

ERROR BOUNDS FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS VIA STRONGLY h -CONVEX FUNCTIONS WITH HIGHER ORDER DERIVATIVES

Man Huang¹ and Jianmiao Ruan^{1,†}

Abstract This paper is devoted to establishing new general identities associated with Riemann–Liouville fractional integrals for n -times differentiable functions. By employing these identities, we derive a series of midpoint- and trapezoid-type error bounds for approximations involving such fractional operators. A key feature of our analysis is the assumption that the q -th power of the n -th derivative of the function under consideration is strongly h -convex. This approach yields estimates that depend not only on the behavior of the function at the endpoints of the interval but also on an interior point, thereby offering enhanced generality. Our results unify and extend several inequalities previously established in the literature for convex, s -convex, and h -convex functions in the context of fractional integrals.

Keywords Strongly h -convex functions, Riemann–Liouville fractional integrals, Hermite–Hadamard type inequalities, error estimations.

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1. Introduction

Let $h : [0, 1] \rightarrow [0, \infty)$ be a given function, and $c > 0$ a constant. A real function f defined on the interval $I \subset \mathbb{R}$ is said to be *strongly h -convex with modulus c* , denoted by $f \in SX(h, c, I)$, if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) - ct(1-t)(x-y)^2 \quad (1.1)$$

holds for all $x, y \in I$ and $t \in (0, 1)$. This class of functions, introduced by Angulo et al. [1], generalizes both strongly convex functions [25] and h -convex functions [34]. Special cases include: Strongly convex functions when $h(t) = t$; strongly s -convex functions (in the second sense) when $h(t) = t^s$ ($s \in (0, 1]$); strongly s -convex functions (in the fourth sense) when $h(t) = t^{1/s}$ ($s \in (0, 1]$); strongly Godunova–Levin functions when $h(t) = 1/t$; and strongly P -functions when $h(t) = 1$. In the limit $c \rightarrow 0$, the strongly h -convex function reduces to the standard h -convex function [34], which itself generalizes convex, s -convex (in the second and fourth sense), Godunova–Levin, and P -functions. These classes play significant roles in mathematical analysis and applications; see, e.g., [2–4, 7, 10, 12–17, 19, 20, 23, 24, 29–31, 37, 38].

A fundamental result in the theory of convex functions is the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad (1.2)$$

[†]The corresponding author.

¹Department of Mathematics, Zhejiang International Studies University, Hangzhou 310014, China
Email: 2436855207@qq.com(M. Huang), rjmath@163.com, rjmath@zisu.edu.cn(J. Ruan)

which has been extended to numerous generalized convexity classes. In this direction, analogues have been established for Godunova–Levin and P -functions by Dragomir et al. [11], for s -convex functions by Dragomir and Fitzpatrick [9], for h -convex functions by Sarikaya et al. [26], and for strongly h -convex functions by Angulo et al. [1].

The study of error estimates associated with the Hermite–Hadamard inequality has likewise received considerable attention. In this context, Dragomir and Agarwal [8] established an identity pertaining to the right-hand side of (1.2) for differentiable functions, which yields trapezoid-type estimates under the assumption that $|f'|$ is convex. Subsequently, Pearce and Pečarić [22] refined these results as follows:

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and suppose that $|f'|^q$ is convex on $[a, b]$ for some $1 \leq q < \infty$. Then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}, \quad (1.3)$$

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}. \quad (1.4)$$

Additional results along these lines may be found in, e.g., [36, 40].

Let f be Lebesgue integrable on $[a, b]$. The left- and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x \in (a, b], \quad (1.5)$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x \in [a, b), \quad (1.6)$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ is the Gamma function.

A significant result concerning the Riemann-Liouville fractional integrals was obtained by Sarikaya et al. as follows.

Theorem 1.2 ([27]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{f(a) + f(b)}{2}. \quad (1.7)$$

In addition, the work in the same paper provided the subsequent identity:

Lemma 1.1 ([27]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and f' be integrable on $[a, b]$. Then*

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] = \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta + (1-t)b) dt.$$

Consequently, error estimates for convex functions were derived. These findings were later generalized to the settings of h -convex functions [32], m -convex functions [35], and n -fold differentiable convex functions [33].

Lemma 1.2 ([33]). *Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Define*

$$\mathfrak{R}_n(f) = \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \quad (1.8)$$

$$+ \sum_{k=1}^{n-1} \frac{(b-a)^k}{\prod_{i=1}^k (\alpha+i)} \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2},$$

with the convention that an empty sum takes the value zero and an empty product takes the value one. Then

$$\mathfrak{R}_n(f) = \frac{(b-a)^n}{2 \prod_{i=1}^{n-1} (\alpha+i)} \int_0^1 [(-1)^{n-1} (1-t)^{\alpha+n-1} - t^{\alpha+n-1}] f^{(n)}(ta + (1-t)b) dt.$$

Here $L([a, b])$ denotes the class of (Lebesgue) integrable functions on $[a, b]$, and $C^n([a, b])$ represents the space of n -fold differentiable functions on $[a, b]$.

As an application, the authors yielded the following estimates for $\mathfrak{R}_n(f)$.

Theorem 1.3 ([33]). Let $f \in C^n([a, b])$, $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$ and suppose that $|f^{(n)}|^q$ is convex for some $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then

$$|\mathfrak{R}_n(f)| \leq \frac{(b-a)^n}{\prod_{i=1}^n (\alpha+i)} \left(1 - \frac{1}{2^{\alpha+n-1}}\right) \left(\frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2}\right)^{\frac{1}{q}}$$

when n is odd and

$$|\mathfrak{R}_n(f)| \leq \frac{(b-a)^n}{\prod_{i=1}^n (\alpha+i)} \left(\frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2}\right)^{\frac{1}{q}}$$

when n is even.

(ii) If $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} |\mathfrak{R}_n(f)| &\leq \frac{(b-a)^n}{2 \prod_{i=1}^{n-1} (\alpha+i)} \left[\frac{1}{p(\alpha+n-1)+1} \left(1 - \frac{1}{2^{p(\alpha+n-1)}}\right) \right]^{1/p} \\ &\quad \times \left[\left(\frac{|f^{(n)}(a)|^q + 3|f^{(n)}(b)|^q}{8}\right)^{1/q} + \left(\frac{3|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{8}\right)^{1/q} \right] \end{aligned}$$

when n is odd and

$$|\mathfrak{R}_n(f)| \leq \frac{(b-a)^n}{2 \prod_{i=1}^{n-1} (\alpha+i)} \left(\frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2}\right)^{1/q}$$

when n is even.

Remark 1.1. In the special case where $n = 1$, Lemma 1.2 simplifies to Lemma 1.1, and Theorem 1.3 (i) collapses to Theorem 3 in [27] with $q = 1$.

On the other hand, another Hermite–Hadamard-type inequality via Riemann–Liouville fractional integrals was given by the following form:

Theorem 1.4 ([28]). Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable convex function. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}. \quad (1.9)$$

A key equality was also formulated by the authors, namely:

Lemma 1.3 ([28]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable with $f' \in L([a, b])$. Then*

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{4} \int_0^1 (1-t)^\alpha \left[f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) - f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right] dt. \end{aligned}$$

This gave rise to error estimates similar to (1.4). The scope of these results was later broadened to encompass twice differentiable s -convex functions [21] as well as n -times differentiable strongly h -convex functions [39].

Lemma 1.4 ([39]). *Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Denote*

$$\begin{aligned} \mathfrak{L}_n\left(f, \frac{a+b}{2}\right) &= \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] - f\left(\frac{a+b}{2}\right) \\ &\quad - \sum_{j=1}^{n-1} \frac{[1+(-1)^j](b-a)^j}{2^{j+1} \prod_{k=1}^j (\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right). \end{aligned} \quad (1.10)$$

Then

$$\begin{aligned} \mathfrak{L}_n\left(f, \frac{a+b}{2}\right) &= \frac{(b-a)^n}{2^{n+1} \prod_{k=1}^{n-1} (\alpha+k)} \left[(-1)^n \int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \right. \\ &\quad \left. + \int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \right]. \end{aligned} \quad (1.11)$$

Remark 1.2. For $n = 1$, Lemma 1.4 simplifies to Lemma 1.3, while the case $n = 2$ recovers Lemma 1 in [21].

Remark 1.3. When $n = 1$ or 2,

$$\mathfrak{L}_n\left(f, \frac{a+b}{2}\right) = \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] - f\left(\frac{a+b}{2}\right). \quad (1.12)$$

From the lemma, the following inequalities pertaining to strongly h -convex functions were obtained.

Theorem 1.5 ([39]). *Let h be a nonnegative integrable function on $[0, 1]$, $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$ for some $1 \leq q < \infty$.*

(i) *If $1 \leq q < \infty$, then*

$$\begin{aligned} \left| \mathfrak{L}_n\left(f, \frac{a+b}{2}\right) \right| &\leq \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^n (\alpha+k)} \\ &\quad \times \left\{ \left[\mathcal{I}_2 |f^{(n)}(a)|^q + \mathcal{I}_1 |f^{(n)}(b)|^q - \frac{c(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right. \\ &\quad \left. + \left[\mathcal{I}_1 |f^{(n)}(a)|^q + \mathcal{I}_2 |f^{(n)}(b)|^q - \frac{c(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right\}, \end{aligned}$$

where

$$\mathcal{I}_1 = \int_0^{1/2} t^{\alpha+n-1} h(t) dt, \quad \mathcal{I}_2 = \int_0^{1/2} t^{\alpha+n-1} h(1-t) dt.$$

(ii) If $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} \left| \mathfrak{L}_n \left(f, \frac{a+b}{2} \right) \right| &\leq \frac{(b-a)^n}{2^{n+1-1/q} \prod_{k=1}^{n-1} (\alpha+k)} \left(\frac{1}{p(\alpha+n-1)+1} \right)^{1/p} \\ &\times \left[\left(\int_{1/2}^1 h(t) dt \left| f^{(n)}(a) \right|^q + \int_0^{1/2} h(t) dt \left| f^{(n)}(b) \right|^q - \frac{c(b-a)^2}{12} \right)^{1/q} \right. \\ &\left. + \left(\int_0^{1/2} h(t) dt \left| f^{(n)}(a) \right|^q + \int_{1/2}^1 h(t) dt \left| f^{(n)}(b) \right|^q - \frac{c(b-a)^2}{12} \right)^{1/q} \right]. \end{aligned}$$

Remark 1.4. By setting $c \rightarrow 0$ in Theorem 1.5, one obtains Theorem 5–6 in [28] when $n = 1$ and $h(t) = t$, together with Theorem 2–4 in [21] when $n = 2$ and $h(t) = t^s$.

