

EXPLORING LOCAL FRACTIONAL INTEGRAL INEQUALITIES THROUGH BIPARAMETRIC ANALYSIS

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Abstract In this paper, we investigate a class of biparametrized integral inequalities in the setting of local fractional calculus on fractal sets. Motivated by earlier results in [27, 39], we derive a new integral identity associated with a three-point Newton–Cotes formula involving two parameters. A noteworthy feature of this formula is that, through suitable choices of the parameters, it unifies a broad family of one-point, two-point, and three-point quadrature rules. Based on this identity, we establish several new inequalities for functions whose first-order local fractional derivatives satisfy generalized s -convexity assumptions. Our results not only extend a number of existing inequalities in the literature, but also recover several known estimates as particular cases. To illustrate the effectiveness of the theoretical findings, we present an example supported by two- and three-dimensional graphical representations, together with some applications related to quadrature formulas and special means.

Keywords Newton–Cotes formula, biparametrized identity, local fractional integral, generalized s -convex functions, fractal set.

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1. Introduction

Convexity is a fundamental concept in mathematics that has many applications in various fields, including optimization, analysis, and geometry [25, 26]. A function is convex if its graph lies below its secant lines. In other words, a function is convex if for any two points on its graph, the line segment connecting those two points lies above the graph. Visually, a convex function has a “smiling” shape, where its graph curves upward.

This notion is crucial in the development of integral inequalities. Many integral inequalities rely on the convexity of the functions involved, and understanding the relationship between convexity and integral inequalities can lead to a deeper understanding of both concepts.

The inequality that is considered the most well-known and important involving convexity is the Hermite–Hadamard inequality. It can be stated as follows: For any convex function $\mathcal{S}$ defined on the interval $[\omega, \sigma]$, we have [26]:

$$\mathcal{S}\left(\frac{\omega + \sigma}{2}\right) \leq \frac{1}{\sigma - \omega} \int_{\omega}^{\sigma} \mathcal{S}(\eta) d\eta \leq \frac{\mathcal{S}(\omega) + \mathcal{S}(\sigma)}{2}. \quad (1.1)$$

For recent developments related to the Hermite–Hadamard inequality (1.1), we refer the reader to [2, 9, 24, 40].

Fractal sets are widely used to describe both natural and artificial objects, such as snowflakes, coastlines, lightning patterns, and ferns. In addition, they play an important role in image and signal processing, computer graphics, and the analysis of complex systems. In most cases, fractals are produced through recursive or iterative constructions based on the repeated application of simple rules. Classical examples include the Mandelbrot set [17], the Koch snowflake [12], and the Sierpiński triangle [32].

Local fractional calculus has emerged as a new tool for studying fractal sets, with researchers leveraging this approach to model processes exhibiting multi-scale behavior. By using local fractional calculus to investigate fractals, researchers have gained fresh insights into the intricate behavior of these complex systems [13, 35].

In 2012, Yang introduced the concept of generalized convexity on fractal sets in [36] as follows:

Definition 1.1. Let $\mathcal{S} : \mathcal{D} \subseteq \mathbb{R} \rightarrow \mathbb{R}^{\bar{\delta}}$. For any $\eta_1, \eta_2 \in \mathcal{D}$ and $\nu \in [0, 1]$, if

$$\mathcal{S}(\nu\eta_1 + (1 - \nu)\eta_2) \leq \nu^{\bar{\delta}} \mathcal{S}(\eta_1) + (1 - \nu)^{\bar{\delta}} \mathcal{S}(\eta_2)$$

holds, then $\mathcal{S}$ is a generalized convex function on $\mathcal{D}$.

Since the notion of generalized convexity on fractal sets was introduced, many authors have focused on extending classical integral inequalities to the framework of local fractional calculus. For instance, Abdeljawad et al. [1] derived several local fractional inequalities for generalized (s, m) -convex functions and presented corresponding applications. Du et al. [7] established integral inequalities for generalized m -convex functions defined on fractal sets, while Du and Yuan [8] studied parameter-dependent fractal integral inequalities and explored their applications. Erden and Sarikaya [11] obtained Bullen-type inequalities in the setting of local fractional integrals, whereas Khan et al. [14] proved trapezium-type inequalities for functions enjoying generalized convexity on fractal sets. In the same area, Lakhdari et al. [15] proposed corrected dual Simpson-type inequalities for differentiable generalized convex functions in the fractal context, and Luo et al. [16] established Fejér–Hermite–Hadamard-type inequalities involving generalized h -convex functions.

From another perspective, Meftah et al. [21] developed local fractional Maclaurin-type inequalities for generalized convex mappings, and in [20] the same authors obtained new Milne-type integral inequalities via generalized convexity arguments. Saleh et al. [28] investigated local fractional integral inequalities associated with the Mittag–Leffler kernel under generalized (E, h) -convexity assumptions. In addition, Sarikaya et al. [30] derived new Simpson-type inequalities for generalized convex functions, while Sun [33] studied local fractional integral inequalities for generalized preinvex functions and applied these results to numerical quadrature. More recently, Xu et al. [34] introduced new parameterized inequalities on fractal sets; Yu and Du [38] established error estimates for Bullen-type integral inequalities in fractal spaces; Eddahbi et al. [10] examined multi-parameter fractal inequalities for fractal (s, P) -convex functions; and Zhou and Du [41] derived Simpson-type inequalities for twice local fractional differentiable generalized (s, P) -convex functions. Among the notable contributions in this direction is the paper by Yu et al. [39], where a parameterized three-point Newton–Cotes formula was introduced and then used to obtain several inequalities related to the midpoint, trapezoidal, Bullen, and Simpson quadrature formulas. For further studies on fractal integral inequalities, we refer the reader to [3–5, 18].

To expand the scope of study and encompass a wider range of functions, several generalizations of convexity have been developed, among which we mention the class of s -convex functions in the second sense introduced by Mo et al. in [23].

Definition 1.2. Let $\mathcal{S} : \mathcal{D} \subseteq \mathbb{R} \rightarrow \mathbb{R}^{\bar{\sigma}}$. For any $\eta_1, \eta_2 \in \mathcal{D}$ and $\nu \in [0, 1]$, if

$$\mathcal{S}(\nu\eta_1 + (1 - \nu)\eta_2) \leq \nu^{s\bar{\sigma}} \mathcal{S}(\eta_1) + (1 - \nu)^{s\bar{\sigma}} \mathcal{S}(\eta_2)$$

holds for some fixed $s \in (0, 1]$, then the function $\mathcal{S}$ is generalized s -convex in the second sense on $\mathcal{D}$.

Motivated by earlier contributions, especially those in [27] and [39], the present work derives a more comprehensive identity based on a biparametric family of formulas. This framework is sufficiently broad to encompass quadrature-type expressions involving one-point, two-point, and three-point rules. By exploiting this identity, we obtain a wide collection of inequalities for locally fractional differentiable s -convex functions, including both novel estimates and several known results as particular cases. Finally, we provide an illustrative example, together with two- and three-dimensional graphical visualizations, to highlight the sharpness of the established results and to present some practical applications.

For $\mathcal{S} : [\omega, \sigma] \rightarrow \mathbb{R}$, $x \in [\omega, \frac{\omega+\sigma}{2}]$ and $\lambda \in [0, 1]$, we consider the following parametrized three-point Newton–Cotes formula:

$$\begin{aligned} \mathcal{I}(\lambda, \omega, \sigma, x, \mathcal{S}) &= \frac{(x - \omega)^{\bar{\sigma}} + \lambda^{\bar{\sigma}}(\omega + \sigma - 2x)^{\bar{\sigma}}}{(\sigma - \omega)^{\bar{\sigma}}} (\mathcal{S}(x) + \mathcal{S}(\omega + \sigma - x)) \\ &\quad + \frac{(1 - 2\lambda)^{\bar{\sigma}}(\omega + \sigma - 2x)^{\bar{\sigma}}}{(\sigma - \omega)^{\bar{\sigma}}} \mathcal{S}\left(\frac{\omega + \sigma}{2}\right). \end{aligned} \quad (1.2)$$

Note that by using formula (1.2) and specific values for the parameters x and λ , we can recover most of the famous formulas, see Table 1.

The rest of the paper is structured as follows. Section 2 presents the basic definitions and preliminary tools needed for the development of our results. The principal findings are established in Section 3. In Section 4, we examine several particular cases corresponding to

Table 1. Derived formulas.

x	λ	$\mathcal{I}(\lambda, \omega, \sigma, x, \mathcal{I})$	Formula
ω	0	$\mathcal{I}\left(\frac{\omega + \sigma}{2}\right)$	Midpoint
	$\frac{7}{30}$	$\left(\frac{1}{30}\right)^{\bar{\theta}} \left(7^{\bar{\theta}} \mathcal{I}(\omega) + 16^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 7^{\bar{\theta}} \mathcal{I}(\sigma)\right)$	Corrected Simpson
	$\frac{1}{6}$	$\left(\frac{1}{6}\right)^{\bar{\theta}} \left(\mathcal{I}(\omega) + 4^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + \mathcal{I}(\sigma)\right)$	Simpson
	$\frac{1}{4}$	$\left(\frac{1}{4}\right)^{\bar{\theta}} \left(\mathcal{I}(\omega) + 2^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + \mathcal{I}(\sigma)\right)$	Bullen
	$\frac{1}{2}$	$\frac{\mathcal{I}(\omega) + \mathcal{I}(\sigma)}{2^{\bar{\theta}}}$	Trapezium
	$\frac{2}{3}$	$\left(\frac{1}{3}\right)^{\bar{\theta}} \left(2^{\bar{\theta}} \mathcal{I}(\omega) - \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 2^{\bar{\theta}} \mathcal{I}(\sigma)\right)$	Milne
$x \in \left[\omega, \frac{\omega + \sigma}{2}\right)$	$\frac{1}{2}$	$\frac{\mathcal{I}(x) + \mathcal{I}(\omega + \sigma - x)}{2^{\bar{\theta}}}$	Companion Ostrowski
$\frac{5\omega + \sigma}{6}$	$\frac{5}{16}$	$\left(\frac{1}{8}\right)^{\bar{\theta}} \left(3^{\bar{\theta}} \mathcal{I}\left(\frac{5\omega + \sigma}{6}\right) + 2^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 3^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + 5\sigma}{6}\right)\right)$	Maclaurin
	$\frac{41}{160}$	$\left(\frac{1}{80}\right)^{\bar{\theta}} \left(27^{\bar{\theta}} \mathcal{I}\left(\frac{5\omega + \sigma}{6}\right) + 26^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 27^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + 5\sigma}{6}\right)\right)$	Corrected Euler-Maclaurin
$\frac{3\omega + \sigma}{4}$	$\frac{5}{6}$	$\left(\frac{1}{3}\right)^{\bar{\theta}} \left(2^{\bar{\theta}} \mathcal{I}\left(\frac{3\omega + \sigma}{4}\right) - \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 2^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + 3\sigma}{4}\right)\right)$	Dual-Simpson
	$\frac{17}{30}$	$\left(\frac{1}{15}\right)^{\bar{\theta}} \left(8^{\bar{\theta}} \mathcal{I}\left(\frac{3\omega + \sigma}{4}\right) - \mathcal{I}\left(\frac{\omega + \sigma}{2}\right) + 8^{\bar{\theta}} \mathcal{I}\left(\frac{\omega + 3\sigma}{4}\right)\right)$	Corrected Dual-Simpson

fixed choices of the parameters x and λ . Section 5 contains an illustrative graphical example, together with some applications, in order to verify the accuracy of the derived results. Finally, Section 6 summarizes the main conclusions of the paper.

2. Preliminaries

In this section, we revisit the manipulation of real numbers within a fractal set, using the approach proposed by Yang in [36] to define the local fractional derivative and local fractional integral.

