

NONLINEAR DYNAMICS OF POLITICAL INSTABILITY IN A REDUCED STRUCTURAL-DEMOGRAPHIC MODEL

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Abstract Political instability often exhibits long-term oscillatory patterns, commonly referred to in Structural-Demographic Theory as secular cycles. Although existing computational approaches, such as the Multi-Path Forecasting framework, can reproduce historical instability trends through high-dimensional simulations, they often lack the analytical tractability needed to clarify the structural mechanisms underlying these oscillations. To address this limitation, we propose a reduced-dimensional continuous-time model consisting of three coupled ordinary differential equations that capture the nonlinear feedback among radicalization, collective violence, and accumulated structural pressure. Using a mean-field approximation for the population reservoir, we derive a minimal dynamical system that preserves the essential structure of the original framework. Stability analysis based on the Routh–Hurwitz criterion shows that equilibrium stability is governed by the rate of exogenous economic decline. We further establish the occurrence of a supercritical Hopf bifurcation, whereby sufficiently severe economic deterioration destabilizes the equilibrium and gives rise to a stable limit cycle representing persistent secular cycles of political violence. Numerical simulations support the analytical results and provide a rigorous mathematical foundation for the endogenous periodicity of social unrest.

Keywords Structural-Demographic Theory, social dynamics, social instability, Hopf bifurcation, limit cycles.

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1. Introduction

Political instability often manifests as collective phenomena, including protests, riots, political violence, and legitimacy crises. Its emergence is not solely triggered by short-term shocks but is often associated with the accumulation of long-term structural pressures. An increasing number of studies [9, 18, 22, 23, 25] suggest that such phenomena exhibit pronounced long-term evolutionary dynamics and quasi-periodic patterns, implying that political order may be understood as

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a complex dynamical system evolving under structural constraints. In this context, characterizing the long-term behaviour of sociopolitical systems, particularly their equilibrium structure, stability, and potential oscillatory mechanisms, has become a core issue in interdisciplinary research.

Structural-Demographic Theory (SDT) provides a systematic theoretical framework for addressing this issue. The theory was first proposed by Goldstone in his comparative study of early modern social revolutions [9]. Its central idea is to conceptualize society as an interacting system composed of three main components: The general population, the elite class, and the state. SDT posits that endogenous feedback arises among population growth, elite expansion, and the state's fiscal and administrative capacity. When these structural variables deviate from a sustainable range over an extended period, pressures accumulate within the social system and may eventually be released as political instability. In 2009, Turchin et al. [21, 25] further showed that this mechanism can generate long-term periodic fluctuations spanning decades or even centuries, with a structure resembling classical dynamical systems in which slow variables drive the system toward instability and regime shifts.

Building on this idea, Turchin [23] explicitly formalized SDT as a dynamical model. He developed age-structured differential equations to describe the temporal evolution of population pressure, elite competition, and indicators of political instability, and introduced the Political Stress Indicator (PSI) as a composite measure of the system's state. Subsequently, Turchin and Korotayev [24] applied this framework to contemporary American society, producing forward-looking predictions of instability risk from the late 2010s to the early 2020s and later assessing the model's empirical performance. A key contribution of this line of research is that it transforms the study of political instability from a primarily empirical problem into one that can be examined within the framework of nonlinear dynamical systems, where equilibrium states, parameter sensitivity, and long-term evolutionary trajectories become mathematically tractable objects of analysis.

However, these predictions have also sparked ongoing academic debate. Georgescu [7] re-examined the core mechanism of SDT using updated data and argued that the statistical relationships among declining relative income, elite competition, and political violence are not always robust in modern societies. In a subsequent comment, Georgescu [8] further emphasized the sensitivity of early predictions to the choice of variables and parameter settings. These criticisms do not reject the theoretical potential of SDT; rather, they suggest that relying solely on trend fitting or composite indicators may be insufficient to fully reveal the internal dynamical structure of sociopolitical systems.

From a mathematical perspective, models such as those proposed by Turchin [23], and Turchin and Korotayev [24] incorporate many interacting components, including age structure, nonlinear feedbacks, and multiple additive effects arising from income inequality, elite overproduction, and the legacy of past violence. As a result, it remains unclear which mechanisms play the dominant role in generating political instability.

Meanwhile, research in related fields has advanced the mathematical modelling of political instability from several perspectives. Political demography, for example, examines the long-term effects of population age structure, educational expansion, and migration on political behaviour [14], providing an empirical foundation for incorporating demographic variables into SDT-based models. Cross-national studies further suggest that SDT is not limited to cases such as the United States or the United Kingdom, but may also have explanatory value in regions such as Latin America [2]. At the methodological level, the introduction of complex systems approaches

and concepts from physics has broadened the analytical framework. Gros [10] examined the role of time delays in democratic instability using delay differential equations, while Pirker-Díaz et al. [18] employed diffusion and stochastic process models to characterize the long-term evolutionary trajectories of political systems in state space. In addition, environmental shocks and population displacement have been identified as important exogenous disturbances that can significantly alter the stability structure of sociopolitical systems [4].

Within the broader context of social physics and sociodynamics, mathematical modelling has developed along two main methodological paths. On the one hand, Agent-Based Models (ABMs) have been widely used to simulate civil violence and rebellion, such as the influential framework proposed by Epstein [5] and subsequent extensions addressing polarization and conflict [5, 13]. While ABMs are effective in capturing emergent phenomena arising from micro-level interactions, they often lack analytical tractability. On the other hand, equation-based approaches, particularly systems of nonlinear ordinary differential equations (ODEs), provide a macroscopic perspective that can reveal the underlying dynamical structure of the system. Classical work by Richardson on conflict dynamics [19], along with more recent studies on reaction-diffusion models of crime and riots [3, 6], has shown that social unrest can exhibit deterministic patterns similar to those observed in biological or chemical systems. However, most existing continuous-time models focus on the short-term spatial propagation of riots rather than on the long-term secular cycles of political instability driven by slow structural variables. This paper addresses this gap by applying bifurcation analysis, commonly used in mathematical biology, to the slow-fast dynamics implied by structural-demographic theory.

The main drivers of social instability remain sufficiently understood in the existing literature. To address this issue, we develop a new mathematical model in Section 2 that captures the key mechanisms proposed by Turchin [24] within a framework of social interactions. From this model, we derive a reduced system that retains the essential mechanisms governing the emergence of instability. In Section 3, we analyze the model and show how these mechanisms can generate sustained oscillations in political instability. Concluding remarks are provided in Section 4.

2. Mathematical modelling and dimensional reduction

In this section, we construct an ordinary differential equation model that captures the main mechanisms of the political instability framework proposed by Turchin [23] based on Structural-Demographic Theory (SDT). We then simplify this model through a systematic dimensional reduction to derive a minimal nonlinear system suitable for mathematical analysis. Despite its reduced form, the simplified model retains the essential dynamical features of the full system.

2.1. Theoretical basis: The Structural-Demographic framework

Our model is grounded in SDT, initially proposed by Goldstone [9] and further developed by Turchin and Nefedov [25]. SDT posits that historical societies behave as complex systems evolving under long-term structural pressures. Turchin later formalized this framework into a dynamical model in which political instability is driven by the interaction of three social subsystems: The general population, the elites, and the state [23].

In Turchin's broader forecasting framework, known as Multi-Path Forecasting (MPF) [26], the state of the system is monitored through the Political Stress Indicator (PSI), which combines three major drivers of instability [24]. The first is mass mobilization potential (MMP), which is associated with the immiseration of the general population, particularly declining relative

income and urbanization pressures. The second is elite mobilization potential (EMP), which arises from elite overproduction, that is, when the number of elite aspirants exceeds the number of available elite positions, intensifying intra-elite competition and fragmentation. The third is state fiscal distress (SFD), which reflects the state's declining capacity to contain unrest under conditions such as debt accumulation and loss of legitimacy.

While this high-dimensional approach effectively captures historical dynamics through simulation, the complexity of the interacting variables (e.g., age-cohort structures such as the “youth bulge” and detailed elite stratifications) makes it difficult to identify which mechanisms are the primary drivers of social instability. To address this issue, we seek a mean-field approximation that retains the core nonlinear feedbacks while reducing the number of degrees of freedom.