Recently, more general identities and error estimates were introduced in [5, 6], where the variable values may be chosen freely and are not restricted solely to the endpoint or midpoint of $[a, b]$.

Lemma 1.5 ([5]). Let $f \in C^1([a, b])$ and $f' \in L([a, b])$. Then $\forall x \in [a, b]$,

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{b-a} \left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x) \right] - f(a+b-x) \\ &= \frac{(x-a)^2}{b-a} \int_0^1 (1-t^{\alpha}) f'(tb + (1-t)(a+b-x)) dt \\ &\quad - \frac{(b-x)^2}{b-a} \int_0^1 (1-t^{\alpha}) f'(ta + (1-t)(a+b-x)) dt, \end{aligned} \quad (1.13)$$

Lemma 1.6 ([6]). Let $f \in C^1([a, b])$ and $f' \in L([a, b])$. Then $\forall x \in [a, b]$,

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{b-a} \left[(b-x)^{1-\alpha} J_{(a+b-x)-}^{\alpha} f(a) + (x-a)^{1-\alpha} J_{(a+b-x)+}^{\alpha} f(b) \right] - f(a+b-x) \\ &= \frac{(x-a)^2}{b-a} \int_0^1 t^{\alpha} f'(t(a+b-x) + (1-t)b) dt \\ &\quad - \frac{(b-x)^2}{b-a} \int_0^1 t^{\alpha} f'(t(a+b-x) + (1-t)a) dt. \end{aligned} \quad (1.14)$$

Remark 1.5. Lemma 1.6 reduces to Lemma 1.3 with $x = (a+b)/2$.

By employing the preceding two lemmas, new error estimates for differentiable convex functions were obtained in [5, 6]. Subsequently, in 2021, these results were extended to differentiable h -convex functions by Mahreen and Budak [18].

Theorem 1.6 ([18]). Let $f \in C^1([a, b])$ and $f' \in L([a, b])$. If $|f'|^q$ is an h -convex function on $[a, b]$, then $\forall x \in [a, b]$, the following fractional integral inequalities hold:

(i) If $1 \leq q < \infty$, then

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{b-a} \left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x) \right] - f(a+b-x) \right| \\ & \leq \frac{1}{b-a} \left(\frac{\alpha}{\alpha+1} \right)^{1-1/q} \left[\frac{(x-a)^2}{b-a} (\mathcal{I}_3 |f'(b)|^q + \mathcal{I}_4 |f'(a+b-x)|^q)^{1/q} \right. \\ & \quad \left. + \frac{(b-x)^2}{b-a} (\mathcal{I}_3 |f'(a)|^q + \mathcal{I}_4 |f'(a+b-x)|^q)^{1/q} \right], \end{aligned}$$

where

$$\mathcal{I}_3 = \int_0^1 (1-t^{\alpha})h(t)dt, \quad \mathcal{I}_4 = \int_0^1 (1-t^{\alpha})h(1-t)dt.$$

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{b-a} \left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x) \right] - f(a+b-x) \right| \\ & \leq \frac{1}{b-a} \left(\frac{\alpha p}{\alpha p + 1} \right)^{1/p} \left[(x-a)^2 (|f'(b)|^q + |f'(a+b-x)|^q)^{1/q} \right. \\ & \quad \left. + (b-x)^2 (|f'(a)|^q + |f'(a+b-x)|^q)^{1/q} \right] \left(\int_0^1 h(t)dt \right)^{1/q}. \end{aligned}$$

Remark 1.6. If we take $h(t) = t$ in the above theorem, then Theorem 1.6 (i) reduces to Theorem 4 and Theorem 1.6 (ii) reduces to Theorem 3 in [5].

Theorem 1.7 ([18]). Let $f \in C^1([a, b])$ and $f' \in L([a, b])$. If $|f'|^q$ is an h -convex function on $[a, b]$, then $\forall x \in [a, b]$, the following fractional integral inequalities hold:

(i) If $1 \leq q < \infty$, then

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{b-a} \left[(b-x)^{1-\alpha} J_{(a+b-x)-}^{\alpha} f(a) + (x-a)^{1-\alpha} J_{(a+b-x)+}^{\alpha} f(b) \right] - f(a+b-x) \right| \\ & \leq \frac{1}{b-a} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \left[(x-a)^2 (\mathcal{I}_5 |f'(a+b-x)|^q + \mathcal{I}_6 |f'(b)|^q)^{1/q} \right. \\ & \quad \left. + (b-x)^2 (\mathcal{I}_5 |f'(a+b-x)|^q + \mathcal{I}_6 |f'(a)|^q)^{1/q} \right], \end{aligned}$$

where

$$\mathcal{I}_5 = \int_0^1 t^{\alpha} h(t)dt, \quad \mathcal{I}_6 = \int_0^1 t^{\alpha} h(1-t)dt.$$

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{b-a} \left[(b-x)^{1-\alpha} J_{(a+b-x)-}^{\alpha} f(a) + (x-a)^{1-\alpha} J_{(a+b-x)+}^{\alpha} f(b) \right] - f(a+b-x) \right| \\ & \leq \frac{1}{b-a} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \left[(x-a)^2 (|f'(a+b-x)|^q + |f'(b)|^q)^{1/q} \right. \\ & \quad \left. + (b-x)^2 (|f'(a+b-x)|^q + |f'(a)|^q)^{1/q} \right] \left(\int_0^1 h(t)dt \right)^{1/q}. \end{aligned}$$

Remark 1.7. By taking $h(t) = t$ in Theorem 1.7, part (i) is reduced to Theorem 2.7 and part (ii) to Theorem 2.5 in [5].

Examining the theorems above, it is evident that Theorem 1.3 and Theorem 1.5 provide trapezoid- and midpoint-type estimates that depend only on $|f^{(n)}(a)|$ and $|f^{(n)}(b)|$, whereas Theorem 1.6 and Theorem 1.7 show that the generalized midpoint-type estimates depend on the values of $|f^{(n)}|$ at both endpoints and at an additional point within (a, b) . On the basis of these motivations, the current work extends these results to n -times differentiable strongly h -convex functions and derives more general error estimates pertaining to these inequalities.

2. New identities for Riemann-Liouville fractional integrals

In this section, we derive several novel identities concerning Riemann-Liouville fractional integrals, which will form the cornerstone of the error estimates presented later.

Let f be an n -times differentiable function on $[a, b]$, $n \in \mathbb{Z}^+$. We define the following auxiliary functions:

$$\begin{aligned} \phi_n(f, x) = & \frac{\Gamma(\alpha + 1)}{b - a} \left[(x - a)^{1-\alpha} J_{b-}^{\alpha} f(a + b - x) + (b - x)^{1-\alpha} J_{a+}^{\alpha} f(a + b - x) \right] - f(a + b - x) \\ & + \frac{1}{b - a} \sum_{i=1}^{n-1} \left(\frac{1}{i!} - \frac{1}{\prod_{k=1}^i (\alpha + k)} \right) \left[(b - x)^{i+1} f^{(i)}(a) - (a - x)^{i+1} f^{(i)}(b) \right], \end{aligned} \quad (2.1)$$

$$\begin{aligned} \varphi_n(f, x) = & \frac{\Gamma(\alpha + 1)}{b - a} \left[(b - x)^{1-\alpha} J_{(a+b-x)-}^{\alpha} f(a) + (x - a)^{1-\alpha} J_{(a+b-x)+}^{\alpha} f(b) \right] - f(a + b - x) \\ & + \frac{1}{b - a} \sum_{i=1}^{n-1} \frac{(x - b)^{i+1} - (x - a)^{i+1}}{\prod_{k=1}^i (\alpha + k)} f^{(i)}(a + b - x). \end{aligned} \quad (2.2)$$

For $n = 1$, the sums vanish, yielding the simpler forms:

$$\begin{aligned} \phi_1(f, x) &= \frac{\Gamma(\alpha + 1)}{b - a} \left[(x - a)^{1-\alpha} J_{b-}^{\alpha} f(a + b - x) + (b - x)^{1-\alpha} J_{a+}^{\alpha} f(a + b - x) \right] - f(a + b - x), \\ \varphi_1(f, x) &= \frac{\Gamma(\alpha + 1)}{b - a} \left[(b - x)^{1-\alpha} J_{(a+b-x)-}^{\alpha} f(a) + (x - a)^{1-\alpha} J_{(a+b-x)+}^{\alpha} f(b) \right] - f(a + b - x), \end{aligned}$$

which mean that

$$\phi_1(f, a) + \phi_1(f, b) = \varphi_1(f, a) + \varphi_1(f, b).$$

Remark 2.1. In view of (1.8), (1.10) and (2.2), one may verify that

$$\varphi_n(f, \frac{a+b}{2}) = \mathfrak{L}_n(f, \frac{a+b}{2}), \quad \varphi_n(f, a) + \varphi_n(f, b) = -2\mathfrak{R}_n(f). \quad (2.3)$$

We now present the main identities of this section.

Lemma 2.1. Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Then $\forall x \in [a, b]$, we have

$$\phi_n(f, x) = \frac{1}{b - a} \left[(a - x)^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1} (\alpha + k)} \right) f^{(n)}(tb + (1-t)(a + b - x)) dt \right]$$

$$-(b-x)^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) f^{(n)}(ta + (1-t)(a+b-x)) dt \Bigg]. \quad (2.4)$$

Proof. The case $n = 1$ corresponds to Lemma (2.4). For $n \geq 2$, we proceed by iterative integration by parts.

$$\begin{aligned} & \int_0^1 (1-t^\alpha) f'(tb + (1-t)(a+b-x)) dt \\ &= \left(1 - \frac{1}{\alpha+1}\right) f'(b) + (a-x) \int_0^1 \left(t - \frac{t^{\alpha+1}}{\alpha+1}\right) f''(tb + (1-t)(a+b-x)) dt \\ &= \left(1 - \frac{1}{\alpha+1}\right) f'(b) + (a-x) \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right) f''(b) \\ &\quad + (a-x)^2 \int_0^1 \left(\frac{t^2}{2} - \frac{t^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right) f'''(tb + (1-t)(a+b-x)) dt \\ &= \dots \\ &= \left(1 - \frac{1}{\alpha+1}\right) f'(b) + (a-x) \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right) f''(b) \\ &\quad + \dots + (a-x)^{n-2} \left(\frac{1}{(n-1)!} - \frac{1}{\prod_{k=1}^{n-1}(\alpha+k)}\right) f^{(n-1)}(b) \\ &\quad + (a-x)^{n-1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)}\right) f^{(n)}(tb + (1-t)(a+b-x)) dt, \end{aligned}$$

which means that

$$\begin{aligned} & \int_0^1 (1-t^\alpha) f'(tb + (1-t)(a+b-x)) dt \\ &= \sum_{i=1}^{n-1} (a-x)^{i-1} \left(\frac{1}{i!} - \frac{1}{\prod_{k=1}^i(\alpha+k)}\right) f^{(i)}(b) \\ &\quad + (a-x)^{n-1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)}\right) f^{(n)}(tb + (1-t)(a+b-x)) dt. \end{aligned} \quad (2.5)$$

