

PARAMETERIZED REGULARIZED DPSS PRECONDITIONERS FOR NON-HERMITIAN SADDLE POINT PROBLEMS*

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Abstract Recently, Cao [*Applied Mathematics Letters*, 2018, 84: 96-102] presented a regularized deteriorated positive and skew-Hermitian splitting (RDPSS) iteration method for solving the large sparse non-Hermitian saddle point problems and studied the convergence of the corresponding RDPSS stationary iteration method. In this paper, based on the RDPSS iteration method, we construct a parameterized RDPSS (PRDPSS) iteration method for solving the large sparse non-Hermitian saddle point problems and theoretically prove that presented method converges to the unique solution of the system of linear equations unconditionally. Finally, one example is provided to confirm the effectiveness.

Keywords Non-Hermitian matrix, regularization, convergence, preconditioner, eigenvalue.

MSC(2010) 65F10, 65F15, 65F50.

1. Introduction

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^* \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \quad (1.1)$$

where $A \in \mathbb{C}^{n \times n}$ is a non-Hermitian positive definite matrix (that is, the Hermitian part of A is

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positive definite), $B \in \mathcal{C}^{m \times n}$ ($m \leq n$) is a rectangular matrix of full row rank, B^* denotes the conjugate transpose of B , and $f \in \mathcal{C}^n$ and $g \in \mathcal{C}^m$ are two given vectors. These assumptions guarantee the existence and uniqueness of the solution of the saddle-point linear system (1.1); see [18].

Such kind of linear systems arise in many problems in scientific computing and engineering applications. For more detailed descriptions, we refer to [1, 5, 20, 25, 28, 29, 35–37, 40–42, 47, 48, 54, 55] and the references therein.

The Hermitian and skew-Hermitian parts of the complex symmetric matrix $A \in \mathcal{C}^{n \times n}$ are given by

$$H = \frac{1}{2}(A + A^*) \text{ and } S = \frac{1}{2}(A - A^*)$$

respectively, hence, $A \in \mathcal{C}^{n \times n}$ is non-Hermitian, but positive definite matrix. Here A^* is used to denote the conjugate transpose of the matrix A . Based on the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S$$

of the matrix $A \in \mathcal{C}^{n \times n}$, Bai et al. [13] gave HSS iteration method, which is as follows:

Algorithm 1: The HSS iteration method [13]. Let $x^{(0)} \in \mathcal{C}^n$ be arbitrary initial guess. For $k = 0, 1, 2, \dots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathcal{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \quad (1.2)$$

where α is a given positive constant and I is the identity matrix.

However, a potential difficulty with the HSS iteration method is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step, which is as difficult as that of the original problem; see [2, 4, 10–14, 17, 47] for more detailed descriptions about the HSS iteration method and its variants. Recently, by making use of the special structure of the coefficient matrix $A \in \mathcal{C}^{n \times n}$, Bai et al. established the following modified HSS iteration (MHSS) method and a preconditioned MHSS (PMHSS) method for solving the complex symmetric linear system (1.1-1.2) in an analogous fashion to the HSS iteration scheme in [8] and [7], respectively. Concerning the convergence of the stationary MHSS iteration method and PMHSS iteration method, Bai et al. [7, 8] analyzed the convergence. In addition, the HSS-like iteration methods for system of linear equations have been extended to other equations and conditions in [3, 43–45, 49–51, 57–59]. In 2013, Based on the ideas of [7] and [31], Li et al. presented a new approach named as the lopsided PMHSS (LPMHSS) iteration method to solve the complex symmetric linear system of linear equation (1.1-1.2). In 2015, Wu investigated several variants of the HSS iterative method in [38]. In 2015, Cao et al. studied two variants of the PMHSS iterative method for a class of complex symmetric indefinite linear systems in [25]. In 2018, Chen and Ma constructed the generalized shift-splitting (GSS) preconditioner, and gave the corresponding theoretical analysis and numerical experiments in [26]. In [47], Wu studied several variants of the Hermitian and skew-Hermitian splitting iteration method. In [56], Zheng et al. proposed a block alternating splitting iteration method. Moreover, Li and Wu in [30] presented the single-step HSS (SHSS) iteration method. In 2017, Bai and Benzi [6] proposed a class of regularized Hermitian and skew-Hermitian splitting methods for saddle-point linear systems (1.1) and established unconditional convergence of the presented stationary iterations.