Assuming that $\eta_1^{\bar{\theta}}$, $\eta_2^{\bar{\theta}}$, and $\eta_3^{\bar{\theta}}$ are within the set $\mathbb{R}^{\bar{\theta}}$ ($0 < \bar{\theta} \leq 1$), then the following statements can be made:

1. $\eta_1^{\bar{\theta}} + \eta_2^{\bar{\theta}}$ and $\eta_1^{\bar{\theta}} \eta_2^{\bar{\theta}}$ belong to the set $\mathbb{R}^{\bar{\theta}}$,
2. $\eta_1^{\bar{\theta}} + \eta_2^{\bar{\theta}} = \eta_2^{\bar{\theta}} + \eta_1^{\bar{\theta}} = (\eta_1 + \eta_2)^{\bar{\theta}} = (\eta_2 + \eta_1)^{\bar{\theta}}$,
3. $\eta_1^{\bar{\theta}} + (\eta_2^{\bar{\theta}} + \eta_3^{\bar{\theta}}) = (\eta_1 + \eta_2)^{\bar{\theta}} + \eta_3^{\bar{\theta}}$,
4. $\eta_1^{\bar{\theta}} \eta_2^{\bar{\theta}} = \eta_2^{\bar{\theta}} \eta_1^{\bar{\theta}} = (\eta_1 \eta_2)^{\bar{\theta}} = (\eta_2 \eta_1)^{\bar{\theta}}$,
5. $\eta_1^{\bar{\theta}} (\eta_2^{\bar{\theta}} \eta_3^{\bar{\theta}}) = (\eta_1^{\bar{\theta}} \eta_2^{\bar{\theta}}) \eta_3^{\bar{\theta}}$,
6. $\eta_1^{\bar{\theta}} (\eta_2^{\bar{\theta}} + \eta_3^{\bar{\theta}}) = \eta_1^{\bar{\theta}} \eta_2^{\bar{\theta}} + \eta_1^{\bar{\theta}} \eta_3^{\bar{\theta}}$,
7. $\eta_1^{\bar{\theta}} + 0^{\bar{\theta}} = 0^{\bar{\theta}} + \eta_1^{\bar{\theta}} = \eta_1^{\bar{\theta}}$ and $\eta_1^{\bar{\theta}} 1^{\bar{\theta}} = 1^{\bar{\theta}} \eta_1^{\bar{\theta}} = \eta_1^{\bar{\theta}}$.

Gao-Yang-Kang's [36, 37] were the ones to propose the concept of local fractional derivative and local fractional integral.

Definition 2.1 ([36]). A function $\mathcal{S} : [\omega, \sigma] \rightarrow \mathbb{R}^{\bar{\theta}}$ is local fractional continuous on $[\omega, \sigma]$, if for any $\kappa > 0$, there exists $\varepsilon > 0$ such that for all η_1 and η_2 satisfying $|\eta_1 - \eta_2| < \varepsilon$ the following inequality

$$|\mathcal{S}(\eta_1) - \mathcal{S}(\eta_2)| < \kappa^{\bar{\theta}}$$

holds true. We denote by $C_{\bar{\theta}}[\omega, \sigma]$ the set that encompasses all functions that are local fractional continuous on $[\omega, \sigma]$.

Definition 2.2 ([36]). The local fractional derivative of $\mathcal{S}(\eta)$ of order $\bar{\theta}$ at $\eta = \eta_0$ is defined by:

$$\mathcal{S}^{(\bar{\theta})}(\eta_0) = \left. \frac{d^{\bar{\theta}} \mathcal{S}(\eta)}{d\eta^{\bar{\theta}}} \right|_{\eta=\eta_0} = \lim_{\eta \rightarrow \eta_0} \frac{\Delta^{\bar{\theta}}(\mathcal{S}(\eta) - \mathcal{S}(\eta_0))}{(\eta - \eta_0)^{\bar{\theta}}},$$

where $\Delta^{\bar{\theta}}(\mathcal{S}(\eta) - \mathcal{S}(\eta_0)) \cong \Gamma(\bar{\theta} + 1)(\mathcal{S}(\eta) - \mathcal{S}(\eta_0))$.

If there exists $\mathcal{S}^{(m+1)\bar{\theta}}(\eta) = \overbrace{D^{\bar{\theta}} D^{\bar{\theta}} \dots D^{\bar{\theta}}}^{(m+1) \text{ times}} \mathcal{S}(\eta)$ for any $t \in I \subseteq \mathbb{R}$, then we say that $\mathcal{S} \in D_{(m+1)\bar{\theta}}(I)$, where $m \in \mathbb{N}$.

Definition 2.3 ([36]). Let $\mathcal{S}(\eta) \in C_{\bar{\theta}}[\omega, \sigma]$. Then the local fractional integral is defined by,

$${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta) = \frac{1}{\Gamma(\bar{\theta} + 1)} \int_{\omega}^{\sigma} \mathcal{S}(\eta) (d\eta)^{\bar{\theta}} = \frac{1}{\Gamma(\bar{\theta} + 1)} \lim_{\Delta\eta \rightarrow 0} \sum_{i=0}^{M-1} \mathcal{S}(\eta_i) (\Delta\eta_i)^{\bar{\theta}},$$

with $\Delta\eta_i = \eta_{i+1} - \eta_i$ and $\Delta\eta = \max\{\Delta\eta_1, \Delta\eta_2, \dots, \Delta\eta_{M-1}\}$, where $[\eta_i, \eta_{i+1}]$, $i = 0, 1, \dots, M-1$ and $\omega = \eta_0 < \eta_1 < \dots < \eta_M = \sigma$ is a partition of the interval $[\omega, \sigma]$.

It can be inferred that, ${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta) = 0$ for $\omega = \sigma$ and ${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta) = -\sigma {}_{\omega} \mathcal{I}_{\omega}^{\bar{\theta}} \mathcal{S}(\eta)$ for $\omega < \sigma$. If for any $\eta \in [\omega, \sigma]$, ${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta)$ exists, then we denote by $\mathcal{S}(\eta) \in \mathcal{Q}_{\eta}^{\bar{\theta}}[\omega, \sigma]$.

Lemma 2.1 ([36]).

1. Let $\mathcal{S}(\eta) = \mathcal{Q}^{(\bar{\theta})}(\eta) \in C_{\bar{\theta}}[\omega, \sigma]$, then we have

$${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta) = \mathcal{Q}(\sigma) - \mathcal{Q}(\omega).$$

2. Let $\mathcal{S}, \mathcal{Q} \in D_{\bar{\theta}}[\omega, \sigma]$ and $\mathcal{S}^{(\bar{\theta})}, \mathcal{Q}^{(\bar{\theta})}(\eta) \in C_{\bar{\theta}}[\omega, \sigma]$, then we have

$${}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}(\eta) \mathcal{Q}^{(\bar{\theta})}(\eta) = \mathcal{S}(\eta) \mathcal{Q}(\eta) \Big|_{\omega}^{\sigma} - {}_{\omega} \mathcal{I}_{\sigma}^{\bar{\theta}} \mathcal{S}^{(\bar{\theta})}(\eta) \mathcal{Q}(\eta).$$

Lemma 2.2 ([36]). For $\mathcal{S}(\eta) = \eta^{m\bar{\theta}}$, we have

$$\begin{aligned} \frac{d^{\bar{\theta}} \eta^{m\bar{\theta}}}{d\eta^{\bar{\theta}}} &= \frac{\Gamma(1 + m\bar{\theta})}{\Gamma(1 + (m-1)\bar{\theta})} \eta^{(m-1)\bar{\theta}}, \\ \frac{1}{\Gamma(1 + \bar{\theta})} \int_{\omega}^{\sigma} \eta^{m\bar{\theta}} (d\eta)^{\bar{\theta}} &= \frac{\Gamma(1 + m\bar{\theta})}{\Gamma(1 + (m+1)\bar{\theta})} \left(\sigma^{(m+1)\bar{\theta}} - \omega^{(m+1)\bar{\theta}} \right), \quad m \in \mathbb{R}. \end{aligned}$$

The generalized Hölder inequality was provided in [6] as follows:

Lemma 2.3. Let $\mathcal{S}, \mathcal{Q} \in C_{\bar{\theta}}[\omega, \sigma]$, then

$$\begin{aligned} &\frac{1}{\Gamma(1 + \bar{\theta})} \int_{\omega}^{\sigma} |\mathcal{S}(\eta) \mathcal{Q}(\eta)| (d\eta)^{\bar{\theta}} \\ &\leq \left(\frac{1}{\Gamma(1 + \bar{\theta})} \int_{\omega}^{\sigma} |\mathcal{S}(\eta)|^p (d\eta)^{\bar{\theta}} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1 + \bar{\theta})} \int_{\omega}^{\sigma} |\mathcal{Q}(\eta)|^q (d\eta)^{\bar{\theta}} \right)^{\frac{1}{q}} \end{aligned}$$

holds, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Principal outcomes

The proof of our results relies on the following lemma.

Lemma 3.1. *Let $\mathcal{S} : [\omega, \sigma] \rightarrow \mathbb{R}^{\bar{\vartheta}}$ be a differentiable function on (ω, σ) such that $\mathcal{S} \in D_{\bar{\vartheta}}[\omega, \sigma]$ and $\mathcal{S}^{(\bar{\vartheta})} \in C_{\bar{\vartheta}}[\omega, \sigma]$ with $0 \leq \omega < \sigma$, then the following equality holds for all $x \in [\omega, \frac{\omega+\sigma}{2}]$*

$$\begin{aligned} & \mathcal{T}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{I}_{\sigma}^{\bar{\vartheta}} \mathcal{S}(\eta) \\ &= \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 \nu^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})}((1 - \nu)\omega + \nu x) (d\nu)^{\bar{\vartheta}} \right. \\ & \quad + \frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 1)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) (d\nu)^{\bar{\vartheta}} \\ & \quad + \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 2\lambda)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})} \left((1 - \nu)x + \nu \frac{\omega + \sigma}{2} \right) (d\nu)^{\bar{\vartheta}} \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 1 + 2\lambda)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})} \left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x) \right) (d\nu)^{\bar{\vartheta}} \right) \right), \end{aligned}$$

where $\mathcal{T}(\lambda, \omega, \sigma, x, \mathcal{S})$ is as defined in (1.2).

Proof. Let

$$\mathcal{L} = \sum_{i=1}^4 \mathcal{L}_i, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{L}_1 &= \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 \nu^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})}((1 - \nu)\omega + \nu x) (d\nu)^{\bar{\vartheta}} \right], \\ \mathcal{L}_2 &= \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 1)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) (d\nu)^{\bar{\vartheta}} \right], \\ \mathcal{L}_3 &= \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 2\lambda)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})} \left((1 - \nu)x + \nu \frac{\omega + \sigma}{2} \right) (d\nu)^{\bar{\vartheta}} \right] \end{aligned}$$

and

$$\mathcal{L}_4 = \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1}{\Gamma(\bar{\vartheta} + 1)} \int_0^1 (\nu - 1 + 2\lambda)^{\bar{\vartheta}} \mathcal{S}^{(\bar{\vartheta})} \left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x) \right) (d\nu)^{\bar{\vartheta}} \right].$$

Using the local fractional integration by parts, we get for $x \in [\omega, \frac{\omega+\sigma}{2}]$

$$\begin{aligned} \mathcal{L}_1 &= \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1^{\bar{\vartheta}}}{(x - \omega)^{\bar{\vartheta}}} \nu^{\bar{\vartheta}} \mathcal{S}((1 - \nu)\omega + \nu x) \Big|_{\nu=0}^{\nu=1} \right. \\ & \quad \left. - \frac{1^{\bar{\vartheta}}}{(x - \omega)^{\bar{\vartheta}} \Gamma(\bar{\vartheta} + 1)} \int_0^1 \Gamma(\bar{\vartheta} + 1) \mathcal{S}((1 - \nu)\omega + \nu x) (d\nu)^{\bar{\vartheta}} \right] \\ &= \frac{(x - \omega)^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\mathcal{S}(x) - \int_0^1 \mathcal{S}((1 - \nu)\omega + \nu x) (d\nu)^{\bar{\vartheta}} \right] \end{aligned}$$

$$= \frac{(x - \omega)^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}(x) - \frac{1^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \int_{\omega}^x \mathcal{S}(\eta) (d\eta)^{\bar{\vartheta}}. \quad (3.2)$$