Similarly,

$$\begin{aligned} & \int_0^1 (1-t^\alpha) f'(ta + (1-t)(a+b-x)) dt \\ &= \sum_{i=1}^{n-1} (b-x)^{i-1} \left(\frac{1}{i!} - \frac{1}{\prod_{k=1}^i(\alpha+k)}\right) f^{(i)}(a) \\ &\quad + (b-x)^{n-1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)}\right) f^{(n)}(ta + (1-t)(a+b-x)) dt. \end{aligned} \quad (2.6)$$

Then we complete the proof by (1.13), (2.1), (2.5) and (2.6). $\square$

Lemma 2.2. Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Then $\forall x \in [a, b]$, the following identity holds:

$$\varphi_n(f, x) = \frac{1}{(b-a) \prod_{k=1}^{n-1}(\alpha+k)} \left[(x-a)^{n+1} \int_0^1 t^{\alpha+n-1} f^{(n)}(t(a+b-x) + (1-t)b) dt \right.$$

$$-(x-b)^{n+1} \int_0^1 t^{\alpha+n-1} f^{(n)}(t(a+b-x) + (1-t)a) dt \Big]. \quad (2.7)$$

Proof. Obviously, (2.7) reduces to (1.14) as $n = 1$. For $n \geq 2$, it follows from integration by parts that

$$\begin{aligned} & \int_0^1 t^\alpha f'(t(a+b-x) + (1-t)b) dt \\ &= \frac{1}{\alpha+1} f'(a+b-x) + \frac{x-a}{\alpha+1} \int_0^1 t^{\alpha+1} f''(t(a+b-x) + (1-t)b) dt \\ &= \frac{1}{\alpha+1} f'(a+b-x) + \frac{x-a}{(\alpha+1)(\alpha+2)} f''(a+b-x) \\ & \quad + \frac{(x-a)^2}{(\alpha+1)(\alpha+2)} \int_0^1 t^{\alpha+2} f'''(t(a+b-x) + (1-t)b) dt \\ &= \dots \\ &= \frac{1}{\alpha+1} f'(a+b-x) + \frac{x-a}{(\alpha+1)(\alpha+2)} f''(a+b-x) \\ & \quad + \frac{(x-a)^2}{(\alpha+1)(\alpha+2)(\alpha+3)} f'''(a+b-x) + \dots + \frac{(x-a)^{n-2}}{\prod_{k=1}^{n-1}(\alpha+k)} f^{(n-1)}(a+b-x) \\ & \quad + \frac{(x-a)^{n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \int_0^1 t^{\alpha+n-1} f^{(n)}(t(a+b-x) + (1-t)b) dt, \end{aligned}$$

which tells us that

$$\begin{aligned} & \int_0^1 t^\alpha f'(t(a+b-x) + (1-t)b) dt \\ &= \sum_{i=1}^{n-1} \frac{(x-a)^{i-1}}{\prod_{k=1}^i(\alpha+k)} f^{(i)}(a+b-x) \\ & \quad + \frac{(x-a)^{n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \int_0^1 t^{\alpha+n-1} f^{(n)}(t(a+b-x) + (1-t)b) dt. \end{aligned} \quad (2.8)$$

By the same way, we have

$$\begin{aligned} & \int_0^1 t^\alpha f'(t(a+b-x) + (1-t)a) dt \\ &= \sum_{i=1}^{n-1} \frac{(x-b)^{i-1}}{\prod_{k=1}^i(\alpha+k)} f^{(i)}(a+b-x) \\ & \quad + \frac{(x-b)^{n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \int_0^1 t^{\alpha+n-1} f^{(n)}(t(a+b-x) + (1-t)a) dt. \end{aligned} \quad (2.9)$$

Thus we finish the proof by (1.14), (2.2), (2.8) and (2.9). $\square$

Remark 2.2. (i) Lemma 2.1 and 2.2 reduce to Lemma 1.5 and 1.6, respectively, when $n = 1$.

(ii) In light of Remark 2.1, Lemma 2.2 generalizes Lemma 1.2.

(iii) Setting $x = (a+b)/2$ in Lemma 2.2 and noting the relationships $\frac{1+t}{2}a + \frac{1-t}{2}b = (1-t)\frac{a+b}{2} + ta$, $\frac{1-t}{2}a + \frac{1+t}{2}b = (1-t)\frac{a+b}{2} + tb$, one can show that Lemma 2.2 yields a variant of Lemma 1.4.

To facilitate parameterized analysis, we set $x = \mu a + (1 - \mu)b$ for $0 \leq \mu \leq 1$ in (2.1) and (2.2) and rewrite the functions $\phi_n(f, x)$, $\varphi_n(f, x)$ respectively as

$$\begin{aligned} \tilde{\phi}_n(f, \mu) = & \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} \left[(1 - \mu)^{1-\alpha} J_{b-}^\alpha f((1 - \mu)a + \mu b) + \mu^{1-\alpha} J_{a+}^\alpha f((1 - \mu)a + \mu b) \right] \\ & - f[(1 - \mu)a + \mu b] \\ & - \sum_{i=1}^{n-1} (b - a)^i \left(\frac{1}{i!} - \frac{1}{\prod_{k=1}^i (\alpha + k)} \right) \left[(-1)^{i+1} (1 - \mu)^{i+1} f^{(i)}(b) - \mu^{i+1} f^{(i)}(a) \right], \end{aligned} \quad (2.10)$$

$$\begin{aligned} \tilde{\varphi}_n(f, \mu) = & \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} \left[\mu^{1-\alpha} J_{((1-\mu)a+\mu b)-}^\alpha f(a) + (1 - \mu)^{1-\alpha} J_{((1-\mu)a+\mu b)+}^\alpha f(b) \right] \\ & - f[(1 - \mu)a + \mu b] \\ & - \sum_{i=1}^{n-1} \frac{(b - a)^i \left[(1 - \mu)^{i+1} - (-\mu)^{i+1} \right]}{\prod_{k=1}^i (\alpha + k)} f^{(i)}[(1 - \mu)a + \mu b]. \end{aligned} \quad (2.11)$$

Remark 2.3. From the definitions and Remark 2.1, it follows that

$$\tilde{\phi}_n\left(f, \frac{1}{2}\right) = \phi_n\left(f, \frac{a+b}{2}\right), \tilde{\varphi}_n\left(f, \frac{1}{2}\right) = \varphi_n\left(f, \frac{a+b}{2}\right) = \mathfrak{L}_n\left(f, \frac{a+b}{2}\right), \quad (2.12)$$

$$\tilde{\varphi}_n(f, 0) + \tilde{\varphi}_n(f, 1) = -2\mathfrak{R}_n(f). \quad (2.13)$$

Substituting $x = \mu a + (1 - \mu)b$ into the identities of (2.4) and (2.7) yields the following lemmas.

Lemma 2.3. Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Then $\forall \mu \in [0, 1]$, we have

$$\begin{aligned} \tilde{\phi}_n(f, \mu) = & (b - a)^n \left\{ (-1)^{n+1} (1 - \mu)^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1} (\alpha + k)} \right) \right. \\ & \times f^{(n)}[(1 - t)(1 - \mu)a + (t - \mu t + \mu)b] dt \\ & \left. - \mu^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1} (\alpha + k)} \right) f^{(n)}[(\mu t - \mu + 1)a + (1 - t)\mu b] dt \right\}. \end{aligned} \quad (2.14)$$

Lemma 2.4. Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Then $\forall \mu \in [0, 1]$, we have

$$\begin{aligned} \tilde{\varphi}_n(f, \mu) = & \frac{(b - a)^n}{\prod_{k=1}^{n-1} (\alpha + k)} \left\{ (1 - \mu)^{n+1} \int_0^1 (1 - t)^{\alpha+n-1} f^{(n)}[(1 - t)(1 - \mu)a + (t - \mu t + \mu)b] dt \right. \\ & \left. + (-1)^n \mu^{n+1} \int_0^1 (1 - t)^{\alpha+n-1} f^{(n)}[(\mu t - \mu + 1)a + (1 - t)\mu b] dt \right\}. \end{aligned} \quad (2.15)$$

Remark 2.4. Setting $\mu = 1/2$ in the proceeding lemma, using (1.11), (2.15) and (2.12), one can check that Lemma 2.4 reduces to Lemma 1.4.

3. Error estimates for higher order differentiable strongly h -convex functions

This section is devoted to deriving error bounds for the fractional integral operators introduced earlier, under the assumption that the q -th power of the n -th derivative is strongly h -convex.

Let $n \in \mathbb{Z}^+$, $0 < \alpha < \infty$, $1 \leq p, q < \infty$ and $\mu \in [0, 1]$. For brevity, we introduce the following notation:

$$\mathcal{A} = \frac{1}{n!} - \frac{1}{\prod_{k=1}^n (\alpha + k)}, \quad \mathcal{B} = \frac{n}{(n+2)!} - \frac{\alpha + n}{\prod_{k=1}^{n+2} (\alpha + k)}, \quad (3.1)$$

$$\mathcal{C} = \frac{1}{(np - p + 1)((n-1)!)^p} - \frac{1}{(\alpha p + np - p + 1) \left(\prod_{k=1}^{n-1} (\alpha + k) \right)^p}, \quad (3.2)$$

$$\mathcal{D} = \left(\frac{1}{\alpha + n} \right)^{1-1/q} \frac{1}{\prod_{k=1}^{n-1} (\alpha + k)}, \quad \mathcal{E} = \frac{1}{(\alpha p + np - p + 1) \left(\prod_{k=1}^{n-1} (\alpha + k) \right)^p}, \quad (3.3)$$

$$\mathcal{J}_1(\mu) = \int_0^\mu t^{\alpha+n-1} h(t) dt, \quad \mathcal{J}_2(\mu) = \int_0^\mu t^{\alpha+n-1} h(1-t) dt. \quad (3.4)$$

