

LYAPUNOV-TYPE INEQUALITIES FOR FRACTIONAL BOUNDARY VALUE PROBLEMS INVOLVING TWO GENERALIZED FRACTIONAL DERIVATIVES WITH RESPECT TO ANOTHER FUNCTION*

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Abstract This paper is devoted to the study of Dirichlet boundary value problems for fractional differential equations involving two types of fractional derivatives defined with respect to another function, introduced by Nieto et al. [22]. By constructing the corresponding generalized Green's functions and analyzing their structural properties, we derive necessary conditions for the existence of nontrivial solutions in the form of generalized Lyapunov-type inequalities. The results extend the classical Lyapunov inequality and contribute to the qualitative analysis of boundary value problems governed by generalized fractional derivatives.

Keywords Generalized fractional derivatives, fractional differential equations, Dirichlet boundary conditions, Lyapunov-type inequalities.

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1. Introduction

The investigation of Lyapunov-type inequalities originates from the seminal work of Lyapunov in 1892 [20]. A classical result states the following: Let $q(\vartheta) \in C([a, b], \mathbb{R})$, and consider the second-order linear differential equation (DE) with Dirichlet boundary conditions (BCs):

$$\begin{cases} z''(\vartheta) + q(\vartheta)z(\vartheta) = 0, & \vartheta \in (a, b), \\ z(a) = 0, & z(b) = 0. \end{cases} \quad (1.1)$$

If the boundary value problem (BVP) (1.1) admits a nontrivial solution on the interval $[a, b]$. Then the function $q(\vartheta)$ satisfies the integral inequality

$$\int_a^b |q(\vartheta)| d\vartheta > \frac{4}{b-a}. \quad (1.2)$$

The inequality (1.2) is referred to as the Lyapunov inequality. This classical result is sharp: The constant “4” appearing therein is optimal, in that no larger constant can replace it [2]. The study of Lyapunov inequality along with its generalized versions, known as Lyapunov-type

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inequalities (LTIs), reveals a profound structural relationship between the zero distribution of solutions and the properties of the associated coefficient functions. Moreover, these inequalities serve as powerful analytical tools for addressing several fundamental problems, such as describing when solutions oscillate or remain non-oscillatory, the derivation of lower bounds for eigenvalue problems, the existence and uniqueness of solutions to BVPs, and the stability analysis of dynamical systems [1, 2, 6, 15].

In recent years, the study of LTIs has been substantially extended to encompass a wide range of mathematical frameworks, including higher-order DEs [29], fractional differential equations (FDEs) [11, 12], difference equations [8], dynamic equations on time scales [16], partial DEs [21], and systems in multiple spatial dimensions [14]. Especially, in the past decade, considerable attention has been devoted to the investigation of LTIs arising in fractional BVPs [7, 10, 13, 18, 19, 30]. A thorough account of recent progress in this direction can be found in [23, 24], which offer a systematic review of developments related to LTIs.

Fractional calculus (FCs), as a natural extension of classical differential and integral calculus, has demonstrated significant advantages in the modeling of nonlocal phenomena, memory effects, and complex dynamical behaviors. With the increasing demand for more versatile mathematical frameworks in applied sciences, traditional fractional derivatives (FDs)—such as the Riemann-Liouville, Caputo, and Hadamard types—have revealed structural limitations in terms of expressiveness and flexibility. This has motivated the researchers to construct generalized fractional operators with enhanced unification and adaptability. For example, Sabzikar, Meerschaert and Chen developed in [26] the framework of tempered FCs, which incorporates exponential tempering into classical fractional operators and thereby yields a more realistic analytical setting for modeling semi-heavy-tailed phenomena. This class of FCs has also been employed in the study of fractional BVPs, as illustrated in [5]. Oliveira and Capelas de Oliveira introduced in [25] the Hilfer-Katugampola FD, which synthesizes the structural features of both Hilfer and Katugampola operators. This formulation enables a unified treatment of multi-parametric systems and encompasses, as particular cases, the classical Riemann-Liouville, Caputo, and Hadamard FDs. The Hilfer-Katugampola operator exhibits favorable analytical properties, especially in the study of initial value problems and integral equations (IEs). In addition, Vanterler da C. Sousa and Capelas de Oliveira introduced the ψ -Hilfer FD in [28], which provides a generalization of various existing fractional operators through the inclusion of an arbitrary monotonic function $\psi(t)$. This construction provides a unified framework for Hilfer, Hadamard, and Katugampola-type FDs. The ψ -Hilfer operator possesses desirable structural attributes, such as semigroup properties and consistency, and has proven to be effective in the analysis of existence and stability of solutions to BVPs. Owing to the broad applicability and structural adaptability of generalized FDs, several researchers have extended LTIs to fractional BVPs that incorporate these operators [9, 27, 31, 32]. For instance, Dien and Nieto [9] established LTI for the following nonlinear sequential fractional BVP formulated within the framework of ψ -Hilfer FDs

$$\begin{cases} ({}^H\mathcal{D}_{a+}^{\iota_1, \varrho_1, \psi} {}^H\mathcal{D}_{a+}^{\iota_2, \varrho_2, \psi} u)(\vartheta) + f(\vartheta, u(\vartheta)) = 0, & \vartheta \in (a, b), \\ u(a) = 0, & {}^H\mathcal{D}_{a+}^{\iota_3, \varrho_3, \psi} u(b) = 0, \end{cases} \quad (1.3)$$

where ${}^H\mathcal{D}_{a+}^{\iota_i, \varrho_i, \psi}$ for $i = 1, 2, 3$ denote the ψ -Hilfer FDs of order ι_i and type ϱ_i with respect to the function ψ . Setting $f(\vartheta, u(\vartheta)) = q(\vartheta)u(\vartheta)$ with $q(\vartheta) \in C([a, b], \mathbb{R})$. If the problem (1.3) has a nontrivial solution, then

$$\int_a^b \psi'(\vartheta)(\psi(b) - \psi(\vartheta))^{\iota_1 + \iota_2 - \iota_3 - 1} |q(\vartheta)| d\vartheta \geq \frac{\Gamma(\iota_1 + \iota_2)}{(\psi(b) - \psi(a))^{\iota_3}}.$$

Thabet and Kedim [27] investigated LTI associated with FDEs governed by the Hilfer-Katugampola operator, supplemented with nonlocal integral BCs:

$$\begin{cases} {}^{\nu, H} \mathcal{D}_{a+}^{\iota, \varrho} u(\vartheta) + \mathfrak{p}(\vartheta)u(\vartheta) = 0, & \vartheta \in (a, b), \\ u(a) = \mathfrak{f}(u), \quad u(b) = \int_a^b (\mathfrak{h}u)(\vartheta) d\vartheta, \end{cases} \quad (1.4)$$

where ${}^{\nu, H} \mathcal{D}_{a+}^{\iota, \varrho}$ denotes the Hilfer-Katugampola FD of order $\iota \in (1, 2]$ and type $\varrho \in [0, 1]$, with $\nu > 0$. The functions $\mathfrak{p}$ and $\mathfrak{h}$ are assumed to be continuous, and the nonlocal term $\mathfrak{f} \in C(\mathbb{R}, \mathbb{R})$ is continuous and satisfies a growth condition of the form $|\mathfrak{f}(u)| \leq \mu \|u\|$, $u \in \mathbb{R}$, for some constant $\mu > 0$. If BVP (1.4) admits a nontrivial solution $u \in C([a, b], \mathbb{R})$, then the coefficient function $\mathfrak{q}$ satisfies the inequality stated below

$$\begin{aligned} & \frac{1 - \int_a^b \frac{(\vartheta^\nu - a^\nu)^{\iota + \varrho(2-\iota) - 1}}{(\mathfrak{b}^\nu - a^\nu)^{\iota + \varrho(2-\iota) - 1}} \mathfrak{h}(\vartheta) d\vartheta}{1 + \int_a^b |\mathfrak{h}(\vartheta)| d\vartheta - \int_a^b \frac{(\vartheta^\nu - a^\nu)^{\iota + \varrho(2-\iota) - 1}}{(\mathfrak{b}^\nu - a^\nu)^{\iota + \varrho(2-\iota) - 1}} \mathfrak{h}(\vartheta) d\vartheta} \\ & < 2\mu + \left[\frac{\iota + \varrho(2-\iota) - 1}{\varrho(2-\iota) - 2} \right]^{\iota + \varrho(2-\iota) - 1} \times \left[\frac{(\mathfrak{b}^\nu - a^\nu)(\iota - 1)}{2\iota + \varrho(2-\iota) - 2} \right]^{\iota - 1} \frac{\max\{a^{\nu-1}, \mathfrak{b}^{\nu-1}\}}{\nu^{\iota-1} \Gamma(\iota)} \int_a^b |\mathfrak{p}(\vartheta)| d\vartheta. \end{aligned}$$

