

HERMITIAN POSITIVE SEMI-DEFINITE SOLUTION OF THE QUATERNION MATRIX EQUATION $X^2 + XB^*X + BX + Q = 0$

Jun Yu Luo¹ and Zhen Chuan Ng^{1,†}

Abstract This paper discusses necessary and sufficient conditions for the existence of Hermitian positive semi-definite solutions of the quadratic matrix equation $X^2 + XB^*X + BX + Q = 0$ ($Q \geq 0$) on the quaternion field and its iterative solution method. Some necessary and sufficient conditions for the existence of Hermitian positive semi-definite solutions of this equation are proved mainly for the characteristics of coefficient matrices by introducing appropriate parameters to establish matrix inequalities and by using the theory of fixed points on convex sets. On this basis, three convergent iteration formats are constructed for different conditions and solution existence intervals, and the selection of the initial matrix is given according to each iteration property. The algorithm is solved in the MATLAB environment by using the complex operator of the quaternion matrix. At the same time, a perturbation analysis is carried out on the solution of the equation, and two perturbation error bounds are obtained by considering the Frobenius norm. Three numerical examples are used to test the effectiveness and feasibility of the given method.

Keywords Nonlinear matrix equation, positive semi-definite solution, Newton's iteration method, perturbation analysis.

MSC(2010) 15A05, 15A09, 15A36, 15A63, 68W30.

1. Introduction

Denote the real number field by $\mathbb{R}$ and the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k : i^2 = j^2 = k^2 = -1, \quad a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$. In this work, we consider the quadratic quaternion matrix equation

$$X^2 + XB^*X + BX + Q = 0, \quad (1.1)$$

and study the unknown matrix $X \in \mathbb{H}^{n \times n}$, for some $B \in \mathbb{H}^{n \times n}$ and $Q \in \mathbb{H}^{n \times n}$, where $Q \geq 0$ is Hermitian positive semi-definite and $B^* \in \mathbb{H}^{n \times n}$ is the conjugate transpose of B .

The quadratic matrix equation (1.1) is an extended form of Lyapunov matrix equation $A^T X + XA = -C$, where $-C > 0$, $C \in \mathbb{H}^{n \times n}$ and A^T denotes the transpose of matrix A , which has a wide range of applications in stability theory, optimal control, high-speed signal processing and digital sampling control (see [5, 13]) and the study of Lyapunov type equations has attracted many researchers (see [6, 11, 14, 16, 17, 24]). In particular, Lu et al. [18] and Guan et al. [8] discusses the properties of the solutions of a class of real quadratic matrix equations

$$X^2 - EX - F = 0, \quad (1.2)$$

[†]The corresponding author.

¹School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
Email: luojunyu@student.usm.my (J. Y. Luo), zhenchuanng@usm.my (Z. C. Ng)

where $E, F \in \mathbb{R}^{n \times n}$, E is a diagonal matrix and F is an M -matrix, related to the M -matrix and their iterative methods; Vorontsov and Ikramov [22] and Adam et al. [1] has studied the real quadratic matrix solutions of the equations $X^T DX + AX + X^T B + C = 0$ and $AXA = XAX$; in 2019, Chen et al. [3] studied the general solution of the real matrix equation $A_0 + A_1 X + A_2 X^2 = X$. In the same year, Huang and Ma [9] applied the structure preserving doubling algorithm to discuss the positive definite solutions of the system of nonlinear equations

$$X - A^* Y^{-1} A = I_n, \quad Y - B^* X^{-1} B = I_n,$$

where $A, B \in \mathbb{H}^{n \times n}$, X and Y are the two unknowns of the system of equations. In 2021, Masoudi and Salemi [19] discussed the existence conditions and iterative methods of Hermitian positive definite solutions for the complex matrix equations $X + A^* X^{-t} A = Q$, where A is an $n \times n$ non-singular matrix and $t > 1$ is a real number. Also in 2021, Huang et al. [10] discussed positive definite solutions of the quaternion quadratic system

$$XAX - BX = P, \tag{1.3}$$

where $B \in \mathbb{H}^{n \times n}$, $A, P > 0$ are known quaternion matrices, under certain conditions by means of transformation into algebraic equations. Authors in the paper set $Y = A^{1/2} X A^{1/2}$, $M = A^{1/2} B A^{-1/2}$, $Q = A^{1/2} P A^{1/2}$, and equation (1.3) is equivalent to

$$Y^2 - MY = Q \quad (Q > 0). \tag{1.4}$$

Clearly, the existence of a positive definite solution to equation (1.4) is equivalent to the existence of a positive definite solution to equation (1.3) [10]. By looking at equations (1.2), (1.3) and (1.4), one can see that different forms of equations can be transformed into each other by substituting variables. It is comprehensively shown that there are few studies on quaternion nonlinear matrix equations, which is mainly due to the non-commutative nature of quaternion multiplication.

Observe the quaternion matrix equations (1.2) and (1.3), we find a close connection between these two types of equations. So we are interested in whether, when generalizing equations (1.2) and (1.3) to obtain equation (1.1), we can obtain its positive semi-definite solution under certain conditions [2, 7, 20, 21]? How to find the solution of the quaternion matrix equation (1.1)? Based on this finding, we further explore the Hermitian positive semi-definite solver for the quadratic quaternion system (1.1) and its perturbation analysis.