For any $\mu \in [0, 1)$, define

$$\mathcal{K}_1(\mu) = \int_\mu^1 \left[\frac{1}{(n-1)!} \left(\frac{t-\mu}{1-\mu} \right)^{n-1} - \frac{1}{\prod_{k=1}^{n-1} (\alpha + k)} \left(\frac{t-\mu}{1-\mu} \right)^{\alpha+n-1} \right] h(t) dt, \quad (3.5)$$

$$\mathcal{K}_2(\mu) = \int_\mu^1 \left[\frac{1}{(n-1)!} \left(\frac{t-\mu}{1-\mu} \right)^{n-1} - \frac{1}{\prod_{k=1}^{n-1} (\alpha + k)} \left(\frac{t-\mu}{1-\mu} \right)^{\alpha+n-1} \right] h(1-t) dt. \quad (3.6)$$

For any $\mu \in (0, 1]$, define

$$\tilde{\mathcal{K}}_1(\mu) = \int_0^\mu \left[\frac{1}{(n-1)!} \left(1 - \frac{t}{\mu} \right)^{n-1} - \frac{1}{\prod_{k=1}^{n-1} (\alpha + k)} \left(1 - \frac{t}{\mu} \right)^{\alpha+n-1} \right] h(t) dt, \quad (3.7)$$

$$\tilde{\mathcal{K}}_2(\mu) = \int_0^\mu \left[\frac{1}{(n-1)!} \left(1 - \frac{t}{\mu} \right)^{n-1} - \frac{1}{\prod_{k=1}^{n-1} (\alpha + k)} \left(1 - \frac{t}{\mu} \right)^{\alpha+n-1} \right] h(1-t) dt. \quad (3.8)$$

Note that

$$\mathcal{K}_1(0) = \tilde{\mathcal{K}}_2(1), \quad \mathcal{K}_2(0) = \tilde{\mathcal{K}}_1(1).$$

When $h(t) = t$, these expressions simplify to:

$$\mathcal{K}_1(\mu) = (1-\mu) \left(\frac{n+\mu}{(n+1)!} - \frac{\alpha+n+\mu}{\prod_{k=1}^{n+1} (\alpha+k)} \right), \quad \mu \in [0, 1), \quad (3.9)$$

$$\mathcal{K}_2(\mu) = (1-\mu)^2 \left(\frac{1}{(n+1)!} - \frac{1}{\prod_{k=1}^{n+1} (\alpha+k)} \right), \quad \mu \in [0, 1), \quad (3.10)$$

$$\tilde{\mathcal{K}}_1(\mu) = \mu^2 \left(\frac{1}{(n+1)!} - \frac{1}{\prod_{k=1}^{n+1} (\alpha+k)} \right), \quad \mu \in (0, 1], \quad (3.11)$$

$$\tilde{\mathcal{K}}_2(\mu) = \mu \left(\frac{n+1-\mu}{(n+1)!} - \frac{\alpha+n+1-\mu}{\prod_{k=1}^{n+1} (\alpha+k)} \right), \quad \mu \in (0, 1]. \quad (3.12)$$

Now we state our main results.

Theorem 3.1. Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then for any $x \in [a, b]$,

$$|\phi_n(f, x)| \leq \frac{\mathcal{A}^{1-1/q}}{b-a} \left\{ (x-a)^{n+1} \left[\mathcal{K}_1(0) |f^{(n)}(b)|^q + \mathcal{K}_2(0) |f^{(n)}(a+b-x)|^q - c\mathcal{B}(x-a)^2 \right]^{1/q} \right. \\ \left. + (b-x)^{n+1} \left[\mathcal{K}_1(0) |f^{(n)}(a)|^q + \mathcal{K}_2(0) |f^{(n)}(a+b-x)|^q - c\mathcal{B}(b-x)^2 \right]^{1/q} \right\}.$$

(ii) If $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$|\phi_n(f, x)| \leq \frac{\mathcal{C}^{1/p}}{b-a} \left\{ (x-a)^{n+1} \left[\left(|f^{(n)}(b)|^q + |f^{(n)}(a+b-x)|^q \right) \int_0^1 h(t) dt - \frac{c(x-a)^2}{6} \right]^{1/q} \right. \\ \left. + (b-x)^{n+1} \left[\left(|f^{(n)}(a)|^q + |f^{(n)}(a+b-x)|^q \right) \int_0^1 h(t) dt - \frac{c(b-x)^2}{6} \right]^{1/q} \right\}.$$

Proof. By Lemma 2.1, we have

$$|\phi_n(f, x)| \leq \frac{1}{b-a} \left[(x-a)^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) |f^{(n)}(tb + (1-t)(a+b-x))| dt \right. \\ \left. + (b-x)^{n+1} \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) |f^{(n)}(ta + (1-t)(a+b-x))| dt \right]. \quad (3.13)$$

(i) If $q = 1$, then we infer from the fact of $|f^{(n)}| \in SX(h, c, [a, b])$ that

$$\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) |f^{(n)}(tb + (1-t)(a+b-x))| dt \\ \leq \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) \\ \times \left[h(t) |f^{(n)}(b)| + h(1-t) |f^{(n)}((a+b-x))| - ct(1-t)(x-a)^2 \right] dt \\ = \mathcal{K}_1(0) |f^{(n)}(b)| + \mathcal{K}_2(0) |f^{(n)}(a+b-x)| - c\mathcal{B}(x-a)^2. \quad (3.14)$$

By a similar argument, we have

$$\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) |f^{(n)}(ta + (1-t)(a+b-x))| dt \\ \leq \mathcal{K}_1(0) |f^{(n)}(a)| + \mathcal{K}_2(0) |f^{(n)}(a+b-x)| - c\mathcal{B}(b-x)^2. \quad (3.15)$$

Thus we complete the proof by (3.13)-(3.15) for $q = 1$.

If $1 < q < \infty$, then the power-mean inequality and the fact of $|f^{(n)}|^q \in SX(h, c, [a, b])$ imply that

$$\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) |f^{(n)}(tb + (1-t)(a+b-x))| dt$$

$$\begin{aligned}
& \leq \left(\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) dt \right)^{1-1/q} \\
& \quad \times \left(\int_0^1 \left| f^{(n)}(tb + (1-t)(a+b-x)) \right|^q \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) dt \right)^{1/q} \\
& \leq \mathcal{A}^{1-1/q} \left\{ \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) h(t) dt \left| f^{(n)}(b) \right|^q \right. \\
& \quad \left. + \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) h(1-t) dt \left| f^{(n)}(a+b-x) \right|^q \right. \\
& \quad \left. - c(x-a)^2 \left[\frac{n}{(n+2)!} - \frac{\alpha+n}{\prod_{k=1}^{n+2}(\alpha+k)} \right] \right\}^{1/q} \\
& = \mathcal{A}^{1-1/q} \left[\mathcal{K}_1(0) \left| f^{(n)}(b) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}(a+b-x) \right|^q - c\mathcal{B}(x-a)^2 \right]^{1/q}. \tag{3.16}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) \left| f^{(n)}(ta + (1-t)(a+b-x)) \right| dt \\
& \leq \mathcal{A}^{1-1/q} \left[\mathcal{K}_1(0) \left| f^{(n)}(a) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}(a+b-x) \right|^q - c\mathcal{B}(x-b)^2 \right]^{1/q}. \tag{3.17}
\end{aligned}$$

Then the required result is obtained by (3.13), (3.16) and (3.17).

(ii) If $1 < q < \infty$, then the Hölder inequality and the fact of $|f^{(n)}|^q \in SX(h, c, [a, b])$ show that

$$\begin{aligned}
& \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) \left| f^{(n)}(tb + (1-t)(a+b-x)) \right| dt \\
& \leq \left(\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right)^p dt \right)^{1/p} \left[\left| f^{(n)}(b) \right|^q \int_0^1 h(t) dt \right. \\
& \quad \left. + \left| f^{(n)}(a+b-x) \right|^q \int_0^1 h(1-t) dt - c(x-a)^2 \int_0^1 t(1-t) dt \right]^{1/q} \\
& \leq \left[\int_0^1 \left(\frac{t^{np-p}}{((n-1)!)^p} - \frac{t^{\alpha p+n p-p}}{(\prod_{k=1}^{n-1}(\alpha+k))^p} \right) dt \right]^{1/p} \\
& \quad \times \left[\left| f^{(n)}(b) \right|^q \int_0^1 h(t) dt + \left| f^{(n)}(a+b-x) \right|^q \int_0^1 h(1-t) dt - \frac{c(x-a)^2}{6} \right]^{1/q} \\
& = \mathcal{C}^{1/p} \left[\left(\left| f^{(n)}(b) \right|^q + \left| f^{(n)}(a+b-x) \right|^q \right) \int_0^1 h(t) dt - \frac{c(x-a)^2}{6} \right]^{1/q}. \tag{3.18}
\end{aligned}$$

Here we use the basic inequality $(X - Y)^p \leq X^p - Y^p, \forall X > Y \geq 0, p \geq 1$. Similarly,

$$\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) \left| f^{(n)}(ta + (1-t)(a+b-x)) \right| dt$$

$$\begin{aligned}
&\leq \left(\int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right)^p dt \right)^{1/p} \\
&\quad \times \left[\left(|f^{(n)}(a)|^q + |f^{(n)}(a+b-x)|^q \right) \int_0^1 h(t)dt - \frac{c(x-b)^2}{6} \right]^{1/q} \\
&\leq \mathcal{C}^{1/p} \left[\left(|f^{(n)}(a)|^q + |f^{(n)}(a+b-x)|^q \right) \int_0^1 h(t)dt - \frac{c(x-b)^2}{6} \right]^{1/q}. \quad (3.19)
\end{aligned}$$

Thus we finish the proof by (3.13), (3.18) and (3.19). $\square$

As a consequence, letting $c \rightarrow 0$ in Theorem 3.1, we obtain the corresponding results for h -convex functions.

Corollary 3.1. *Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q$ is h -convex on $[a, b]$, $1 \leq q < \infty$.*

(i) *If $1 \leq q < \infty$, then for any $x \in [a, b]$,*

$$\begin{aligned}
|\phi_n(f, x)| &\leq \frac{\mathcal{A}^{1-1/q}}{b-a} \left\{ (x-a)^{n+1} \left[\mathcal{K}_1(0) |f^{(n)}(b)|^q + \mathcal{K}_2(0) |f^{(n)}(a+b-x)|^q \right]^{1/q} \right. \\
&\quad \left. + (b-x)^{n+1} \left[\mathcal{K}_1(0) |f^{(n)}(a)|^q + \mathcal{K}_2(0) |f^{(n)}(a+b-x)|^q \right]^{1/q} \right\}.
\end{aligned}$$

(ii) *If $1 < q < \infty$ and $1/p + 1/q = 1$, then*

$$\begin{aligned}
|\phi_n(f, x)| &\leq \frac{\mathcal{C}^{1/p}}{b-a} \left\{ (x-a)^{n+1} \left[|f^{(n)}(b)|^q + |f^{(n)}(a+b-x)|^q \right]^{1/q} \right. \\
&\quad \left. + (b-x)^{n+1} \left[|f^{(n)}(a)|^q + |f^{(n)}(a+b-x)|^q \right]^{1/q} \right\} \left(\int_0^1 h(t)dt \right)^{1/q}.
\end{aligned}$$

Remark 3.1. (i) If $n = 1$, then Corollary 3.1 reduces to Theorem 1.6.