2.2. Initial compartmental framework

To capture the micro-level kinetic dynamics of sociopolitical instability, we adopt a compartmental modelling approach analogous to the Susceptible-Infected-Recovered (SIR) model in epidemiology [1]. This approach is consistent with the framework of social contagion proposed by Turchin [24]. Rather than modelling biological contagion [17], the compartmental structure is adapted here to describe the spread of radical ideology and political mobilization.

Let $N(t)$ denote the total population size. We partition the population into three primary compartments: Naive individuals (S), who are politically inactive but susceptible to radicalization; radicals (I), who are actively engaged in destabilizing activities; and moderates (R), who oppose violence and prioritize social order, thereby acting as a stabilizing force.

To explicitly incorporate the structural-demographic drivers, particularly economic immiseration and elite overproduction, we further subdivide the susceptible population S . Based on social stratification and labor market status, S is decomposed into four distinct sub-compartments: Employed commoners (S_{C_j}), unemployed commoners (S_{C_n}), employed elites (S_{E_j}), and unemployed elites (S_{E_n}).

The core mechanisms governing state transitions between compartments are:

- **Labour market transitions.** Individuals may move between employment and unemployment within their social class. Employed commoners (S_{C_j}) may lose their jobs and enter the unemployed group (S_{C_n}) as the relative income of commoners (w) declines, while unemployed commoners may regain employment and return to S_{C_j} . Similarly, employed elites (S_{E_j}) may become unemployed (S_{E_n}), reflecting competition for elite positions, and unemployed elites may return to employment. In addition, unemployed elites (S_{E_n}) may transition to the unemployed commoner class (S_{C_n}) through interaction with moderates (R). This transition represents surplus elites who, influenced by moderates before being recruited by radicals, abandon elite aspirations and accept downward social mobility.
- **Upward social mobility.** Some employed commoners may move into elite positions through education, administrative promotion, or political recruitment. This process is represented by the transition from S_{C_j} to S_{E_j} .
- **Radicalization processes ($S \rightarrow I$).** Susceptible individuals may become radicalized through three distinct pathways. First, unemployed commoners (S_{C_n}) and surplus elites (S_{E_n}) may experience spontaneous radicalization driven by structural grievances, at rates σ_C and σ_E , respectively. Second, commoners may be recruited into radicalization through contact with existing radicals (I), with a baseline conversion rate α_0 . For unemployed commoners (S_{C_n}), this recruitment is further intensified by economic decline through the

additional term $\alpha_w(w_0 - w)$. Third, surplus elites may also be recruited through contact with radicals, at a rate α_e . These pathways allow the model to distinguish between spontaneous grievance-based radicalization and contact-driven ideological contagion.

- **Deradicalization** ($I \rightarrow R$). Radical individuals may disengage from extremist activities through interaction with moderates (R) or as a result of exposure to violence and repression. Individuals leaving the radical state enter the moderate compartment (R), where they no longer contribute to radical recruitment.

This model also incorporates demographic factors, particularly the youth bulge effect, represented by the proportion of young adults aged 15–24, A_{20} . A larger youth cohort implies greater pressure on the labour market as more young people compete for employment, which can contribute to rising social stress. The model also tracks the distribution of employed and unemployed individuals in the population [23]. In addition, the model includes a variable V representing the level of violent incidents. This variable captures both the accumulation of violent events and the gradual backlash they generate, where τ denotes the characteristic timescale over which exposure to violence produces this response.

Under these assumptions, the population dynamics are described by the following system of differential equations.

$$\begin{aligned}
 S'_{C_n} = & \underbrace{\varepsilon_C(w_0 - w)S_{C_j}}_{S_{C_j} \text{ lose jobs}} + \underbrace{\beta\gamma S_{E_n} \frac{R}{N}}_{\text{surplus elites are influenced by moderates}} - \underbrace{\lambda_C S_{C_n}}_{S_{C_n} \text{ find jobs}} \\
 & - \underbrace{\sigma_C S_{C_n}}_{S_{C_n} \text{ spontaneously become radicals}} - \underbrace{[\alpha_0 + \alpha_w(w_0 - w)]S_{C_n} \frac{I}{N}}_{S_{C_n} \text{ become radical through contact with other radicals}} \quad (2.1a) \\
 & + \underbrace{(b_0 C_n + A_{20} C)N}_{\text{birth \& Youth bulge effect}} - \underbrace{dS_{C_n}}_{\text{death}},
 \end{aligned}$$

$$\begin{aligned}
 S'_{C_j} = & \underbrace{-\varepsilon_C(w_0 - w)S_{C_j}}_{S_{C_j} \text{ lose jobs}} + \underbrace{\lambda_C S_{C_n}}_{S_{C_n} \text{ find jobs}} - \underbrace{\alpha_0 S_{C_j} \frac{I}{N}}_{S_{C_n} \text{ become radical through contact with other radicals}} \quad (2.1b) \\
 & - \underbrace{\phi u S_{C_j}}_{S_{C_j} \text{ become } S_{E_j}} + \underbrace{b_0 N C_j}_{\text{birth}} - \underbrace{dS_{C_j}}_{\text{death}},
 \end{aligned}$$

$$\begin{aligned}
 S'_{E_n} = & \underbrace{\varepsilon_E S_{E_j}}_{S_{E_j} \text{ lose jobs}} - \underbrace{\beta\gamma S_{E_n} \frac{R}{N}}_{\text{surplus elites are influenced by moderates}} - \underbrace{\lambda_E S_{E_n}}_{S_{E_n} \text{ find jobs}} \\
 & + \underbrace{\sigma_E S_{E_n}}_{S_{E_n} \text{ spontaneously become radicals}} - \underbrace{\alpha_e S_{E_n} \frac{I}{N}}_{S_{E_n} \text{ become radical through contact with other radicals}} \quad (2.1c) \\
 & + \underbrace{(b_0 E_n + A_{20} E)N}_{\text{birth \& Youth bulge effect}} - \underbrace{dS_{E_n}}_{\text{death}},
 \end{aligned}$$

$$\begin{aligned}
 S'_{E_j} = & \underbrace{-\varepsilon_E S_{E_j}}_{S_{E_j} \text{ lose jobs}} + \underbrace{\lambda_E S_{E_n}}_{S_{E_n} \text{ find jobs}} + \underbrace{\phi u S_{C_j}}_{S_{C_j} \text{ become } S_{E_j}} + \underbrace{b_0 N E_j}_{\text{birth}} - \underbrace{dS_{E_j}}_{\text{death}}, \quad (2.1d)
 \end{aligned}$$

$$I' = I_{\text{enter}} - \underbrace{\beta\gamma R \frac{I}{N}}_{\text{Conversion of radicals by moderates}} - \underbrace{\delta V \frac{I}{N}}_{\text{Conversion of radicals by violence}} - \underbrace{dI}_{\text{death}}, \quad (2.1e)$$

$$R' = \beta\gamma R \frac{I}{N} + \delta V \frac{I}{N} - dR, \quad (2.1f)$$

$$V' = I_{\text{enter}} - V/\tau, \quad (2.1g)$$

where

$$C_n = \frac{S_{C_n}}{N}, E_n = \frac{S_{E_n}}{N}, C = \frac{S_{C_n} + S_{C_j}}{N}, E = \frac{S_{E_n} + S_{E_j}}{N},$$

and

$$I_{\text{enter}} = \sigma_C S_{C_n} + \sigma_E S_{E_n} + \alpha_w(w_0 - w)S_{C_n} \frac{I}{N} + \alpha_e S_{E_n} \frac{I}{N} + \alpha_0(S_{C_n} + S_{C_j}) \frac{I}{N}.$$

Table 1 summarizes the definitions of all parameters used in the model.

Table 1. Parameter definitions for the full sociopolitical instability model (2.1).

Parameter	Definition
ε_C (ε_E)	Unemployment rate of commoners (elites)
w	Relative income of commoners
w_0	Initial value of w
β	Contact rate
γ	Recovery rate
λ_C (λ_E)	Employment rate of commoners (elites)
σ_C (σ_E)	Rate at which unemployed commoners (elites) spontaneously become radical
α_0	Base conversion rate
α_w (α_e)	Radicalization rate of commoners (elites) through contact with radicals
b_0	Birth rate
A_{20}	Youth bulge effect
d	Death rate
ϕ	Conversion rate of commoners to elites
μ	Coefficient capturing the balance of upward and downward social mobility
τ	Timescale of the backlash to violence

2.3. Dimensional reduction and formal derivation

The full compartmental system (2.1) described in the MPF framework is high-dimensional and structurally complex. To derive a tractable dynamical system suitable for mathematical analysis, we perform a systematic simplification.