More recently, a unified framework for handling various fractional operators, such as those of Katugampola and Caputo-Katugampola type—was recently developed by Nieto et al. [22] who systematically examined the theory of fractional integrals and derivatives with respect to an auxiliary function $\varsigma(t)$. By introducing a monotonic function $\varsigma(t)$, they defined generalized fractional integral and derivative operators that possess desirable properties such as semigroup structure and inverse relations. Moreover, these operators are well-suited for analysis within weighted Lebesgue spaces and generalized absolutely continuous function spaces. Within this FCs framework, the present work is devoted to investigating LTIs for the following two classes of fractional Dirichlet BVPs:

$$\begin{cases} {}^{\nu} \mathcal{D}_{a+}^{\iota, \varsigma} \mathfrak{x}(t) + \mathfrak{q}(t)\mathfrak{x}(t) = 0, & t \in (a, b), \\ \mathfrak{x}(a) = \mathfrak{x}(b) = 0, \end{cases} \quad (1.5)$$

and

$$\begin{cases} {}^C_{\nu} \mathcal{D}_{a+}^{\iota, \varsigma} \mathfrak{x}(t) + \mathfrak{q}(t)\mathfrak{x}(t) = 0, & t \in (a, b), \\ \mathfrak{x}(a) = \mathfrak{x}(b) = 0, \end{cases} \quad (1.6)$$

where ${}^{\nu} \mathcal{D}_{a+}^{\iota, \varsigma}$ and ${}^C_{\nu} \mathcal{D}_{a+}^{\iota, \varsigma}$ denote, respectively, the ς -Katugampola and ς -Caputo-Katugampola FDs of order $\iota \in (1, 2)$, as detailed in Definitions 2.2 and 2.3.

LTIs play a fundamental role in the qualitative analysis of FDEs. They not only provide necessary conditions for the existence and uniqueness of solutions, but also serve as essential tools in the study of stability, oscillation, and eigenvalue estimation. In recent years, the framework of fractional integrals and derivatives of a function with respect to another function (see [22]) has offered a new perspective for unifying different classes of fractional operators. Within this framework, the ς -Katugampola and ς -Caputo-Katugampola FDs encompass several classical FDs as particular cases, thereby allowing the investigation of LTIs in a more general setting of FCs. Furthermore, the motivation for considering BVPs (1.5) and (1.6) lies in the fact that, by employing the generalized FDs ς -Katugampola and ς -Caputo-Katugampola, one can not only compare the differences and connections among LTIs associated with various FDs, but also reveal

the influence of the choice of the variable function on the inequalities. Such a generalization provides more flexible theoretical tools and establishes a solid foundation for subsequent studies on nonlocal BVPs and stability analysis.

The main innovations of this study are summarized as follows.

- Extension of LTIs via a new generalized FCs framework: This study is the first to investigate LTIs for fractional BVPs within the framework of generalized FCs, which includes the ς -Katugampola and ς -Caputo-Katugampola FDs, thereby extending the scope of such inequalities to more general operator structures.
- The obtained results are general and structurally refined: The LTIs established in the present paper formally extend classical conclusions available in the literature, and reduce exactly to known results under specific conditions, which reflects the theoretical coherence and broad applicability of the approach.
- Extension of fractional BVPs analysis via generalized FCs: This work employs novel generalized FD operators to study linear fractional BVPs, and effectively reformulates them as equivalent IEs involving Green's functions. Through a more refined analysis, we determine the properties of the associated Green's functions. This then enables us to formulate LTIs for BVPs (1.5) and (1.6). These findings lay a theoretical foundation for further research on nonlinear fractional BVPs involving ς -Katugampola or ς -Caputo-Katugampola FDs, and offer a new analytical perspective for the study of BVPs governed by generalized FDEs.

The remainder of this paper is structured as follows. Section 2 offers a comprehensive exposition of the core definitions and fundamental properties associated with the ς -Katugampola and ς -Caputo-Katugampola FDs, thus laying the essential groundwork for the investigations that follow. Subsequently, Section 3 re-expresses BVP (1.5) as a matching IE utilizing a Green's function. Based on a detailed analysis, we then determine the structural properties of this equation and propose a set of sufficient conditions that guarantee the existence of non-trivial solutions. Section 4 treats BVP (1.6) in a similar manner, constructing the corresponding Green's function and analyzing its behavior, which leads to the derivation of a LTI for the problem. To validate the proposed method, Section 5 provides illustrative applications that apply the theoretical findings. The paper then concludes with Section 6, which summarizes the main work and outlines potential directions for future research.

2. Preliminaries

Given the constants $\iota > 0$ and $\nu > 0$, let $\varsigma : [\mathfrak{a}, \mathfrak{b}] \rightarrow \mathbb{R}^+$ be an increasing, monotonic function on $(\mathfrak{a}, \mathfrak{b})$ with a continuous derivative on $(\mathfrak{a}, \mathfrak{b})$, and such that $\varsigma'(t) \neq 0$ for every $t \in (\mathfrak{a}, \mathfrak{b})$. Now, take $f \in \mathcal{X}_{\varsigma; \varsigma}^{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b})$, where $\mathcal{X}_{\varsigma; \varsigma}^{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b})$ denotes the weighted Lebesgue space endowed with the following norm [22]:

$$\|f\|_{\mathcal{X}_{\varsigma; \varsigma}^{\mathfrak{p}}} = \begin{cases} \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |(\varsigma(t))^{\mathfrak{c}} f(t)|^{\mathfrak{p}} \frac{\varsigma'(t)}{\varsigma(t)} dt \right)^{1/\mathfrak{p}} < \infty, & 1 \leq \mathfrak{p} < \infty, \mathfrak{c} \in \mathbb{R}, \\ \text{ess sup}_{t \in (\mathfrak{a}, \mathfrak{b})} [(\varsigma(t))^{\mathfrak{c}} |f(t)|], & \mathfrak{p} = \infty. \end{cases}$$

Definition 2.1 ([22]). (ς -Katugampola fractional integral) The left and right generalized fractional integrals of order $\iota > 0$ and parameter $\nu > 0$ of the function f with respect to another function ς on $[a, b]$, denoted by ${}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} f$ and ${}^\nu \mathcal{J}_{b-}^{\iota, \varsigma} f$ are defined respectively by

$$({}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} f)(t) = \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^t (\varsigma(s))^{\nu-1} \varsigma'(s) [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-1} f(s) ds,$$

and

$$({}^\nu \mathcal{J}_{b-}^{\iota, \varsigma} f)(t) = \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_t^b (\varsigma(s))^{\nu-1} \varsigma'(s) [(\varsigma(s))^\nu - (\varsigma(t))^\nu]^{\iota-1} f(s) ds.$$

Definition 2.2 ([22]). (ς -Katugampola FD) Let $\iota > 0$, $n = [\iota] + 1$ if $\iota \notin \mathbb{N}$, and $n = \iota$ if $\iota \in \mathbb{N}$. Let $\nu > 0$, and $t \in [a, b]$. The left and right generalized fractional derivatives of order ι and parameter ν of the function f with respect to another function ς on $[a, b]$, associated with the generalized fractional integrals ${}^\nu \mathcal{J}_{a+}^{\iota, \varsigma}$ and ${}^\nu \mathcal{J}_{b-}^{\iota, \varsigma}$, respectively, are defined by

$$\begin{aligned} ({}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} f)(t) &= \left(\frac{(\varsigma(t))^{1-\nu}}{\varsigma'(t)} \frac{d}{dt} \right)^n ({}^\nu \mathcal{J}_{a+}^{n-\iota, \varsigma} f)(t) \\ &= \frac{\nu^{\iota-n+1}}{\Gamma(n-\iota)} \left(\frac{(\varsigma(t))^{1-\nu}}{\varsigma'(t)} \frac{d}{dt} \right)^n \int_a^t \frac{(\varsigma(s))^{\nu-1} \varsigma'(s)}{[(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-n+1}} f(s) ds, \end{aligned}$$

and

$$\begin{aligned} ({}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} f)(t) &= \left(-\frac{(\varsigma(t))^{1-\nu}}{\varsigma'(t)} \frac{d}{dt} \right)^n ({}^\nu \mathcal{J}_{b-}^{n-\iota, \varsigma} f)(t) \\ &= \frac{\nu^{\iota-n+1}}{\Gamma(n-\iota)} \left(-\frac{(\varsigma(t))^{1-\nu}}{\varsigma'(t)} \frac{d}{dt} \right)^n \int_t^b \frac{(\varsigma(s))^{\nu-1} \varsigma'(s)}{[(\varsigma(s))^\nu - (\varsigma(t))^\nu]^{\iota-n+1}} f(s) ds. \end{aligned}$$

Remark 2.1. The ς -Katugampola FD can be reduced to several classical FDs under appropriate choices of the function $\varsigma(t)$. Specifically, if $\varsigma(t) = [\nu\psi(t)]^{1/\nu}$, then the ς -Katugampola FD coincides with the ψ -Riemann-Liouville FD. If $\varsigma(t) = t$, it reduces to the Katugampola FD, whereas for $\varsigma(t) \equiv (\nu t)^{1/\nu}$, it reduces to the Riemann-Liouville FD.