Similarly,

$$\begin{aligned} \mathcal{L}_2 &= \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{1^{\bar{\vartheta}}}{(x - \omega)^{\bar{\vartheta}}} (\nu - 1)^{\bar{\vartheta}} \mathcal{S}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) \right]_{\nu=0}^{\nu=1} \\ &\quad - \frac{1^{\bar{\vartheta}}}{(x - \omega)^{\bar{\vartheta}} \Gamma(\bar{\vartheta} + 1)} \int_0^1 \Gamma(\bar{\vartheta} + 1) \mathcal{S}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) (d\nu)^{\bar{\vartheta}} \Big] \\ &= \frac{(x - \omega)^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}(\omega + \sigma - x) - \frac{1^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \int_{\omega + \sigma - x}^{\sigma} \mathcal{S}(\eta) (d\eta)^{\bar{\vartheta}}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \mathcal{L}_3 &= \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}} (\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{2^{\bar{\vartheta}}}{(\omega + \sigma - 2x)^{\bar{\vartheta}}} (\nu - 2\lambda)^{\bar{\vartheta}} \mathcal{S}\left((1 - \nu)x + \nu \frac{\omega + \sigma}{2}\right) \right]_{\nu=0}^{\nu=1} \\ &\quad - \frac{2^{\bar{\vartheta}}}{(\omega + \sigma - 2x)^{\bar{\vartheta}} \Gamma(\bar{\vartheta} + 1)} \int_0^1 \Gamma(\bar{\vartheta} + 1) \mathcal{S}\left((1 - \nu)x + \nu \frac{\omega + \sigma}{2}\right) (d\nu)^{\bar{\vartheta}} \Big] \\ &= \frac{(1 - 2\lambda)^{\bar{\vartheta}} (\omega + \sigma - 2x)^{\bar{\vartheta}}}{2^{\bar{\vartheta}} (\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}\left(\frac{\omega + \sigma}{2}\right) + \frac{\lambda^{\bar{\vartheta}} (\omega + \sigma - 2x)^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}(x) \\ &\quad - \frac{1^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \int_x^{\frac{\omega + \sigma}{2}} \mathcal{S}(\eta) (d\eta)^{\bar{\vartheta}} \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \mathcal{L}_4 &= \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}} (\sigma - \omega)^{\bar{\vartheta}}} \left[\frac{2^{\bar{\vartheta}}}{(\omega + \sigma - 2x)^{\bar{\vartheta}}} (\nu - 1 + 2\lambda)^{\bar{\vartheta}} \mathcal{S}\left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x)\right) \right]_{\nu=0}^{\nu=1} \\ &\quad - \frac{2^{\bar{\vartheta}}}{(\omega + \sigma - 2x)^{\bar{\vartheta}} \Gamma(\bar{\vartheta} + 1)} \int_0^1 \Gamma(\bar{\vartheta} + 1) \mathcal{S}\left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x)\right) (d\nu)^{\bar{\vartheta}} \Big] \\ &= \frac{\lambda^{\bar{\vartheta}} (\omega + \sigma - 2x)^{\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}(\omega + \sigma - x) + \frac{(1 - 2\lambda)^{\bar{\vartheta}} (\omega + \sigma - 2x)^{\bar{\vartheta}}}{2^{\bar{\vartheta}} (\sigma - \omega)^{\bar{\vartheta}}} \mathcal{S}\left(\frac{\omega + \sigma}{2}\right) \\ &\quad - \frac{1^{\bar{\vartheta}}}{(\sigma - \omega)^{2\bar{\vartheta}}} \int_{\frac{\omega + \sigma}{2}}^{\omega + \sigma - x} \mathcal{S}(\eta) (d\eta)^{\bar{\vartheta}}. \end{aligned} \quad (3.5)$$

Substituting (3.2)-(3.5) in (3.1), we get the desired result. $\square$

Theorem 3.1. Let us assume that the conditions outlined in Lemma 3.1 are fully met. If $|\mathcal{S}^{(\bar{\vartheta})}|$ is generalized s -convex in the second sense on $[\omega, \sigma]$, then for all $\lambda \in [0, \frac{1}{2}]$ we have

$$\begin{aligned} &\left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{S}(\eta) \right| \\ &\leq \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left[\left(\frac{\Gamma(1 + s\bar{\vartheta})}{\Gamma(1 + (s + 1)\bar{\vartheta})} - \frac{\Gamma(1 + (s + 1)\bar{\vartheta})}{\Gamma(1 + (s + 2)\bar{\vartheta})} \right) (|\mathcal{S}^{(\bar{\vartheta})}(\omega)| + |\mathcal{S}^{(\bar{\vartheta})}(\sigma)|) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\Gamma(1+(s+1)\bar{\vartheta})}{\Gamma(1+(s+2)\bar{\vartheta})} \left(\left| \mathcal{J}^{(\bar{\vartheta})}(x) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\omega + \sigma - x) \right| \right) \Bigg] \\
& + \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left[\left(2^{\bar{\vartheta}}(1-2\lambda)^{(s+2)\bar{\vartheta}} \left(\frac{\Gamma(1+s\bar{\vartheta})}{\Gamma(1+(s+1)\bar{\vartheta})} - \frac{\Gamma(1+(s+1)\bar{\vartheta})}{\Gamma(1+(s+2)\bar{\vartheta})} \right) \right. \right. \\
& + \left. \frac{\Gamma(1+(s+1)\bar{\vartheta})}{\Gamma(1+(s+2)\bar{\vartheta})} - (1-2\lambda)^{\bar{\vartheta}} \frac{\Gamma(1+s\bar{\vartheta})}{\Gamma(1+(s+1)\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})}(x) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\omega + \sigma - x) \right| \right) \\
& + \left(4^{\bar{\vartheta}}(2\lambda)^{(s+2)\bar{\vartheta}} \left(\frac{\Gamma(1+s\bar{\vartheta})}{\Gamma(1+(s+1)\bar{\vartheta})} - \frac{\Gamma(1+(s+1)\bar{\vartheta})}{\Gamma(1+(s+2)\bar{\vartheta})} \right) 2^{\bar{\vartheta}} \frac{\Gamma(1+(s+1)\bar{\vartheta})}{\Gamma(1+(s+2)\bar{\vartheta})} \right. \\
& \left. \left. - (4\lambda)^{\bar{\vartheta}} \frac{\Gamma(1+s\bar{\vartheta})}{\Gamma(1+(s+1)\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right].
\end{aligned}$$

Proof. From Lemma 3.1 and properties of modulus, we have

$$\begin{aligned}
& \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\vartheta}+1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 \nu^{\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}((1-\nu)\omega + \nu x) \right| (d\nu)^{\bar{\vartheta}} \right. \\
& \quad + \frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 (1-\nu)^{\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}((1-\nu)(\omega + \sigma - x) + \nu\sigma) \right| (d\nu)^{\bar{\vartheta}} \Bigg) \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 |\nu - 2\lambda|^{\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}\left((1-\nu)x + \nu\frac{\omega + \sigma}{2}\right) \right| (d\nu)^{\bar{\vartheta}} \right. \\
& \quad \left. + \frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 |\nu - (1-2\lambda)|^{\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}\left((1-\nu)\frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x)\right) \right| (d\nu)^{\bar{\vartheta}} \right). \quad (3.6)
\end{aligned}$$

Using the generalized s -convexity of $|\mathcal{J}^{(\bar{\vartheta})}|$, inequality (3.6) yields

$$\begin{aligned}
& \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\vartheta}+1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 \nu^{\bar{\vartheta}} \left((1-\nu)^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \nu^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(x) \right| \right) (d\nu)^{\bar{\vartheta}} \right. \\
& \quad + \frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 (1-\nu)^{\bar{\vartheta}} \left((1-\nu)^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(\omega + \sigma - x) \right| + \nu^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) (d\nu)^{\bar{\vartheta}} \Bigg) \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\bar{\vartheta}}}{4^{\bar{\vartheta}}(\sigma - \omega)^{\bar{\vartheta}}} \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 |\nu - 2\lambda|^{\bar{\vartheta}} \left((1-\nu)^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(x) \right| + \nu^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right) (d\nu)^{\bar{\vartheta}} \right. \\
& \quad + \frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 |\nu - 1 + 2\lambda|^{\bar{\vartheta}} \left((1-\nu)^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| + \nu^{s\bar{\vartheta}} \left| \mathcal{J}^{(\bar{\vartheta})}(\omega + \sigma - x) \right| \right) (d\nu)^{\bar{\vartheta}} \Bigg) \\
& = \frac{(x - \omega)^{2\bar{\vartheta}}}{(\sigma - \omega)^{\bar{\vartheta}}} \left(\left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 \nu^{\bar{\vartheta}} (1-\nu)^{s\bar{\vartheta}} (d\nu)^{\bar{\vartheta}} \right) \left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| \right. \\
& \quad + \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 \nu^{(s+1)\bar{\vartheta}} (d\nu)^{\bar{\vartheta}} \right) \left| \mathcal{J}^{(\bar{\vartheta})}(x) \right| \\
& \quad + \left(\frac{1}{\Gamma(\bar{\vartheta}+1)} \int_0^1 (1-\nu)^{(s+1)\bar{\vartheta}} (d\nu)^{\bar{\vartheta}} \right) \left| \mathcal{J}^{(\bar{\vartheta})}(\omega + \sigma - x) \right|
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 (1 - \nu)^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \right) \left| \mathcal{J}^{(\tilde{\sigma})}(\sigma) \right| \\
& + \frac{(\omega + \sigma - 2x)^{2\tilde{\sigma}}}{4^{\tilde{\sigma}} (\sigma - \omega)^{\tilde{\sigma}}} \left(\left(\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - 2\lambda|^{\tilde{\sigma}} (1 - \nu)^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \right) \left| \mathcal{J}^{(\tilde{\sigma})}(x) \right| \right. \\
& + \left(\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - 2\lambda|^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} + \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - (1 - 2\lambda)|^{\tilde{\sigma}} (1 - \nu)^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \right) \\
& \times \left| \mathcal{J}^{(\tilde{\sigma})} \left(\frac{\omega + \sigma}{2} \right) \right| + \left. \left(\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - (1 - 2\lambda)|^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \right) \left| \mathcal{J}^{(\tilde{\sigma})}(\omega + \sigma - x) \right| \right) \\
& = \frac{(x - \omega)^{2\tilde{\sigma}}}{(\sigma - \omega)^{\tilde{\sigma}}} \left[\left(\frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} - \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \right) \left(\left| \mathcal{J}^{(\tilde{\sigma})}(\omega) \right| + \left| \mathcal{J}^{(\tilde{\sigma})}(\sigma) \right| \right) \right. \\
& + \left. \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \left(\left| \mathcal{J}^{(\tilde{\sigma})}(x) \right| + \left| \mathcal{J}^{(\tilde{\sigma})}(\omega + \sigma - x) \right| \right) \right] \\
& + \frac{(\omega + \sigma - 2x)^{2\tilde{\sigma}}}{4^{\tilde{\sigma}} (\sigma - \omega)^{\tilde{\sigma}}} \left[\left(2^{\tilde{\sigma}} (1 - 2\lambda)^{(s+2)\tilde{\sigma}} \left(\frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} - \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \right) \right. \right. \\
& + \left. \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} - (1 - 2\lambda)^{\tilde{\sigma}} \frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} \right) \left(\left| \mathcal{J}^{(\tilde{\sigma})}(x) \right| + \left| \mathcal{J}^{(\tilde{\sigma})}(\omega + \sigma - x) \right| \right) \\
& + \left(4^{\tilde{\sigma}} (2\lambda)^{(s+2)\tilde{\sigma}} \left(\frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} - \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \right) 2^{\tilde{\sigma}} \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \right. \\
& \left. \left. - (4\lambda)^{\tilde{\sigma}} \frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} \right) \left| \mathcal{J}^{(\tilde{\sigma})} \left(\frac{\omega + \sigma}{2} \right) \right| \right],
\end{aligned}$$