2.3.1. Simplifying the population compartments

While the full compartmental system captures detailed social interactions, it contains too many variables for explicit mathematical analysis. Therefore, we simplify the system based on practical

empirical observations.

In real-world sociopolitical systems, the active radical population $I(t)$ typically represents a very small fraction of the total population N (usually less than 1%). Consequently, the vast susceptible population $S(t)$ remains relatively stable. The reduction in $S(t)$ due to individuals becoming radicals is negligible, allowing the susceptible compartments (S_{C_n}, S_{E_n} , etc.) to be treated as constants rather than dynamic variables. Furthermore, natural demographic processes such as births, deaths, and retirements occur on much slower time scales than the rapid escalation of political violence. These slow background processes can therefore be omitted from the model.

By treating S as constant, we can substantially simplify the complex radical influx term (I_{enter}). In particular, we neglect spontaneous radicalization and ignore the baseline conversion rate, assuming that recruitment into the radical class is dominated by contact-driven mobilization under structural pressure. Under this assumption, the influx into I is represented solely by a term proportional to both the current radical population and the level of structural pressure.

Additionally, the process of radicals (I) transforming into moderates (R) through personal contact is minimal compared to the direct demobilization caused by state suppression or violence. Therefore, we can also remove the explicit R compartment.

Applying these straightforward simplifications reduces the full model to the following streamlined equations:

$$\begin{aligned} I' &= (w_0 - w)(\alpha_w S_{C_n} + \alpha_e S_{E_n}) \frac{I}{N} - \delta V \frac{I}{N}, \\ V' &= (w_0 - w)(\alpha_w S_{C_n} + \alpha_e S_{E_n}) \frac{I}{N} - \frac{V}{\tau}. \end{aligned}$$

2.3.2. Aggregation of structural pressures

The generalized radicalization rate in system (2.1) depends on multiple structural variables. To facilitate mathematical analysis, we introduce a composite state variable $W(t)$, representing *Social Unhappiness* or *Structural Pressure*. We define this transformation based on the dominant driver: Economic immiseration.

$$W(t) \propto (w_0 - w(t)). \quad (2.2)$$

Substituting this relationship into the recruitment term of the system (2.1), and assuming that the contagion effect is modulated by structural pressure (i.e., radical ideology spreads more rapidly when social unhappiness is high), we approximate the complex influx term J_{in} as:

$$J_{in} \approx \sigma \cdot W(t) \cdot U \cdot I, \quad (2.3)$$

where $U \approx S/N$ denotes the effective susceptible density, for example, the unemployment rate or the proportion of underemployed youth, and is treated as a quasi-static parameter, while σ combines several reaction coefficients, such as α_w and α_e , into a single mobilization parameter.

2.3.3. The dynamics of unhappiness

Rather than treating the relative income of commoners as a static algebraic term, we model the time evolution of social unhappiness $W(t)$. The dynamics of $W(t)$ are driven by macroeconomic trends, but may also be shaped by social instability. This is described by

$$\frac{dW}{dt} = \underbrace{-\alpha \frac{dw}{dt}}_{\text{Exogenous Driver}} - \underbrace{\gamma VW}_{\text{Endogenous Relief}}, \quad (2.4)$$

where $\frac{dw}{dt}$ denotes the exogenous rate of change in the relative income of commoners (e.g., changes in economic well-being), $\alpha > 0$ measures the sensitivity of social unhappiness to changes in relative income, and $\gamma > 0$ quantifies the strength of the feedback through which violence reduces structural pressure.

Let $s = \frac{dw}{dt}$. When $s < 0$, corresponding to declining relative income, the term $-\alpha s$ becomes positive and social unhappiness increases. The term $-\gamma VW$ represents an endogenous feedback mechanism through which high-intensity violence (V) releases structural pressure, for example, through elite concessions, redistribution, or emigration, thereby slowing the accumulation of W .

2.4. The reduced minimal model

Based on the reduction steps above, we obtain a condensed three-dimensional nonlinear system. The model describes the interplay among three state variables: The density of radicals (I), the intensity of violence (V), and the level of social unhappiness (W). For notational simplicity, constant scaling factors are absorbed into the effective parameters; in particular, the factor $1/N$ in the suppression term is incorporated into δ , i.e., $\delta \leftarrow \delta/N$. The governing ordinary differential equations are

$$\dot{I} = \sigma WUI - \delta IV, \quad (2.5a)$$

$$\dot{V} = \sigma WUI - \frac{1}{\tau}V, \quad (2.5b)$$

$$\dot{W} = -\alpha s(t) - \gamma VW. \quad (2.5c)$$

System (2.5) captures a feedback structure linking radicalization, violence, and social dissatisfaction. In (2.5a), the term σWUI represents radicalization as a self-reinforcing process, where the growth of the radical population depends on the existing number of radicals and is amplified by both unemployment and social unhappiness. The term $-\delta IV$ introduces a negative feedback effect, reflecting the demobilization of radicals under heightened violence, for example, through repression, exhaustion, or declining public support.

As shown in (2.5b), violence is generated by the same mobilization process that drives radicalization, while decaying at a rate $1/\tau$. The parameter τ therefore characterizes the average persistence of a violent episode before tensions temporarily subside.

In (2.5c), social unhappiness evolves under the combined influence of external economic conditions and internal social responses. The forcing term $-\alpha s(t)$ captures the effect of exogenous economic deterioration. In particular, when $s(t) < 0$, corresponding for example to declining relative income, this term becomes positive and drives an increase in W . The term $-\gamma VW$ represents a pressure-release mechanism, whereby heightened violence induces institutional responses or social adjustments that temporarily reduce accumulated dissatisfaction.

System (2.5) retains the central feedback loop of the SDT framework, social unhappiness $\rightarrow$ radicalization $\rightarrow$ violence $\rightarrow$ pressure relief, while remaining sufficiently simple for rigorous stability analysis in the following sections.

3. Dynamical analysis of the model

3.1. Empirical motivation from spectral analysis

Before proceeding with the formal mathematical analysis of system (2.5), it is crucial to empirically validate the underlying assumption that sociopolitical instability exhibits endogenous

cyclical dynamics distinct from pure macroeconomic fluctuations. To this end, we performed a Fast Fourier Transform (FFT) on historical time series of collective violence and the unemployment rate in the United States over the period 1954-1993, corresponding to a total of 40 years of annual data [20, 27].

Since the FFT produces complex-valued output containing both amplitude and phase information, we computed the modulus of the Fourier coefficients to obtain the spectral amplitude. The resulting smoothed periodograms are shown in Figure 1.

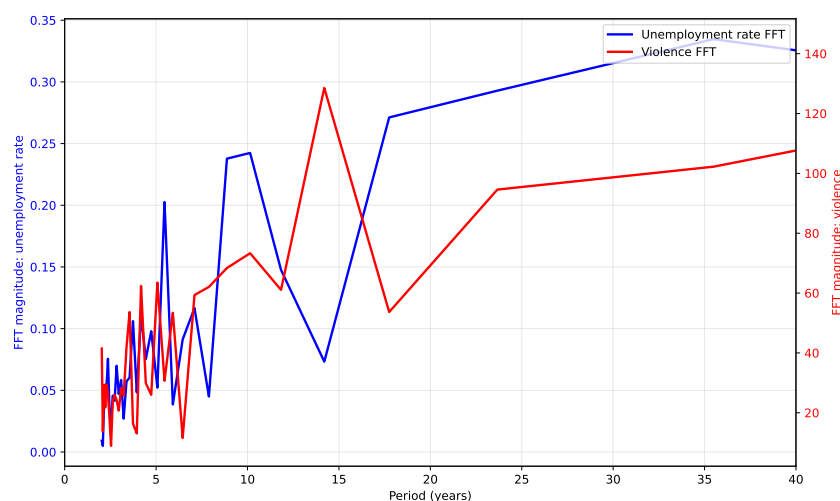


Figure 1. Spectral amplitude of collective violence and the unemployment rate in the United States over the period 1954-1993 [20, 27]. The vertical axis represents the oscillation amplitude obtained from the FFT, while the horizontal axis represents the corresponding cycle period. Oscillations in violence trace those in the unemployment rate, except for a 14-year cycle that appears in the violence data but not in the unemployment data.