Definition 2.3 ([22]). (ς -Caputo-Katugampola FD) Let $\iota > 0$, $n = [\iota] + 1$, $\varsigma \in C^1[a, b]$ be a strictly increasing function with continuous derivative, and let $f \in AC_{\delta; \varsigma}^n[a, b]$, where the function space $AC_{\delta; \varsigma}^n[a, b]$ is defined by

$$AC_{\delta; \varsigma}^n = \left\{ f : [a, b] \rightarrow \mathbb{R} : \delta_\varsigma^{n-1} f \in AC([a, b], \mathbb{R}), \delta_\varsigma = \frac{(\varsigma(t))^{1-\nu}}{\varsigma'(t)} \frac{d}{dt} \right\}.$$

The left and right Caputo-type generalized FDs of f with respect to ς , denoted respectively by ${}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} f(t)$ and ${}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} f(t)$, are defined as follows:

(1) If $\iota \neq n$, the left and right ς -Caputo-type generalized FDs of f are given by:

$$\begin{aligned} {}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} f(t) &= \frac{1}{\Gamma(n-\iota)} \int_a^t (\varsigma(s))^{\nu-1} \varsigma'(s) \left(\frac{(\varsigma(t))^\nu - (\varsigma(s))^\nu}{\nu} \right)^{n-\iota-1} (\delta_\varsigma^n f)(s) ds, \\ {}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} f(t) &= \frac{1}{\Gamma(n-\iota)} \int_t^b (\varsigma(s))^{\nu-1} \varsigma'(s) \left(\frac{(\varsigma(s))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{n-\iota-1} (-1)^n (\delta_\varsigma^n f)(s) ds. \end{aligned}$$

(2) If $\iota = n$:

$${}^C \mathcal{D}_{a+}^{\iota, \varsigma} f(t) = (\delta_\varsigma^n f)(t), \quad {}^C \mathcal{D}_{b-}^{\iota, \varsigma} f(t) = (-1)^n (\delta_\varsigma^n f)(t),$$

and for $\iota = 0$:

$${}^C \mathcal{D}_{a+}^{0, \varsigma} f(t) = f(t), \quad {}^C \mathcal{D}_{b-}^{0, \varsigma} f(t) = f(t).$$

Remark 2.2. The ς -Caputo-Katugampola FD can be reduced to several classical FDs under suitable choices of the function $\varsigma(t)$. In particular, if $\varsigma(t) = [\nu\psi(t)]^{1/\nu}$, then the ς -Caputo-Katugampola FD coincides with the ψ -Caputo FD. If $\varsigma(t) = t$, it reduces to the Caputo-Katugampola FD, whereas for $\varsigma(t) \equiv (\nu t)^{1/\nu}$, it reduces to the Caputo FD.

Lemma 2.1 ([22]). Let $1 \leq p \leq \infty$, $0 < a < b < \infty$, and let $\nu, c \in \mathbb{R}$ with $\nu \geq c$. For $f \in \mathcal{X}_{c; \varsigma}^p(a, b)$, then the generalized fractional integral operators with respect to ς satisfy the following semigroup properties

$$\begin{aligned} {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{a+}^{\varrho, \varsigma} f)(t) &= {}^\nu \mathcal{J}_{a+}^{\iota+\varrho, \varsigma} f(t), \quad \text{for all } \iota > 0, \varrho > 0, \\ {}^\nu \mathcal{J}_{b-}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{b-}^{\varrho, \varsigma} f)(t) &= {}^\nu \mathcal{J}_{b-}^{\iota+\varrho, \varsigma} f(t), \quad \text{for all } \iota > 0, \varrho > 0. \end{aligned}$$

Lemma 2.2 ([22]). Let $f \in \mathcal{X}_{c; \varsigma}^p(a, b)$, $0 < \iota < \varrho$, $\nu > 0$, and $0 < a < b < \infty$. Then the following composition identities hold

$$\begin{aligned} {}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{a+}^{\varrho, \varsigma} f)(t) &= {}^\nu \mathcal{J}_{a+}^{\varrho-\iota, \varsigma} f(t), \\ {}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{b-}^{\varrho, \varsigma} f)(t) &= {}^\nu \mathcal{J}_{b-}^{\varrho-\iota, \varsigma} f(t). \end{aligned}$$

In particular, when $\iota = \varrho$, the generalized FDs and integrals are inverse operators:

$$\begin{aligned} {}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} f)(t) &= f(t), \\ {}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} ({}^\nu \mathcal{J}_{b-}^{\iota, \varsigma} f)(t) &= f(t). \end{aligned}$$

Lemma 2.3 ([22]). Let $\iota_1, \iota_2 > 0$, then

$$\begin{aligned} {}^\nu \mathcal{J}_{a+}^{\iota_1, \varsigma} \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota_2-1} &= \frac{\Gamma(\iota_2)}{\Gamma(\iota_1 + \iota_2)} \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota_1 + \iota_2 - 1}, \\ {}^\nu \mathcal{J}_{b-}^{\iota_1, \varsigma} \left(\frac{(\varsigma(b))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{\iota_2-1} &= \frac{\Gamma(\iota_2)}{\Gamma(\iota_1 + \iota_2)} \left(\frac{(\varsigma(b))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{\iota_1 + \iota_2 - 1}, \end{aligned}$$

and

$$\begin{aligned} {}^\nu \mathcal{D}_{a+}^{\iota_1, \varsigma} \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota_2-1} &= \frac{\Gamma(\iota_2)}{\Gamma(\iota_2 - \iota_1)} \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota_2 - \iota_1 - 1}, \\ {}^\nu \mathcal{D}_{b-}^{\iota_1, \varsigma} \left(\frac{(\varsigma(b))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{\iota_2-1} &= \frac{\Gamma(\iota_2)}{\Gamma(\iota_2 - \iota_1)} \left(\frac{(\varsigma(b))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{\iota_2 - \iota_1 - 1}. \end{aligned}$$

Lemma 2.4 ([22]). Let $\iota > 0$, $n = [\iota] + 1$. Suppose $f \in L[a, b]$, and ${}^\nu \mathcal{J}_{a+}^{n-\iota, \varsigma} f \in AC_{\delta; \varsigma}^n[a, b]$ (${}^\nu \mathcal{J}_{b-}^{n-\iota, \varsigma} f \in AC_{\delta; \varsigma}^n[a, b]$). Then the composition of the generalized fractional integral and derivative operators satisfies

$$\begin{aligned} {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} {}^\nu \mathcal{D}_{a+}^{\iota, \varsigma} f(t) &= f(t) - \sum_{j=1}^n \frac{({}^\nu \mathcal{D}_{a+}^{\iota-j, \varsigma} f)(a)}{\Gamma(\iota - j + 1)} \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota-j}, \\ {}^\nu \mathcal{J}_{b-}^{\iota, \varsigma} {}^\nu \mathcal{D}_{b-}^{\iota, \varsigma} f(t) &= f(t) - \sum_{j=1}^n \frac{(-1)^j ({}^\nu \mathcal{D}_{b-}^{\iota-j, \varsigma} f)(b)}{\Gamma(\iota - j + 1)} \left(\frac{(\varsigma(b))^\nu - (\varsigma(t))^\nu}{\nu} \right)^{\iota-j}. \end{aligned}$$