where we have used the facts that

$$\begin{aligned}
\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 \nu^{\tilde{\sigma}} (1 - \nu)^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} &= \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 (1 - \nu)^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \\
&= \frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} - \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})},
\end{aligned} \tag{3.7}$$

$$\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 \nu^{(s+1)\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} = \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 (1 - \nu)^{(s+1)\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} = \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})}, \tag{3.8}$$

$$\begin{aligned}
& \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - 2\lambda|^{\tilde{\sigma}} (1 - \nu)^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \\
&= \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |1 - 2\lambda - \nu|^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \\
&= \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^{1-2\lambda} (1 - 2\lambda - \nu)^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} + \frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_{1-2\lambda}^1 (\nu - 1 + 2\lambda)^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}} \\
&= 2^{\tilde{\sigma}} (1 - 2\lambda)^{(s+2)\tilde{\sigma}} \left(\frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})} - \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} \right) \\
&+ \frac{\Gamma(1 + (s + 1)\tilde{\sigma})}{\Gamma(1 + (s + 2)\tilde{\sigma})} - (1 - 2\lambda)^{\tilde{\sigma}} \frac{\Gamma(1 + s\tilde{\sigma})}{\Gamma(1 + (s + 1)\tilde{\sigma})}
\end{aligned} \tag{3.9}$$

and

$$\frac{1}{\Gamma(\tilde{\sigma} + 1)} \int_0^1 |\nu - 2\lambda|^{\tilde{\sigma}} \nu^{s\tilde{\sigma}} (d\nu)^{\tilde{\sigma}}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 |\nu - 1 + 2\lambda|^{\tilde{\theta}} (1 - \nu)^{s\tilde{\theta}} (d\nu)^{\tilde{\theta}} \\
&= \frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^{2\lambda} (2\lambda - \nu)^{\tilde{\theta}} \nu^{s\tilde{\theta}} (d\nu)^{\tilde{\theta}} + \frac{1}{\Gamma(\tilde{\theta} + 1)} \int_{2\lambda}^1 (\nu - 2\lambda)^{\tilde{\theta}} \nu^{s\tilde{\theta}} (d\nu)^{\tilde{\theta}} \\
&= 2^{\tilde{\theta}} (2\lambda)^{(s+2)\tilde{\theta}} \left(\frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s+1)\tilde{\theta})} - \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} \right) \\
&\quad + \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} - (2\lambda)^{\tilde{\theta}} \frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s+1)\tilde{\theta})}.
\end{aligned} \tag{3.10}$$

The proof is completed. $\square$

Corollary 3.1. *Setting $s = 1$ in Theorem 3.1, we obtain*

$$\begin{aligned}
&\left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\tilde{\theta} + 1)}{(\sigma - \omega)^{\tilde{\theta}}} \omega \mathcal{J}_{\sigma}^{\tilde{\theta}} \mathcal{J}(\eta) \right| \\
&\leq \frac{(x - \omega)^{2\tilde{\theta}}}{(\sigma - \omega)^{\tilde{\theta}}} \left[\left(\frac{\Gamma(1 + \tilde{\theta})}{\Gamma(1 + 2\tilde{\theta})} - \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} \right) \left(\left| \mathcal{J}^{(\tilde{\theta})}(\omega) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\sigma) \right| \right) \right. \\
&\quad \left. + \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} \left(\left| \mathcal{J}^{(\tilde{\theta})}(x) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\omega + \sigma - x) \right| \right) \right] \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\tilde{\theta}}}{4^{\tilde{\theta}} (\sigma - \omega)^{\tilde{\theta}}} \left[\left(2^{\tilde{\theta}} (1 - 2\lambda)^{3\tilde{\theta}} \left(\frac{\Gamma(1 + \tilde{\theta})}{\Gamma(1 + 2\tilde{\theta})} - \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} \right) \right. \right. \\
&\quad \left. \left. + \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} - (1 - 2\lambda)^{\tilde{\theta}} \frac{\Gamma(1 + \tilde{\theta})}{\Gamma(1 + 2\tilde{\theta})} \right) \left(\left| \mathcal{J}^{(\tilde{\theta})}(x) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\omega + \sigma - x) \right| \right) \right. \\
&\quad \left. + \left(4^{\tilde{\theta}} (2\lambda)^{3\tilde{\theta}} \left(\frac{\Gamma(1 + \tilde{\theta})}{\Gamma(1 + 2\tilde{\theta})} - \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} \right) + 2^{\tilde{\theta}} \frac{\Gamma(1 + 2\tilde{\theta})}{\Gamma(1 + 3\tilde{\theta})} - (4\lambda)^{\tilde{\theta}} \frac{\Gamma(1 + \tilde{\theta})}{\Gamma(1 + 2\tilde{\theta})} \right) \left| \mathcal{J}^{(\tilde{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right].
\end{aligned}$$

Theorem 3.2. *Let us assume that the conditions outlined in Theorem 3.1 are fully met. Then, for $\lambda \in [\frac{1}{2}, 1]$ the following inequality holds*

$$\begin{aligned}
&\left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\tilde{\theta} + 1)}{(\sigma - \omega)^{\tilde{\theta}}} \omega \mathcal{J}_{\sigma}^{\tilde{\theta}} \mathcal{J}(\eta) \right| \\
&\leq \frac{(x - \omega)^{2\tilde{\theta}}}{(\sigma - \omega)^{\tilde{\theta}}} \left[\left(\frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s+1)\tilde{\theta})} - \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} \right) \left(\left| \mathcal{J}^{(\tilde{\theta})}(\omega) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\sigma) \right| \right) \right. \\
&\quad \left. + \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} \left(\left| \mathcal{J}^{(\tilde{\theta})}(x) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\omega + \sigma - x) \right| \right) \right] \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\tilde{\theta}}}{4^{\tilde{\theta}} (\sigma - \omega)^{\tilde{\theta}}} \left[\left((2\lambda - 1)^{\tilde{\theta}} \frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s+1)\tilde{\theta})} + \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} \right) \right. \\
&\quad \times \left(\left| \mathcal{J}^{(\tilde{\theta})}(x) \right| + \left| \mathcal{J}^{(\tilde{\theta})}(\omega + \sigma - x) \right| \right) \\
&\quad \left. + \left((4\lambda)^{\tilde{\theta}} \frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s+1)\tilde{\theta})} - 2^{\tilde{\theta}} \frac{\Gamma(1 + (s+1)\tilde{\theta})}{\Gamma(1 + (s+2)\tilde{\theta})} \right) \left| \mathcal{J}^{(\tilde{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right].
\end{aligned}$$

Proof. From Lemma 3.1 and properties of modulus, we have

$$\left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\tilde{\theta} + 1)}{(\sigma - \omega)^{\tilde{\theta}}} \omega \mathcal{J}_{\sigma}^{\tilde{\theta}} \mathcal{J}(\eta) \right|$$

$$\begin{aligned}
&\leq \frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 \nu^{\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}((1-\nu)\omega + \nu x) \right| (d\nu)^{\bar{\theta}} \right. \\
&\quad + \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (1-\nu)^{\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}((1-\nu)(\omega + \sigma - x) + \nu\sigma) \right| (d\nu)^{\bar{\theta}} \Big) \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma-\omega)^{\bar{\theta}}} \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 |\nu - 2\lambda|^{\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})} \left((1-\nu)x + \nu \frac{\omega + \sigma}{2} \right) \right| (d\nu)^{\bar{\theta}} \right. \\
&\quad \left. + \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 |\nu - (1-2\lambda)|^{\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})} \left((1-\nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x) \right) \right| (d\nu)^{\bar{\theta}} \right). \quad (3.11)
\end{aligned}$$

Using the generalized s -convexity of $|\mathcal{J}^{(\bar{\theta})}|$, inequality (3.11) yields

$$\begin{aligned}
&\left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta}+1)}{(\sigma-\omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\eta) \right| \\
&\leq \frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 \nu^{\bar{\theta}} \left((1-\nu)^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(\omega) \right| + \nu^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(x) \right| \right) (d\nu)^{\bar{\theta}} \right. \\
&\quad + \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (1-\nu)^{\bar{\theta}} \left((1-\nu)^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| + \nu^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right| \right) (d\nu)^{\bar{\theta}} \Big) \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma-\omega)^{\bar{\theta}}} \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (2\lambda - \nu)^{\bar{\theta}} \left((1-\nu)^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(x) \right| + \nu^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right) (d\nu)^{\bar{\theta}} \right. \\
&\quad + \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (\nu - (1-2\lambda))^{\bar{\theta}} \left((1-\nu)^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| + \nu^{s\bar{\theta}} \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \right) (d\nu)^{\bar{\theta}} \Big) \\
&= \frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 \nu^{\bar{\theta}} (1-\nu)^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(\omega) \right| + \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 \nu^{(s+1)\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(x) \right| \right. \\
&\quad + \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (1-\nu)^{(s+1)\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \\
&\quad + \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (1-\nu)^{\bar{\theta}} \nu^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right| \Big) \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma-\omega)^{\bar{\theta}}} \left(\left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (2\lambda - \nu)^{\bar{\theta}} (1-\nu)^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(x) \right| \right. \\
&\quad + \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (2\lambda - \nu)^{\bar{\theta}} \nu^{s\bar{\theta}} (d\nu)^{\bar{\theta}} + \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (\nu - (1-2\lambda))^{\bar{\theta}} (1-\nu)^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \\
&\quad \times \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| + \left(\frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (\nu - (1-2\lambda))^{\bar{\theta}} \nu^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \right) \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \Big) \\
&= \frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left[\left(\frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} - \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})} \right) \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right| \right) \right. \\
&\quad + \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})} \left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \right) \Big] \\
&\quad + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma-\omega)^{\bar{\theta}}} \left[\left((2\lambda - 1)^{\bar{\theta}} \frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} + \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})} \right) \right. \\
&\quad \times \left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \right) \Big]
\end{aligned}$$

$$+ \left((4\lambda)^{\bar{\theta}} \frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} - 2^{\bar{\theta}} \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})} \right) \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| \Bigg],$$

where we have used (3.7), (3.8) and

$$\begin{aligned} \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (2\lambda - \nu)^{\bar{\theta}} (1 - \nu)^{s\bar{\theta}} (d\nu)^{\bar{\theta}} &= \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (\nu + (2\lambda - 1))^{\bar{\theta}} \nu^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \\ &= (2\lambda - 1)^{\bar{\theta}} \frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} + \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})} \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (2\lambda - \nu)^{\bar{\theta}} \nu^{s\bar{\theta}} (d\nu)^{\bar{\theta}} &= \frac{1}{\Gamma(\bar{\theta}+1)} \int_0^1 (\nu - (1 - 2\lambda))^{\bar{\theta}} (1 - \nu)^{s\bar{\theta}} (d\nu)^{\bar{\theta}} \\ &= (2\lambda)^{\bar{\theta}} \frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} - \frac{\Gamma(1+(s+1)\bar{\theta})}{\Gamma(1+(s+2)\bar{\theta})}. \end{aligned} \quad (3.13)$$

The proof is completed. $\square$

Corollary 3.2. *Setting $s = 1$ in Theorem 3.2, we obtain*

$$\begin{aligned} & \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta}+1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\mathfrak{h}) \right| \\ & \leq \frac{(x - \omega)^{2\bar{\theta}}}{(\sigma - \omega)^{\bar{\theta}}} \left[\left(\frac{\Gamma(1+\bar{\theta})}{\Gamma(1+2\bar{\theta})} - \frac{\Gamma(1+2\bar{\theta})}{\Gamma(1+3\bar{\theta})} \right) \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right| \right) \right. \\ & \quad \left. + \frac{\Gamma(1+2\bar{\theta})}{\Gamma(1+3\bar{\theta})} \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| + \left| \mathcal{J}^{(\bar{\theta})}(x) \right| \right) \right] \\ & \quad + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}} (\sigma - \omega)^{\bar{\theta}}} \left(\left((2\lambda - 1)^{\bar{\theta}} \frac{\Gamma(1+\bar{\theta})}{\Gamma(1+2\bar{\theta})} + \frac{\Gamma(1+2\bar{\theta})}{\Gamma(1+3\bar{\theta})} \right) \left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right| \right) \right. \\ & \quad \left. + \left((4\lambda)^{\bar{\theta}} \frac{\Gamma(1+\bar{\theta})}{\Gamma(1+2\bar{\theta})} - 2^{\bar{\theta}} \frac{\Gamma(1+2\bar{\theta})}{\Gamma(1+3\bar{\theta})} \right) \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right). \end{aligned}$$