As shown in Figure 1, the trajectories of violence and the unemployment rate exhibit synchronized resonance at certain periodicities (e.g., sharing common low-frequency peaks). This alignment reflects the well-documented exogenous mechanism whereby economic downturns act as a catalyst for social unrest. However, the violence data show a string 14-year cycle that is not present in the unemployment data. This indicates that short- and medium-term oscillations in violence are largely driven by external economic shocks, represented here by unemployment. At the same time, the violence data also shows an intrinsic oscillation that cannot be explained by unemployment alone. Instead, the data strongly imply the existence of an endogenous nonlinear amplification mechanism.

The existence of these independent, structurally driven cycles justifies the formulation of the nonlinear dynamical system in (2.5). We hypothesize that these periodic patterns emerge mathematically from the nonlinear cathartic effect of cumulative violence. To rigorously demonstrate that these secular cycles can arise as an inherent property of the sociopolitical system, we next compute the equilibrium points of (2.5) and drive the conditions for a Hopf bifurcation.

3.2. Equilibrium points and local stability

To systematically investigate the dynamical behavior of system (2.5), we consider an autonomous snapshot in which the effective susceptible density U is treated as a constant, and based on the theoretical and related dynamic knowledge that Turchin employed when establishing the

macroscopic mathematical model in his early work [12, 21], we can change the rate of economic $s = w'(t)$ as an exogenous constant parameter. The steady states are obtained by setting $\dot{I} = \dot{V} = \dot{W} = 0$:

$$\begin{cases} 0 = \sigma W^* U I^* - \delta I^* V^*, \\ 0 = \sigma W^* U I^* - \frac{1}{\tau} V^*, \\ 0 = -\alpha s - \gamma V^* W^*. \end{cases} \quad (3.1)$$

Assuming a nondegenerate interior equilibrium $E^*(I^*, V^*, W^*)$, where all state variables are strictly positive, algebraic manipulation of the nullclines yields the unique positive steady state:

$$I^* = \frac{1}{\delta\tau}, \quad V^* = \frac{\sigma U}{\delta} W^*, \quad W^* = \sqrt{-\frac{\alpha\delta}{\gamma\sigma U} s}. \quad (3.2)$$

From the expression for W^* , it follows that the existence of a real, positive interior equilibrium requires $s < 0$. This is consistent with the theoretical premise that persistent structural unhappiness is driven by deteriorating macroeconomic conditions.

To determine the local asymptotic stability of E^* , we evaluate the Jacobian matrix $J(E^*)$. Using the steady-state relations, the Jacobian simplifies to:

$$J(E^*) = \begin{pmatrix} 0 & -\delta I^* & \sigma U I^* \\ \sigma W^* U & -\frac{1}{\tau} & \sigma U I^* \\ 0 & -\gamma W^* & -\gamma V^* \end{pmatrix}. \quad (3.3)$$

The characteristic equation is given by $\det(\lambda I - J) = \lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0$. Substituting the equilibrium relations yields coefficients:

$$\begin{aligned} b_1 &= \frac{1}{\tau} + \gamma V^*, \\ b_2 &= \frac{\sigma U W^*}{\tau} \left(1 + \frac{2\gamma}{\delta} \right), \\ b_3 &= \frac{2\alpha\sigma U(-s)}{\tau}. \end{aligned} \quad (3.4)$$

According to the Routh-Hurwitz criterion, all roots of the characteristic polynomial have negative real parts, ensuring local asymptotic stability, if and only if

$$b_1 > 0, \quad b_3 > 0, \quad \Delta(s) = b_1 b_2 - b_3 > 0.$$

Since all model parameters $(\sigma, \delta, \gamma, \alpha, \tau, U)$ are strictly positive and $s < 0$, it is trivial to see that $b_1 > 0$ and $b_3 > 0$. Therefore, the stability of the equilibrium is determined entirely by the condition $\Delta(s) > 0$.

Substituting the scaling relations $V^* \propto \sqrt{-s}$ and $W^* \propto \sqrt{-s}$ into the coefficients shows that the product $b_1 b_2$ grows sublinearly with respect to $(-s)$, whereas b_3 grows linearly. Specifically:

$$\Delta(s) = \underbrace{\frac{C_W}{\tau^2} \sqrt{U} \sqrt{-s}}_{\text{Slow-growing term}} + \underbrace{\left(\frac{C_V C_W}{\tau} - C_S \right) U(-s)}_{\text{Fast-growing term}} > 0, \quad (3.5)$$

where C_W, C_V, C_S are positive parametric constants.

Remark 3.1 (Socio-economic interpretation). Equation (3.5) illustrates how exogenous economic shocks affect social stability. For a small magnitude of s , corresponding to a slow economic decline, the condition $\Delta(s) > 0$ is satisfied, and the equilibrium remains stable. However, when the magnitude of the economic deterioration becomes sufficiently large, b_3 exceeds b_1b_2 , thereby violating the Routh–Hurwitz stability condition [23]. Sociologically, this suggests that rapid economic immiseration can push the system beyond a critical threshold, destabilizing the social equilibrium and allowing persistent structural oscillations to emerge.

This loss of stability induced by the bifurcation parameter s motivates a further investigation of the post-instability dynamics. In the next section, we show that crossing this threshold leads to the emergence of limit cycles via a Hopf bifurcation.

3.3. Hopf bifurcation analysis

Having established the conditions for local asymptotic stability, we now investigate how the system undergoes a dynamical transition as the exogenous economic driver s crosses a critical threshold. In particular, we show the existence of a supercritical Hopf bifurcation, thereby providing a mathematical mechanism for the emergence of secular cycles.

3.3.1. Theoretical conditions for Hopf bifurcation

Consider the characteristic polynomial of the Jacobian matrix

$$P(\lambda) = \lambda^3 + b_1(s)\lambda^2 + b_2(s)\lambda + b_3(s) = 0, \quad (3.6)$$

where the coefficients b_1, b_2 , and b_3 are smooth functions of the bifurcation parameter s . According to the Poincaré–Andronov–Hopf Bifurcation Theorem [15], a simple Hopf bifurcation occurs at a critical value $s = s_c$ when the characteristic equation admits a pair of purely imaginary roots, $\lambda_{1,2} = \pm i\omega_0$ with $\omega_0 > 0$, while the third eigenvalue satisfies $\lambda_3 < 0$. In addition, the transversality condition must hold, meaning that the real part of the complex eigenvalues crosses the imaginary axis with nonzero speed as s passes through s_c . That is,

$$\left. \frac{d}{ds} \operatorname{Re}(\lambda(s)) \right|_{s=s_c} \neq 0. \quad (3.7)$$

3.3.2. Proof of existence

From the Routh–Hurwitz criterion, the stability boundary is determined by $\Delta(s) = b_1(s)b_2(s) - b_3(s) = 0$. Let s_c denote the critical value such that:

$$b_1(s_c)b_2(s_c) = b_3(s_c), \quad \text{with } b_1(s_c) > 0, b_3(s_c) > 0. \quad (3.8)$$

Substituting $b_3 = b_1b_2$ into the characteristic equation (3.6) gives

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_1b_2 = 0. \quad (3.9)$$

Factoring by grouping terms yields

$$\lambda^2(\lambda + b_1) + b_2(\lambda + b_1) = (\lambda^2 + b_2)(\lambda + b_1) = 0. \quad (3.10)$$

Therefore, the roots are:

$$\lambda_1 = -b_1(s_c), \quad \lambda_{2,3} = \pm i\sqrt{b_2(s_c)}. \quad (3.11)$$

Since $b_1 > 0$ and $b_2 = b_3/b_1 > 0$ under strictly positive model parameters, λ_1 is real and negative, while $\lambda_{2,3}$ are purely imaginary with frequency $\omega_0 = \sqrt{b_2(s_c)}$. Hence, the first condition of the Hopf bifurcation theorem is satisfied.

3.3.3. Verification of the transversality condition

To ensure that the bifurcation is nondegenerate, we must verify that the complex eigenvalues do not merely touch the imaginary axis, but cross it. Let the complex eigenvalues in the neighbourhood of s_c be written as

$$\lambda(s) = \mu(s) + i\omega(s),$$

where $\mu(s_c) = 0$ and $\omega(s_c) = \sqrt{b_2(s_c)}$.

