive equilibrium point Π_3 of the system (1.4).

Theorem 2.7. *Assume that $0 < a < 1$, and $H1 \neq 0, H2 \neq 0$, then the positive equilibrium point Π_3 undergoes a flip bifurcation at $d = d^*$.*

2.3. Turing bifurcation

In this section, we discuss the Turing bifurcation at the positive equilibrium point Π_3 of (1.4). Turing's theory postulates that diffusion can destabilize an otherwise stable equilibrium in a reaction-diffusion system, thereby giving rise to non-uniform spatial patterns.

To analyze the Turing bifurcation, a prerequisite is to establish the stability condition at the equilibrium point Π_3 in the non-diffusive system.

$$\begin{cases} u(t+1) = u(t) \exp \left[r - u(t) - av(t) \right], \\ v(t+1) = v(t) \exp \left[r - au(t) - v(t) \right], \quad t \in Z^+. \end{cases} \quad (2.5)$$

Through simple calculations, the system (2.5) at the positive equilibrium point Π_3 is asymptotically stable if and only if $0 < a < 1, 0 < r < 2$.

The linearized form of (1.4) is

$$wx_i^{t+1} = Jw_i^t + D\nabla^2 w_i^t \quad (2.6)$$

with Neumann boundary conditions

$$w_0^t = w_m^t, w_1^t = w_{m+1}^t, \quad (2.7)$$

where

$$D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix},$$

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 - \frac{r}{1+a} & -\frac{ar}{1+a} \\ -\frac{ar}{1+a} & 1 - \frac{r}{1+a} \end{pmatrix},$$

$$w_i^t = \begin{pmatrix} u_i^t - u^* \\ v_i^t - v^* \end{pmatrix} = \begin{pmatrix} x_i^t \\ y_i^t \end{pmatrix}.$$

Assume that $(x_i(k), y_i(k))$ is a solution of the system (2.6)-(2.7). According to Able lemma and the eigenvalue problem (2.2), we have

$$\begin{aligned} \sum_{i=1}^m \varphi_i^{(j)} x_i(k+1) &= a_{11} \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + a_{12} \sum_{i=1}^m \varphi_i^{(j)} y_i(k) + d \sum_{i=1}^m \varphi_i^{(j)} \nabla^2 x_i(k) \\ &= a_{11} \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + a_{12} \sum_{i=1}^m \varphi_i^{(j)} y_i(k) + d \sum_{i=1}^m x_i(k) \nabla^2 \varphi_i^{(j)} \\ &= (a_{11} - d\mu_j) \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + a_{12} \sum_{i=1}^m \varphi_i^{(j)} y_i(k), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^m \varphi_i^{(j)} y_i(k+1) &= a_{21} \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + a_{22} \sum_{i=1}^m \varphi_i^{(j)} y_i(k) + d \sum_{i=1}^m \varphi_i^{(j)} \nabla^2 y_i(k) \\ &= a_{21} \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + a_{22} \sum_{i=1}^m \varphi_i^{(j)} y_i(k) + d \sum_{i=1}^m y_i(k) \nabla^2 \varphi_i^{(j)} \\ &= a_{21} \sum_{i=1}^m \varphi_i^{(j)} x_i(k) + (a_{22} - d\mu_j) \sum_{i=1}^m \varphi_i^{(j)} y_i(k). \end{aligned}$$

Let $U(t) = \sum_{i=1}^m \varphi_i^{(j)} x_i^t$, $V(t) = \sum_{i=1}^m \varphi_i^{(j)} y_i^t$, and incorporate the Neumann boundary conditions (2.7), the following expression can be derived

$$\begin{cases} U(t+1) = (a_{11} - d\mu_j)U(t) + a_{12}V(t), \\ V(t+1) = a_{21}U(t) + (a_{22} - d\mu_j)V(t). \end{cases} \quad (2.8)$$

And, the following fact can be obtained.

Proposition 2.1. Assume that (u_i^t, v_i^t) is a solution of the system (2.7), then

$$(U(t) = \sum_{i=1}^m \varphi_i^{(j)} x_i^t, V(t) = \sum_{i=1}^m \varphi_i^{(j)} y_i^t)$$

is a solution of the system (2.8). For some eigenvalue μ_j , if $(U(t), V(t))$ is a solution of the system (2.8), then (u_i^t, v_i^t) is also a solution of the system (2.7).