Theorem 3.3. *Let us assume that the conditions outlined in Lemma 3.1 are fully met. If $|\mathcal{J}^{(\bar{\theta})}|^q$ is generalized s -convex on $[\omega, \sigma]$, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $\lambda \in [0, \frac{1}{2}]$ we have*

$$\begin{aligned} & \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta}+1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\mathfrak{h}) \right| \\ & \leq \left(\frac{\Gamma(1+p\bar{\theta})}{\Gamma(1+(p+1)\bar{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+s\bar{\theta})}{\Gamma(1+(s+1)\bar{\theta})} \right)^{\frac{1}{q}} \\ & \quad \times \left[\frac{(x - \omega)^{2\bar{\theta}}}{(\sigma - \omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & \quad \left. + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}} (\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} + (1 - 2\lambda)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

Proof. From Lemma 3.1, properties of modulus together with generalized Hölder's inequality, we have

$$\begin{aligned}
& \left| \mathcal{I}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\tilde{\theta} + 1)}{(\sigma - \omega)^{\tilde{\theta}}} \omega \mathcal{I}_{\sigma}^{\tilde{\theta}} \mathcal{S}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\tilde{\theta}}}{(\sigma - \omega)^{\tilde{\theta}}} \left(\left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \nu^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left| \mathcal{S}^{(\tilde{\theta})}((1 - \nu)\omega + \nu x) \right|^q (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 (1 - \nu)^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left| \mathcal{S}^{(\tilde{\theta})}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) \right|^q (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \Bigg) \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\tilde{\theta}}}{4^{\tilde{\theta}} (\sigma - \omega)^{\tilde{\theta}}} \left(\left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 |\nu - 2\lambda|^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left| \mathcal{S}^{(\tilde{\theta})} \left((1 - \nu)x + \nu \frac{\omega + \sigma}{2} \right) \right|^q (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \\
& \quad + \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 |\nu - (1 - 2\lambda)|^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left| \mathcal{S}^{(\tilde{\theta})} \left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x) \right) \right|^q (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \right).
\end{aligned}$$

Since $|\mathcal{S}^{(\tilde{\theta})}|^q$ is generalized s -convex, we have

$$\begin{aligned}
& \left| \mathcal{I}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\tilde{\theta} + 1)}{(\sigma - \omega)^{\tilde{\theta}}} \omega \mathcal{I}_{\sigma}^{\tilde{\theta}} \mathcal{S}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\tilde{\theta}}}{(\sigma - \omega)^{\tilde{\theta}}} \left(\frac{\Gamma(1 + p\tilde{\theta})}{\Gamma(1 + (p + 1)\tilde{\theta})} \right)^{\frac{1}{p}} \\
& \quad \times \left(\left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left((1 - \nu)^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(\omega) \right|^q + \nu^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(x) \right|^q \right) (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left((1 - \nu)^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(\omega + \sigma - x) \right|^q + \nu^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(\sigma) \right|^q \right) (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \Bigg) \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\tilde{\theta}}}{4^{\tilde{\theta}} (\sigma - \omega)^{\tilde{\theta}}} \left(\left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 |\nu - 2\lambda|^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left((1 - \nu)^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(x) \right|^q + \nu^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right|^q \right) (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \\
& \quad + \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 |\nu - (1 - 2\lambda)|^{p\tilde{\theta}} (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\frac{1}{\Gamma(\tilde{\theta} + 1)} \int_0^1 \left((1 - \nu)^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})} \left(\frac{\omega + \sigma}{2} \right) \right|^q + \nu^{s\tilde{\theta}} \left| \mathcal{S}^{(\tilde{\theta})}(\omega + \sigma - x) \right|^q \right) (d\nu)^{\tilde{\theta}} \right)^{\frac{1}{q}} \right) \\
& = \left(\frac{\Gamma(1 + p\tilde{\theta})}{\Gamma(1 + (p + 1)\tilde{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + s\tilde{\theta})}{\Gamma(1 + (s + 1)\tilde{\theta})} \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned} & \times \left[\frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} + (1 - 2\lambda)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \\ & \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right], \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \frac{1}{\Gamma(\bar{\theta} + 1)} \int_0^1 |\nu - 2\lambda|^{p\bar{\theta}} (d\nu)^{\bar{\theta}} &= \frac{1}{\Gamma(\bar{\theta} + 1)} \int_0^1 |\nu - (1 - 2\lambda)|^{p\bar{\theta}} (d\nu)^{\bar{\theta}} \\ &= \frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p + 1)\bar{\theta})} \left((2\lambda)^{(p+1)\bar{\theta}} + (1 - 2\lambda)^{(p+1)\bar{\theta}} \right). \end{aligned}$$

The proof is completed. $\square$

Corollary 3.3. *Setting $s = 1$ in Theorem 3.3, we obtain*

$$\begin{aligned} & \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta} + 1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\eta) \right| \\ & \leq \left(\frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p + 1)\bar{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + \bar{\theta})}{\Gamma(1 + 2\bar{\theta})} \right)^{\frac{1}{q}} \\ & \times \left[\frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} + (1 - 2\lambda)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \\ & \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

Theorem 3.4. *Assume that all the assumptions of Theorem 3.3 are satisfied. Then, for $\lambda \in [\frac{1}{2}, 1]$ the following inequality holds*

$$\begin{aligned} & \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta} + 1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\eta) \right| \\ & \leq \left(\frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p + 1)\bar{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + s\bar{\theta})}{\Gamma(1 + (s + 1)\bar{\theta})} \right)^{\frac{1}{q}} \\ & \times \left[\frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} - (2\lambda - 1)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \\ & \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

Proof. From Lemma 3.1, properties of modulus together with generalized Hölder inequality, we have

$$\begin{aligned}
& \left| \mathcal{T}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\vartheta + 1)}{(\sigma - \omega)^\vartheta} \omega \mathcal{I}_\sigma^\vartheta \mathcal{S}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\vartheta}}{(\sigma - \omega)^\vartheta} \left(\left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \nu^{p\vartheta} (d\nu)^\vartheta \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left| \mathcal{S}^{(\vartheta)}((1 - \nu)\omega + \nu x) \right|^q (d\nu)^\vartheta \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 (1 - \nu)^{p\vartheta} (d\nu)^\vartheta \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left| \mathcal{S}^{(\vartheta)}((1 - \nu)(\omega + \sigma - x) + \nu\sigma) \right|^q (d\nu)^\vartheta \right)^{\frac{1}{q}} \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\vartheta}}{4^\vartheta (\sigma - \omega)^\vartheta} \left(\left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 (2\lambda - \nu)^{p\vartheta} (d\nu)^\vartheta \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left| \mathcal{S}^{(\vartheta)}\left((1 - \nu)x + \nu \frac{\omega + \sigma}{2}\right) \right|^q (d\nu)^\vartheta \right)^{\frac{1}{q}} \\
& \quad + \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 (\nu - (1 - 2\lambda))^{p\vartheta} (d\nu)^\vartheta \right)^{\frac{1}{p}} \\
& \quad \times \left. \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left| \mathcal{S}^{(\vartheta)}\left((1 - \nu) \frac{\omega + \sigma}{2} + \nu(\omega + \sigma - x)\right) \right|^q (d\nu)^\vartheta \right)^{\frac{1}{q}} \right).
\end{aligned}$$

Since $|\mathcal{S}^{(\vartheta)}|^q$ is generalized s -convex, we have

$$\begin{aligned}
& \left| \mathcal{T}(\lambda, \omega, \sigma, x, \mathcal{S}) - \frac{\Gamma(\vartheta + 1)}{(\sigma - \omega)^\vartheta} \omega \mathcal{I}_\sigma^\vartheta \mathcal{S}(\eta) \right| \\
& \leq \frac{(x - \omega)^{2\vartheta}}{(\sigma - \omega)^\vartheta} \left(\frac{\Gamma(1 + p\vartheta)}{\Gamma(1 + (p + 1)\vartheta)} \right)^{\frac{1}{p}} \\
& \quad \times \left(\left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left((1 - \nu)^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(\omega) \right|^q + \nu^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(x) \right|^q \right) (d\nu)^\vartheta \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left((1 - \nu)^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(\omega + \sigma - x) \right|^q + \nu^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(\sigma) \right|^q \right) (d\nu)^\vartheta \right)^{\frac{1}{q}} \\
& \quad + \frac{(\omega + \sigma - 2x)^{2\vartheta}}{4^\vartheta (\sigma - \omega)^\vartheta} \left(\left(\frac{\Gamma(1 + p\vartheta)}{\Gamma(1 + (p + 1)\vartheta)} \left((2\lambda)^{(p+1)\vartheta} - (2\lambda - 1)^{(p+1)\vartheta} \right) \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left((1 - \nu)^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(x) \right|^q + \nu^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right) (d\nu)^\vartheta \right)^{\frac{1}{q}} \right. \\
& \quad + \left. \left(\frac{1}{\Gamma(\vartheta + 1)} \int_0^1 \left((1 - \nu)^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \nu^{s\vartheta} \left| \mathcal{S}^{(\vartheta)}(\omega + \sigma - x) \right|^q \right) (d\nu)^\vartheta \right)^{\frac{1}{q}} \right) \\
& = \left(\frac{\Gamma(1 + p\vartheta)}{\Gamma(1 + (p + 1)\vartheta)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + s\vartheta)}{\Gamma(1 + (s + 1)\vartheta)} \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned} & \times \left[\frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} - (2\lambda - 1)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \\ & \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right], \end{aligned}$$

where we have used

$$\begin{aligned} \frac{1}{\Gamma(\bar{\theta} + 1)} \int_0^1 (2\lambda - \nu)^{p\bar{\theta}} (d\nu)^{\bar{\theta}} &= \frac{1}{\Gamma(\bar{\theta} + 1)} \int_0^1 (\nu - (1 - 2\lambda))^{p\bar{\theta}} (d\nu)^{\bar{\theta}} \\ &= \frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p + 1)\bar{\theta})} \left((2\lambda)^{(p+1)\bar{\theta}} - (2\lambda - 1)^{(p+1)\bar{\theta}} \right). \end{aligned}$$

The proof is completed. $\square$

Corollary 3.4. *Setting $s = 1$ in Theorem 3.4, we obtain*

$$\begin{aligned} & \left| \mathcal{J}(\lambda, \omega, \sigma, x, \mathcal{J}) - \frac{\Gamma(\bar{\theta} + 1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\eta) \right| \\ & \leq \left(\frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p + 1)\bar{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + \bar{\theta})}{\Gamma(1 + 2\bar{\theta})} \right)^{\frac{1}{q}} \\ & \times \left[\frac{(x-\omega)^{2\bar{\theta}}}{(\sigma-\omega)^{\bar{\theta}}} \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right|^q \right)^{\frac{1}{q}} \right) \right. \\ & + \frac{(\omega + \sigma - 2x)^{2\bar{\theta}}}{4^{\bar{\theta}}(\sigma - \omega)^{\bar{\theta}}} \left((2\lambda)^{(p+1)\bar{\theta}} - (2\lambda - 1)^{(p+1)\bar{\theta}} \right)^{\frac{1}{p}} \\ & \left. \times \left(\left(\left| \mathcal{J}^{(\bar{\theta})}(x) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right|^q + \left| \mathcal{J}^{(\bar{\theta})}(\omega + \sigma - x) \right|^q \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

4. Special cases

This section is dedicated to presenting specific cases derived from Theorems 3.1 and 3.2 by fixing the parameters x and λ . These cases include numerous new results, as well as some previously established ones.