Substituting $\lambda(s)$ into the characteristic equation and differentiating with respect to s gives

$$(3\lambda^2 + 2b_1\lambda + b_2)\frac{d\lambda}{ds} + (b'_1\lambda^2 + b'_2\lambda + b'_3) = 0, \quad (3.12)$$

where b'_1 , b'_2 , and b'_3 denote derivatives with respect to s . Evaluating at $s = s_c$, where $\lambda = i\omega_0$, allows us to solve for $\frac{d\lambda}{ds}$. Following the standard derivation for third-order systems [11], the sign of the derivative of the real part is determined by the derivative of the Routh-Hurwitz determinant $\Delta(s)$. Specifically,

$$\left. \frac{d\text{Re}(\lambda)}{ds} \right|_{s=s_c} \text{ has the same sign as } \left. \frac{d}{ds}(b_1b_2 - b_3) \right|_{s=s_c}. \quad (3.13)$$

In our model, the term b_3 depends linearly on $-s$ (since $b_3 \propto -s$), while b_1b_2 has a sublinear dependence ($\propto \sqrt{-s}$). As s decreases and becomes more negative, b_3 increases faster than b_1b_2 , causing $\Delta(s)$ to change sign from positive to negative. Therefore:

$$\frac{d}{ds}(b_1b_2 - b_3) \neq 0. \quad (3.14)$$

This verifies the transversality condition. Consequently, a periodic solution (limit cycle) bifurcates from the equilibrium point E^* as s passes through s_c . Detailed calculations are provided in Appendix B.1.

3.3.4. Numerical simulation and empirical validation

To assess the structural robustness of the model, it is important to determine whether the system inevitably destabilizes under economic stress or whether some parameter configurations can prevent oscillations entirely. As shown in Appendix B.2, evaluating the stability discriminant and identifying a valid critical state on the stability boundary yields the fundamental structural condition for the existence of a Hopf bifurcation:

$$\frac{2\gamma}{\delta} < 1. \quad (3.15)$$

To verify this theoretical result and illustrate the resulting dynamics, we perform numerical simulations using a parameter set that satisfies this condition.

We choose the parameter values: $\sigma = 35.84$, $\delta = 0.033$, $\gamma = 0.0075$, $\alpha = 10.64$, $\tau = 2.67$, and $U = 0.05$. Under this configuration, $2\gamma/\delta \approx 0.45 < 1$, which satisfies the necessary condition for the emergence of a limit cycle.

Substituting these parameter values into the stability boundary equation $b_1(s)b_2(s) - b_3(s) = 0$, we numerically obtain the critical bifurcation value:

$$s_c \approx -0.2304. \quad (3.16)$$

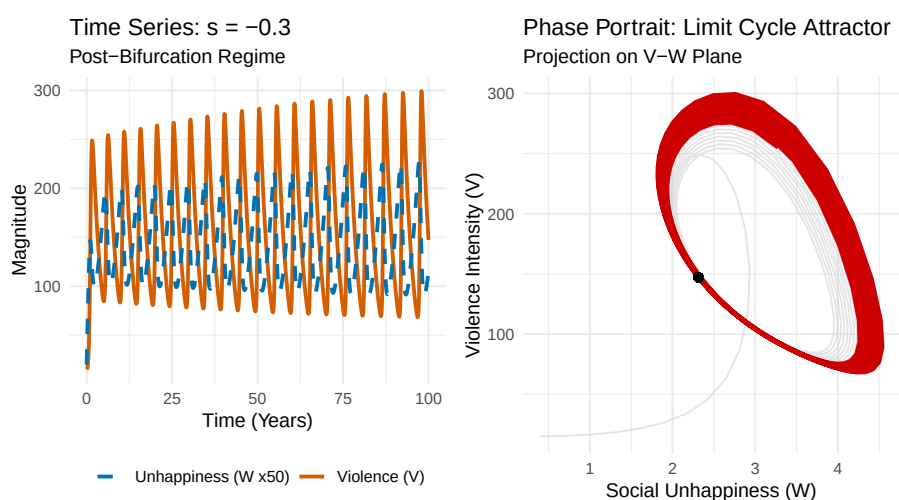


Figure 2. Numerical simulation of the Hopf bifurcation for $s = -0.3$. Left: Time series showing sustained oscillations. Right: Projection of the trajectory onto the (W, V) plane. The transient trajectory (gray) approaches a stable limit cycle (red), while the black dot denotes the unstable equilibrium.

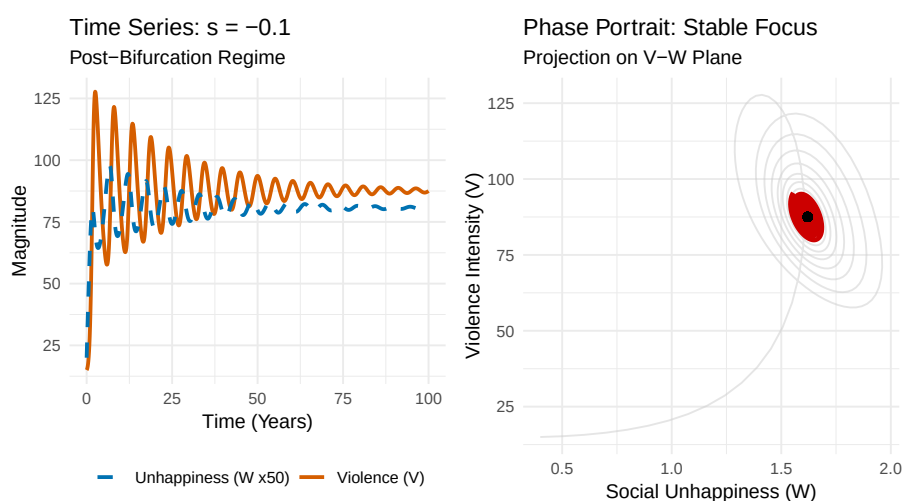


Figure 3. Numerical simulation for $s = -0.1$. Left: Time series showing damped oscillations converging to equilibrium. Right: Projection of the trajectory onto the (W, V) plane, showing a spiral approach to the stable equilibrium E^* (black dot).

To illustrate the change in system dynamics across the bifurcation threshold, we simulate trajectories under two levels of economic stress, as shown in Figures 2 and 3.

In the oscillatory regime ($s < s_c$), illustrated by $s = -0.3$, the more negative value of s represents a more severe deterioration in economic conditions. The equilibrium is unstable, and trajectories within the basin of attraction of the stable limit cycle approach this cycle after an initial transient. The system therefore exhibits persistent self-sustained oscillations.

In the stable regime ($s > s_c$), illustrated by $s = -0.1$, the less negative value of s represents a milder deterioration in economic conditions. The equilibrium E^* is a stable focus. Nearby trajectories exhibit damped oscillations and eventually converge to E^* , as shown in Figure 3.

The emergence of a stable limit cycle in the oscillatory regime provides a model-based mechanism for recurring cycles of political violence. In this regime, accumulating structural pressure promotes radicalization and violence; the resulting violence subsequently relieves some of that pressure, creating the conditions for another cycle of accumulation and release. This oscillatory behaviour is qualitatively consistent with the periodic components detected earlier using the fast Fourier transform. However, the spectral analysis alone does not establish that the observed periodicity is generated endogenously.

3.4. The importance of cathartic effects for oscillation

To demonstrate that the emergence of secular cycles is fundamentally linked to the violence compartment (V), we conduct an ablation study using a proof by contradiction. Specifically, we consider a hypothetical reduced two-dimensional subsystem obtained by eliminating the violence variable. In this counterfactual setting, we assume that the radicalized population (I) directly induces social concessions that relieve structural unhappiness (W), without the intermediate stage of kinetic violence.

The resulting ablated system is obtained by replacing V with I in the pressure-relief term and introducing a baseline demobilization rate δ_0 to prevent unbounded growth of radicals:

$$\begin{cases} \dot{I} = \sigma WUI - \delta_0 I, \\ \dot{W} = -\alpha s - \gamma_0 IW. \end{cases} \quad (3.17)$$

Here, γ_0 denotes the rate at which direct radical mobilization relieves systemic unhappiness.