In 2018, Cao [21] presented a regularized deteriorated positive and skew-Hermitian splitting (RDPSS) iteration method for saddle-point linear systems (1.1) and studied the convergence of the corresponding RDPSS stationary iteration method, which is as follows:

Algorithm 2: The RDPSS iteration method [21]. Let $x^{(0)} \in \mathcal{C}^n$ be arbitrary initial guess. For $k = 0, 1, 2, \dots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathcal{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + \mathcal{P}_+)x^{(k+\frac{1}{2})} = (\alpha I - \mathcal{S}_-)x^{(k)} + b, \\ (\alpha I + \mathcal{S}_+)x^{(k+1)} = (\alpha I - \mathcal{P}_-)x^{(k+\frac{1}{2})} + b, \end{cases} \quad (1.3)$$

where α is a given positive constant and I is the identity matrix.

Here,

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} 0 & B^* \\ -B & -Q \end{pmatrix} \\ &= \mathcal{P}_+ + \mathcal{S}_- \\ &= \begin{pmatrix} 0 & B^* \\ -B & Q \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -Q \end{pmatrix} \\ &= \mathcal{S}_+ + \mathcal{P}_-, \end{aligned} \quad (1.4)$$

where $Q \in \mathcal{C}^{m \times m}$ is a Hermitian and positive semidefinite matrix, and $\alpha > 0$ is a parameter and I is the identity matrix with suitable dimension. By assumptions, both the matrices $\alpha I + \mathcal{P}_+$ and $\alpha I + \mathcal{S}_+$ are nonsingular.

To further generalize the RDPSS iteration method and accelerate its convergence rate, in this paper, we propose a parameterized RDPSS iteration method (named as PRDPSS) for saddle-point linear systems (1.1). Furthermore, we study the convergence property of the PRDPSS method and derive the clustering property of the eigenvalues of the PRDPSS preconditioned matrix.

The organization of the paper is as follows. In Section 2 we provide a parameterized RDPSS iteration method (named as PRDPSS) for solving the large sparse non-Hermitian positive definite linear system (1.1). In Section 3, we establish the convergence of the PRDPSS method. Finally, one example is provided to confirm the effectiveness.

2. The parameterized RDPSS iteration method

In this section, based on the RDPSS iteration method, we propose a parameterized RDPSS iteration method (named as PRDPSS) for solving the large sparse non-Hermitian positive definite linear system (1.1), which is as follows:

Algorithm 3: The parameterized RDPSS iteration method. Let $x^{(0)} \in \mathcal{C}^n$ be arbitrary initial guess. For $k = 0, 1, 2, \dots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathcal{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + \mathcal{P}_+)x^{(k+\frac{1}{2})} = (\alpha I - \mathcal{S}_-)x^{(k)} + b, \\ (\beta I + \mathcal{S}_+)x^{(k+1)} = (\beta I - \mathcal{P}_-)x^{(k+\frac{1}{2})} + b, \end{cases} \quad (2.1)$$

where α, β are two given positive constant and I is the identity matrix.

Remark 2.1. In Algorithm 3, when $\alpha = \beta$, then the PRDPSS iteration method reduces to the RDPSS iteration method; The regularization matrix Q is a zero matrix and $\alpha = \beta$, then the PRDPSS iteration method reduces to the DPSS iteration method studied in [34]. So, the PRDPSS iteration method is the generalization of existing iterative algorithm.