Proposition 2.2. *Provided that there exist a positive number d and an eigenvalue μ_j satisfying the following condition*

$$d\mu_j > 2 + \frac{(a-1)r}{a+1},$$

the system (1.4) at the positive equilibrium point Π_3 is unstable.

Proof. From (2.8), we can obtain

$$J = \begin{pmatrix} 1 - \frac{r}{1+a} - d\mu_j & -\frac{ar}{1+a} \\ -\frac{ar}{1+a} & 1 - \frac{r}{1+a} - d\mu_j \end{pmatrix}, \quad (2.9)$$

and the eigenvalues of (2.9) are

$$\lambda_1 = 1 + \frac{(a-1)r}{a+1} - d\mu_j, \lambda_2 = 1 - r - d\mu_j.$$

If $|\lambda_1| > 1$ or $|\lambda_2| > 1$ namely, $d\mu_j > 2 + \frac{(a-1)r}{a+1}$ holds, the problem (1.4) at the equilibrium point Π_3 is unstable. The proof is over. $\square$

Then we have the following result immediately.

Theorem 2.8. *Assume that $0 < a < 1$, $0 < r < 2$ and $d\mu_j > 2 + \frac{(a-1)r}{a+1}$ hold, the system (1.4) at the equilibrium point Π_3 is diffusion-driven unstable or Turing unstable, then Turing bifurcation will emerge.*

3. Control of bifurcations

In certain scenarios, it is necessary to either delay or advance the bifurcation in a chaotic system while ensuring the stability of the bifurcation solution. This requires the design of a controller capable of modifying the system's bifurcation characteristics. From our perspective, bifurcation control strategies can be broadly classified into two categories: Direct control and indirect control. Direct control involves introducing control variables into the system without altering its equilibrium points, as exemplified by methods such as state feedback control [4] and hybrid control [28]. In contrast, indirect control is applied when the existing equilibrium of the dynamical model is undesirable or unsustainable, and a lower equilibrium value is preferred. In such cases, the model structure must be adjusted to stabilize the system at a reduced equilibrium level [7, 26], leading to an indirect control formulation.

For system (1.4), a direct controller, specifically, a hybrid control strategy, is incorporated, resulting in the following controlled system.

$$\begin{cases} u_i^{t+1} = s_1(u_i^t \exp[r - u_i^t - av_i^t] + d\nabla^2 u_i^t) + (1 - s_1)u_i^t, \\ v_i^{t+1} = s_2(v_i^t \exp[r - au_i^t - v_i^t] + d\nabla^2 v_i^t) + (1 - s_2)v_i^t, \\ \Delta u_0^t = \Delta u_m^t = \Delta v_0^t = \Delta v_m^t = 0, i = 1, 2, \dots, m, t \in Z^+, \end{cases} \quad (3.1)$$

here s_1 and s_2 the control parameter. If an indirect controller is added into the system (1.4), the following control system can be obtained.

$$\begin{cases} u_i^{t+1} = u_i^t \exp \left[r - u_i^t - av_i^t - b_1 \omega_i^t \right] + d \nabla^2 u_i^t, \\ v_i^{t+1} = v_i^t \exp \left[r - au_i^t - v_i^t - b_2 \phi_i^t \right] + d \nabla^2 v_i^t, \\ \omega_i^{t+1} = (1 - e_1) \omega_i^t + \eta_1 u_i^t, \\ \phi_i^{t+1} = (1 - e_2) \phi_i^t + \eta_2 v_i^t, \\ \Delta u_0^t = \Delta u_m^t = \Delta v_0^t = \Delta v_m^t = 0, \\ \Delta \omega_0^t = \Delta \omega_m^t = \Delta \phi_0^t = \Delta \phi_m^t = 0, i = 1, 2, \dots, m, t \in Z^+, \end{cases} \quad (3.2)$$

where ω and ϕ are feedback control variables and the parameters $b_1, b_2, e_1, e_2, \eta_1$ and η_2 are positive constants.

3.1. Direct control

Here, we apply a hybrid control method that combines state feedback with parameter perturbation to stabilize the unstable periodic orbits, which yields the controlled system (3.1) that maintains stability over a wide parameter range. By virtue of the linear stability theory, the following result can be derived.