Corollary 4.1. *Setting $x = \frac{\omega + \sigma}{2}$ in Corollary 3.1, we obtain the following midpoint inequality for generalized convex local fractional derivatives*

$$\begin{aligned} & \left| \mathcal{J}\left(\frac{\omega + \sigma}{2}\right) - \frac{\Gamma(\bar{\theta} + 1)}{(\sigma - \omega)^{\bar{\theta}}} \omega \mathcal{J}_{\sigma}^{\bar{\theta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\theta}}}{4^{\bar{\theta}}} \left(\left(\frac{\Gamma(1 + \bar{\theta})}{\Gamma(1 + 2\bar{\theta})} - \frac{\Gamma(1 + 2\bar{\theta})}{\Gamma(1 + 3\bar{\theta})} \right) \left(\left| \mathcal{J}^{(\bar{\theta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\theta})}(\sigma) \right| \right) \right. \\ & \quad \left. + 2^{\bar{\theta}} \frac{\Gamma(1 + 2\bar{\theta})}{\Gamma(1 + 3\bar{\theta})} \left| \mathcal{J}^{(\bar{\theta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right). \end{aligned}$$

Corollary 4.2. *Setting $x = \omega$ in Corollary 3.1, we obtain three point Newton–Cotes inequality for generalized convex local fractional derivatives*

$$\begin{aligned} & \left| \lambda^{\bar{\vartheta}} \mathcal{J}(\omega) + (1 - 2\lambda)^{\bar{\vartheta}} \mathcal{J}\left(\frac{\omega + \sigma}{2}\right) + \lambda^{\bar{\vartheta}} \mathcal{J}(\sigma) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\left(2^{\bar{\vartheta}} (1 - 2\lambda)^{3\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \right. \right. \\ & \quad \left. \left. + \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - (1 - 2\lambda)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \right. \\ & \quad \left. + \left(4^{\bar{\vartheta}} (2\lambda)^{3\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - (4\lambda)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right). \end{aligned}$$

Besides, if we choose

- $\lambda = \frac{1}{6}$, then we obtain the following inequality related to Simpson's rule via generalized convexity

$$\begin{aligned} & \left| \frac{1^{\bar{\vartheta}}}{6^{\bar{\vartheta}}} \left(\mathcal{J}(\omega) + 4^{\bar{\vartheta}} \mathcal{J}\left(\frac{\omega + \sigma}{2}\right) + \mathcal{J}(\sigma) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\left(\left(\frac{16}{27} \right)^{\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) + \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - \left(\frac{2}{3} \right)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \right. \\ & \quad \times \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \\ & \quad \left. + \left(\left(\frac{4}{27} \right)^{\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) + 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - \left(\frac{2}{3} \right)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \right. \\ & \quad \left. \times \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right). \end{aligned}$$

- $\lambda = \frac{1}{4}$, then we obtain the following inequality related to Bullen's quadrature rule via generalized convexity

$$\begin{aligned} & \left| \frac{1^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\mathcal{J}(\omega) + 2^{\bar{\vartheta}} \mathcal{J}\left(\frac{\omega + \sigma}{2}\right) + \mathcal{J}(\sigma) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\left(\left(\frac{1}{4} \right)^{\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) + \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - \left(\frac{1}{2} \right)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \right. \\ & \quad \times \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \\ & \quad \left. + \left(\left(\frac{1}{2} \right)^{\bar{\vartheta}} \left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) + 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} - \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})}\left(\frac{\omega + \sigma}{2}\right) \right| \right). \end{aligned}$$

- $\lambda = \frac{7}{30}$, then we obtain the following corrected Simpson inequality for generalized convex local fractional derivatives

$$\left| \frac{1^{\bar{\vartheta}}}{30^{\bar{\vartheta}}} \left(7^{\bar{\vartheta}} \mathcal{J}(\omega) + 16^{\bar{\vartheta}} \mathcal{J}\left(\frac{\omega + \sigma}{2}\right) + 7^{\bar{\vartheta}} \mathcal{J}(\sigma) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right|$$

$$\leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\left(\left(\frac{2351}{3375} \right)^{\bar{\vartheta}} \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} - \left(\frac{776}{3375} \right)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \right. \\ \left. + \left(\left(\frac{5378}{3375} \right)^{\bar{\vartheta}} \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} - \left(\frac{1778}{3375} \right)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right).$$

Remark 4.1. By setting $\lambda = \frac{1}{2}$ in Corollary 4.2, we retrieve the result of Corollary 3.1 related to trapezium formula, established by Set et al. in [31].

Remark 4.2. When $\lambda = \frac{1}{2}$, Theorem 3.1 simplifies to Theorem 1 from [29], which concerns the companion of Ostrowski formula.

Corollary 4.3. Setting $x = \frac{5\omega + \sigma}{6}$ in Corollary 3.1, we obtain

$$\left| \mathcal{J} \left(\lambda, \omega, \sigma, \frac{5\omega + \sigma}{6}, \mathcal{J} \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{36^{\bar{\vartheta}}} \left(\left(\frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} - \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \right. \\ \left. + \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \left(\left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{5\omega + \sigma}{6} \right) \right| + \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + 5\sigma}{6} \right) \right| \right) \right) \\ + \frac{(\sigma - \omega)^{\bar{\vartheta}}}{9^{\bar{\vartheta}}} \left(\left(2^{\bar{\vartheta}} (1 - 2\lambda)^{3\bar{\vartheta}} \left(\frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} - \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \right) \right. \right. \\ \left. + \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} - (1 - 2\lambda)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{5\omega + \sigma}{6} \right) \right| + \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + 5\sigma}{6} \right) \right| \right) \\ \left. + \left(4^{\bar{\vartheta}} (2\lambda)^{3\bar{\vartheta}} \left(\frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} - \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \right) + 2^{\bar{\vartheta}} \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \right. \right. \\ \left. \left. - (4\lambda)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right).$$

Besides, if we take

- $\lambda = \frac{5}{16}$, then we obtain the following Maclaurin inequality for generalized convex local fractional derivatives

$$\left| \left(\frac{1}{8} \right)^{\bar{\vartheta}} \left(3^{\bar{\vartheta}} \mathcal{J} \left(\frac{5\omega + \sigma}{6} \right) + 2^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + \sigma}{2} \right) + 3^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + 5\sigma}{6} \right) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} \omega \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{36^{\bar{\vartheta}}} \left(\left(\frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} - \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})}(\omega) \right| + \left| \mathcal{J}^{(\bar{\vartheta})}(\sigma) \right| \right) \right. \\ \left. + \left(\left(\frac{293}{64} \right)^{\bar{\vartheta}} \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} - \left(\frac{69}{64} \right)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{5\omega + \sigma}{6} \right) \right| + \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + 5\sigma}{6} \right) \right| \right) \right) \\ \left. + \left(\left(\frac{131}{32} \right)^{\bar{\vartheta}} \frac{\Gamma(1+2\bar{\vartheta})}{\Gamma(1+3\bar{\vartheta})} - \left(\frac{35}{32} \right)^{\bar{\vartheta}} \frac{\Gamma(1+\bar{\vartheta})}{\Gamma(1+2\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right).$$

- $\lambda = \frac{41}{160}$, then we obtain the following Corrected Euler-Maclaurin inequality for generalized

convex local fractional derivatives

$$\begin{aligned} & \left| \frac{1}{(80)^{\bar{\vartheta}}} \left(27^{\bar{\vartheta}} \mathcal{J} \left(\frac{5\omega + \sigma}{6} \right) + 26^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + \sigma}{2} \right) + 27^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + 5\sigma}{6} \right) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} {}_{\omega} \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{36^{\bar{\vartheta}}} \left(\left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(|\mathcal{J}^{(\bar{\vartheta})}(\omega)| + |\mathcal{J}^{(\bar{\vartheta})}(\sigma)| \right) \right. \\ & \quad + \left(\left(\frac{39}{20} \right)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} + 5^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{5\omega + \sigma}{6} \right) \right| + \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + 5\sigma}{6} \right) \right| \right) \\ & \quad \left. + \left(\left(\frac{41}{10} \right)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - 8^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right). \end{aligned}$$

Corollary 4.4. Setting $x = \omega$ in Corollary 3.2, we obtain

$$\begin{aligned} & \left| \lambda^{\bar{\vartheta}} \mathcal{J}(\omega) + (1 - 2\lambda)^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + \sigma}{2} \right) + \lambda^{\bar{\vartheta}} \mathcal{J}(\sigma) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} {}_{\omega} \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{4^{\bar{\vartheta}}} \left(\left((2\lambda - 1)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} + \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(|\mathcal{J}^{(\bar{\vartheta})}(\omega)| + |\mathcal{J}^{(\bar{\vartheta})}(\sigma)| \right) \right. \\ & \quad \left. + \left((4\lambda)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right). \end{aligned}$$

Remark 4.3. By setting $\lambda = \frac{2}{3}$ in Corollary 4.4, we retrieve the result of Theorem 6 related to Milne's formula, established by Meftah et al. in [20].

Corollary 4.5. Setting $x = \frac{3\omega + \sigma}{4}$ in Corollary 3.2, we obtain

$$\begin{aligned} & \left| \left(\frac{1 + 2\lambda}{4} \right)^{\bar{\vartheta}} \mathcal{J} \left(\frac{3\omega + \sigma}{4} \right) + \left(\frac{1 - 2\lambda}{2} \right)^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + \sigma}{2} \right) \right. \\ & \quad \left. + \left(\frac{1 + 2\lambda}{4} \right)^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + 3\sigma}{4} \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} {}_{\omega} \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{16^{\bar{\vartheta}}} \left(\left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(|\mathcal{J}^{(\bar{\vartheta})}(\omega)| + |\mathcal{J}^{(\bar{\vartheta})}(\sigma)| \right) \right. \\ & \quad + \left((2\lambda - 1)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} + 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(\left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{3\omega + \sigma}{4} \right) \right| + \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + 3\sigma}{4} \right) \right| \right) \\ & \quad \left. + \left((4\lambda)^{\bar{\vartheta}} \frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - 2^{\bar{\vartheta}} \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left| \mathcal{J}^{(\bar{\vartheta})} \left(\frac{\omega + \sigma}{2} \right) \right| \right). \end{aligned}$$

Besides, if we take $\lambda = \frac{5}{6}$, then we obtain the following dual Simpson inequality for generalized convex local fractional derivatives

$$\begin{aligned} & \left| \left(\frac{1}{3} \right)^{\bar{\vartheta}} \left(2^{\bar{\vartheta}} \mathcal{J} \left(\frac{3\omega + \sigma}{4} \right) - \mathcal{J} \left(\frac{\omega + \sigma}{2} \right) + 2^{\bar{\vartheta}} \mathcal{J} \left(\frac{\omega + 3\sigma}{4} \right) \right) - \frac{\Gamma(\bar{\vartheta} + 1)}{(\sigma - \omega)^{\bar{\vartheta}}} {}_{\omega} \mathcal{J}_{\sigma}^{\bar{\vartheta}} \mathcal{J}(\eta) \right| \\ & \leq \frac{(\sigma - \omega)^{\bar{\vartheta}}}{16^{\bar{\vartheta}}} \left(\left(\frac{\Gamma(1 + \bar{\vartheta})}{\Gamma(1 + 2\bar{\vartheta})} - \frac{\Gamma(1 + 2\bar{\vartheta})}{\Gamma(1 + 3\bar{\vartheta})} \right) \left(|\mathcal{J}^{(\bar{\vartheta})}(\omega)| + |\mathcal{J}^{(\bar{\vartheta})}(\sigma)| \right) \right. \end{aligned}$$

$$\begin{aligned}
& + \left(\left(\frac{2}{3} \right)^{\eth} \frac{\Gamma(1+\eth)}{\Gamma(1+2\eth)} + 2^{\eth} \frac{\Gamma(1+2\eth)}{\Gamma(1+3\eth)} \right) \left(\left| \mathcal{J}^{(\eth)} \left(\frac{3\omega + \sigma}{4} \right) \right| + \left| \mathcal{J}^{(\eth)} \left(\frac{\omega + 3\sigma}{4} \right) \right| \right) \\
& + \left(\left(\frac{10}{3} \right)^{\eth} \frac{\Gamma(1+\eth)}{\Gamma(1+2\eth)} - 2^{\eth} \frac{\Gamma(1+2\eth)}{\Gamma(1+3\eth)} \right) \left| \mathcal{J}^{(\eth)} \left(\frac{\omega + \sigma}{2} \right) \right|.
\end{aligned}$$

Remark 4.4. By setting $\lambda = \frac{17}{30}$ in Corollary 4.5, we retrieve the result of Theorem 5 from [15], related to Dual-Simpson formula.