Lemma 2.5 ([22]). *Let $\iota, \lambda > 0$, then*

$$\begin{aligned} {}^C_{\nu} \mathcal{D}_{a+}^{\iota, \varsigma} \left(\frac{(\varsigma(t))^{\nu} - (\varsigma(a))^{\nu}}{\nu} \right)^{\lambda-1} &= \frac{\Gamma(\lambda)}{\Gamma(\lambda - \iota)} \left(\frac{(\varsigma(t))^{\nu} - (\varsigma(a))^{\nu}}{\nu} \right)^{\lambda - \iota - 1}, \\ {}^C_{\nu} \mathcal{D}_{b-}^{\iota, \varsigma} \left(\frac{(\varsigma(b))^{\nu} - (\varsigma(t))^{\nu}}{\nu} \right)^{\lambda-1} &= \frac{\Gamma(\lambda)}{\Gamma(\lambda - \iota)} \left(\frac{(\varsigma(b))^{\nu} - (\varsigma(t))^{\nu}}{\nu} \right)^{\lambda - \iota - 1}, \end{aligned}$$

for $\lambda > n$, where $n = [\iota] + 1$. In particular, for $k = 0, 1, 2, \dots, n-1$, it follows

$${}^C_{\nu} \mathcal{D}_{a+}^{\iota, \varsigma} \left(\frac{(\varsigma(t))^{\nu} - (\varsigma(a))^{\nu}}{\nu} \right)^k = 0, \quad {}^C_{\nu} \mathcal{D}_{b-}^{\iota, \varsigma} \left(\frac{(\varsigma(b))^{\nu} - (\varsigma(t))^{\nu}}{\nu} \right)^k = 0.$$

Lemma 2.6 ([22]). *Let $\iota > 0$ and $n = [\iota] + 1$. For $f \in C[a, b]$, then the following identities hold*

$${}^C_{\nu} \mathcal{D}_{a+}^{\iota, \varsigma \nu} \mathcal{J}_{a+}^{\iota, \varsigma} f(t) = f(t), \quad {}^C_{\nu} \mathcal{D}_{b-}^{\iota, \varsigma \nu} \mathcal{J}_{b-}^{\iota, \varsigma} f(t) = f(t).$$

Lemma 2.7 ([22]). *Let $\iota > 0$ and $n = [\iota] + 1$. Suppose that $f \in AC_{\delta; \varsigma}^n[a, b]$ or $f \in C_{\delta; \varsigma}^n[a, b]$. Then the following identities hold*

$$\begin{aligned} {}^{\nu} \mathcal{J}_{a+}^{\iota, \varsigma C} \mathcal{D}_{a+}^{\iota, \varsigma} f(t) &= f(t) - \sum_{k=0}^{n-1} \frac{(\delta_{\varsigma}^k f)(a)}{k!} \left(\frac{(\varsigma(t))^{\nu} - (\varsigma(a))^{\nu}}{\nu} \right)^k, \\ {}^{\nu} \mathcal{J}_{b-}^{\iota, \varsigma C} \mathcal{D}_{b-}^{\iota, \varsigma} f(t) &= f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k (\delta_{\varsigma}^k f)(b)}{k!} \left(\frac{(\varsigma(b))^{\nu} - (\varsigma(t))^{\nu}}{\nu} \right)^k. \end{aligned}$$

In particular, for $0 < \iota < 1$, the identities simplify to

$${}^{\nu} \mathcal{J}_{a+}^{\iota, \varsigma C} \mathcal{D}_{a+}^{\iota, \varsigma} f(t) = f(t) - f(a), \quad {}^{\nu} \mathcal{J}_{b-}^{\iota, \varsigma C} \mathcal{D}_{b-}^{\iota, \varsigma} f(t) = f(t) - f(b).$$

Lemma 2.8 ([12]). *Let $\varpi \in (1, 2)$. Then the following inequality holds*

$$(2 - \varpi)(\varpi - 1)^{\frac{\varpi-1}{2-\varpi}} \leq \frac{(\varpi - 1)^{\varpi-1}}{\varpi^{\varpi}}. \quad (2.1)$$

Remark 2.3. Inequality (2.1) remains valid in the particular case $\varpi \rightarrow 2$.

Proof. In fact, for $\varpi \rightarrow 2$, we have

$$\begin{aligned} \lim_{\varpi \rightarrow 2-} \frac{\ln(\varpi-1)}{2-\varpi} &= -1, \quad \lim_{\varpi \rightarrow 2-} (\varpi - 1)^{\frac{\varpi-1}{2-\varpi}} = e^{\lim_{\varpi \rightarrow 2-} (\varpi-1) \frac{\ln(\varpi-1)}{2-\varpi}} = e^{-1}, \\ \lim_{\varpi \rightarrow 2-} (2-\varpi)(\varpi-1)^{\frac{\varpi-1}{2-\varpi}} &= \lim_{\varpi \rightarrow 2-} (2-\varpi) e^{(\varpi-1) \frac{\ln(\varpi-1)}{2-\varpi}} = 0 \leq \lim_{\varpi \rightarrow 2-} \frac{(\varpi - 1)^{\varpi-1}}{\varpi^{\varpi}} = \frac{1}{4}. \end{aligned}$$

Hence, the remark is proved. $\square$

3. Main result for problem (1.5)

3.1. Integral formulation corresponding to problem (1.5)

In this subsection, BVP (1.5) is studied within the framework of the ς -Katugampola FCs. By utilizing its fundamental properties, the problem is first recast equivalently as an IE involving a Green's function. An upper bound estimate for the Green's function then follows.

Lemma 3.1. For $1 < \iota \leq 2$. Then $\mathfrak{x}(t) \in C[a, b]$ is a solution to the BVP (1.5) if it fulfills the following IE

$$\mathfrak{x}(t) = \int_a^b \mathcal{T}(s) \mathcal{G}(t, s) q(s) \mathfrak{x}(s) ds, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{T}(s) &= \frac{\nu^{1-\iota}}{\Gamma(\iota)} (\varsigma(s))^{\nu-1} \varsigma'(s), \quad s \in [a, b], \\ \mathcal{G}(t, s) &= \begin{cases} \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1}, & a \leq t \leq s \leq b, \\ \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1} - [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-1}, & a \leq s \leq t \leq b. \end{cases} \end{aligned}$$

Proof. Applying Lemma 2.4, and acting with the operator ${}^\nu \mathcal{J}_{a+}^{\iota, \varsigma}$ on both sides of equation (1.5), we derive

$$\mathfrak{x}(t) = -{}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t) + c_1 \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota-1} + c_2 \left(\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{\iota-2}, \quad c_1, c_2 \in \mathbb{R},$$

and upon incorporating the BCs $\mathfrak{x}(a) = \mathfrak{x}(b) = 0$, we arrive at

$$c_2 = 0, \quad c_1 = \left(\frac{(\varsigma(b))^\nu - (\varsigma(a))^\nu}{\nu} \right)^{1-\iota} {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t) \Big|_{t=b}.$$

Hence,

$$\begin{aligned} \mathfrak{x}(t) &= \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t) \Big|_{t=b} - {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t) \\ &= \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^b (\varsigma(s))^{\nu-1} \varsigma'(s) [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1} q(s) \mathfrak{x}(s) ds \\ &\quad - \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^t (\varsigma(s))^{\nu-1} \varsigma'(s) [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-1} q(s) \mathfrak{x}(s) ds \\ &= \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^t (\varsigma(s))^{\nu-1} \varsigma'(s) \left\{ \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1} \right. \\ &\quad \left. - [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-1} \right\} q(s) \mathfrak{x}(s) ds \\ &\quad + \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_t^b \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} (\varsigma(s))^{\nu-1} \varsigma'(s) [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1} q(s) \mathfrak{x}(s) ds \\ &= \int_a^b \mathcal{T}(s) \mathcal{G}(t, s) q(s) \mathfrak{x}(s) ds. \end{aligned}$$

The converse direction follows directly from Lemmas 2.2 and 2.3, which imply that $\mathfrak{x}(t)$ satisfies equation (1.5) along with the associated BCs. This establishes the desired result. $\square$

Lemma 3.2. The Green's function $\mathcal{G}$ as introduced in Lemma 3.1, is characterized by the following properties:

(i) $\mathcal{G}(\mathbf{t}, \mathbf{s}) \geq 0$ for all $\mathbf{a} \leq \mathbf{t}, \mathbf{s} \leq \mathbf{b}$.