On analyzing the algorithm carefully, we can rewrite the PRDPSS iteration method as a standard stationary iteration scheme as

$$x^{(k+1)} = \mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{N}_{\text{PRDPSS}} x^{(k)} + \mathcal{M}_{\text{PRDPSS}}^{-1} b, \quad k = 0, 1, 2, \dots \quad (2.2)$$

where

$$\begin{aligned} \mathcal{M}_{\text{PRDPSS}} &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} I & 0 \\ 0 & (\beta I + Q)^{-1} \end{pmatrix} (\alpha I + \mathcal{P}_+) (\beta I + \mathcal{S}_+) \\ &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} (\alpha I + A) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha I & B^* \\ -B & \beta I + Q \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} (\alpha I + A) & \frac{1}{\alpha} (\alpha I + A) B^* \\ -B & \beta I + Q \end{pmatrix} \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \mathcal{N}_{\text{PRDPSS}} &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} I & 0 \\ 0 & (\beta I + Q)^{-1} \end{pmatrix} (\alpha I - \mathcal{P}_-) (\beta I - \mathcal{S}_-) \\ &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} (\alpha I - A) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha I & -B^* \\ B & \beta I + Q \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \frac{1}{\alpha} (\alpha I - A) & -\frac{1}{\alpha} (\alpha I - A) B^* \\ B & \beta I + Q \end{pmatrix}. \end{aligned} \quad (2.4)$$

Remark 2.2. Note that $\mathcal{T}_{\text{PRDPSS}} = \mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{N}_{\text{PRDPSS}}$ is the iteration matrix of the PRDPSS iteration method. From this equivalent reformulation we see that the PRDPSS iteration method can also be induced by the matrix splitting $\mathcal{A} = \mathcal{M}_{\text{PRDPSS}} - \mathcal{N}_{\text{PRDPSS}}$. The splitting matrix $\mathcal{M}_{\text{PRDPSS}}$ can be employed to precondition the saddle-point matrix $\mathcal{A}$, and will be referred to as the PRDPSS preconditioner.

Note that the iteration matrix of the PRDPSS method is $\mathcal{T}_{\text{PRDPSS}}$ and coefficient matrix $\mathcal{A}$ is a nonsingular matrix, we can see that the iteration (2.2) is convergent if and only if $\rho(\mathcal{T}_{\text{PRDPSS}}) < 1$. To improve the efficiency of the PRDPSS iteration method, we may compute the inexact PRDPSS iteration method, which employs some Krylov subspace methods as its inner iteration processes at each step of the outer PRDPSS method. So, the inexact PRDPSS iteration method for solving the system of linear equations (1.1) is as follows:

Algorithm 4: The inexact parameterized RDPSS iteration method (named as IPRDPSS). Let $\bar{x}^{(0)} \in \mathbb{C}^n$ be arbitrary initial guess. For $k = 0, 1, 2, \dots$ until the sequence

of iterates $\{\bar{x}^{(k)}\}_{k=0}^{\infty} \subset \mathcal{C}^n$ converges, compute $\bar{x}^{(k+\frac{1}{2})}$ approximately from:

$$(\alpha I + \mathcal{P}_+) \bar{x}^{(k+\frac{1}{2})} \approx (\alpha I - \mathcal{S}_-) \bar{x}^{(k)} + b, \quad (2.5)$$

by employing an inner iteration (e.g., the CG method) with $\bar{x}^{(k)}$ as the initial guess; then solve $\bar{x}^{(k+1)}$ approximately from

$$(\beta I + \mathcal{S}_+) \bar{x}^{(k+1)} \approx (\beta I - \mathcal{P}_-) \bar{x}^{(k+\frac{1}{2})} + b, \quad (2.6)$$

by employing an inner iteration (e.g., some Krylov subspace method) with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where α and β are two given positive constants and I is the identity matrix.

3. Convergence of PRDPSS method

In this section, we analyze the convergence of the PRDPSS iteration method, which is motivated by the corresponding results in [21] and study the eigenvalue distribution of the preconditioned matrix $\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A}$ with respect to the PRDPSS preconditioner.

Theorem 3.1. *Let $A \in \mathcal{C}^{n \times n}$ be non-Hermitian positive definite, $B \in \mathcal{C}^{m \times n}$ have full row rank, $Q \in \mathcal{C}^{m \times m}$ be a Hermitian positive semidefinite matrix and α, β be two given positive constants. Then the spectral radius of the PRDPSS iteration matrix satisfies*

$$\rho(\mathcal{T}_{\text{PRDPSS}}) < 1, \quad \forall \alpha, \beta > 0, \quad (3.1)$$

i.e., the PRDPSS iteration method converges unconditionally to the exact solution of the non-Hermitian saddle point problems (1.1).