(ii) If we take $h(t) = t^s$ in Corollary 3.1, then the error estimates are obtained for the s -convex function in the second sense with

$$\begin{aligned}
\int_0^1 h(t)dt &= \int_0^1 t^s dt = \frac{1}{s+1}, \\
\mathcal{K}_1(0) &= \frac{1}{(n+s)(n-1)!} - \frac{1}{(\alpha+n+s) \prod_{k=1}^{n-1}(\alpha+k)}, \quad \mathcal{K}_2(0) = \frac{B(n, s+1)}{(n+1)!} - \frac{B(\alpha+n, s+1)}{\prod_{k=1}^{n-1}(\alpha+k)},
\end{aligned}$$

where $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the usual Beta function. Furthermore, if $s = 1$, i.e. $h(t) = t$, we have the error estimates for the convex function with

$$\mathcal{K}_1(0) = \frac{n}{(n+1)!} - \frac{\alpha+n}{\prod_{k=1}^{n+1}(\alpha+k)}, \quad \mathcal{K}_2(0) = \frac{1}{(n+1)!} - \frac{1}{\prod_{k=1}^{n+1}(\alpha+k)}.$$

Setting $x = (a+b)/2$ in Theorem 3.1 gives the midpoint-type estimates.

Corollary 3.2. *Let $f \in C^n([a, b])$ and $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.*

(i) If $1 \leq q < \infty$, then

$$\begin{aligned} \left| \phi_n(f, \frac{a+b}{2}) \right| &\leq \frac{(b-a)^n \mathcal{A}^{1-1/q}}{2^{n+1}} \left\{ \left[\mathcal{K}_1(0) \left| f^{(n)}(b) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q - \frac{c\mathcal{B}(b-a)^2}{4} \right]^{1/q} \right. \\ &\quad \left. + \left[\mathcal{K}_1(0) \left| f^{(n)}(a) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q - \frac{c\mathcal{B}(b-a)^2}{4} \right]^{1/q} \right\} \\ &\leq \frac{(b-a)^n \mathcal{A}^{1-1/q}}{2^{n+1}} \left\{ \mathcal{K}_1(0)^{1/q} \left[\left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right| \right] + 2\mathcal{K}_2(0)^{1/q} \left| f^{(n)}\left(\frac{a+b}{2}\right) \right| \right. \\ &\quad \left. + 2 \left[\frac{c\mathcal{B}(b-a)^2}{4} \right]^{1/q} \right\}. \end{aligned}$$

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} \left| \phi_n(f, \frac{a+b}{2}) \right| &\leq \frac{(b-a)^n \mathcal{C}^{1/p}}{2^{n+1}} \left\{ \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q \right) \int_0^1 h(t) dt - \frac{c(b-a)^2}{24} \right]^{1/q} \right. \\ &\quad \left. + \left[\left(\left| f^{(n)}(b) \right|^q + \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q \right) \int_0^1 h(t) dt - \frac{c(b-a)^2}{24} \right]^{1/q} \right\} \\ &\leq \frac{(b-a)^n \mathcal{C}^{1/p}}{2^{n+1}} \left\{ \left[\left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right| + 2 \left| f^{(n)}\left(\frac{a+b}{2}\right) \right| \right] \left(\int_0^1 h(t) dt \right)^{1/q} \right. \\ &\quad \left. + 2 \left[\frac{c(b-a)^2}{24} \right]^{1/q} \right\}. \end{aligned}$$

Theorem 3.2. Let $f \in C^n([a, b])$, $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then for any $x \in [a, b]$,

$$\begin{aligned} |\varphi_n(f, x)| &\leq \frac{\mathcal{D}}{b-a} \left\{ (x-a)^{n+1} \left[\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(b) \right|^q \right. \right. \\ &\quad \left. \left. - \frac{c(x-a)^2}{(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right. \\ &\quad \left. + (b-x)^{n+1} \left[\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(a) \right|^q \right. \right. \\ &\quad \left. \left. - \frac{c(b-x)^2}{(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right\}. \end{aligned} \quad (3.20)$$

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} |\varphi_n(f, x)| &\leq \frac{\mathcal{E}^{1/p}}{b-a} \left\{ (x-a)^{n+1} \left[\left(\left| f^{(n)}(a+b-x) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6}c(x-a)^2 \right]^{1/q} \right. \\ &\quad \left. + (b-x)^{n+1} \left[\left(\left| f^{(n)}(a+b-x) \right|^q + \left| f^{(n)}(a) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6}c(b-x)^2 \right]^{1/q} \right\}. \end{aligned} \quad (3.21)$$

Proof. By Lemma 2.2, we have

$$|\varphi_n(f, x)| \leq \frac{1}{(b-a) \prod_{k=1}^{n-1} (\alpha+k)} \left[(x-a)^{n+1} \int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)b) \right| dt \right. \\ \left. + (b-x)^{n+1} \int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)a) \right| dt \right]. \quad (3.22)$$

(i) Without loss of generality, we will prove the inequality for $1 < q < \infty$. It follows from the power mean inequality and $|f^{(n)}|^q \in SX(h, c, [a, b])$ that

$$\int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)b) \right| dt \\ \leq \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left(\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(b) \right|^q \right. \\ \left. - \frac{c(x-a)^2}{(\alpha+n+1)(\alpha+n+2)} \right)^{1/q}, \quad (3.23)$$

and

$$\int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)a) \right| dt \\ \leq \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left(\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(a) \right|^q \right. \\ \left. - \frac{c(b-x)^2}{(\alpha+n+1)(\alpha+n+2)} \right)^{1/q}. \quad (3.24)$$

Then we complete the proof of the inequality (3.20) by (3.22)-(3.25).

(ii) By virtue of the Hölder inequality and the fact of $|f^{(n)}|^q \in SX(h, c, [a, b])$, we have

$$\int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)b) \right| dt \\ \leq \left(\frac{1}{(\alpha+n-1)p+1} \right)^{1/p} \left(\left| f^{(n)}(a+b-x) \right|^q \int_0^1 h(t) dt \right. \\ \left. + \left| f^{(n)}(b) \right|^q \int_0^1 h(1-t) dt - \frac{c(x-a)^2}{6} \right)^{1/q}, \quad (3.25)$$

and

$$\int_0^1 t^{\alpha+n-1} \left| f^{(n)}(t(a+b-x) + (1-t)a) \right| dt \\ \leq \left(\frac{1}{(\alpha+n-1)p+1} \right)^{1/p} \left(\left| f^{(n)}(a+b-x) \right|^q \int_0^1 h(t) dt \right. \\ \left. + \left| f^{(n)}(a) \right|^q \int_0^1 h(1-t) dt - \frac{c(b-x)^2}{6} \right)^{1/q}. \quad (3.26)$$

Thus we finish the proof of (3.21) by (3.22), (3.25) and (3.26). $\square$

Letting $c \rightarrow 0$ in the proceeding theorem, we have the following conclusion.

Corollary 3.3. Let $f \in C^n([a, b]), f^{(n)} \in L([a, b]), n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q$ is h -convex on $[a, b]$, $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then

$$|\varphi_n(f, x)| \leq \frac{\mathcal{D}}{b-a} \left\{ (x-a)^{n+1} \left[\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(b) \right|^q \right]^{1/q} \right. \\ \left. + (b-x)^{n+1} \left[\mathcal{J}_1(1) \left| f^{(n)}(a+b-x) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(a) \right|^q \right]^{1/q} \right\}.$$

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$|\varphi_n(f, x)| \leq \frac{\mathcal{E}^{1/p}}{b-a} \left\{ (x-a)^{n+1} \left[\left| f^{(n)}(a+b-x) \right|^q + \left| f^{(n)}(b) \right|^q \right]^{1/q} \right. \\ \left. + (b-x)^{n+1} \left[\left| f^{(n)}(a+b-x) \right|^q + \left| f^{(n)}(a) \right|^q \right]^{1/q} \right\} \left(\int_0^1 h(t) dt \right)^{1/q}.$$

Remark 3.2. (i) If $n = 1$, then Corollary 3.3 reduces to Theorem 1.7.

(ii) If we take $h(t) = t^s$ in Corollary 3.3, the error estimation for s -convex functions in the second sense is obtained, in which

$$\int_0^1 h(t) dt = \frac{1}{s+1}, \quad \mathcal{J}_1(1) = \frac{1}{\alpha+n+s}, \quad \mathcal{J}_2(1) = B(\alpha+n, s+1).$$

Moreover, if $s = 1$, we have the error estimates for convex functions with

$$\int_0^1 h(t) dt = \frac{1}{2}, \quad \mathcal{J}_1(1) = \frac{1}{\alpha+n+1}, \quad \mathcal{J}_2(1) = \frac{1}{(\alpha+n)(\alpha+n+1)}.$$

Taking $x = (a+b)/2$ in Theorem 3.2, by Remark 2.1, we obtain a new modified estimate different from that in Theorem 1.5 for $\mathfrak{L}_n(f, \frac{a+b}{2})$ as follows.

Corollary 3.4. Under the same assumption in Theorem 3.2 with $x = \frac{a+b}{2}$, the following inequalities hold:

(i) If $1 \leq q < \infty$, then

$$\left| \mathfrak{L}_n \left(f, \frac{a+b}{2} \right) \right| \\ = \left| \varphi_n \left(f, \frac{a+b}{2} \right) \right| \\ \leq \frac{(b-a)^n \mathcal{D}}{2^{n+1}} \left\{ \left[\mathcal{J}_1(1) \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(b) \right|^q - \frac{c(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right. \\ \left. + \left[\mathcal{J}_1(1) \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^q + \mathcal{J}_2(1) \left| f^{(n)}(a) \right|^q - \frac{c(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)} \right]^{1/q} \right\}.$$

(ii) If $1 \leq q < \infty$, then

$$\left| \mathfrak{L}_n \left(f, \frac{a+b}{2} \right) \right|$$

$$\begin{aligned}
&= \left| \varphi_n \left(f, \frac{a+b}{2} \right) \right| \\
&\leq \frac{(b-a)^n \mathcal{E}^{1/p}}{2^{n+1}} \left\{ \left[\left(\left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6} c \left(\frac{b-a}{2} \right)^2 \right]^{1/q} \right. \\
&\quad \left. + \left[\left(\left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^q + \left| f^{(n)}(a) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6} c \left(\frac{b-a}{2} \right)^2 \right]^{1/q} \right\}.
\end{aligned}$$

As an application of Lemma 2.2, we obtain the error estimates of $\mathfrak{R}_n(f)$ for strongly h -convex functions.