(ii) For fixed $\mathbf{t} \in [\mathbf{a}, \mathbf{b}]$, the function $\mathcal{G}(\mathbf{t}, \mathbf{s})$ attains its maximum at $\mathbf{s} = \mathbf{t}$, i.e.,

$$\max_{\mathbf{s} \in [\mathbf{a}, \mathbf{b}]} \mathcal{G}(\mathbf{t}, \mathbf{s}) = \mathcal{G}(\mathbf{t}, \mathbf{t}).$$

(iii) The function $\mathcal{G}(\mathbf{t}, \mathbf{t})$ attains a unique global maximum at

$$\mathbf{t} = \varsigma^{-1} \left\{ \left[\frac{(\varsigma(\mathbf{b}))^\nu + (\varsigma(\mathbf{a}))^\nu}{2} \right]^{1/\nu} \right\},$$

and

$$\max_{\mathbf{t} \in [\mathbf{a}, \mathbf{b}]} \mathcal{G}(\mathbf{t}, \mathbf{t}) = \left[\frac{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu}{4} \right]^{\iota-1}.$$

Proof. (i) **Positivity of $\mathcal{G}(\mathbf{t}, \mathbf{s})$.** We introduce two auxiliary functions as follows:

$$\begin{aligned} g_1(\mathbf{t}, \mathbf{s}) &= \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{\iota-1} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1} \\ &\quad - [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1}, \quad \text{for } \mathbf{a} \leq \mathbf{s} \leq \mathbf{t} \leq \mathbf{b}, \\ g_2(\mathbf{t}, \mathbf{s}) &= \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{\iota-1} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1}, \quad \text{for } \mathbf{a} \leq \mathbf{t} \leq \mathbf{s} \leq \mathbf{b}. \end{aligned}$$

Clearly, $g_2(\mathbf{t}, \mathbf{s}) \geq 0$ since all terms are nonnegative. For $g_1(\mathbf{t}, \mathbf{s})$, $\mathbf{a} \leq \mathbf{s} \leq \mathbf{t} \leq \mathbf{b}$, we can find that

$$\begin{aligned} g_1(\mathbf{t}, \mathbf{s}) &= \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{\iota-1} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1} - [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1} \\ &= \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{\iota-1} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1} - \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{\iota-1} \\ &\quad \times \left\{ (\varsigma(\mathbf{b}))^\nu - \left[(\varsigma(\mathbf{a}))^\nu + \frac{((\varsigma(\mathbf{s}))^\nu - (\varsigma(\mathbf{a}))^\nu)((\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu)}{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right] \right\}^{\iota-1}. \end{aligned}$$

Consider the fact that

$$\begin{aligned} &(\varsigma(\mathbf{a}))^\nu + \frac{[(\varsigma(\mathbf{s}))^\nu - (\varsigma(\mathbf{a}))^\nu][(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]}{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu} \geq (\varsigma(\mathbf{s}))^\nu \\ \Leftrightarrow &\frac{(\varsigma(\mathbf{a}))^\nu[(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu] + [(\varsigma(\mathbf{s}))^\nu - (\varsigma(\mathbf{a}))^\nu][(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]}{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu} \geq (\varsigma(\mathbf{s}))^\nu \\ \Leftrightarrow &[(\varsigma(\mathbf{s}))^\nu - (\varsigma(\mathbf{a}))^\nu][(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{t}))^\nu] \geq 0 \\ \Leftrightarrow &(\varsigma(\mathbf{s}))^\nu \geq (\varsigma(\mathbf{a}))^\nu. \end{aligned}$$

Thus, from this, and $\iota > 1$, we see that, $g_1(\mathbf{t}, \mathbf{s})$ is nonnegative. Therefore, $\mathcal{G}(\mathbf{t}, \mathbf{s}) \geq 0$ for all $\mathbf{t}, \mathbf{s} \in [\mathbf{a}, \mathbf{b}]$.

(ii) **Maximum of $\mathcal{G}(\mathbf{t}, \mathbf{s})$ in $\mathbf{s}$.** Consider the behavior of $\mathcal{G}(\mathbf{t}, \mathbf{s})$ as a function of $\mathbf{s}$ for fixed $\mathbf{t} \in [\mathbf{a}, \mathbf{b}]$. Then, for $\mathbf{a} \leq \mathbf{s} \leq \mathbf{t} \leq \mathbf{b}$, we differentiate $g_1(\mathbf{t}, \mathbf{s})$ with respect to $\mathbf{t}$ for each fixed $\mathbf{s}$, we

obtain

$$\begin{aligned}
 \frac{\partial g_1(t, s)}{\partial s} &= \nu \varsigma'(s)(\varsigma(s))^{\nu-1}(\iota-1) \left\{ [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-2} \right. \\
 &\quad \left. - \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-2} \right\} \\
 &= \nu \varsigma'(s)(\varsigma(s))^{\nu-1}(\iota-1) [(\varsigma(t))^\nu - (\varsigma(s))^\nu]^{\iota-2} \\
 &\quad \times \left\{ 1 - \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} \left[\frac{(\varsigma(b))^\nu - (\varsigma(s))^\nu}{(\varsigma(t))^\nu - (\varsigma(s))^\nu} \right]^{\iota-2} \right\} \\
 &\geq 0.
 \end{aligned}$$

Consequently, $g_1(t, s)$ increases monotonically with respect to s , attaining its maximum at $s = t$. For $a \leq t \leq s \leq b$, one finds that $g_2(t, s)$ is nonincreasing with respect to the variable s , with its maximum likewise located at $s = t$. Therefore,

$$\max_{s \in [a, b]} \mathcal{G}(t, s) = \mathcal{G}(t, t).$$

(iii) **Maximum of $\mathcal{G}(t, t)$.** Define

$$g(t) = \mathcal{G}(t, t) = \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-1} [(\varsigma(b))^\nu - (\varsigma(t))^\nu]^{\iota-1}, \quad t \in [a, b].$$

Taking the derivative of $g(t)$ with respect to the variable t on (a, b) yields

$$\begin{aligned}
 g'(t) &= \frac{\nu(\varsigma(t))^{\nu-1}\varsigma'(t)[(\varsigma(b))^\nu - (\varsigma(t))^\nu]^{\iota-2}}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \left[\frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} \right]^{\iota-2} \\
 &\quad \times (\iota-1) \{ (\varsigma(b))^\nu - 2(\varsigma(t))^\nu + (\varsigma(a))^\nu \}.
 \end{aligned}$$

Let $g'(t) = 0$. Observe that $\varsigma(t)$ is increasing on (a, b) , and that $\varsigma(t) \neq 0, \varsigma'(t) \neq 0$ for all $t \in (a, b)$. Consequently, $g'(t) = 0$ for $t \in (a, b)$ if and only if

$$(\varsigma(b))^\nu - 2(\varsigma(t))^\nu + (\varsigma(a))^\nu = 0.$$

It follows that the function $g(t)$ admits a unique stationary point $t^* \in (a, b)$ satisfying

$$(\varsigma(t^*))^\nu = \frac{(\varsigma(b))^\nu + (\varsigma(a))^\nu}{2}.$$

Furthermore, by exploiting the monotonicity of $\varsigma(t)$, it is readily seen that $g(t) \geq 0$ for all $t \in (a, b)$, and that $g(a) = g(b) = 0$. Since the function $g : [a, b] \rightarrow \mathbb{R}$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , its maximum value necessarily exists. Moreover, such a maximum can only be attained either at the endpoints a, b , or at an interior point $t \in (a, b)$ satisfying $g'(t) = 0$. Hence, we deduce that the function $g(t)$ attains its maximum on (a, b) at the unique point $t = t^*$. The corresponding maximum value is given by

$$\max_{t \in (a, b)} g(t) = \left(\frac{(\varsigma(b))^\nu - (\varsigma(a))^\nu}{4} \right)^{\iota-1}.$$

Lemma 3.2 has now been proved. $\square$

3.2. Lyapunov-type inequality for problem (1.5)

With Lemmas 3.1 and 3.2 at our disposal, this subsection establishes a necessary condition—namely, a LTI for BVP (1.5) to admit nontrivial continuous solutions. Consider the Banach space $\mathcal{X} = C[\mathfrak{a}, \mathfrak{b}]$ equipped with the supremum norm $\|\mathfrak{r}\|_\infty = \max_{t \in [\mathfrak{a}, \mathfrak{b}]} |\mathfrak{r}(t)|$ for $\mathfrak{r}(t) \in \mathcal{X}$.