3.4.1. Equilibrium and stability of the ablated system

Setting $\dot{I} = \dot{W} = 0$, we obtain the interior equilibrium $E_{2D}^*(I^*, W^*)$:

$$W^* = \frac{\delta_0}{\sigma U}, \quad I^* = -\frac{\alpha \sigma U}{\gamma_0 \delta_0} s. \quad (3.18)$$

For economically deteriorating conditions ($s < 0$), this equilibrium is uniquely defined and positive. To determine its local stability, we evaluate the Jacobian matrix at E_{2D}^* :

$$J_{2D}(E_{2D}^*) = \begin{pmatrix} \sigma U W^* - \delta_0 & \sigma U I^* \\ -\gamma_0 W^* & -\gamma_0 I^* \end{pmatrix} = \begin{pmatrix} 0 & \sigma U I^* \\ -\gamma_0 W^* & -\gamma_0 I^* \end{pmatrix}. \quad (3.19)$$

For a two-dimensional continuous-time system, local asymptotic stability is determined by the trace and determinant of the Jacobian matrix:

$$\begin{aligned} \text{Tr}(J_{2D}) &= 0 - \gamma_0 I^* = -\gamma_0 I^* < 0, \\ \text{Det}(J_{2D}) &= 0 - (\sigma U I^*)(-\gamma_0 W^*) = \gamma_0 \sigma U I^* W^* > 0. \end{aligned} \quad (3.20)$$

Therefore, the equilibrium E_{2D}^* is locally asymptotically stable.

3.4.2. Global impossibility of oscillation: Bendixson-Dulac criterion

Local stability alone is not sufficient to exclude sustained oscillations. To establish the non-existence of closed orbits in the entire biologically and sociologically meaningful phase space, namely the positive orthant $\mathbb{R}_+^2 = \{(I, W) \mid I > 0, W > 0\}$, we apply the Bendixson-Dulac criterion [16].

Theorem 3.1 (Nonexistence of periodic solutions). *The ablated two-dimensional system (3.17) admits no nonconstant periodic solutions in the interior of the first quadrant.*

Proof. Let the vector field be defined by

$$\begin{aligned}\dot{I} &= f_1(I, W) = \sigma W U I - \delta_0 I, \\ \dot{W} &= f_2(I, W) = -\alpha s - \gamma_0 I W.\end{aligned}$$

Consider the continuously differentiable Dulac multiplier

$$B(I, W) = \frac{1}{I}, \quad I > 0,$$

on the simply connected region

$$D = \{(I, W) \in \mathbb{R}^2 : I > 0, W > 0\}.$$

The divergence of the modified vector field (Bf_1, Bf_2) is

$$\begin{aligned}\nabla \cdot (B\mathbf{f}) &= \frac{\partial(Bf_1)}{\partial I} + \frac{\partial(Bf_2)}{\partial W} \\ &= \frac{\partial}{\partial I} (\sigma W U - \delta_0) + \frac{\partial}{\partial W} \left(-\frac{\alpha s}{I} - \gamma_0 W \right).\end{aligned}$$

Treating U as independent of I in the reduced system gives

$$\frac{\partial(Bf_1)}{\partial I} = 0, \quad \frac{\partial(Bf_2)}{\partial W} = -\gamma_0.$$

Therefore,

$$\nabla \cdot (B\mathbf{f}) = -\gamma_0 < 0 \quad \text{for all } (I, W) \in D,$$

because $\gamma_0 > 0$. The Bendixson-Dulac criterion consequently implies that no nonconstant periodic orbit can be entirely contained in D . $\square$

Remark 3.2 (Role of the violence compartment). Theorem 3.1 shows that the ablated system, which contains only structural pressure W and radicalization I , cannot sustain periodic oscillations in the interior of the positive quadrant. In the full model, the introduction of the violence compartment V creates an additional feedback mechanism through which violence reduces accumulated structural pressure. This feedback provides the dynamical structure that allows a Hopf bifurcation and a stable limit cycle to occur. Thus, within the proposed model, collective violence plays a central role in generating endogenous socio-political cycles.

4. Concluding remarks

This study formalizes the macroscopic dynamics of sociopolitical instability by linking empirical spectral analysis with nonlinear dynamical systems theory. Grounded in Structural Demographic Theory, we derived a minimal and tractable three-dimensional continuous-time model that captures the nonlinear feedback among radicalization, collective violence, and accumulated structural pressure.

Using U.S. social violence data from 1950-2023, published in [22, 24], we found that social violence and the unemployment rate exhibit strikingly similar oscillatory patterns for periods shorter than 35 years. One notable exception occurs at the period of 14.2 years, where the two series differ substantially. For longer-period oscillations, the discrepancies may reflect broader socioeconomic and political transformations, such as the rise of the internet or the collapse of the Soviet Union. Overall, these findings suggest that much of the observed social instability is associated with unemployment, even before explicitly accounting for class structures such as elites and commoners. At the same time, social interactions between radicals and susceptible individuals may generate additional oscillatory patterns that are not explained by unemployment alone.

To investigate this mechanism, we developed a simple mathematical model in which radicals are recruited from unemployed susceptible individuals whose unhappiness is largely driven by changes in relative income, while a cathartic effect arises from high levels of accumulated violence in the recent past. Through rigorous mathematical analysis, we showed that this model can generate stable oscillations in violence. In particular, the cathartic effect is essential for the emergence of these oscillations, even when all parameters are held constant. By contrast, when this effect is removed, the system converges to a stable equilibrium and cannot sustain oscillations under constant parameters. This was shown by analyzing a counterfactual two-dimensional subsystem that excludes the cathartic effect. These results formalize an important sociopolitical insight: Collective violence is not merely a byproduct or final-stage symptom of instability, but an essential delayed feedback mechanism that destabilizes equilibrium and sustains secular cycles. These analytical results are further supported by numerical simulations.

This unified framework shows that recurrent surges of political violence can arise as deterministic and intrinsic features of sociopolitical systems under structural stress, rather than as mere statistical artifacts. By linking empirical patterns with the underlying dynamical structure of the model, this work provides a mathematical foundation for understanding, anticipating, and potentially mitigating the endogenous cyclicity of modern social unrest.

The model is intentionally minimal and does not explicitly incorporate important social mechanisms such as class heterogeneity, network structure, institutional responses, or exogenous political shocks. In addition, the theoretical analysis is based on a deterministic system with constant parameters, whereas real sociopolitical systems are shaped by time-varying policies, economic conditions, and historical contingencies. For future research, an extension would be to incorporate richer population structure, particularly the distinction between elites and commoners emphasized in SDT, and to examine how such heterogeneity alters the bifurcation structure of the system. Another important direction is to introduce time-dependent parameters that reflect socioeconomic and political change. The model may also be calibrated to historical data to assess its explanatory and predictive performance.

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A. Analysis of the equilibrium point

A.1. Equilibrium solution

We consider the autonomous version of the system in which the unemployment rate U is treated as a constant parameter. Setting $\dot{I} = \dot{V} = \dot{W} = 0$ yields the steady-state algebraic system

$$0 = \sigma W^* U I^* - \delta I^* V^*, \quad (\text{A.1})$$

$$0 = \sigma W^* U I^* - \frac{1}{\tau} V^*, \quad (\text{A.2})$$

$$0 = -\alpha s - \gamma V^* W^*. \quad (\text{A.3})$$

Assuming a nondegenerate interior equilibrium with $I^* > 0$, $V^* > 0$, and $W^* > 0$, the first equation implies

$$V^* = \frac{\sigma U}{\delta} W^*.$$

From the second equation, we obtain

$$V^* = \sigma W^* U I^* \tau.$$

Equating these two expressions and cancelling the common factor $\sigma W^* U$ gives

$$I^* = \frac{1}{\delta \tau}.$$

Substituting this relation into the third equation yields

$$\gamma \frac{\sigma U}{\delta} W^{*2} = -\alpha s,$$

and hence

$$W^{*2} = -\frac{\alpha \delta}{\gamma \sigma U} s. \quad (\text{A.4})$$

Therefore, the existence of a real positive interior equilibrium requires $s < 0$, assuming all parameters are positive.

Degenerate or semi-trivial equilibria, such as $(I^*, V^*, W^*) = (I^*, 0, 0)$ or $(0, 0, W^*)$, must be considered separately, since the derivation above relies on division by nonzero quantities.