Theorem 3.3. Let $f \in C^n([a, b])$, $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.

(i) Let $1 \leq q < \infty$. If n is odd, then

$$\begin{aligned}
|\mathfrak{R}_n(f)| &= \frac{|\varphi_n(f, a) + \varphi(f, b)|}{2} \\
&\leq \frac{(b-a)^n}{\prod_{k=1}^n (\alpha + k)} \left(\frac{\alpha + n}{2} \right)^{1/q} \left(1 - \frac{1}{2^{\alpha+n-1}} \right)^{1-1/q} \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) \right. \\
&\quad \left. \times \int_0^{1/2} ((1-t)^{\alpha+n-1} - t^{\alpha+n-1}) (h(t) + h(1-t)) dt - 2c(b-a)^2 \mathcal{F} \right]^{1/q}, \quad (3.27)
\end{aligned}$$

and if n is even, then

$$\begin{aligned}
|\mathfrak{R}_n(f)| &= \frac{|\varphi_n(f, a) + \varphi(f, b)|}{2} \\
&\leq \frac{(b-a)^n}{\prod_{k=1}^n (\alpha + k)} \left(\frac{\alpha + n}{2} \right)^{1/q} \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) (\mathcal{J}_1(1) + \mathcal{J}_2(1)) \right. \\
&\quad \left. - \frac{2c(b-a)^2}{(\alpha + n + 1)(\alpha + n + 2)} \right]^{1/q}, \quad (3.28)
\end{aligned}$$

where

$$\mathcal{F} = \frac{1}{\alpha + n + 1} \left(1 - \frac{1}{2^{\alpha+n}} \right) - \frac{1}{\alpha + n + 2} \left(1 - \frac{1}{2^{\alpha+n+1}} \right).$$

(ii) Let $1 < q < \infty$ and $1/p + 1/q = 1$. If n is odd, then

$$\begin{aligned}
|\mathfrak{R}_n(f)| &= \frac{|\varphi_n(f, a) + \varphi(f, b)|}{2} \\
&\leq \frac{(b-a)^n \mathcal{E}^{1/p}}{2 \prod_{k=1}^{n-1} (\alpha + k)} \left[\left(\left| f^{(n)}(a) \right|^q \int_0^{1/2} h(t) dt + \left| f^{(n)}(b) \right|^q \int_{1/2}^1 h(t) dt - \frac{c(b-a)^2}{12} \right)^{1/q} \right. \\
&\quad \left. + \left(\left| f^{(n)}(a) \right|^q \int_{1/2}^1 h(t) dt + \left| f^{(n)}(b) \right|^q \int_0^{1/2} h(t) dt - \frac{c(b-a)^2}{12} \right)^{1/q} \right], \quad (3.29)
\end{aligned}$$

and if n is even, then

$$|\mathfrak{R}_n(f)| = \frac{|\varphi_n(f, a) + \varphi(f, b)|}{2}$$

$$\leq \frac{(b-a)^n}{2 \prod_{k=1}^{n-1} (\alpha+k)} \left[\left(|f^{(n)}(a)|^q + |f^{(n)}(b)|^q \right) \int_0^1 h(t) dt - \frac{c(b-a)^2}{6} \right]^{1/q}. \quad (3.30)$$

Proof. According to Lemma 1.2 or Remark 2.1 and Lemma 2.2, we can obtain

$$\begin{aligned} & |\Re_n(f)| \\ &= \frac{|\varphi_n(f, a) + \varphi(f, b)|}{2} \\ &\leq \frac{(b-a)^n}{2 \prod_{k=1}^{n-1} (\alpha+k)} \int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| |f^{(n)}(ta + (1-t)b)| dt. \end{aligned} \quad (3.31)$$

(i) Without loss of generality, we prove the inequalities for $1 < q < \infty$. Then power-mean inequality implies that

$$\begin{aligned} & \int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| |f^{(n)}(ta + (1-t)b)| dt \\ &\leq \left(\int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \right)^{1-1/q} \\ &\quad \times \left(\int_0^1 |f^{(n)}(ta + (1-t)b)|^q |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \right)^{1/q}. \end{aligned} \quad (3.32)$$

When n is odd, it is easy to see that

$$\begin{aligned} & \int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \\ &= \int_0^{1/2} [(1-t)^{\alpha+n-1} - t^{\alpha+n-1}] dt + \int_{1/2}^1 [t^{\alpha+n-1} - (1-t)^{\alpha+n-1}] dt \\ &= \frac{2}{\alpha+n} \left(1 - \frac{1}{2^{\alpha+n-1}} \right), \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} & \int_0^1 |f^{(n)}(ta + (1-t)b)|^q |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \\ &\leq |f^{(n)}(a)|^q \int_0^{1/2} ((1-t)^{\alpha+n-1} - t^{\alpha+n-1}) h(t) dt \\ &\quad + |f^{(n)}(b)|^q \int_0^{1/2} ((1-t)^{\alpha+n-1} - t^{\alpha+n-1}) h(1-t) dt \\ &\quad + |f^{(n)}(a)|^q \int_{1/2}^1 (t^{\alpha+n-1} - (1-t)^{\alpha+n-1}) h(t) dt \\ &\quad + |f^{(n)}(b)|^q \int_{1/2}^1 (t^{\alpha+n-1} - (1-t)^{\alpha+n-1}) h(1-t) dt \\ &\quad - c(b-a)^2 \int_0^1 t(1-t) |(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \\ &= \left(|f^{(n)}(a)|^q + |f^{(n)}(b)|^q \right) \int_0^{1/2} ((1-t)^{\alpha+n-1} - t^{\alpha+n-1}) (h(t) + h(1-t)) dt \end{aligned}$$

$$-2c(b-a)^2 \left[\frac{1}{\alpha+n+1} \left(1 - \frac{1}{2^{\alpha+n}} \right) - \frac{1}{\alpha+n+2} \left(1 - \frac{1}{2^{\alpha+n+1}} \right) \right]. \quad (3.34)$$

Then we finish the proof of (3.27) by (3.31)- (3.34).

When n is even,

$$\int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt = \int_0^1 [(1-t)^{\alpha+n-1} + t^{\alpha+n-1}] dt = \frac{2}{\alpha+n},$$

and

$$\begin{aligned} & \int_0^1 \left| f^{(n)}(ta + (1-t)b) \right|^q |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| dt \\ & \leq \left| f^{(n)}(a) \right|^q \int_0^1 ((1-t)^{\alpha+n-1} + t^{\alpha+n-1}) h(t) dt \\ & \quad + \left| f^{(n)}(b) \right|^q \int_0^1 ((1-t)^{\alpha+n-1} + t^{\alpha+n-1}) h(1-t) dt - \frac{2c(b-a)^2}{(\alpha+n+1)(\alpha+n+2)}. \end{aligned} \quad (3.35)$$

Thus, the inequality (3.28) is obtained by (3.31), (3.32), (3.35) and (3.35).

(ii) Let $1 < q < \infty$. If n is odd, then the Hölder inequality shows that

$$\begin{aligned} & \int_0^1 |(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}| \left| f^{(n)}(ta + (1-t)b) \right| dt \\ & \leq \left(\int_0^{1/2} [(1-t)^{\alpha+n-1} - t^{\alpha+n-1}]^p dt \right)^{1/p} \left(\int_0^{1/2} \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{1/q} \\ & \quad + \left(\int_{1/2}^1 [t^{\alpha+n-1} - (1-t)^{\alpha+n-1}]^p dt \right)^{1/p} \left(\int_{1/2}^1 \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{1/q}. \end{aligned} \quad (3.36)$$

On the other hand,

$$\begin{aligned} \int_0^{1/2} [(1-t)^{\alpha+n-1} - t^{\alpha+n-1}]^p dt & \leq \int_0^{1/2} [(1-t)^{(\alpha+n-1)p} - t^{(\alpha+n-1)p}] dt \\ & = \frac{1}{p(\alpha+n-1)+1} \left(1 - \frac{1}{2^{p(\alpha+n-1)}} \right), \end{aligned} \quad (3.37)$$

$$\begin{aligned} \int_{1/2}^1 [t^{\alpha+n-1} - (1-t)^{\alpha+n-1}]^p dt & \leq \int_{1/2}^1 [t^{(\alpha+n-1)p} - (1-t)^{(\alpha+n-1)p}] dt \\ & = \frac{1}{p(\alpha+n-1)+1} \left(1 - \frac{1}{2^{p(\alpha+n-1)}} \right). \end{aligned} \quad (3.38)$$

And the assumption of $|f^{(n)}|^q \in SX(h, c, [a, b])$ tells us that

$$\begin{aligned} & \int_0^{1/2} \left| f^{(n)}(ta + (1-t)b) \right|^q dt \\ & \leq \left| f^{(n)}(a) \right|^q \int_0^{1/2} h(t) dt + \left| f^{(n)}(b) \right|^q \int_{1/2}^1 h(t) dt - \frac{c(b-a)^2}{12}, \end{aligned} \quad (3.39)$$

$$\begin{aligned} & \int_{1/2}^1 \left| f^{(n)}(ta + (1-t)b) \right|^q dt \\ & \leq \left| f^{(n)}(a) \right|^q \int_{1/2}^1 h(t) dt + \left| f^{(n)}(b) \right|^q \int_0^{1/2} h(t) dt - \frac{c(b-a)^2}{12}. \end{aligned} \quad (3.40)$$

Then we complete the proof of (3.29) by (3.31), (3.36)-(3.40).

When n is even, we have

$$\begin{aligned} & \int_0^1 \left| (-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right| \left| f^{(n)}(ta + (1-t)b) \right| dt \\ & \leq \left(\int_0^1 [(1-t)^{\alpha+n-1} + t^{\alpha+n-1}]^p dt \right)^{1/p} \left(\int_0^1 \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{1/q} \\ & \leq \left(\left| f^{(n)}(a) \right|^q \int_0^1 h(t) dt + \left| f^{(n)}(b) \right|^q \int_0^1 h(1-t) dt - \frac{c(b-a)^2}{6} \right)^{1/q}. \end{aligned} \quad (3.41)$$

Thus we finish the proof of (3.30) by (3.31) and (3.41). $\square$

Remark 3.3. Letting $c \rightarrow 0$ in Theorem 3.3, we get the error estimates of $\mathfrak{R}_n(f)$ for h -convex functions. Furthermore, taking $h(t) = t$, it reduces to Theorem 1.3.