Theorem 3.1. *Let $\iota \in (1, 2]$ and $\mathfrak{q} \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$. If the fractional BVP (1.5) possesses a nontrivial solution $\mathfrak{r} \not\equiv 0$, then the following inequality holds*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \Gamma(\iota) \left(\frac{(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu}{4\nu} \right)^{1-\iota}. \quad (3.2)$$

Proof. Lemma 3.1 implies that every solution $\mathfrak{r}(t) \in C[\mathfrak{a}, \mathfrak{b}]$ to BVP (1.5) fulfills the IE

$$\mathfrak{r}(t) = \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) \mathfrak{q}(\mathfrak{s}) \mathfrak{r}(\mathfrak{s}) d\mathfrak{s}.$$

Taking norms on both sides of the IE, yields

$$\begin{aligned} \|\mathfrak{r}\|_\infty &\leq \max_{t \in [\mathfrak{a}, \mathfrak{b}]} \left| \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) \mathfrak{q}(\mathfrak{s}) \mathfrak{r}(\mathfrak{s}) d\mathfrak{s} \right| \\ &\leq \|\mathfrak{r}\|_\infty \max_{t \in [\mathfrak{a}, \mathfrak{b}]} \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) \mathfrak{q}(\mathfrak{s})| d\mathfrak{s}. \end{aligned}$$

Since $\mathfrak{r} \not\equiv 0$, we may divide both sides by $\|\mathfrak{r}\|_\infty$, one has

$$1 \leq \max_{t \in [\mathfrak{a}, \mathfrak{b}]} \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) \mathfrak{q}(\mathfrak{s})| d\mathfrak{s}.$$

Furthermore, by Lemma 3.2, we have that

$$1 \leq \left(\frac{(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu}{4} \right)^{\iota-1} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{T}(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s},$$

which implies

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{T}(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \left(\frac{(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu}{4} \right)^{1-\iota},$$

that is,

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \frac{\nu^{1-\iota}}{\Gamma(\iota)} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \left(\frac{(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu}{4} \right)^{1-\iota}.$$

This yields the desired inequality (3.2). Thus, the assertion is proved. $\square$

Corollary 3.1. *Consider the FDE below under Dirichlet BCs*

$$\begin{cases} \mathcal{D}_{\mathfrak{a}+}^{\iota, \psi} \mathfrak{r}(t) + \mathfrak{q}(t) \mathfrak{r}(t) = 0, & t \in (\mathfrak{a}, \mathfrak{b}), \\ \mathfrak{r}(\mathfrak{a}) = \mathfrak{r}(\mathfrak{b}) = 0, \end{cases} \quad (3.3)$$

where $\mathfrak{q}(t) \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$, and $\mathcal{D}_{\mathfrak{a}+}^{\iota, \psi}$ stands for the ψ -Riemann-Liouville FD of order ι [17]. If equation (3.3) possesses a nontrivial continuous solution, then the following holds

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \Gamma(\iota) \left(\frac{\psi(\mathfrak{b}) - \psi(\mathfrak{a})}{4} \right)^{1-\iota}. \quad (3.4)$$

Proof. Utilizing Theorem 3.1 with $\varsigma(t) = [\nu\psi(t)]^{1/\nu}$, we immediately obtain inequality (3.4). $\square$

Remark 3.1. Theorem 3.1 recovers ([19], Theorem 5) in the case $\varsigma(t) = t$, $t \in [a, b]$, and further reduces to ([11], Theorem 2.1) as $\varsigma(t) \equiv (\nu t)^{1/\nu}$, $t \in [a, b]$.

4. Main result for problem (1.6)

4.1. Integral formulation corresponding to problem (1.6)

This subsection relies on the fundamental properties of the ς -Caputo-Katugampola FD to recast BVP (1.6) equivalently as an IE that contains a Green's function. We also establish an upper bound for this Green's function.

Lemma 4.1. *For $1 < \iota \leq 2$. Then, a necessary and sufficient condition for a function $\mathfrak{x}(t) \in C[a, b]$ to be a solution of BVP (1.6) is that it meets the following IE*

$$\mathfrak{x}(t) = \int_a^b \mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) q(\mathfrak{s}) \mathfrak{x}(\mathfrak{s}) d\mathfrak{s}, \quad (4.1)$$

where

$$\mathcal{G}(t, \mathfrak{s}) = \begin{cases} \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(\mathfrak{s}))^\nu]^{\iota-1}, & a \leq t \leq \mathfrak{s} \leq b, \\ \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(\mathfrak{s}))^\nu]^{\iota-1} - [(\varsigma(t))^\nu - (\varsigma(\mathfrak{s}))^\nu]^{\iota-1}, & a \leq \mathfrak{s} \leq t \leq b. \end{cases}$$

Proof. By Lemma 2.7, applying the operator ${}^\nu \mathcal{J}_{a+}^{\iota, \varsigma}$ to both sides of equation (1.6), we deduce that

$$\mathfrak{x}(t) = -{}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t) + \mathfrak{x}(a) + \frac{(\varsigma(a))^{1-\nu}}{\varsigma'(a)} \cdot \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{\nu} \mathfrak{x}'(a),$$

which, together with the BCs $\mathfrak{x}(a) = 0$ and $\mathfrak{x}(b) = 0$, yields

$$\mathfrak{x}'(a) = \left[\frac{(\varsigma(a))^{1-\nu}}{\varsigma'(a)} \cdot \frac{(\varsigma(b))^\nu - (\varsigma(a))^\nu}{\nu} \right]^{-1} {}^\nu \mathcal{J}_{a+}^{\iota, \varsigma} q(t) \mathfrak{x}(t)|_{t=b}.$$

Consequently, we derive the following integral representation

$$\begin{aligned} \mathfrak{x}(t) &= \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^b \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) q(\mathfrak{s}) \mathfrak{x}(\mathfrak{s}) [(\varsigma(b))^\nu - (\varsigma(\mathfrak{s}))^\nu]^{\iota-1} d\mathfrak{s} \\ &\quad - \frac{\nu^{1-\iota}}{\Gamma(\iota)} \int_a^t (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) q(\mathfrak{s}) \mathfrak{x}(\mathfrak{s}) [(\varsigma(t))^\nu - (\varsigma(\mathfrak{s}))^\nu]^{\iota-1} d\mathfrak{s}, \end{aligned}$$

which can be equivalently written as

$$\mathfrak{x}(t) = \int_a^b \mathcal{T}(\mathfrak{s}) \mathcal{G}(t, \mathfrak{s}) q(\mathfrak{s}) \mathfrak{x}(\mathfrak{s}) d\mathfrak{s}.$$

Conversely, it can be readily verified via Lemmas 2.5 and 2.6 that $\mathfrak{x}(t)$ fulfills the BVP (1.6). We have thus finished the argument. $\square$

Lemma 4.2. *The Green's function $\mathcal{G}(\mathbf{t}, \mathbf{s})$, as introduced in Lemma 4.1, admits the following property*

$$|\mathcal{G}(\mathbf{t}, \mathbf{s})| \leq \frac{(\iota - 1)^{\iota-1}}{\iota^\iota} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-1}, \quad (\mathbf{t}, \mathbf{s}) \in [\mathbf{a}, \mathbf{b}] \times [\mathbf{a}, \mathbf{b}]. \quad (4.2)$$

Proof. To establish the stated bound, we analyze the two cases separately.

Case 1. For $\mathbf{a} \leq \mathbf{s} \leq \mathbf{t} \leq \mathbf{b}$. Define

$$\mathbf{g}_1(\mathbf{t}, \mathbf{s}) = \frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1} - [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-1}.$$

For each fixed $\mathbf{t} \in [\mathbf{a}, \mathbf{b}]$, differentiation of $\mathbf{g}_1(\mathbf{t}, \mathbf{s})$ with respect to $\mathbf{s}$ yields

$$\begin{aligned} \frac{\partial \mathbf{g}_1(\mathbf{t}, \mathbf{s})}{\partial \mathbf{s}} &= \nu \varsigma'(\mathbf{s}) (\varsigma(\mathbf{s}))^{\nu-1} (\iota-1) \left\{ [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-2} \right. \\ &\quad \left. - \frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-2} \right\} \\ &= \nu \varsigma'(\mathbf{s}) (\varsigma(\mathbf{s}))^{\nu-1} (\iota-1) [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-2} \\ &\quad \times \left\{ 1 - \frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \cdot \frac{[(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-2}}{[(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu]^{\iota-2}} \right\}. \end{aligned}$$

Using the inequalities

$$\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} \leq 1, \quad \left[\frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{s}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{s}))^\nu} \right]^{2-\iota} \leq 1,$$

we deduce that

$$\frac{\partial \mathbf{g}_1(\mathbf{t}, \mathbf{s})}{\partial \mathbf{s}} \geq 0.$$

Hence, $\mathbf{g}_1(\mathbf{t}, \mathbf{s})$ is increasing in $\mathbf{s} \in [\mathbf{a}, \mathbf{t}]$. Therefore,

$$\mathbf{g}_1(\mathbf{t}, \mathbf{a}) \leq \mathbf{g}_1(\mathbf{t}, \mathbf{s}) \leq \mathbf{g}_1(\mathbf{t}, \mathbf{t}).$$