A.2. Jacobian matrix and characteristic polynomial

Linearizing the system around a generic equilibrium (I^*, V^*, W^*) yields the Jacobian matrix

$$J = \begin{pmatrix} \sigma W^* U - \delta V^* & -\delta I^* & \sigma U I^* \\ \sigma W^* U & -1/\tau & \sigma U I^* \\ 0 & -\gamma W^* & -\gamma V^* \end{pmatrix}. \quad (\text{A.5})$$

The characteristic polynomial is

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0,$$

where

$$b_1 = -\operatorname{tr}(J), \quad b_2 = M_{11} + M_{22} + M_{33}, \quad b_3 = -\det(J).$$

Let

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

denote the Jacobian evaluated at an equilibrium (I^*, V^*, W^*) . Then

$$\begin{aligned} b_1 &= -(a_{11} + a_{22} + a_{33}), \\ b_2 &= \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} + \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\ b_3 &= -\det(J). \end{aligned}$$

For the Jacobian matrix (A.5), the coefficients can be obtained by direct expansion. However, they simplify considerably at a nondegenerate interior equilibrium, since the steady-state relations imply

$$\begin{aligned} \dot{I} = 0 &\Rightarrow \sigma W^* U - \delta V^* = 0, \\ \dot{V} = 0 &\Rightarrow I^* = \frac{1}{\delta\tau}, \\ \sigma W^* U - \delta V^* = 0 &\Rightarrow V^* = \frac{\sigma U}{\delta} W^*. \end{aligned}$$

Under these relations, the coefficients reduce to the following explicit forms.

(i) The coefficient b_1 .

$$b_1 = -\operatorname{tr}(J) = -(0 - \frac{1}{\tau} - \gamma V^*) = \frac{1}{\tau} + \gamma V^*.$$

(ii) The coefficient b_2 . The three principal 2×2 minors are

$$\begin{aligned} M_{11} &:= \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = \left(-\frac{1}{\tau}\right)(-\gamma V^*) - (\sigma U I^*)(-\gamma W^*) = \frac{\gamma V^*}{\tau} + \gamma \sigma U I^* W^*, \\ M_{22} &:= \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} = a_{11} a_{33} - a_{13} a_{31} = 0 \quad (\text{since } a_{11} = 0, a_{31} = 0), \\ M_{33} &:= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0 \cdot a_{22} - (-\delta I^*)(\sigma W^* U) = \delta I^* \sigma W^* U. \end{aligned}$$

Hence

$$b_2 = M_{11} + M_{22} + M_{33} = \frac{\gamma V^*}{\tau} + \gamma \sigma U I^* W^* + \delta \sigma U I^* W^*.$$

Using $I^* = \frac{1}{\delta\tau}$ and $V^* = \frac{\sigma U}{\delta} W^*$, this simplifies to

$$b_2 = \frac{\sigma U W^*}{\tau} \left(1 + \frac{2\gamma}{\delta} \right).$$

(iii) **The coefficient b_3 .** A direct expansion gives

$$\det(J) = -\gamma \sigma W^* U I^* (\delta V^* + \sigma U W^*).$$

Therefore,

$$b_3 = -\det(J) = \gamma \sigma W^* U I^* (\delta V^* + \sigma U W^*).$$

Using $I^* = \frac{1}{\delta\tau}$ and $V^* = \frac{\sigma U}{\delta} W^*$, we have $\delta V^* + \sigma U W^* = 2\sigma U W^*$, so that

$$b_3 = 2\gamma \sigma^2 U^2 \frac{1}{\delta\tau} W^{*2}.$$

Finally, substituting the interior equilibrium relation

$$W^{*2} = -\frac{\alpha\delta}{\gamma\sigma U} s$$

the dependence on W^* is eliminated and b_3 reduces to

$$b_3 = \frac{2\alpha\sigma U(-s)}{\tau}.$$

Since an interior equilibrium requires $s < 0$, it follows that $b_3 > 0$ under the usual positive-parameter assumptions.

Summary for the interior equilibrium. Collecting the above expressions, we obtain

$$b_1 = \frac{1}{\tau} + \gamma V^* = \frac{1}{\tau} + \gamma \frac{\sigma U}{\delta} W^*,$$

$$b_2 = \frac{\sigma U W^*}{\tau} \left(1 + \frac{2\gamma}{\delta} \right),$$

$$b_3 = \frac{2\alpha\sigma U(-s)}{\tau},$$

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = \lambda^3 + \left(\frac{1}{\tau} + \gamma \frac{\sigma U}{\delta} W^* \right) \lambda^2 + \left(\frac{\sigma U W^*}{\tau} \left(1 + \frac{2\gamma}{\delta} \right) \right) \lambda + \frac{2\alpha\sigma U(-s)}{\tau} = 0.$$

These explicit formulas are convenient for applying the Routh–Hurwitz criterion:

$$b_1 > 0, \quad b_3 > 0, \quad b_1 b_2 > b_3.$$

These expressions also make clear how the parameters s and γ affect local stability. The coefficient b_3 is proportional to $-s$ and no longer depends explicitly on W^* after substitution. Hence, the exogenous driver s directly controls the constant term of the characteristic polynomial. In particular, if s changes sign, the interior equilibrium disappears, and b_3 is no longer positive.

The coefficients b_1 and b_2 , on the other hand, depend on W^* , and therefore indirectly on $-s$, as well as on γ . Increasing γ raises b_1 through the term γV^* and modifies b_2 through the factor $2\gamma/\delta$. Thus, both s and γ play a central role in determining the stability boundary.

For fixed parameter values, local asymptotic stability can therefore be checked by computing W^* from (A.4), evaluating b_1 , b_2 , and b_3 , and verifying the Routh–Hurwitz conditions.

A.3. Sensitivity and asymptotic breakdown of the stability boundary

Having derived explicit expressions for the coefficients of the characteristic polynomial, we now investigate the robustness of the interior equilibrium under exogenous economic shocks. In particular, we study how the stability boundary depends on the exogenous rate of change in the relative income of commoners, denoted by s .

A.3.1. Sensitivity of the characteristic polynomial to parameter changes

The local asymptotic stability of the system is determined by the roots of the characteristic equation evaluated at the interior equilibrium:

$$\lambda^3 + \left(\frac{1}{\tau} + \gamma \frac{\sigma U}{\delta} W^*\right) \lambda^2 + \left(\frac{\sigma U W^*}{\tau} \left(1 + \frac{2\gamma}{\delta}\right)\right) \lambda + \frac{2\alpha \sigma U(-s)}{\tau} = 0.$$

The coefficients of this polynomial are not fixed constants; rather, they depend on the equilibrium values W^* , V^* , and I^* , which themselves depend on the exogenous driver s . To ensure the existence of a real positive interior equilibrium, we restrict attention to the regime $s < 0$. The corresponding equilibrium relations are

$$W^{*2} = -\frac{\alpha \delta}{\gamma \sigma U} s, \quad V^* = \frac{\sigma U}{\delta} W^*, \quad I^* = \frac{1}{\delta \tau}.$$

A.3.2. Scaling analysis of coefficient dependence

To understand how the system transitions from a stable to an unstable regime, it is not enough to note that the coefficients depend on s ; we must also quantify how rapidly they change as the economic shock intensifies, that is, as $-s$ increases.

From the equilibrium relations, we obtain the following scaling laws:

$$W^* \propto \sqrt{\frac{-s}{U}}, \quad V^* \propto \sqrt{U} \sqrt{-s}, \quad I^* = \text{constant}.$$

Substituting these relations into the characteristic coefficients yields the asymptotic forms

$$b_1 \sim \frac{1}{\tau} + \gamma \sqrt{U} \sqrt{-s}, \quad b_2 \sim \frac{1}{\tau} C_W \sqrt{U} \sqrt{-s}, \quad b_3 \sim C_S U(-s),$$

where C_V , C_W , and C_S are positive constants determined by the model parameters: $C_V = \gamma$, $C_W = \sigma(1 + \frac{2\gamma}{\delta})$, and $C_S = \frac{2\alpha\sigma}{\tau}$. These asymptotic forms reveal an important structural asymmetry: The coefficients b_1 and b_2 grow sublinearly, at order $\sqrt{-s}$, whereas b_3 grows linearly in $-s$.