5. Example and applications

This section aims to verify the accuracy and effectiveness of the obtained results. We start with an example that includes graphical representations to demonstrate the precision of our findings. Following this, we present several applications, including the estimation of the error in a given quadrature formula and some inequalities involving special means.

5.1. Example supporting study results

To further illustrate and verify the results established in this work, we provide an example covering several cases, accompanied by two-dimensional and three-dimensional plots that highlight the effectiveness and precision of the derived inequalities.

It is worth noting that all of the forthcoming figures were produced using MATLAB. In these plots, the red curves or surfaces correspond to the right-hand side (*RHS*), whereas the blue curves or surfaces represent the left-hand side (*LHS*) of the associated inequalities.

Example 5.1. Let $\mathcal{J} : [0, 1] \rightarrow \mathbb{R}^{\eth}$ be a function defined for a fixed value $s \in (0, 1]$ by $\mathcal{J}(\eta) = \frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} \eta^{(s+1)\eth}$. Hence, the derivative $|\mathcal{J}^{(\eth)}(\eta)| = \eta^{s\eth}$ is a generalized s -convex function, which satisfies the fundamental condition that underlies this study.

From Theorems 3.1 and 3.2, we obtain the following multi-parametrized three-point Newton–Cotes type inequalities related to the function under consideration.

- For $x \in [0, \frac{1}{2}]$ and $0 \leq \lambda \leq \frac{1}{2}$

$$\begin{aligned}
& \left| \mathcal{J}(\lambda, 0, 1, x, \mathcal{J}) - \Gamma(\eth + 1) {}_0\mathcal{J}_1^{\eth} \mathcal{J}(\eta) \right| \\
& \leq x^{2\eth} \left(\frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} x^{s\eth} + \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} (1-x)^{s\eth} \right. \\
& \quad \left. + \left(\frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} - \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} \right) \right) \\
& \quad + \frac{(1-2x)^{2\eth}}{4^{\eth}} \left(\left(2^{\eth} (1-2\lambda)^{(s+2)\eth} \left(\frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} - \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} \right) \right. \right. \\
& \quad \left. \left. + \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} - (1-2\lambda)^{\eth} \frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} \right) (x^{s\eth} + (1-x)^{s\eth}) \right. \\
& \quad \left. + \left(\frac{1}{2} \right)^{s\eth} \left(4^{\eth} (2\lambda)^{(s+2)\eth} \left(\frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} - \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} \right) 2^{\eth} \frac{\Gamma(1+(s+1)\eth)}{\Gamma(1+(s+2)\eth)} \right. \right. \\
& \quad \left. \left. - (4\lambda)^{\eth} \frac{\Gamma(1+s\eth)}{\Gamma(1+(s+1)\eth)} \right) \right). \tag{5.1}
\end{aligned}$$

- For $x \in [0, \frac{1}{2}]$ and $\frac{1}{2} < \lambda \leq 1$

$$\begin{aligned}
& \left| \mathcal{J}(\lambda, 0, 1, x, \mathcal{S}) - \Gamma(\bar{\sigma} + 1) {}_0\mathcal{J}_1^{\bar{\sigma}} \mathcal{S}(\eta) \right| \\
& \leq x^{2\bar{\sigma}} \left(\frac{\Gamma(1 + (s+1)\bar{\sigma})}{\Gamma(1 + (s+2)\bar{\sigma})} x^{s\bar{\sigma}} + \frac{\Gamma(1 + (s+1)\bar{\sigma})}{\Gamma(1 + (s+2)\bar{\sigma})} (1-x)^{s\bar{\sigma}} \right. \\
& \quad \left. + \left(\frac{\Gamma(1 + s\bar{\sigma})}{\Gamma(1 + (s+1)\bar{\sigma})} - \frac{\Gamma(1 + (s+1)\bar{\sigma})}{\Gamma(1 + (s+2)\bar{\sigma})} \right) \right) \\
& \quad + \frac{(1-2x)^{2\bar{\sigma}}}{4^{\bar{\sigma}}(\sigma - \omega)^{\bar{\sigma}}} \left(\left((2\lambda - 1)^{\bar{\sigma}} \frac{\Gamma(1 + s\bar{\sigma})}{\Gamma(1 + (s+1)\bar{\sigma})} + \frac{\Gamma(1 + (s+1)\bar{\sigma})}{\Gamma(1 + (s+2)\bar{\sigma})} \right) (x^{s\bar{\sigma}} + (1-x)^{s\bar{\sigma}}) \right. \\
& \quad \left. + \left((2\lambda)^{\bar{\sigma}} \frac{\Gamma(1 + s\bar{\sigma})}{\Gamma(1 + (s+1)\bar{\sigma})} - \frac{\Gamma(1 + (s+1)\bar{\sigma})}{\Gamma(1 + (s+2)\bar{\sigma})} \right) \right), \tag{5.2}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}(\lambda, 0, 1, x, \mathcal{S}) = & \frac{\Gamma(1 + s\bar{\sigma})}{\Gamma(1 + (s+1)\bar{\sigma})} \left[\left(x^{\bar{\sigma}} + \lambda^{\bar{\sigma}} (1-2x)^{\bar{\sigma}} \right) x^{(s+1)\bar{\sigma}} \right. \\
& \left. + \frac{(1-2\lambda)^{\bar{\sigma}} (1-2x)^{\bar{\sigma}}}{2^{(s+1)\bar{\sigma}}} + \left(x^{\bar{\sigma}} + \lambda^{\bar{\sigma}} (1-2x)^{\bar{\sigma}} \right) (1-x)^{(s+1)\bar{\sigma}} \right].
\end{aligned}$$

If we set $\bar{\sigma} = 1$, then we can clearly see that the result depends on 3 parameters. Next, we will fix one parameter at a time and plot the result as a function of the remaining two.

Case 1. Let us begin by fixing $x = \omega = 0$. Consequently, from (5.1) and (5.2) we obtain

- For $0 \leq \lambda \leq \frac{1}{2}$

$$\begin{aligned}
& \left| \frac{1}{s+1} \left(\frac{(1-2\lambda)}{2^{s+1}} + \lambda - \frac{1}{s+2} \right) \right| \\
& \leq \frac{1}{4} \left[\left(\frac{2(1-2\lambda)^{s+2}}{(s+1)(s+2)} + \frac{1}{s+2} - \frac{1-2\lambda}{s+1} \right) + \frac{1}{2^s} \left(\frac{4(2\lambda)^{s+2}}{(s+1)(s+2)} + \frac{2}{s+2} - \frac{4\lambda}{s+1} \right) \right].
\end{aligned}$$

- For $\frac{1}{2} < \lambda \leq 1$

$$\left| \frac{1}{s+1} \left(\frac{(1-2\lambda)}{2^{s+1}} + \lambda - \frac{1}{s+2} \right) \right| \leq \frac{1}{4} \left[\frac{2\lambda - 1}{s+1} + \frac{1}{s+2} + \frac{1}{2^s} \left(\frac{4\lambda}{s+1} - \frac{2}{s+2} \right) \right].$$

The previous result is illustrated in Figure 1. Figure 2, on the other hand, depicts the special cases for $s = \frac{1}{2}$, $\lambda \in [0, 1]$ and $\lambda = 0$, $s \in (0, 1]$, respectively.

Case 2. Now, let us focus on the Companion Ostrowski formula. To do so, we fix $\lambda = \frac{1}{2}$ and plot the results with respect to $x \in [0, \frac{1}{2}]$ and $s \in (0, 1]$. From (5.1) and (5.2), we have

$$\left| \frac{1}{s+1} \left(\frac{x^{s+1} + (1-x)^{s+1}}{2} - \frac{1}{s+2} \right) \right| \leq \frac{4x^2 + (1-2x)^2}{4(s+2)} (x^s + (1-x)^s) + \frac{2^{s+1}x^2 + (1-2x)^2}{2^{s+1}(s+2)}.$$

Figure 3 represents the previous result. In contrast, Figure 4 shows the specific case for $s = 1$.

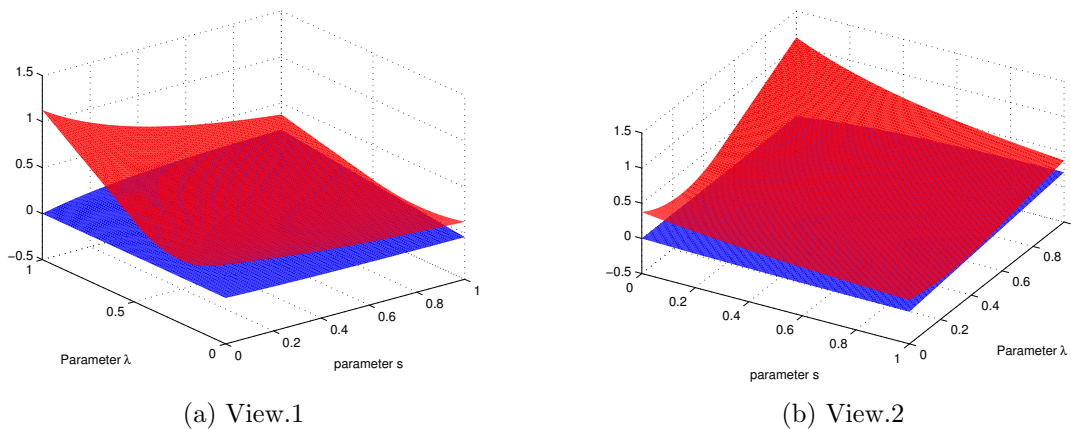


Figure 1. Case 1. $x = \omega$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

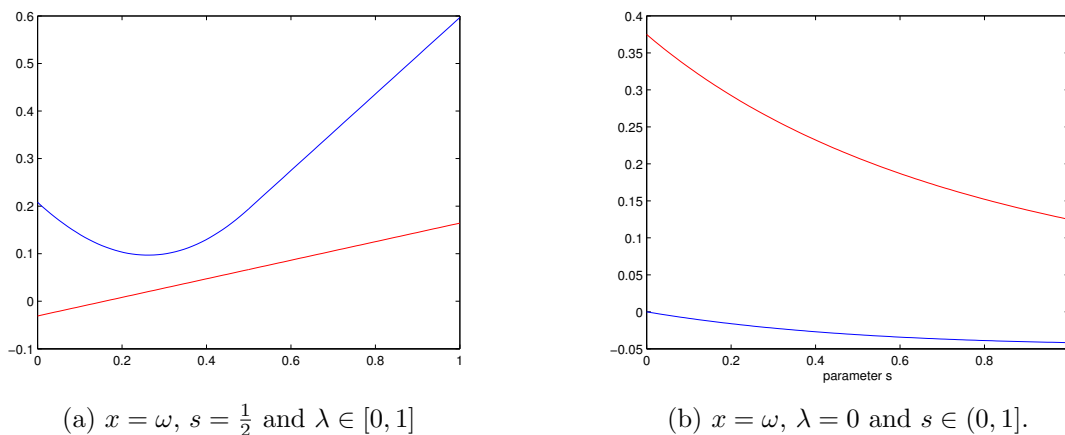


Figure 2. Special cases.