Theorem 3.1. *The discretized non-linear system (3.1) is said to be locally asymptotically stable if and only if $1 - \text{tr}(J)\lambda + \det(J) > 0$, $1 + \text{tr}(J)\lambda + \det(J) > 0$ and $\det(J) < 1$ hold, here*

$$\begin{aligned} \text{tr}(J) &= 2 - \frac{(s_1 + s_2)r}{1 + a} - (s_1 + s_2)d\mu_j, \\ \det(J) &= 1 + s_1 s_2 d\mu_j \left(\frac{2r}{1 + a} + d\mu_j \right) - (s_1 + s_2) \left(\left(\frac{r}{1 + a} + d\mu_j \right) \right) + \frac{1 - a}{1 + a} s_1 s_2 r^2. \end{aligned}$$

Proof. The Jacobian matrix for the linearized form of (3.1) at the positive equilibrium point Π_3 is

$$J = \begin{pmatrix} 1 - \frac{s_1 r}{1 + a} - s_1 d\mu_j & -\frac{a s_1 r}{1 + a} \\ -\frac{a s_2 r}{1 + a} & 1 - \frac{s_2 r}{1 + a} - d s_2 \mu_j \end{pmatrix}, \quad (3.3)$$

and the characteristic polynomial corresponding to the Jacobian matrix is represented by

$$F(\lambda) = \lambda^2 - \text{tr}(J)\lambda + \det(J),$$

where

$$\begin{aligned} \text{tr}(J) &= 2 - \frac{(s_1 + s_2)r}{1 + a} - (s_1 + s_2)d\mu_j, \\ \det(J) &= 1 + s_1 s_2 d\mu_j \left(\frac{2r}{1 + a} + d\mu_j \right) - (s_1 + s_2) \left(\left(\frac{r}{1 + a} + d\mu_j \right) \right) + \frac{1 - a}{1 + a} s_1 s_2 r^2. \end{aligned}$$

By means of $|\lambda_{1,2}| < 1$, the stability conditions of the system (3.1) at the positive equilibrium point Π_3 can be obtained:

$$F(1) > 0, F(-1) > 0, \det(J) < 1,$$

namely, $1 - \text{tr}(J)\lambda + \det(J) > 0$, $1 + \text{tr}(J)\lambda + \det(J) > 0$ and $\det(J) < 1$. $\square$

In particular, let $s_1 = s_2 = s$, we can obtain the following asymptotic stability conditions for the discrete-time control system.

$$\begin{cases} u_i^{t+1} = s(u_i^t \exp[r - u_i^t - av_i^t] + d\nabla^2 u_i^t) + (1-s)u_i^t, \\ v_i^{t+1} = s(v_i^t \exp[r - au_i^t - v_i^t] + d\nabla^2 v_i^t) + (1-s)v_i^t, \\ \Delta u_0^t = \Delta u_m^t = \Delta v_0^t = \Delta v_m^t = 0, i = 1, 2, \dots, m, k \in Z^+. \end{cases} \quad (3.4)$$

Corollary 3.1. *The discretized non-linear system (3.4) is said to be locally asymptotically stable if $0 < a < 1$, $ds\mu_j < 2 - rs$.*

3.2. Indirect control

As discussed in Introduction part, an indirect controlled system is obtained as (3.2).

The fixed points of system (3.2) can be obtained by solving the following system:

$$\begin{cases} u = u \exp[r - u - av - b_1\omega] + d\nabla^2 u, \\ v = v \exp[r - au - v - b_2\phi] + d\nabla^2 v, \\ \omega = (1 - e_1)\omega + \eta_1 u, \\ \phi = (1 - e_2)\phi + \eta_2 v. \end{cases} \quad (3.5)$$

By simple computation, we can obtain its positive fixed points $E_1^*(u^*, v^*, \omega^*, \phi^*)$ and

$$\begin{aligned} u^* &= \frac{(e_1 e_2 + b_1 \eta_1 e_2 - a e_1 e_2)r}{(e_1 + b_1 \eta_1)(e_2 + b_2 \eta_2) - e_1 e_2 a^2}, \\ v^* &= \frac{(e_1 e_2 + b_2 \eta_2 e_1 - a e_1 e_2)r}{(e_1 + b_1 \eta_1)(e_2 + b_2 \eta_2) - e_1 e_2 a^2}, \\ \omega^* &= \frac{\eta_1}{e_1} u^*, \\ \phi^* &= \frac{\eta_2}{e_2} v^*, \end{aligned}$$

if $0 < a < 1$ holds.