A.3.3. Stability inequality and qualitative behaviour

According to the Routh-Hurwitz criterion, local asymptotic stability requires

$$b_1 > 0, \quad b_3 > 0, \quad b_1 b_2 > b_3.$$

Using the asymptotic forms above, we obtain

$$b_1 b_2 - b_3 \approx \underbrace{\frac{C_W}{\tau} \sqrt{U} \sqrt{-s}}_{(A)} + \underbrace{(C_V C_W - C_S) U(-s)}_{(B)-(C)}.$$

Indeed,

$$\begin{aligned} b_1 b_2 &= \left(\frac{1}{\tau} + \gamma \sqrt{U} \sqrt{-s} \right) \left(\frac{1}{\tau} C_W \sqrt{U} \sqrt{-s} \right) \\ &= \frac{1}{\tau} \frac{C_W}{\tau} \sqrt{U} \sqrt{-s} + C_V \sqrt{U} \sqrt{-s} \frac{C_W}{\tau} \sqrt{U} \sqrt{-s} \\ &= \frac{C_W}{\tau^2} \sqrt{U} \sqrt{-s} + \frac{C_V C_W}{\tau} U(-s). \end{aligned}$$

This decomposition allows the following qualitative interpretation. For small $|s|$, corresponding to weak or slowly varying social stress, the first term (A) dominates, and the stability condition $b_1 b_2 > b_3$ is satisfied. In this regime, the interior equilibrium remains locally stable. As $|s|$ increases, however, the second term becomes dominant because it grows linearly in $-s$, while the first term grows only like $\sqrt{-s}$. If

$$C_S > \frac{C_V C_W}{\tau},$$

then the coefficient of the linear term is negative, so $b_1 b_2 - b_3$ eventually becomes negative. In that case, the Routh-Hurwitz stability condition is violated, and the equilibrium loses stability.

Therefore, the model admits a critical threshold in the exogenous stress parameter s (for fixed U), beyond which the interior equilibrium destabilizes. This asymptotic imbalance between the growth rates of $b_1 b_2$ and b_3 provides the mathematical basis for the breakdown of the stability boundary and the onset of oscillatory dynamics.

B. Hopf analysis

B.1. Verify the transversality condition

To verify the transversality condition, we must show that the real part of the complex conjugate eigenvalues crosses the imaginary axis with nonzero speed as the bifurcation parameter s passes through the critical value s_c . Let the relevant eigenvalues in a neighbourhood of s_c be written as $\lambda(s) = \mu(s) + i\omega(s)$, where $\mu(s_c) = 0$ and $\omega(s_c) = \omega_0 = \sqrt{b_2(s_c)}$.

We begin by implicitly differentiating the characteristic equation $P(\lambda(s), s) = \lambda^3 + b_1(s)\lambda^2 + b_2(s)\lambda + b_3(s) = 0$ with respect to s :

$$3\lambda^2 \frac{d\lambda}{ds} + \frac{db_1}{ds} \lambda^2 + 2b_1 \lambda \frac{d\lambda}{ds} + \frac{db_2}{ds} \lambda + b_2 \frac{d\lambda}{ds} + \frac{db_3}{ds} = 0. \quad (\text{B.1})$$

Rearranging gives

$$\frac{d\lambda}{ds} = -\frac{\frac{db_1}{ds}\lambda^2 + \frac{db_2}{ds}\lambda + \frac{db_3}{ds}}{3\lambda^2 + 2b_1\lambda + b_2}. \quad (\text{B.2})$$

Evaluating this expression at $s = s_c$, where $\lambda = i\omega_0$ and $\lambda^2 = -\omega_0^2 = -b_2$, yields

$$\left. \frac{d\lambda}{ds} \right|_{s=s_c} = -\frac{-b_2 \frac{db_1}{ds} + i\omega_0 \frac{db_2}{ds} + \frac{db_3}{ds}}{-3b_2 + 2ib_1\omega_0 + b_2} = \frac{\left(b_2 \frac{db_1}{ds} - \frac{db_3}{ds}\right) - i\omega_0 \frac{db_2}{ds}}{-2b_2 + 2i\omega_0 b_1}. \quad (\text{B.3})$$

To compute the crossing speed $\frac{d\mu}{ds}$, we take the real part of this complex fraction. Multiplying the numerator and denominator by the complex conjugate of the denominator, $(-2b_2 - 2i\omega_0 b_1)$, gives

$$\text{Re} \left(\frac{d\lambda}{ds} \right) \Big|_{s=s_c} = \frac{-2b_2 \left(b_2 \frac{db_1}{ds} - \frac{db_3}{ds}\right) - 2\omega_0^2 b_1 \frac{db_2}{ds}}{(-2b_2)^2 + (2\omega_0 b_1)^2}. \quad (\text{B.4})$$

Since $\omega_0^2 = b_2$, this simplifies to

$$\left. \frac{d\mu}{ds} \right|_{s=s_c} = -\frac{2b_2}{4b_2^2 + 4b_2 b_1^2} \left(b_1 \frac{db_2}{ds} + b_2 \frac{db_1}{ds} - \frac{db_3}{ds}\right) = -\frac{1}{2(b_2 + b_1^2)} \left. \frac{d\Delta(s)}{ds} \right|_{s=s_c}. \quad (\text{B.5})$$

Because $b_2 > 0$ and $b_1^2 > 0$, the prefactor is nonzero. Therefore, the transversality condition reduces to

$$\left. \frac{d\Delta(s)}{ds} \right|_{s=s_c} \neq 0.$$

As shown in Appendix A.2, the coefficient b_3 grows linearly in $-s$, whereas $b_1 b_2$ grows more slowly. Consequently, $\Delta(s) = b_1 b_2 - b_3$ crosses zero with a nonzero slope at the critical point. Hence,

$$\left. \frac{d\Delta(s)}{ds} \right|_{s=s_c} \neq 0,$$

which implies

$$\left. \frac{d\mu}{ds} \right|_{s=s_c} \neq 0.$$

Therefore, the complex eigenvalues cross the imaginary axis transversely as s passes through s_c , verifying the transversality condition required for Hopf bifurcation.

B.2. Parametric conditions for endogenous cycles

An important theoretical question is whether the sociopolitical system necessarily develops oscillations under sufficiently strong economic stress, or whether certain structural parameter regimes can prevent them. For a physically meaningful Hopf bifurcation point $s_c < 0$ to exist, the stability boundary

$$\Delta(s_c) = b_1(s_c)b_2(s_c) - b_3(s_c) = 0$$

must admit a solution in the region $s < 0$.

To analyze this condition, we introduce the variable

$$k = \sqrt{-s},$$

so that $k > 0$ measures the magnitude of the economic shock. Using the equilibrium relations derived above, in particular $W^* \propto k$ and $V^* \propto k$, we can express the characteristic coefficients as functions of k :

$$\begin{aligned} b_1(k) &\approx \frac{1}{\tau} + \gamma \sqrt{\frac{\alpha\delta}{\gamma\sigma U}} k, \\ b_2(k) &= \frac{\sigma U}{\tau} \sqrt{\frac{\alpha\delta}{\gamma\sigma U}} k \left(1 + \frac{2\gamma}{\delta}\right), \\ b_3(k) &= \frac{2\alpha\sigma U}{\tau} k^2. \end{aligned} \tag{B.6}$$

Substituting these expressions into the stability discriminant gives

$$\Delta(k) = b_1(k)b_2(k) - b_3(k) = Pk + Qk^2, \tag{B.7}$$

where

$$\begin{aligned} P &= \frac{1}{\tau^2} \sqrt{\frac{\alpha\delta\sigma U}{\gamma}} \left(1 + \frac{2\gamma}{\delta}\right) > 0, \\ Q &= \frac{\alpha\sigma U}{\tau} \left(\frac{2\gamma}{\delta} - 1\right). \end{aligned} \tag{B.8}$$

For a Hopf bifurcation to occur, the equation $\Delta(k) = Pk + Qk^2 = 0$ must admit a nontrivial positive real root k_c . Solving for k_c gives

$$k_c = -\frac{P}{Q}. \tag{B.9}$$

Since $P > 0$, the condition $k_c > 0$ requires $Q < 0$. Therefore,

$$Q < 0 \implies \frac{2\gamma}{\delta} - 1 < 0. \tag{B.10}$$

This yields a purely algebraic condition for the existence of a physically relevant stability boundary. In particular, endogenous oscillations can arise only when

$$\frac{2\gamma}{\delta} < 1.$$

Hence, the emergence of cyclical instability is not a numerical artifact, but a structural consequence of the balance between the pressure-relief effect of violence, measured by γ , and the suppression of radicals, measured by δ . This condition also guides the numerical simulations, since only parameter sets satisfying this inequality can produce a Hopf bifurcation for $s < 0$.

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