Proof. The iteration matrix $\mathcal{T}_{\text{PRDPSS}}$ of the RDPSS iteration scheme (2.2) is defined as

$$\begin{aligned} \mathcal{T}_{\text{PRDPSS}} &= \mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{N}_{\text{PRDPSS}} \\ &= \begin{pmatrix} \alpha I & B^* \\ -B & \beta I + Q \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + A & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha I & -B^* \\ B & \beta I + Q \end{pmatrix} \end{aligned} \quad (3.2)$$

which is similar to

$$\tilde{\mathcal{T}}_{\text{PRDPSS}} = \begin{pmatrix} \alpha I + A & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha I & -B^* \\ B & \beta I + Q \end{pmatrix} \begin{pmatrix} \alpha I & B^* \\ -B & \beta I + Q \end{pmatrix}^{-1}. \quad (3.3)$$

Consider additionally the block diagonal matrix

$$\mathcal{D} = \begin{pmatrix} I & 0 \\ 0 & \sqrt{\alpha}(\beta I + Q)^{-\frac{1}{2}} \end{pmatrix}. \quad (3.4)$$

Then the matrix $\tilde{\mathcal{T}}_{\text{PRDPSS}}$ is similar to

$$\hat{\mathcal{T}}_{\text{PRDPSS}} = \mathcal{D} \begin{pmatrix} \alpha I + A & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - A & 0 \\ 0 & I \end{pmatrix} \mathcal{D}^{-1} \mathcal{D}$$

$$\begin{aligned}
& \times \begin{pmatrix} \alpha I & -B^* \\ B & \beta I + Q \end{pmatrix} \mathcal{D} \mathcal{D}^{-1} \begin{pmatrix} \alpha I & B^* \\ -B & \beta I + Q \end{pmatrix}^{-1} \mathcal{D}^{-1} \\
& = \begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - A & 0 \\ 0 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I - \hat{B}^* \\ \hat{B} & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & \hat{B}^* \\ -\hat{B} & \alpha I \end{pmatrix}^{-1}, \quad (3.5)
\end{aligned}$$

where $\hat{B} = \sqrt{\alpha}(\alpha I + Q)^{-\frac{1}{2}}B$.

Since A is positive definite and $\hat{B}$ has full row rank, the matrix $\hat{\mathcal{T}}_{PRDPSS}$ can be viewed as the iteration matrix of the DPSS iteration method [34] used to solve the non-Hermitian saddle point problem with the coefficient matrix

$$\hat{\mathcal{A}} = \begin{pmatrix} A & \hat{B} \\ -\hat{B} & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{B} \\ -\hat{B} & 0 \end{pmatrix}. \quad (3.6)$$

From [22], Theorem 2.1], we know that $\rho(\hat{\mathcal{T}}_{PRDPSS}) < 1, \forall \alpha, \beta > 0$. Due to the similarity of the matrices $\hat{\mathcal{T}}_{PRDPSS}$ and $\mathcal{T}_{PRDPSS}$, we immediately have

$$\rho(\mathcal{T}_{PRDPSS}) < 1, \forall \alpha, \beta > 0,$$

i.e., the PRDPSS iteration method converges unconditionally to the exact solution of the non-Hermitian saddle point problems (1.1). $\square$

Remark 3.1. From Theorem 3.1, we know that the PRDPSS shift-splitting iterative method is convergent unconditionally. However, the convergence of the stationary iteration is typically too slow for the method to be competitive. For this reason, we propose using Krylov subspace methods to accelerate the convergence of the iteration. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\{b, \mathcal{A}b, \mathcal{A}^2b, \dots, \mathcal{A}^{n-1}b, \dots\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are all common iterative Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices.

4. Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated based on a two-dimensional time-harmonic Maxwell equations in mixed form, respectively. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e., when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-6}\|b\|_2$).

4.1. Example 1

In this section, to further assess the effectiveness of the new block triangular preconditioned matrix $\mathcal{M}_{PRDPSS}^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical

examples which are based on a two-dimensional time-harmonic Maxwell equations in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$. For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where $\text{nz}(A)$ denote the nonzero elements of matrix A .