The Jacobian matrix of the system (3.2) at $E_1^*(u^*, v^*, \omega^*, \phi^*)$ is

$$J_{E_1^*} = \begin{pmatrix} 1 - u^* - d\mu_j & -au^* & -b_1 u^* & 0 \\ -av^* & 1 - v^* - d\mu_j & 0 & -b_2 v^* \\ \eta_1 & 0 & 1 - e_1 & 0 \\ 0 & \eta_2 & 0 & 1 - e_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & b_{22} & 0 & b_{24} \\ b_{31} & 0 & b_{33} & 0 \\ 0 & b_{42} & 0 & b_{44} \end{pmatrix}, \quad (3.6)$$

and its corresponding characteristic equation is

$$\lambda^4 + c_3 \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0,$$

where

$$\begin{aligned} c_0 &= b_{11}b_{22}b_{33}b_{44} - b_{24}b_{42}b_{11}b_{33} - b_{13}b_{31}b_{22}b_{44} - b_{12}b_{21}b_{33}b_{44} + b_{13}b_{31}b_{24}b_{42}, \\ c_1 &= b_{24}b_{42}(b_{11} + b_{33}) + b_{13}b_{31}(b_{22} + b_{44}) + b_{12}b_{21}(b_{33} + b_{44}) - b_{11}b_{22}(b_{33} + b_{44}) \\ &\quad - b_{33}b_{44}(b_{11} + b_{22}), \\ c_2 &= (b_{11} + b_{22})(b_{33} + b_{44}) + b_{11}b_{22} + b_{33}b_{44} - b_{12}b_{21} - b_{13}b_{31} - b_{24}b_{42}, \\ c_3 &= -(b_{11} + b_{22} + b_{33} + b_{44}). \end{aligned}$$

According to the Jury criterion, we can draw the following conclusion.

Theorem 3.2. *The positive homogeneous steady state $E_1^*(u^*, v^*, \omega^*, \phi^*)$ is asymptotically stable if*

$$P(1) > 0, P(-1) > 0, |c_0| < 1, |c_0^2 - 1| > |c_0c_2 - c_1|,$$

and

$$|b_0^2 - b_3^2| > |b_0b_2 - b_1b_3|$$

hold, here $b_0 = c_1 - c_0c_3$, $b_1 = c_2 - c_0c_2$, $b_2 = c_{31} - c_0c_1$, $b_3 = 1 - c_0^2$.

4. Numerical simulations

This section presents numerical examples to support and extend the theoretical results obtained in the former sections.

4.1. Flip bifurcation and its control

Substituting $m = 2$ into system (1.4) yields the following model.

$$\begin{cases} u_1^{t+1} = u_1^t \exp \left[r - u_1^t - av_1^t \right] + d(-u_1^t + u_2^t), \\ u_2^{t+1} = u_2^t \exp \left[r - u_2^t - av_2^t \right] + d(u_1^t - u_2^t), \\ v_1^{t+1} = v_1^t \exp \left[r - au_1^t - v_1^t \right] + d(-v_1^t + v_2^t), \\ v_2^{t+1} = v_2^t \exp \left[r - au_2^t - v_2^t \right] + d(v_1^t - v_2^t). \end{cases} \quad (4.1)$$

To investigate the impact of the parameter d on the dynamics of the system, we first fix the other parameters as $r = 1.95, a = 0.76$. We can verify the conditions H_1 and H_2 are satisfied and can obtain Figure 1(a) and observe the existence of a flip bifurcation in the equilibrium point $(1.108, 1.108, 1.108, 1.108)$, which is in accordance with the result in Theorem 2.3. The maximum Lyapunov exponents corresponding to Figure 1(a) is shown in Figure 1(b). Figure 1(c) and Figure 1(d) are bifurcation diagrams when d varies in $[0.22, 0.29]$ and $[0.2670, 0.2705]$. Figure 2 show the 3-dimensional bifurcation diagram for (u_1, d, a) and (u_1, d, r) when a or r varies in $[0.30, 0.95]$ and $[1.50, 1.95]$ which can show the change of the bifurcation diagram with the parameter a or r .