Note that

$$\begin{aligned} \mathbf{g}_1(\mathbf{t}, \mathbf{a}) &= [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu] [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-2} - [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-1} \\ &= [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu] [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-2} \left\{ 1 - \left[\frac{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu} \right]^{2-\iota} \right\} \\ &\leq 0. \end{aligned}$$

As a consequence of this,

$$|\mathbf{g}_1(\mathbf{t}, \mathbf{s})| \leq \max \left\{ \max_{\mathbf{t} \in [\mathbf{a}, \mathbf{b}]} -\mathbf{g}_1(\mathbf{t}, \mathbf{a}), \max_{\mathbf{t} \in [\mathbf{a}, \mathbf{b}]} \mathbf{g}_1(\mathbf{t}, \mathbf{t}) \right\}. \quad (4.3)$$

Let us define

$$\begin{aligned} \ell_1(\mathbf{t}) &= \mathbf{g}_1(\mathbf{t}, \mathbf{t}) = \frac{(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu}{(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu} [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{t}))^\nu]^{\iota-1}, \quad \mathbf{t} \in [\mathbf{a}, \mathbf{b}], \\ \ell_2(\mathbf{t}) &= -\mathbf{g}_1(\mathbf{t}, \mathbf{a}) \\ &= [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-1} \\ &\quad - [(\varsigma(\mathbf{t}))^\nu - (\varsigma(\mathbf{a}))^\nu] [(\varsigma(\mathbf{b}))^\nu - (\varsigma(\mathbf{a}))^\nu]^{\iota-2}, \quad \mathbf{t} \in [\mathbf{a}, \mathbf{b}]. \end{aligned}$$

Differentiating $\ell_1(t)$ on (a, b) , we arrive at

$$\begin{aligned}\ell_1'(t) &= \frac{\nu \varsigma'(t)(\varsigma(t))^{\nu-1}}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(t))^\nu]^{\iota-2} \\ &\quad \times [(\varsigma(b))^\nu - \iota(\varsigma(t))^\nu + (\iota-1)(\varsigma(a))^\nu], \quad t \in (a, b).\end{aligned}$$

Observe that $\varsigma(t)$ is increasing on (a, b) , and that $\varsigma(t) \neq 0, \varsigma'(t) \neq 0$ for all $t \in (a, b)$. Hence, the function $\ell_1'(t)$ admits a zero in (a, b) if and only if

$$(\varsigma(b))^\nu - \iota(\varsigma(t))^\nu + (\iota-1)(\varsigma(a))^\nu = 0,$$

which is attained at the point $t = t^* \in (a, b)$, characterized by the identity

$$(\varsigma(t^*))^\nu = \frac{(\iota-1)(\varsigma(a))^\nu + (\varsigma(b))^\nu}{\iota}.$$

It is evident that $\ell_1(t) \geq 0$ for all $t \in [a, b]$, and that $\ell_1(a) = \ell_1(b) = 0$. Moreover, since its derivative $\ell_1'(t)$ admits a unique zero within the open interval (a, b) , the unimodal structure of $\ell_1(t)$, together with its boundary behavior, ensures that the global maximum is attained precisely at the point $t = t^* \in (a, b)$. Consequently,

$$\max_{t \in [a, b]} \ell_1(t) = \ell_1(t^*) = \frac{(\iota-1)^{\iota-1}}{\iota^\iota} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1}. \quad (4.4)$$

Analogously, differentiating $\ell_2(t)$ on (a, b) , yields

$$\ell_2'(t) = \nu \varsigma'(t)(\varsigma(t))^{\nu-1} \{(\iota-1)[(\varsigma(t))^\nu - (\varsigma(a))^\nu]^{\iota-2} - [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-2}\}, \quad t \in (a, b).$$

A direct computation reveals that $\ell_2'(t)$ admits a unique zero occurs at the point $t = t^{**}$, which is implicitly defined by the relation

$$(\varsigma(t^{**}))^\nu = (\varsigma(a))^\nu + (\iota-1)^{\frac{1}{2-\iota}} [(\varsigma(b))^\nu - (\varsigma(a))^\nu].$$

In a similar manner, one has

$$\begin{aligned}\max_{t \in [a, b]} \ell_2(t) &= \ell_2(t^{**}) \\ &= [(\iota-1)^{\frac{\iota-1}{2-\iota}} - (\iota-1)^{\frac{1}{2-\iota}}] [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1} \\ &= (2-\iota)(\iota-1)^{\frac{\iota-1}{2-\iota}} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1}.\end{aligned} \quad (4.5)$$

Using (4.4) and Lemma 2.8, we deduce from (4.5) that, for $\iota \in (1, 2]$,

$$\begin{aligned}\max_{t \in [a, b]} \ell_2(t) &= (2-\iota)(\iota-1)^{\frac{\iota-1}{2-\iota}} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1} \\ &\leq \frac{(\iota-1)^{\iota-1}}{\iota^\iota} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1} \\ &= \max_{t \in [a, b]} \ell_1(t),\end{aligned}$$

which combined with (4.3), gives

$$|g_1(t, s)| \leq \frac{(\iota - 1)^{\iota-1}}{\iota^\iota} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1}, \quad (t, s) \in [a, b] \times [a, b].$$

Case 2. For $a \leq t \leq s \leq b$. Define

$$g_2(t, s) := \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1}.$$

Clearly, this function is nonnegative and monotonic in t , and its maximum on $[a, b]$ for fixed s occurs at $t = s$. Thus,

$$\begin{aligned} g_2(t, s) &\leq g_2(s, s) \\ &= \frac{(\varsigma(s))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(s))^\nu]^{\iota-1}, \quad s \in [a, b], \\ &= \frac{(\varsigma(t))^\nu - (\varsigma(a))^\nu}{(\varsigma(b))^\nu - (\varsigma(a))^\nu} [(\varsigma(b))^\nu - (\varsigma(t))^\nu]^{\iota-1} \\ &= g_2(t, t), \quad t \in [a, b]. \end{aligned}$$

Obviously, we have $g_1(t, t) = g_2(t, t)$. It follows that

$$\max_{t \in [a, b]} g_2(t, t) = \frac{(\iota - 1)^{\iota-1}}{\iota^\iota} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{\iota-1}.$$

Consequently, recalling Case 1 and Case 2, we conclude inequality (4.2) holds, and Lemma 4.2 is proved. $\square$

4.2. Lyapunov-type inequality for problem (1.6)

Working in the Banach space $\mathcal{X}$ this subsection invokes Lemmas 4.1 and 4.2 to prove a LTI. This result provides a necessary condition for BVP (1.6) to admit nontrivial continuous solutions.

Theorem 4.1. *Let $\iota \in (1, 2]$ and $q \in C([a, b], \mathbb{R})$. If the fractional BVP (1.6) possesses a nontrivial continuous solution $x \not\equiv 0$, then the inequality stated below is satisfied*

$$\int_a^b (\varsigma(s))^{\nu-1} \varsigma'(s) |q(s)| ds \geq \frac{\iota^\iota \Gamma(\iota)}{(\iota - 1)^{\iota-1} \nu^{1-\iota}} [(\varsigma(b))^\nu - (\varsigma(a))^\nu]^{1-\iota}. \quad (4.6)$$

Proof. In view of Lemma 4.1, every solution $x(t) \in C[a, b]$ to the BVP (1.6) fulfills the following IE

$$x(t) = \int_a^b \mathcal{T}(s) \mathcal{G}(t, s) q(s) x(s) ds. \quad (4.7)$$

By taking norms on both sides of the IE (4.7) yields

$$\begin{aligned} \|x\|_\infty &\leq \max_{t \in [a, b]} \left| \int_a^b \mathcal{T}(s) \mathcal{G}(t, s) q(s) x(s) ds \right| \\ &\leq \|x\|_\infty \max_{t \in [a, b]} \int_a^b |\mathcal{T}(s) \mathcal{G}(t, s) q(s)| ds. \end{aligned}$$

Given that $\mathfrak{x} \not\equiv 0$, division by $\|\mathfrak{x}\|_\infty$ is admissible, and thus

$$1 \leq \max_{\mathfrak{t} \in [\mathfrak{a}, \mathfrak{b}]} \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{T}(\mathfrak{s}) \mathcal{G}(\mathfrak{t}, \mathfrak{s}) \mathfrak{q}(\mathfrak{s})| d\mathfrak{s}.$$