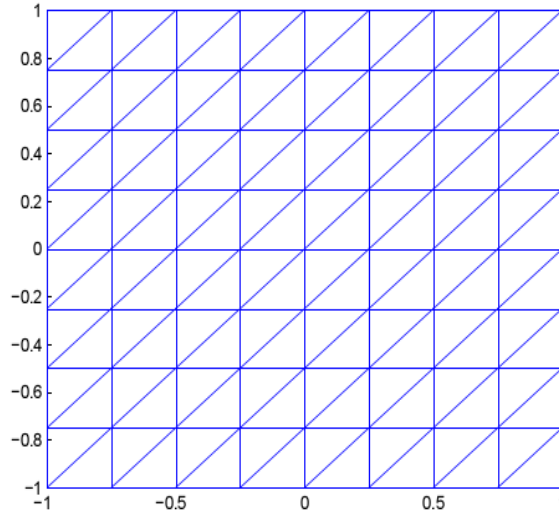


Figure 1. A uniform mesh with $h = \frac{\sqrt{2}}{4}$.

Since the new preconditioners have two parameters, in numerical experiments we will test different values. Numerical experiments show the spectrum of the PRDPSS iteration matrix $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{N}_{\text{PRDPSS}}$ and the RDPSS iteration matrix $\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{N}_{\text{RDPSS}}$ when choosing different parameters, which coincides with theoretical analysis.

In Figures 2, 4 and 6 we display the eigenvalues of the PRDPSS iteration matrix $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{N}_{\text{PRDPSS}}$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. In Figures 3, 5 and 7 we display the eigenvalues of the RDPSS iteration matrix $\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{N}_{\text{RDPSS}}$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. Figures 2 ~ 7 show that the distribution of eigenvalues of the iteration matrix confirms our above theoretical analysis. In Table 2 we show iteration counts about preconditioned matrices $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A}$ and $\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{A}$, when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{\text{BICGSTAB}}(\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A})$ and $Res_{\text{BICGSTAB}}(\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A})$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A}$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{\text{GMRES}}(\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A})$ and $Res_{\text{GMRES}}(\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A})$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{A}$ when applying to GMRES Krylov subspace iterative methods, respectively. $It_{\text{BICGSTAB}}(\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{A})$, $Res_{\text{BICGSTAB}}(\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{A})$, $It_{\text{GMRES}}(\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{A})$, $Res_{\text{GMRES}}(\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{A})$ are similar definitions.

Remark 4.1. From the above figures and tables, we know that the PRDPSS method and the RDPSS method are convergent when choosing the optimal parameters.

Remark 4.2. From Table 2, it is very easy to see that the preconditioner $\mathcal{M}_{\text{PRDPSS}}$ and $\mathcal{M}_{\text{RDPSS}}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solve two-dimensional time-harmonic Maxwell equations by choosing different parameters.

Table 1. Data sheet for different grids.

Grid	m	n	$\text{nz}(A)$	$\text{nz}(B)$	$\text{nz}(W)$	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

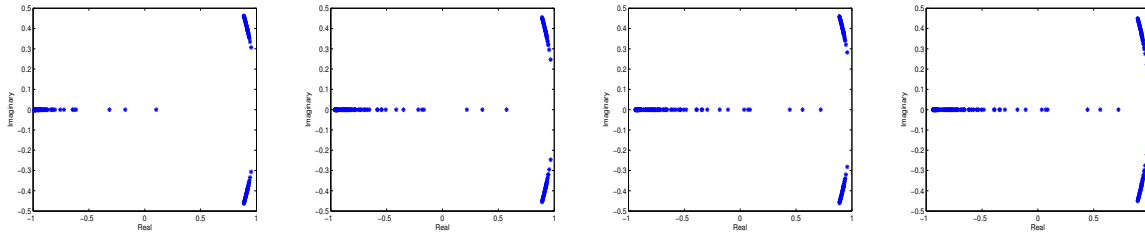


Figure 2. The eigenvalue distribution for the PRDPSS iteration matrix $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{N}_{\text{PRDPSS}}$ when $\alpha = 0.1, \beta = 0.2$ (first), $\alpha = 0.3, \beta = 0.5$ (second), $\alpha = 0.5, \beta = 0.3$ (third) and $\alpha = 0.5, \beta = 0.7$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