Next, we will see the effectiveness of the two control methods by selecting values for specific parameters within the chaotic region. To achieve this, we first apply the hybrid control method

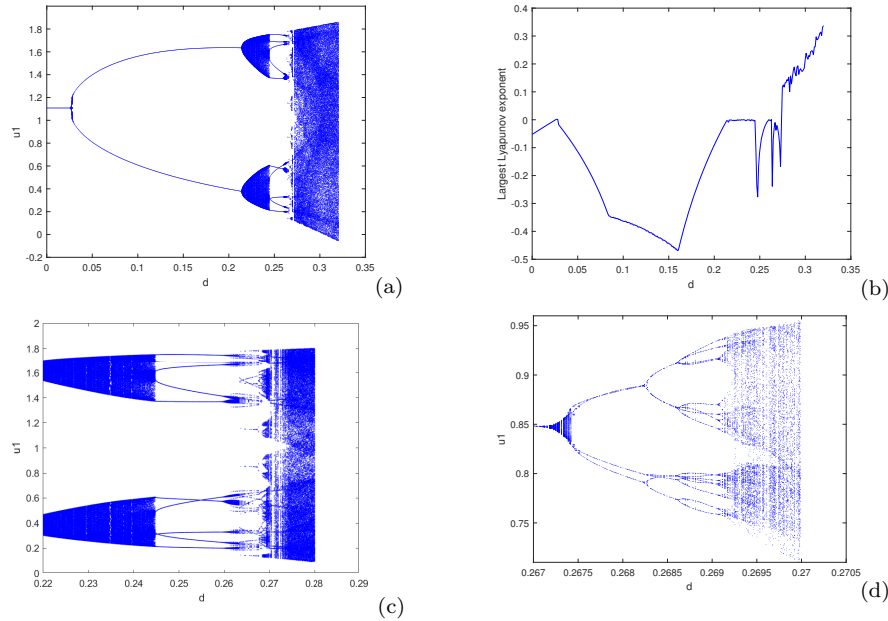


Figure 1. Flip bifurcation diagram and maximum Lyapunov exponent for u_1 on system (4.1).

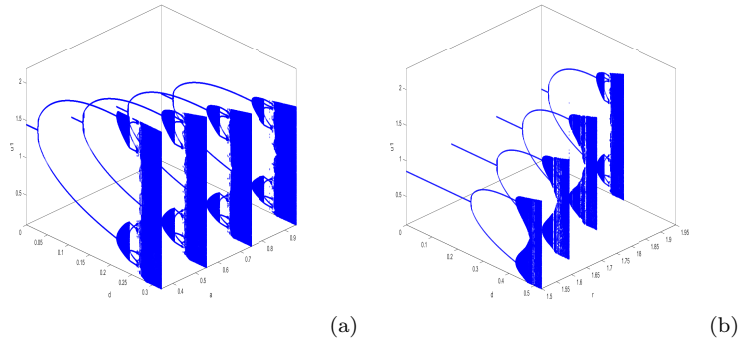


Figure 2. The 3-dimensional bifurcation diagram for (u_1, d, a) and (u_1, d, r) of system (4.1).

to system (4.1) as follows

$$\begin{cases} u_1^{t+1} = s_1(u_1^t \exp[r - u_1^t - av_1^t] + d(-u_1^t + u_2^t)) + (1 - s_1)u_1^t, \\ u_2^{t+1} = s_1(u_2^t \exp[r - u_2^t - av_2^t] + d(u_1^t - u_2^t)) + (1 - s_1)u_2^t, \\ v_1^{t+1} = s_2(v_1^t \exp[r - au_1^t - v_1^t] + d(-v_1^t + v_2^t)) + (1 - s_2)v_1^t, \\ v_2^{t+1} = s_2(v_2^t \exp[r - au_2^t - v_2^t] + d(v_1^t - v_2^t)) + (1 - s_2)v_2^t. \end{cases} \quad (4.2)$$

By setting $r = 1.95, a = 0.76$ and varying s_1 and s_2 , we first select $s_1 = 0.5, s_2 = 0.1$, it is easy to verify that stability condition is satisfied, falling within the stable region which meaning no bifurcation will occur, which is confirmed in Figure 3(a). Further adjusting $s_2 = 0.45, 0.53, 0.56, 0.62, 0.82$, the stability condition is no longer met, but the bifurcation time occurs

earlier, as shown in Figure 3(b)–Figure 3(f). We found that initially, as s gradually increases, chaos begins to emerge progressively. However, upon further increasing s , the chaotic phenomena gradually diminish.