In addition, Lemma 4.2 ensures that

$$1 \leq \frac{(\iota - 1)^{\iota-1} \nu^{1-\iota}}{\iota \Gamma(\iota)} [(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu]^{\iota-1} \int_{\mathfrak{a}}^{\mathfrak{b}} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s},$$

from which it follows that the desired inequality holds, and the assertion of Theorem 4.1 is proved. $\square$

Corollary 4.1. *Consider the FDE below, subject to Dirichlet BCs*

$$\begin{cases} {}^C \mathcal{D}_{\mathfrak{a}+}^{\iota, \psi} \mathfrak{x}(\mathfrak{t}) + \mathfrak{q}(\mathfrak{t}) \mathfrak{x}(\mathfrak{t}) = 0, & \mathfrak{t} \in (\mathfrak{a}, \mathfrak{b}), \\ \mathfrak{x}(\mathfrak{a}) = \mathfrak{x}(\mathfrak{b}) = 0, \end{cases} \quad (4.8)$$

where $\mathfrak{q}(\mathfrak{t}) \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$, ${}^C \mathcal{D}_{\mathfrak{a}+}^{\iota, \psi}$ denotes the ψ -Caputo FD of order ι [3]. If equation (4.8) admits a nontrivial continuous solution, then

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \frac{\iota \Gamma(\iota)}{(\iota - 1)^{\iota-1}} [\psi(\mathfrak{b}) - \psi(\mathfrak{a})]^{1-\iota}. \quad (4.9)$$

Proof. Applying Theorem 4.1 with $\varsigma(\mathfrak{t}) = [\nu \psi(\mathfrak{t})]^{1/\nu}$, $\mathfrak{t} \in [\mathfrak{a}, \mathfrak{b}]$, inequality (4.9) follows directly. $\square$

Corollary 4.2. *Consider the Dirichlet problem for the FDE given below*

$$\begin{cases} {}^C \mathcal{D}_{\mathfrak{a}+}^{\iota} \mathfrak{x}(\mathfrak{t}) + \mathfrak{q}(\mathfrak{t}) \mathfrak{x}(\mathfrak{t}) = 0, & \mathfrak{t} \in (\mathfrak{a}, \mathfrak{b}), \\ \mathfrak{x}(\mathfrak{a}) = \mathfrak{x}(\mathfrak{b}) = 0, \end{cases} \quad (4.10)$$

where $\mathfrak{q}(\mathfrak{t}) \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$, ${}^C \mathcal{D}_{\mathfrak{a}+}^{\iota}$ denotes the Caputo-Katugampola FD of order ι [4]. If equation (4.10) allows for a nontrivial continuous solution, then

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \geq \frac{\iota \Gamma(\iota)}{(\iota - 1)^{\iota-1} \max\{\mathfrak{a}^{\nu-1}, \mathfrak{b}^{\nu-1}\}} \left(\frac{\mathfrak{b}^\nu - \mathfrak{a}^\nu}{\nu} \right)^{1-\iota}. \quad (4.11)$$

Proof. By Theorem 4.1 with $\varsigma(\mathfrak{t}) \equiv \mathfrak{t}$, $\mathfrak{t} \in [\mathfrak{a}, \mathfrak{b}]$, inequality (4.11) is obtained immediately. $\square$

Remark 4.1. If the auxiliary function is taken as $\varsigma(\mathfrak{t}) = (\nu \mathfrak{t})^{1/\nu}$, $\mathfrak{t} \in [\mathfrak{a}, \mathfrak{b}]$, then the LTI (4.6) presented in Theorem 4.1 coincides with the result established in ([12], Theorem 1), corresponding to the Caputo fractional calculus framework.

5. Applications to eigenvalue problems

Example 5.1. The fractional BVP below is considered:

$$\begin{cases} {}^\nu \mathcal{D}_{1+}^{3/2, \varsigma} \mathfrak{x}(\mathfrak{t}) + \lambda \mathfrak{x}(\mathfrak{t}) = 0, & \mathfrak{t} \in (1, 2), \\ \mathfrak{x}(1) = \mathfrak{x}(2) = 0. \end{cases} \quad (5.1)$$

Relative to BVP (1.5), BVP (5.1) is characterized by the parameter choice

$$\mathfrak{a} = 1, \mathfrak{b} = 2, \iota = 3/2, \mathfrak{q}(\mathfrak{t}) \equiv \lambda, \mathfrak{t} \in [1, 2].$$

Under this specification, an application of Theorem 3.1 ensures the existence of a nontrivial solution $\mathfrak{x} \not\equiv 0$ to BVP (5.1), that is, if λ is an eigenvalue of (5.1), then the following estimate is valid:

$$\begin{aligned} \frac{(\varsigma(2))^\nu - (\varsigma(1))^\nu}{\nu} |\lambda| &= \int_1^2 (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\lambda| d\mathfrak{s} \\ &= \int_{\mathfrak{a}}^{\mathfrak{b}} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \\ &\geq \Gamma(\iota) \left(\frac{(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu}{4\nu} \right)^{1-\iota} \\ &= \Gamma(3/2) \left(\frac{(\varsigma(2))^\nu - (\varsigma(1))^\nu}{4\nu} \right)^{-1/2}. \end{aligned}$$

It follows that the eigenvalue parameter must satisfies

$$|\lambda| \geq \sqrt{\pi} \left(\frac{\nu}{(\varsigma(2))^\nu - (\varsigma(1))^\nu} \right)^{3/2}.$$

Example 5.2. The fractional BVP below is considered:

$$\begin{cases} {}^C_{\nu} \mathcal{D}_{1+}^{3/2, \varsigma} \mathfrak{x}(\mathfrak{t}) + \mu \mathfrak{x}(\mathfrak{t}) = 0, & \mathfrak{t} \in (1, 2), \\ \mathfrak{x}(1) = \mathfrak{x}(2) = 0. \end{cases} \quad (5.2)$$

Relative to BVP (1.6), BVP (5.2) is specified by the parameter choice

$$\mathfrak{a} = 1, \mathfrak{b} = 2, \iota = 3/2, \mathfrak{q}(\mathfrak{t}) \equiv \mu, \mathfrak{t} \in [1, 2].$$

Under this specification, Theorem 4.1 ensures that BVP (5.2) admits a nontrivial continuous solution $\mathfrak{x} \not\equiv 0$, equivalently, μ is an eigenvalue of (5.2), then one necessarily has

$$\begin{aligned} \frac{(\varsigma(2))^\nu - (\varsigma(1))^\nu}{\nu} |\mu| &= \int_1^2 (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mu| d\mathfrak{s} \\ &= \int_{\mathfrak{a}}^{\mathfrak{b}} (\varsigma(\mathfrak{s}))^{\nu-1} \varsigma'(\mathfrak{s}) |\mathfrak{q}(\mathfrak{s})| d\mathfrak{s} \\ &\geq \frac{\iota^\iota \Gamma(\iota)}{(\iota-1)^{\iota-1} \nu^{1-\iota}} [(\varsigma(\mathfrak{b}))^\nu - (\varsigma(\mathfrak{a}))^\nu]^{1-\iota} \\ &= \frac{(3/2)^{3/2} \Gamma(3/2)}{(1/2)^{1/2} \nu^{-1/2}} [(\varsigma(2))^\nu - (\varsigma(1))^\nu]^{-1/2}, \end{aligned}$$

that is, the eigenvalue parameter μ must satisfy

$$|\mu| \geq \frac{3\sqrt{3}\pi}{4} \left(\frac{\nu}{(\varsigma(2))^\nu - (\varsigma(1))^\nu} \right)^{3/2}.$$

6. Conclusions

The present study has addressed two classes of Dirichlet BVPs, namely (1.5) and (1.6), within the generalized fractional calculus framework recently introduced by Nieto et al. [22], which is formulated in terms of the ς -Katugampola and ς -Caputo-Katugampola FDs. The application of rigorous analytical techniques has led to LTIs that ensure the presence of non-trivial continuous solutions for each of the two problems. These results have provided a theoretical foundation for further investigations into nonlinear BVPs involving the aforementioned generalized fractional derivatives.

Based on the findings presented here, subsequent investigations will concentrate on nonlocal BVPs described by FDEs that involve generalized FDs. Meanwhile, a key challenge in the discipline continues to be the reconciliation of distinct FD definitions. Inspired by the function-based operator construction proposed by Nieto et al., we plan to integrate the structural features of the Hilfer-Katugampola derivative in order to unify the definitions of the ς -Katugampola and ς -Caputo-Katugampola derivatives. The resulting operator, to be referred to as the ς -Hilfer-Katugampola FD, will be systematically examined with respect to its fundamental properties and potential applications.

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