Theorem 3.4. Let $f \in C^n([a, b])$, $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then

$$\begin{aligned} \left| \tilde{\phi}_n(f, 1) \right| & \leq \mathcal{A}^{1-1/q}(b-a)^n \left[\mathcal{K}_1(0) \left| f^{(n)}(a) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}(b) \right|^q - c(b-a)^2 \mathcal{B} \right]^{1/q}, \\ \left| \tilde{\phi}_n(f, 0) \right| & \leq \mathcal{A}^{1-1/q}(b-a)^n \left[\mathcal{K}_2(0) \left| f^{(n)}(a) \right|^q + \mathcal{K}_1(0) \left| f^{(n)}(b) \right|^q - c(b-a)^2 \mathcal{B} \right]^{1/q}, \end{aligned}$$

and

$$\begin{aligned} & \left| \tilde{\phi}_n(f, \mu) \right| \\ & \leq \mathcal{A}^{1-1/q}(b-a)^n \\ & \times \left\{ (1-\mu)^{n+1} \left[\frac{\tilde{\mathcal{K}}_1(\mu)}{1-\mu} \left| f^{(n)}(a) \right|^q + \frac{\tilde{\mathcal{K}}_2(\mu)}{1-\mu} \left| f^{(n)}(b) \right|^q - c(b-a)^2(1-\mu)(\mu\mathcal{A} + (1-\mu)\mathcal{B}) \right]^{1/q} \right. \\ & \left. + \mu^{n+1} \left[\frac{\mathcal{K}_1(\mu)}{\mu} \left| f^{(n)}(a) \right|^q + \frac{\mathcal{K}_2(\mu)}{\mu} \left| f^{(n)}(b) \right|^q - c(b-a)^2\mu((1-\mu)\mathcal{A} + \mu\mathcal{B}) \right]^{1/q} \right\}, \end{aligned}$$

holds for any $0 < \mu < 1$.

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} \left| \tilde{\phi}_n(f, 1) \right| & \leq \mathcal{C}^{1/p}(b-a)^n \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6}c(b-a)^2 \right]^{1/q}, \\ \left| \tilde{\phi}_n(f, 0) \right| & \leq \mathcal{C}^{1/p}(b-a)^n \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{1}{6}c(b-a)^2 \right]^{1/q}, \end{aligned}$$

and

$$\begin{aligned} \left| \tilde{\phi}_n(f, \mu) \right| &\leq C^{1/p}(b-a)^n \left\{ (1-\mu)^{n+1} \left[\frac{1}{1-\mu} \left(\left| f^{(n)}(a) \right|^q \int_0^{1-\mu} h(t) dt + \left| f^{(n)}(b) \right|^q \int_\mu^1 h(t) dt \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{6} c(b-a)^2 (1-\mu)(2\mu+1) \right]^{1/q} \right. \\ &\quad \left. + \mu^{n+1} \left[\frac{1}{\mu} \left(\left| f^{(n)}(a) \right|^q \int_{1-\mu}^1 h(t) dt + \left| f^{(n)}(b) \right|^q \int_0^\mu h(t) dt \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{6} c(b-a)^2 \mu(3-2\mu) \right]^{1/q} \right\} \end{aligned}$$

holds for any $0 < \mu < 1$.

Proof. According to Lemma 2.3, we have

$$\begin{aligned} \left| \tilde{\phi}_n(f, \mu) \right| &\leq (b-a)^n \left\{ (1-\mu)^{n+1} \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \right. \\ &\quad \times \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \\ &\quad \left. + \mu^{n+1} \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \left| f^{(n)}[(\mu t - \mu + 1)a + (1-t)\mu b] \right| dt \right\}. \end{aligned} \quad (3.42)$$

(i) Let $1 \leq q < \infty$. If $\mu = 1$, then power-mean inequality and the fact of $|f|^q \in SX(h, c, [a, b])$ imply that

$$\begin{aligned} &\left| \tilde{\phi}_n(f, 1) \right| \\ &\leq (b-a)^n \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \left| f^{(n)}(ta + (1-t)b) \right| dt \\ &\leq (b-a)^n \left(\frac{1}{n!} - \frac{1}{\prod_{k=1}^n(\alpha+k)} \right)^{1-1/q} \left\{ \left| f^{(n)}(a) \right|^q \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h(t) dt \right. \\ &\quad \left. + \left| f^{(n)}(b) \right|^q \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h(1-t) dt \right. \\ &\quad \left. - c(b-a)^2 \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] t(1-t) dt \right\}^{1/q} \\ &= (b-a)^n \mathcal{A}^{1-1/q} \left[\mathcal{K}_1(0) \left| f^{(n)}(a) \right|^q + \mathcal{K}_2(0) \left| f^{(n)}(b) \right|^q - c(b-a)^2 \mathcal{B} \right]^{1/q}. \end{aligned}$$

By the same way, we can achieve the conclusion for $\mu = 0$ and leave the proof to readers. If $0 < \mu < 1$, we have

$$\begin{aligned} &\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \\ &\leq \mathcal{A}^{1-1/q} \left\{ \left| f^{(n)}(a) \right|^q \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h((1-t)(1-\mu)) dt \right. \end{aligned} \quad (3.43)$$

$$+ \left| f^{(n)}(b) \right|^q \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h(t - \mu t + \mu) dt \\ - c(b-a)^2(1-\mu) \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] (1-t)(t - \mu t + \mu) dt \Big\}^{1/q}.$$

With a change of variables, it is not difficult to show that

$$\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h((1-t)(1-\mu)) dt \\ = \frac{1}{1-\mu} \int_0^{1-\mu} \left[\frac{1}{(n-1)!} \left(1 - \frac{t}{1-\mu} \right)^{n-1} - \frac{1}{\prod_{k=1}^{n-1}(\alpha+k)} \left(1 - \frac{t}{1-\mu} \right)^{\alpha+n-1} \right] h(t) dt \\ = \frac{\tilde{\mathcal{K}}_1(\mu)}{1-\mu}, \\ \int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] h(t - \mu t + \mu) dt = \frac{\tilde{\mathcal{K}}_2(\mu)}{1-\mu}.$$

And a direct calculate implies that

$$\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] (1-t)(t - \mu t + \mu) dt = \mu \mathcal{A} + (1-\mu) \mathcal{B}.$$

Thus, we have

$$\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \left| f^{(n)}[(1-t)(1-\mu)a + (t - \mu t + \mu)b] \right| dt \quad (3.44) \\ \leq \mathcal{A}^{1-1/q} \left[\frac{\tilde{\mathcal{K}}_1(\mu)}{1-\mu} \left| f^{(n)}(a) \right|^q + \frac{\tilde{\mathcal{K}}_2(\mu)}{1-\mu} \left| f^{(n)}(b) \right|^q - c(b-a)^2(1-\mu)(\mu \mathcal{A} + (1-\mu) \mathcal{B}) \right]^{1/q}.$$

Similarly,

$$\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right] \left| f^{(n)}[(\mu t - \mu + 1)a + (1-t)\mu b] \right| dt \quad (3.45) \\ \leq \mathcal{A}^{1-1/q} \left[\frac{\mathcal{K}_1(\mu)}{\mu} \left| f^{(n)}(a) \right|^q + \frac{\mathcal{K}_2(\mu)}{\mu} \left| f^{(n)}(b) \right|^q - c(b-a)^2 \mu ((1-\mu) \mathcal{A} + \mu \mathcal{B}) \right]^{1/q}.$$

Therefore, we complete the proof of (i) by (3.42)-(3.45).

(ii) Let $1 < q < \infty$. When $\mu = 1$, the Hölder inequality and the fact of $|f|^q \in SX(h, c, [a, b])$ show that

$$\left| \tilde{\phi}(f, 1) \right| \\ \leq (b-a)^n \left(\int_0^1 \left[\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right]^p dt \right)^{1/p} \left(\int_0^1 \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{1/q}$$

$$\leq \mathcal{C}^{1/p}(b-a)^n \left[\left(|f^{(n)}(a)|^q + |f^{(n)}(b)|^q \right) \int_0^1 h(t)dt - \frac{1}{6}c(b-a)^2 \right]^{1/q}.$$

When $\mu = 0$, a similar argument leads to the desired result.

When $0 < \mu < 1$, using the Hölder inequality again, we have

$$\begin{aligned} & \int_0^1 \left(\frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right) \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \\ & \leq \mathcal{C}^{1/p} \left(\int_0^1 \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right|^q dt \right)^{1/q} \\ & \leq \mathcal{C}^{1/p} \left[|f^{(n)}(a)|^q \int_0^1 h((1-t)(1-\mu))dt \right. \\ & \quad \left. + |f^{(n)}(b)|^q \int_0^1 h(t-\mu t + \mu)dt - \frac{1}{6}c(b-a)^2(1-\mu)(2\mu+1) \right]^{1/q} \\ & = \mathcal{C}^{1/p} \left[\frac{1}{1-\mu} |f^{(n)}(a)|^q \int_0^{1-\mu} h(t)dt \right. \\ & \quad \left. + \frac{1}{1-\mu} |f^{(n)}(b)|^q \int_{\mu}^1 h(t)dt - \frac{1}{6}c(b-a)^2(1-\mu)(2\mu+1) \right]^{1/q}. \end{aligned} \quad (3.46)$$

Similarly,

$$\begin{aligned} & \int_0^1 \left| \frac{t^{n-1}}{(n-1)!} - \frac{t^{\alpha+n-1}}{\prod_{k=1}^{n-1}(\alpha+k)} \right| \left| f^{(n)}[(\mu t - \mu + 1)a + (1-t)\mu b] \right| dt \\ & \leq \mathcal{C}^{1/p} \left[\frac{1}{\mu} |f^{(n)}(a)|^q \int_{1-\mu}^1 h(t)dt \right. \\ & \quad \left. + \frac{1}{\mu} |f^{(n)}(b)|^q \int_0^{\mu} h(t)dt - \frac{1}{6}c(b-a)^2\mu(3-2\mu) \right]^{1/q}. \end{aligned} \quad (3.47)$$

Thus we finish the proof by (3.42), (3.46) and (3.47). $\square$

Remark 3.4. If taking $h(t) = t$ in Theorem 3.4, then the error estimates hold for strongly convex functions with $\mathcal{K}_1(\mu)$, $\mathcal{K}_2(\mu)$, $\tilde{\mathcal{K}}_1(\mu)$ and $\tilde{\mathcal{K}}_2(\mu)$ denoted by (3.9)-(3.12), respectively.