5.2. Application to quadrature formula

Let Ω be the partition of the points $\omega = u_0 < u_1 < \dots < u_m = \sigma$ of the interval $[\omega, \sigma]$, and consider the quadrature formula

$$\frac{1}{\Gamma(\vartheta + 1)} \int_{\omega}^{\sigma} \mathcal{S}(u) (\mathfrak{D}u)^{\vartheta} = \mathfrak{L}(\mathcal{S}, \Omega) + \mathcal{R}(\mathcal{S}, \Omega),$$

where

$$\mathfrak{L}(\mathcal{S}, \Omega) = \sum_{j=0}^{m-1} \frac{(u_{j+1} - u_j)^{\vartheta}}{\Gamma(\vartheta + 1)} \left(\lambda^{\vartheta} \mathcal{S}(u_j) + (1 - 2\lambda)^{\vartheta} \mathcal{S}\left(\frac{u_j + u_{j+1}}{2}\right) + \lambda^{\vartheta} \mathcal{S}(u_{j+1}) \right),$$

and $\mathcal{R}(\mathcal{S}, \Omega)$ denotes the associated approximation error.

Proposition 5.1. Let $m \in \mathbb{N}$, $\lambda \in [0, \frac{1}{2}]$ and $\mathcal{S} : [\omega, \sigma] \rightarrow \mathbb{R}^{\vartheta}$ be a differentiable function on (ω, σ) with $0 \leq \omega < \sigma$ and $\mathcal{S}^{(\vartheta)} \in C_{\vartheta}[\omega, \sigma]$. If $|\mathcal{S}^{(\vartheta)}|$ is genaralized s -convex function, then we

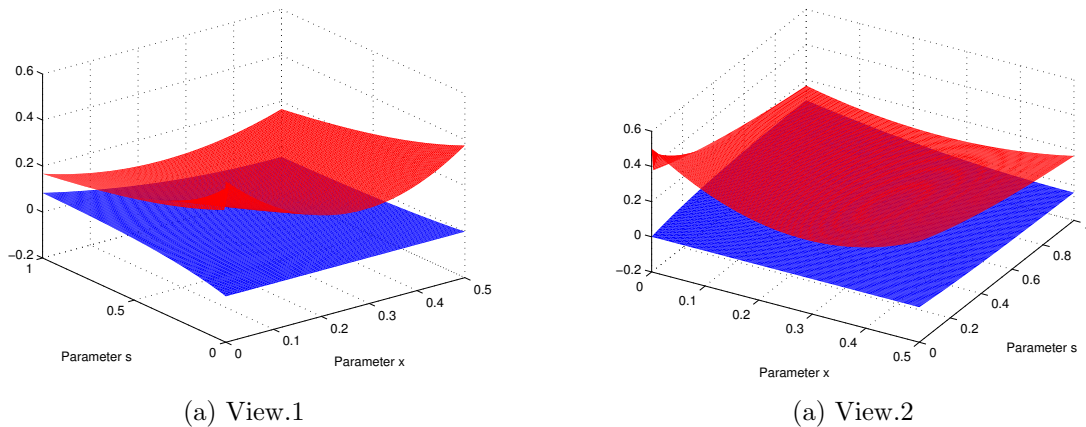


Figure 3. Case 2. $\lambda = \frac{1}{2}$, $x \in [0, \frac{1}{2})$ and $s \in (0, 1]$.

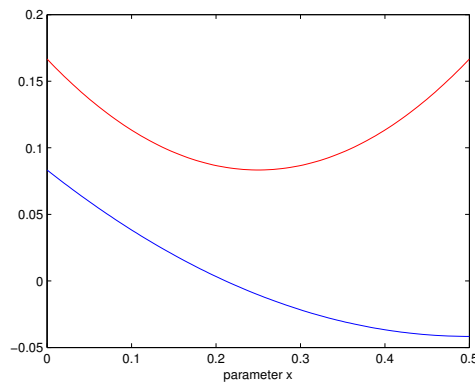


Figure 4. $s = 1$ and $x \in [0, \frac{1}{2}]$.

have

$$\begin{aligned}
 & |\mathcal{R}(\mathcal{S}, \Omega)| \\
 & \leq \frac{1}{\Gamma(1+\vartheta)} \sum_{j=0}^{m-1} \frac{(u_{j+1} - u_j)^{2\vartheta}}{4^\vartheta} \left(\left(2^\vartheta (1-2\lambda)^{(s+2)\vartheta} \left(\frac{\Gamma(1+s\vartheta)}{\Gamma(1+(s+1)\vartheta)} - \frac{\Gamma(1+(s+1)\vartheta)}{\Gamma(1+(s+2)\vartheta)} \right) \right. \right. \\
 & \quad \left. \left. + \frac{\Gamma(1+(s+1)\vartheta)}{\Gamma(1+(s+2)\vartheta)} - (1-2\lambda)^\vartheta \frac{\Gamma(1+s\vartheta)}{\Gamma(1+(s+1)\vartheta)} \right) \left(|\mathcal{J}^{(\vartheta)}(u_j)| + |\mathcal{J}^{(\vartheta)}(u_{j+1})| \right) \right. \\
 & \quad \left. + \left(4^\vartheta (2\lambda)^{(s+2)\vartheta} \left(\frac{\Gamma(1+s\vartheta)}{\Gamma(1+(s+1)\vartheta)} - \frac{\Gamma(1+(s+1)\vartheta)}{\Gamma(1+(s+2)\vartheta)} \right) + 2^\vartheta \frac{\Gamma(1+(s+1)\vartheta)}{\Gamma(1+(s+2)\vartheta)} \right. \right. \\
 & \quad \left. \left. - (4\lambda)^\vartheta \frac{\Gamma(1+s\vartheta)}{\Gamma(1+(s+1)\vartheta)} \right) \left| \mathcal{J}^{(\vartheta)} \left(\frac{u_j + u_{j+1}}{2} \right) \right| \right).
 \end{aligned}$$

Proof. Applying Theorem 3.1 with $x = \omega$ on the subintervals $[u_j, u_{j+1}]$ ($j = 0, 1, \dots, m-1$) of the partition Ω , we get

$$\left| \left(\lambda^\vartheta \mathcal{J}(u_j) + (1-2\lambda)^\vartheta \mathcal{J} \left(\frac{u_j + u_{j+1}}{2} \right) + \lambda^\vartheta \mathcal{J}(u_{j+1}) \right) - \frac{\Gamma(\vartheta+1)}{(u_{j+1} - u_j)^\vartheta} u_j \mathcal{J}_{u_{j+1}}^\vartheta \mathcal{J}(u) \right|$$

$$\begin{aligned}
&\leq \frac{(u_{j+1} - u_j)^{\bar{\theta}}}{4^{\bar{\theta}}} \left(\left(2^{\bar{\theta}} (1 - 2\lambda)^{(s+2)\bar{\theta}} \left(\frac{\Gamma(1 + s\bar{\theta})}{\Gamma(1 + (s+1)\bar{\theta})} - \frac{\Gamma(1 + (s+1)\bar{\theta})}{\Gamma(1 + (s+2)\bar{\theta})} \right) \right. \right. \\
&\quad \left. \left. + \frac{\Gamma(1 + (s+1)\bar{\theta})}{\Gamma(1 + (s+2)\bar{\theta})} - (1 - 2\lambda)^{\bar{\theta}} \frac{\Gamma(1 + s\bar{\theta})}{\Gamma(1 + (s+1)\bar{\theta})} \right) \left(\left| \mathcal{J}^{(\bar{\theta})}(u_j) \right| + \left| \mathcal{J}^{(\bar{\theta})}(u_{j+1}) \right| \right) \right. \\
&\quad \left. + \left(4^{\bar{\theta}} (2\lambda)^{(s+2)\bar{\theta}} \left(\frac{\Gamma(1 + s\bar{\theta})}{\Gamma(1 + (s+1)\bar{\theta})} - \frac{\Gamma(1 + (s+1)\bar{\theta})}{\Gamma(1 + (s+2)\bar{\theta})} \right) + 2^{\bar{\theta}} \frac{\Gamma(1 + (s+1)\bar{\theta})}{\Gamma(1 + (s+2)\bar{\theta})} \right. \right. \\
&\quad \left. \left. - (4\lambda)^{\bar{\theta}} \frac{\Gamma(1 + s\bar{\theta})}{\Gamma(1 + (s+1)\bar{\theta})} \right) \left| \mathcal{J}^{(\bar{\theta})} \left(\frac{u_j + u_{j+1}}{2} \right) \right| \right).
\end{aligned}$$

Multiplying both sides of the given inequality by $\frac{1}{\Gamma(1+\bar{\theta})} (u_{j+1} - u_j)^{\bar{\theta}}$, and then summing these inequalities for all $j = 0, 1, \dots, m-1$, followed by the application of the triangle inequality, yields the desired result. $\square$

5.3. Application to special means

Let $\omega, \sigma \in \mathbb{R}$. The generalized Arithmetic mean $\mathcal{A}(\omega, \sigma)$ and The generalized p -Logarithmic mean $\mathcal{L}_p(\omega, \sigma)$ are given respectively by:

$$\mathcal{A}(\omega, \sigma) = \frac{\omega^{\bar{\theta}} + \sigma^{\bar{\theta}}}{2^{\bar{\theta}}}$$

and

$$\mathcal{L}_p(\omega, \sigma) = \left[\frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p+1)\bar{\theta})} \left(\frac{\sigma^{(p+1)\bar{\theta}} - \omega^{(p+1)\bar{\theta}}}{(\sigma - \omega)^{\bar{\theta}}} \right) \right]^{\frac{1}{p}}, \quad \omega, \sigma \in \mathbb{R}, \omega \neq \sigma \text{ and } p \in \mathbb{Z} \setminus \{-1, 0\}.$$

Proposition 5.2. Let $\omega, \sigma \in \mathbb{R}$ with $0 < \omega < \sigma$, $q > 1$ and $m \geq 2$, then we have

$$\begin{aligned}
&|\mathcal{A}^m(\omega, \sigma) - \Gamma(\bar{\theta} + 1) \mathcal{L}_m^m(\omega, \sigma)| \\
&\leq \frac{(\sigma - \omega)^{\bar{\theta}}}{4^{\bar{\theta}}} \left(\frac{\Gamma(1 + p\bar{\theta})}{\Gamma(1 + (p+1)\bar{\theta})} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1 + \bar{\theta})}{\Gamma(1 + 2\bar{\theta})} \right)^{\frac{1}{q}} \left(\frac{\Gamma(1 + m\bar{\theta})}{\Gamma(1 + (m-1)\bar{\theta})} \right)^{\frac{1}{q}} \\
&\quad \times \left(\left(\omega^{q(m-1)\bar{\theta}} + \left(\frac{\omega + \sigma}{2} \right)^{q(m-1)\bar{\theta}} \right)^{\frac{1}{q}} + \left(\left(\frac{\omega + \sigma}{2} \right)^{q(m-1)\bar{\theta}} + \sigma^{q(m-1)\bar{\theta}} \right)^{\frac{1}{q}} \right).
\end{aligned}$$

Proof. This follows from Corollary 3.4 with $x = \frac{\omega + \sigma}{2}$, applied to the function $\mathcal{J}(u) = u^{m\bar{\theta}}$ where $\mathcal{J} : (0, +\infty) \rightarrow \mathbb{R}^{\bar{\theta}}$. $\square$

6. Conclusion

In this paper, we established a new biparametrized identity associated with a three-point Newton–Cotes formula in the framework of local fractional calculus on fractal sets. One of the main advantages of this identity is its unifying character: By selecting appropriate values of the parameters, it generates a wide range of classical quadrature formulas involving one, two, or three points.

By combining this identity with generalized s -convexity assumptions, we derived several new local fractional integral inequalities for differentiable functions. These results extend and

refine a number of earlier contributions in the literature, while also yielding previously known inequalities as particular cases. In this sense, the present work provides a broader and more flexible framework for the study of quadrature-type inequalities on fractal sets.

To support the theoretical developments, we included an illustrative example together with two- and three-dimensional graphical representations, which show the consistency and applicability of the obtained results. We also discussed some consequences for numerical quadrature formulas and special means.

We believe that the approach developed here may be useful for further investigations in local fractional calculus, especially in the study of new classes of convexity, more general integral operators, and other parameter-dependent quadrature inequalities in fractal settings.

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