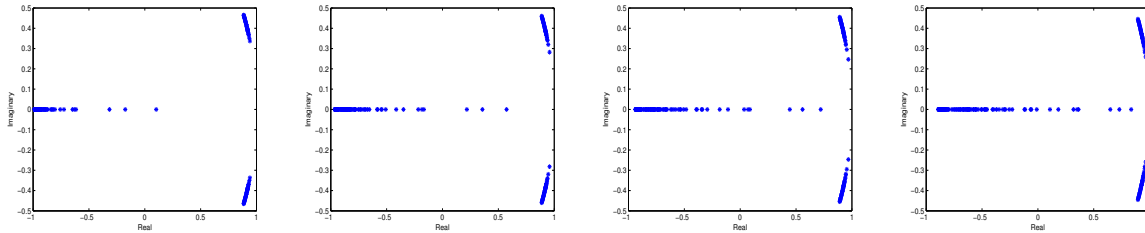


Figure 3. The eigenvalue distribution for the RDPSS iteration matrix $\mathcal{M}_{\text{RDPSS}}^{-1}\mathcal{N}_{\text{RDPSS}}$ when $\alpha = 0.1$ (first), $\alpha = 0.3$ (second), $\alpha = 0.5$ (third) and $\alpha = 0.9$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

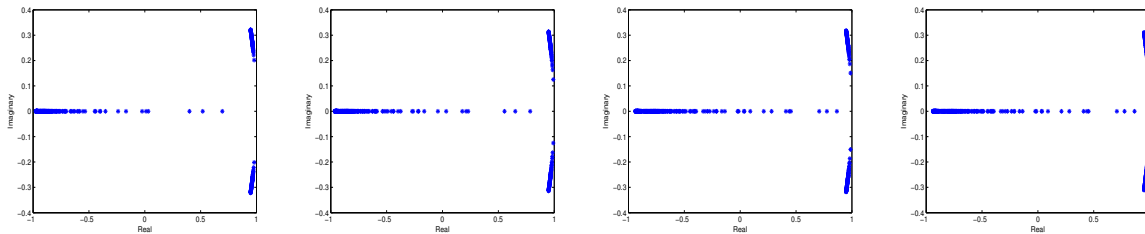


Figure 4. The eigenvalue distribution for the PRDPSS iteration matrix $\mathcal{M}_{\text{PRDPSS}}^{-1}\mathcal{N}_{\text{PRDPSS}}$ when $\alpha = 0.1, \beta = 0.2$ (first), $\alpha = 0.3, \beta = 0.5$ (second), $\alpha = 0.5, \beta = 0.3$ (third) and $\alpha = 0.5, \beta = 0.7$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

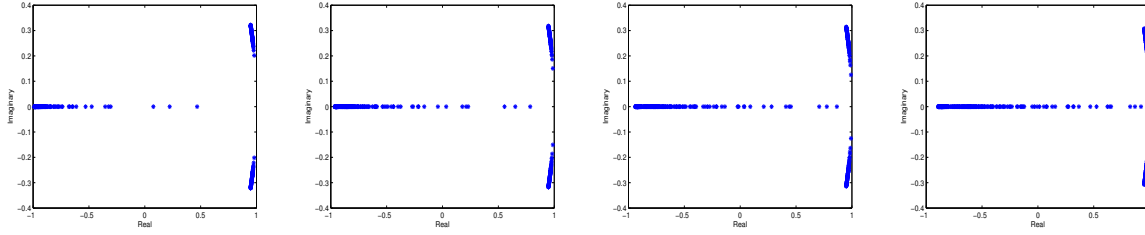


Figure 5. The eigenvalue distribution for the RDPSS iteration matrix $\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{N}_{\text{RDPSS}}$ when $\alpha = 0.1$ (first), $\alpha = 0.3$ (second), $\alpha = 0.5$ (third) and $\alpha = 0.9$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