Theorem 3.5. Let $f \in C^n([a, b])$, $f^{(n)} \in L([a, b])$, $n \in \mathbb{Z}^+$. Suppose that $|f^{(n)}|^q \in SX(h, c, [a, b])$, $1 \leq q < \infty$.

(i) If $1 \leq q < \infty$, then

$$\begin{aligned} |\tilde{\varphi}_n(f, 1)| & \leq \mathcal{D}(b-a)^n \left[\mathcal{J}_1(1) |f^{(n)}(b)|^q + \mathcal{J}_2(1) |f^{(n)}(a)|^q - \frac{c(b-a)^2}{(\alpha+n+1)(\alpha+n+2)} \right]^{1/q}, \\ |\tilde{\varphi}_n(f, 0)| & \leq \mathcal{D}(b-a)^n \left[\mathcal{J}_1(1) |f^{(n)}(a)|^q + \mathcal{J}_2(1) |f^{(n)}(b)|^q - \frac{c(b-a)^2}{(\alpha+n+1)(\alpha+n+2)} \right]^{1/q}, \end{aligned}$$

and

$$|\tilde{\varphi}_n(f, \mu)| \leq \mathcal{D}(b-a)^n \left\{ (1-\mu)^{n+1} \left[\frac{1}{(1-\mu)^{\alpha+n}} \left(\mathcal{J}_1(1-\mu) |f^{(n)}(a)|^q + \mathcal{J}_2(1-\mu) |f^{(n)}(b)|^q \right) \right. \right.$$

$$\begin{aligned} & -c(b-a)^2(1-\mu) \left(\frac{1}{\alpha+n+1} - \frac{1-\mu}{\alpha+n+2} \right) \Big]^{1/q} \\ & + \mu^{n+1} \left[\frac{1}{\mu^{\alpha+n}} \left(\mathcal{J}_2(\mu) \left| f^{(n)}(a) \right|^q + \mathcal{J}_1(\mu) \left| f^{(n)}(b) \right|^q \right) \right. \\ & \left. - c(b-a)^2 \mu \left(\frac{1}{\alpha+n+1} - \frac{\mu}{\alpha+n+2} \right) \right]^{1/q} \Big\} \end{aligned}$$

holds for any $0 < \mu < 1$.

(ii) If $1 < q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} |\tilde{\varphi}_n(f, 1)| & \leq \mathcal{E}^{1/p}(b-a)^n \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{c(b-a)^2}{6} \right]^{1/q}, \\ |\tilde{\varphi}_n(f, 0)| & \leq \mathcal{E}^{1/p}(b-a)^n \left[\left(\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right) \int_0^1 h(t) dt - \frac{c(b-a)^2}{6} \right]^{1/q}, \end{aligned}$$

and

$$\begin{aligned} & |\tilde{\varphi}_n(f, \mu)| \\ & \leq \mathcal{E}^{1/p}(b-a)^n \left\{ (1-\mu)^{n+1} \left[\frac{1}{1-\mu} \left(\left| f^{(n)}(a) \right|^q \int_0^{1-\mu} h(t) dt + \left| f^{(n)}(b) \right|^q \int_0^{1-\mu} h(1-t) dt \right) \right. \right. \\ & \quad \left. \left. - \frac{1}{6} c(b-a)^2 (1-\mu)(2\mu+1) \right]^{1/q} \right. \\ & \quad \left. + \mu^{n+1} \left[\frac{1}{\mu} \left(\left| f^{(n)}(a) \right|^q \int_0^\mu h(1-t) dt + \left| f^{(n)}(b) \right|^q \int_0^\mu h(t) dt \right) \right. \right. \\ & \quad \left. \left. - \frac{1}{6} c(b-a)^2 \mu(3-2\mu) \right]^{1/q} \right\} \end{aligned}$$

holds for all $0 < \mu < 1$.

Proof. Lemma 2.4 tells us that, for any $\mu \in [0, 1]$,

$$\begin{aligned} |\tilde{\varphi}_n(f, \mu)| & \leq \frac{(b-a)^n}{\prod_{k=1}^{n-1} (\alpha+k)} \left[(1-\mu)^{n+1} \int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \right. \\ & \quad \left. + \mu^{n+1} \int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(\mu t - \mu + 1)a + (1-t)\mu b] \right| dt \right]. \end{aligned} \quad (3.48)$$

(i) Let $1 \leq q < \infty$. Since the bound of $|\tilde{\varphi}_n(f, 1)|$ and $|\tilde{\varphi}_n(f, 0)|$ can be derived by the same argument as in Theorem 3.4, we turn to prove the inequality for $0 < \mu < 1$ and $1 < q < \infty$.

$$\begin{aligned} & \int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \\ & \leq \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left[\left| f^{(n)}(a) \right|^q \int_0^1 (1-t)^{\alpha+n-1} h((1-t)(1-\mu)) dt \right. \\ & \quad \left. + \left| f^{(n)}(b) \right|^q \int_0^1 (1-t)^{\alpha+n-1} h(t-\mu t + \mu) dt \right] \end{aligned} \quad (3.49)$$

$$\left. -c(b-a)^2(1-\mu) \int_0^1 (1-t)^{\alpha+n}(t-\mu t+\mu)dt \right]^{1/q}.$$

It is easy to see that

$$\begin{aligned} \int_0^1 (1-t)^{\alpha+n}(t-\mu t+\mu)dt &= \int_0^1 t^{\alpha+n}(\mu t-t+1)dt \\ &= \int_0^1 [(\mu-1)t^{\alpha+n+1} + t^{\alpha+n}] dt \\ &= \frac{\mu-1}{\alpha+n+2} + \frac{1}{\alpha+n+1}. \end{aligned} \quad (3.50)$$

Let $\nu = (1-t)(1-\mu)$, i.e. $t = 1 - \frac{\nu}{1-\mu}$. Then

$$\begin{aligned} \int_0^1 (1-t)^{\alpha+n-1}h((1-t)(1-\mu))dt &= \int_{1-\mu}^0 \left(\frac{\nu}{1-\mu}\right)^{\alpha+n-1} h(\nu)d\left(1 - \frac{\nu}{1-\mu}\right) \\ &= \frac{1}{(1-\mu)^{\alpha+n}} \int_0^{1-\mu} t^{\alpha+n-1}h(t)dt \\ &= \frac{\mathcal{J}_1(1-\mu)}{(1-\mu)^{\alpha+n}}. \end{aligned} \quad (3.51)$$

Let $\omega = t - \mu t + \mu$, i.e. $t = \frac{\omega-\mu}{1-\mu}$. Then

$$\begin{aligned} \int_0^1 (1-t)^{\alpha+n-1}h(t-\mu t+\mu)dt &= \frac{1}{(1-\mu)^{\alpha+n}} \int_{\mu}^1 (1-t)^{\alpha+n-1}h(t)dt \\ &= \frac{\mathcal{J}_2(1-\mu)}{(1-\mu)^{\alpha+n}}. \end{aligned} \quad (3.52)$$

Therefore, it follows from (3.49)-(3.52) that

$$\begin{aligned} &\int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t+\mu)b] \right| dt \\ &\leq \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left[\frac{\mathcal{J}_1(1-\mu)}{(1-\mu)^{\alpha+n}} \left| f^{(n)}(a) \right|^q + \frac{\mathcal{J}_2(1-\mu)}{(1-\mu)^{\alpha+n}} \left| f^{(n)}(b) \right|^q \right. \\ &\quad \left. -c(b-a)^2(1-\mu) \left(\frac{1}{\alpha+n+1} - \frac{1-\mu}{\alpha+n+2} \right) \right]^{1/q}. \end{aligned} \quad (3.53)$$

Similarly,

$$\begin{aligned} &\int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(\mu t-\mu+1)a + (1-t)\mu b] \right| dt \\ &\leq \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left[\frac{\mathcal{J}_2(\mu)}{\mu^{\alpha+n}} \left| f^{(n)}(a) \right|^q + \frac{\mathcal{J}_1(\mu)}{\mu^{\alpha+n}} \left| f^{(n)}(b) \right|^q \int_0^{\mu} t^{\alpha+n-1}h(t)dt \right. \\ &\quad \left. -c(b-a)^2\mu \left(\frac{1}{\alpha+n+1} - \frac{\mu}{\alpha+n+2} \right) \right]^{1/q}. \end{aligned} \quad (3.54)$$

Then we obtain the desired results of (i) by (3.48), (3.53), (3.54).

(ii) If $1 < q < \infty$, without loss of generality, we just prove the results for $\mu \in (0, 1)$ and leave the details of $\tilde{\varphi}_n(f, 1)$ and $\tilde{\varphi}_n(f, 0)$ to readers. The Hölder inequality and the fact of $|f|^q \in SX(h, c, [a, b])$ show that

$$\begin{aligned}
 & \int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(1-t)(1-\mu)a + (t-\mu t + \mu)b] \right| dt \\
 & \leq \left(\frac{1}{\alpha p + np - p + 1} \right)^{1/p} \left[\left| f^{(n)}(a) \right|^q \int_0^1 h((1-t)(1-\mu)) dt \right. \\
 & \quad \left. + \left| f^{(n)}(b) \right|^q \int_0^1 h(t-\mu t + \mu) dt - c(b-a)^2 \int_0^1 (1-t)(1-\mu)(t-\mu t + \mu) dt \right]^{1/q} \\
 & = \left(\frac{1}{\alpha p + np - p + 1} \right)^{1/p} \left[\frac{1}{1-\mu} \left(\left| f^{(n)}(a) \right|^q \int_0^{1-\mu} h(t) dt \right. \right. \\
 & \quad \left. \left. + \left| f^{(n)}(b) \right|^q \int_0^{1-\mu} h(1-t) dt \right) - \frac{1}{6} c(b-a)^2 (1-\mu)(2\mu+1) \right]^{1/q}. \tag{3.55}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \int_0^1 (1-t)^{\alpha+n-1} \left| f^{(n)}[(\mu t - \mu + 1)a + (1-t)\mu b] \right| dt \\
 & \leq \left(\frac{1}{\alpha p + np - p + 1} \right)^{1/p} \left[\frac{1}{\mu} \left(\left| f^{(n)}(a) \right|^q \int_0^\mu h(1-t) dt \right. \right. \\
 & \quad \left. \left. + \left| f^{(n)}(b) \right|^q \int_0^\mu h(t) dt \right) - \frac{1}{6} c(b-a)^2 \mu(3-2\mu) \right]^{1/q}. \tag{3.56}
 \end{aligned}$$

Thus we finish the proof of (ii) by (3.48), (3.55) and (3.56). $\square$

Remark 3.5. If taking $\mu = \frac{1}{2}$ in Theorem 3.5, then it reduces to Theorem 1.5.

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