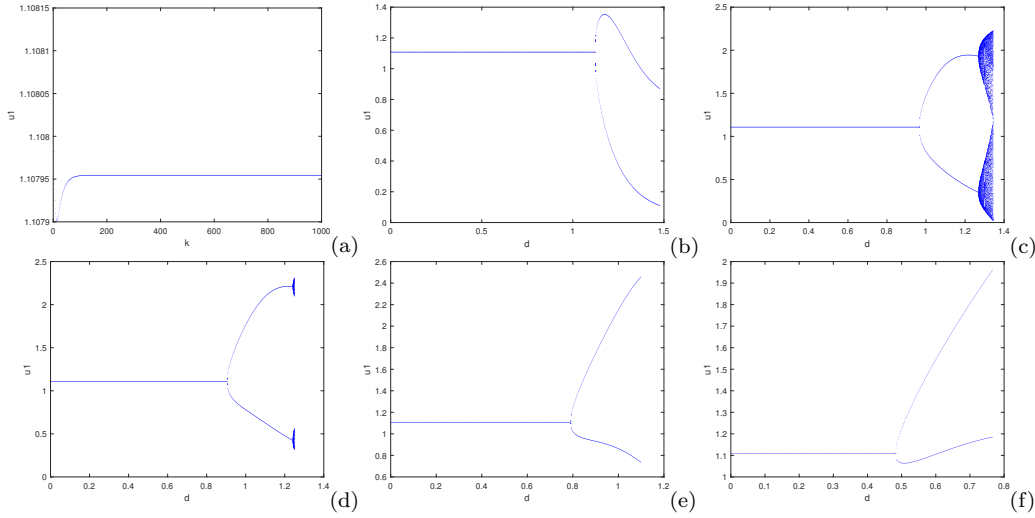


Figure 3. The sequence diagram or bifurcation diagram on direct control for u_1 when $r = 1.95, a = 0.76$. (a) The sequence diagram when $s_1 = 0.2, s_2 = 0.2$. (b) The bifurcation diagram when $s_1 = 0.5, s_2 = 0.45$. (c) The bifurcation diagram when $s_1 = 0.5, s_2 = 0.53$. (d) The bifurcation diagram when $s_1 = 0.5, s_2 = 0.56$. (e) The bifurcation diagram when $s_1 = 0.5, s_2 = 0.62$. (f) The bifurcation diagram when $s_1 = 0.5, s_2 = 0.82$.

Next, we consider the case of indirect control on the system (4.1) as follows.

$$\begin{cases} u_1^{t+1} = u_1^t \exp \left[r - u_1^t - av_1^t - b_1\omega_1^t \right] + d(-u_1^t + u_2^t), \\ u_2^{t+1} = u_2^t \exp \left[r - u_2^t - av_2^t - b_1\omega_2^t \right] + d(u_1^t - u_2^t), \\ v_1^{t+1} = v_1^t \exp \left[r - au_1^t - v_1^t - b_2\phi_1^t \right] + d(-v_1^t + v_2^t), \\ v_2^{t+1} = v_2^t \exp \left[r - au_2^t - v_2^t - b_2\phi_2^t \right] + d(v_1^t - v_2^t), \\ \omega_1^{t+1} = (1 - e_1)\omega_1^t + \eta_1 u_1^t, \\ \omega_2^{t+1} = (1 - e_1)\omega_2^t + \eta_1 u_2^t, \\ \phi_1^{t+1} = (1 - e_2)\phi_1^t + \eta_2 v_1^t, \\ \phi_2^{t+1} = (1 - e_2)\phi_2^t + \eta_2 v_2^t. \end{cases} \quad (4.3)$$

When selecting parameter $b_1 = 0.11, b_2 = 0.1, e_1 = 0.1, e_2 = 0.1, \eta_1 = 0.1, \eta_2 = 0.1$, a flip bifurcation occurs. However, the equilibrium point changes, as shown in Figure 4(a). By fixing parameters $b_1, e_1, e_2, \eta_1, \eta_2$, and a while varying b_2 , Figures 4(b) – (c) demonstrate the variation patterns of the bifurcation diagrams under the influence of parameter b_2 .