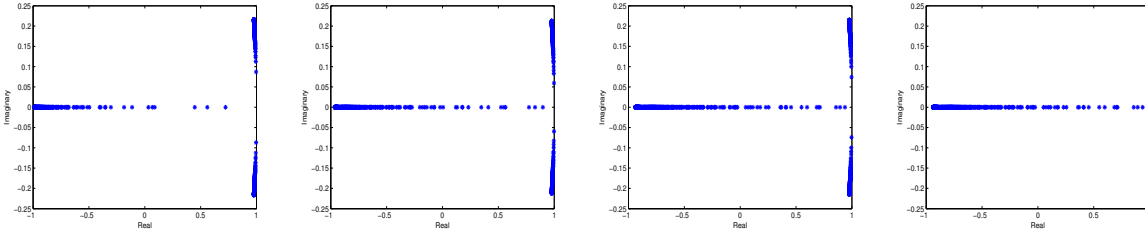


Figure 6. The eigenvalue distribution for the PRDPSS iteration matrix $\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{N}_{\text{PRDPSS}}$ when $\alpha = 0.1, \beta = 0.2$ (first), $\alpha = 0.3, \beta = 0.5$ (second), $\alpha = 0.5, \beta = 0.3$ (third) and $\alpha = 0.5, \beta = 0.7$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

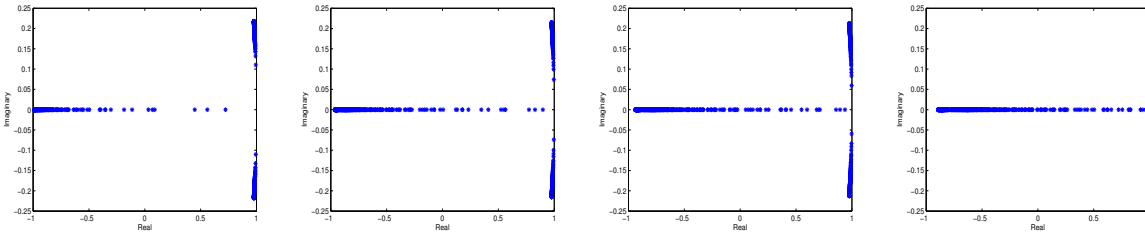


Figure 7. The eigenvalue distribution for the RDPSS iteration matrix $\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{N}_{\text{RDPSS}}$ when $\alpha = 0.1$ (first), $\alpha = 0.3$ (second), $\alpha = 0.5$ (third) and $\alpha = 0.9$ (fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

Table 2. Iteration counts and relative residual about preconditioned matrices $\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A}$ and $\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are divergent, respectively. Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

α	β	$It_{\text{BICGSTAB}(\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A})}$	$Res_{\text{BICGSTAB}(\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A})}$	α	$It_{\text{BICGSTAB}(\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{A})}$	$Res_{\text{BICGSTAB}(\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{A})}$
0.1	0.2	65	1.7275×10^{-5}	0.1	78	6.6541×10^{-6}
0.3	0.5	112.5	4.6279×10^{-7}	0.3	82.5	6.0283×10^{-7}
0.5	0.3	67.5	6.3038×10^{-7}	0.5	83.5	5.4987×10^{-7}
0.5	0.7	58	9.6076×10^{-5}	0.9	105	4.0003×10^{-5}
α	β	$It_{\text{GMRES}(\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A})}$	$Res_{\text{GMRES}(\mathcal{M}_{\text{PRDPSS}}^{-1} \mathcal{A})}$	α	$It_{\text{GMRES}(\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{A})}$	$Res_{\text{GMRES}(\mathcal{M}_{\text{RDPSS}}^{-1} \mathcal{A})}$
0.1	0.2	23(1)	8.6672×10^{-7}	0.1	21(1)	8.3994×10^{-6}
0.3	0.5	29(1)	6.1182×10^{-7}	0.3	27(1)	9.1876×10^{-7}
0.5	0.3	31(1)	4.2012×10^{-7}	0.5	33(1)	9.3831×10^{-7}
0.5	0.7	34(1)	9.1828×10^{-7}	0.9	45(1)	8.0763×10^{-5}

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