4.2. Turing bifurcation and its control

Based on the theoretical conditions determined above, numerical simulations are performed to demonstrate the pattern formation with a system size of $m = 100$, and the initial condition is

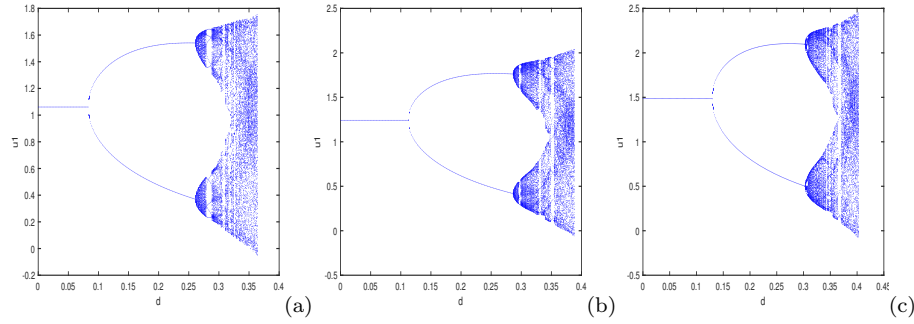


Figure 4. The bifurcation diagram on indirect control for u_1 when $b_1 = 0.1, e_1 = 0.1, e_2 = 0.1, eta_1 = 0.1, \eta_2 = 0.1$. (a) $b_2 = 0.11$. (b) $b_2 = 0.31$. (c) $b_2 = 0.75$.

always a small perturbation around the steady state and is selected as

$$\begin{cases} u_i^0 = \frac{r}{1+a} + 0.01 \times \cos \frac{(j-1)(2i-1)\pi}{2m}, \\ v_i^0 = \frac{r}{1+a} + 0.01 \times \cos \frac{(j-1)(2i-1)\pi}{2m}. \end{cases} \quad (4.4)$$

When different parameters are selected within the Turing instability region, the system exhibits a wealth of fascinating patterns, as depicted in Figure 5. In the graph, the horizontal axis represents the grid point positions, and the vertical axis represents the number of iterations. The same applies to the subsequent graphs.

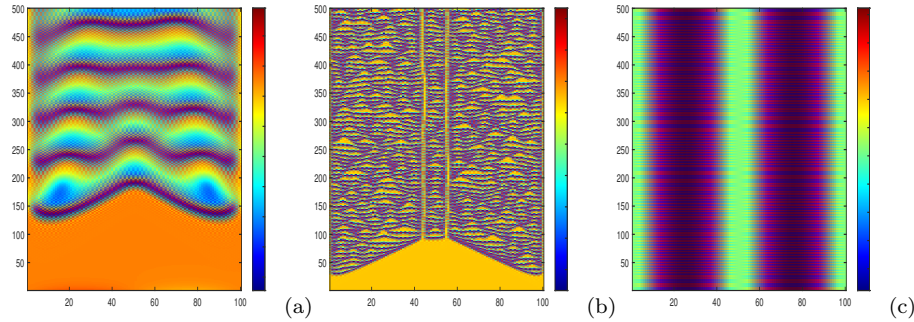


Figure 5. The selected patterns. (a) $r = 0.1, a = 0.1, d = 0.5$. (b) $r = 1.3, a = 0.2, d = 0.3$. (c) $r = 1.99, a = 0.3, d = 0.01$.

Next, we numerically verify the effectiveness of direct control and observe the influence of control parameter variations on pattern formation. Keeping $r = 0.1, a = 0.1, d = 0.5$, let $s_1 = s_2 = 0.1$, it is easy to verify that the condition $ds\mu_j < 2 - rs$ is satisfied, then Turing bifurcation can not emerge, and a single color will gradually spread and fill the space, which can be observed in Figure 6. Varying $s_1 = s_2 = 0.71, 0.81, 0.88$ in turn, we observe that pattern formation initiates at increasingly earlier time points with larger control parameters, as shown in Figure 7. Subsequently, we analyze the impact of indirect control strategies. By selecting parameters according to Table 1, the corresponding evolving patterns are shown in Figure 8. Different control intensities lead to distinct pattern formations.

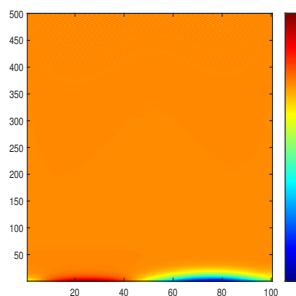


Figure 6. The stable patterns on direct control when $r = 0.1, a = 0.1, d = 0.5$ and $s_1 = s_2 = 0.1$.

Table 1. The selected parameters for indirect control strategies on Turing patterns.

	b_1	b_2	e_1	e_2	η_1	η_2
1	0.5	0.5	0.3	0.3	1	1
2	1	0.32	0.4	0.1	1	1
3	0.8	0.8	0.1	0.1	0.9	0.9

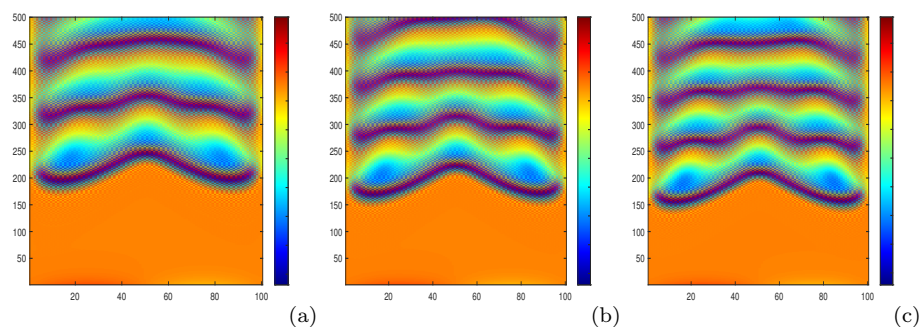


Figure 7. The selected patterns on direct control when $r = 0.1, a = 0.1, d = 0.5$. (a) $s_1 = s_2 = 0.71$. (b) $s_1 = s_2 = 0.81$. (c) $s_1 = s_2 = 0.88$.

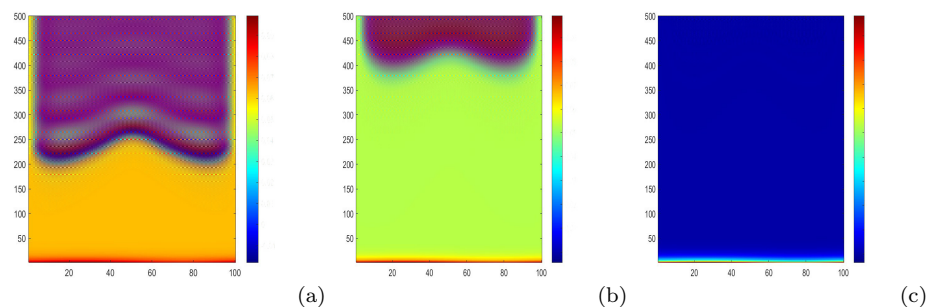


Figure 8. The selected patterns on indirect control when $r = 0.1, a = 0.1, d = 0.5$, other parameters are selected according to Table 1.

5. Conclusions

This paper investigates the complex dynamics of a discrete space-time Lotka–Volterra diffusion equation under Neumann boundary conditions. Using center manifold reduction and normal form theory, we first derive the conditions for a flip bifurcation while showing that fold and Neimark–Sacker bifurcations can not occur. Subsequently, we establish criteria for the emergence of Turing bifurcation in the parameter regime $0 < a < 1$. Based on these analytical results, we apply both direct and indirect control strategies to the discrete system to steer its evolution toward desired outcomes. Numerical simulations confirm the validity of the theoretical analysis and demonstrate that:

- (1) Flip bifurcation emerges as predicted.
- (2) Intricate spatiotemporal patterns arise.
- (3) Controlled evolution is successfully achieved through the proposed strategies.

Furthermore, we develop a general methodology for bifurcation analysis in discrete diffusion systems via center manifold and normal form approaches, which can be extended to systems with other types of boundary conditions.

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