The main contributions of this paper are as follows:

- 1) Under the condition that the matrix Q is positive semi-definite, the matrix inequalities (3.3), (3.6) and (3.9) are established for different cases of the coefficient matrix Q by introducing appropriate parameters $\delta_i > 0$ ($i = 1, 2, 3$), and thus applying the theory of immovable points on a compact convex set to obtain the equations (1.1) necessary and sufficient conditions for the existence of a positive definite solution to Hermitian.
- 2) The selection of the corresponding parameters $\delta_i > 0$ $i = 1, 2, 3$ in the three matrix inequalities is given. The convergent iterative formulas (3.15), (3.17) and (3.19) are constructed for the three cases that the quaternion matrix B is negative semi-definite, the matrix $B + I$ is positive definite, and $B^* + I$ is negative definite, respectively.
- 3) The solution of equation (1.1) was analyzed by perturbation analysis, and two upper bounds for the perturbation error ΔX are given.
- 4) Three numerical examples test the effectiveness of the given iterative algorithms and compare their advantages and disadvantages.

2. Preliminaries

Throughout this paper, let I and 0 denote the identity matrix and zero matrix, respectively. If $A = (a_{ij}) \in \mathbb{H}^{n \times n}$, then $\bar{A} = (\bar{a}_{ij}) \in \mathbb{H}^{n \times n}$ and $A^* = (\bar{a}_{ji}) \in \mathbb{H}^{n \times n}$ are the conjugate transpose of A , respectively, where $\bar{a}_{ij}$ is the conjugate of the quaternion a_{ij} . We call that A is Hermitian or self-conjugate if $A = A^*$, and the class of all $n \times n$ Hermitian quaternion matrices are denoted by $\mathcal{H}(\mathbb{H}) \subset \mathbb{H}^{n \times n}$. The maximum and minimum eigenvalues of the Hermitian matrix A are denoted by $\lambda_{\max}(A), \lambda_{\min}(A)$ separately.

Let $\|A\|_F$ denote the Frobenius norm of matrix $A = (a_{ij})$. Then $\|A\|_F = [\text{tr}(A^*A)]^{1/2} = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$ where $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ denotes the trace of the matrix A , and $\rho(A)$ represents the spectral radius of matrix A . Note that the Frobenius norm is unitarily invariant, that is, $\|A\| = \|UAV\|$ for all unitary matrices U, V .

The notation $X \geq 0$ means that X is a positive semi-definite Hermitian matrix. For the sake of convenience of discussion, hereafter Hermitian positive definitions are simply referred to as positive definitions. If matrices A and B be Hermitian, $A > B$ is used to indicate that $A - B > 0$, that is $A - B$ be positive definite.

Definition 2.1. [12] For a quaternion matrix $A = A_1 + A_2j \in \mathbb{H}^{n \times n}$ ($A_1, A_2 \in \mathbb{C}^{n \times n}$), the matrix

$$A^\sigma = \begin{pmatrix} A_1 - A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \in \mathbb{C}^{2n \times 2n},$$

is the complex representation of A .

Note that for any $A, B \in \mathbb{H}^{n \times n}$, the following properties hold:

- (1) $(A \pm B)^\sigma = A^\sigma \pm B^\sigma$,
- (2) $(AB)^\sigma = A^\sigma B^\sigma$,
- (3) $(A^{-1})^\sigma = (A^\sigma)^{-1}$,
- (4) $\|A^\sigma\|_F = \sqrt{2} \|A\|_F$,
- (5) $\text{rank}(A^\sigma) = 2 \text{rank}(A)$.

Theorem 2.1 (Fixed point theorem on convex set [4]). *If $E \in \mathbb{R}$ is a non-empty compact convex set and T is a mapping from E to E , then there exist at least one of $x \in E$, such that $T(x) = x$ holds.*

Theorem 2.2 (Hermitian matrix eigenvalue inequality [15]). *Let $A, B \in \mathcal{H}(\mathbb{H})$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A and B respectively. Then*

$$\lambda_t(A) + \lambda_n(B) \leq \lambda_t(A + B) \leq \lambda_t(A) + \lambda_t(B), t = 1, \dots, n.$$

Theorem 2.3 (Iterative local convergence theorem [23]). *Let $x = \varphi(x)$ be the iterative form of the functional equation $f(x) = 0$, and $\tilde{x}$ be the fixed point of the equation $\varphi(x)$. If $\varphi(x)$ is continuous at a certain neighborhood $U(\tilde{x})$ and satisfies $|\varphi'(\tilde{x})| < 1$, then the iterative method $x = \varphi(x)$ is called locally convergent.*

Theorem 2.4 (See [23]). *For the iterative process $x_{k+1} = \varphi(x_k)$ and the positive integer p , if $\varphi^{(p)}(x)$ is continuously close to the sought root $\tilde{x}$ and satisfies*

$$\begin{aligned}\varphi'(\tilde{x}) &= \varphi''(\tilde{x}) = \cdots = \varphi^{(p-1)}(\tilde{x}) = 0, \quad p > 1, \\ \varphi^{(p)}(\tilde{x}) &\neq 0,\end{aligned}\tag{2.1}$$

then this iterative process is p -order convergent at a certain neighborhood $U(\tilde{x})$.

3. The main results

Our main results are following theorems, iterative format construction and the perturbation analysis of positive semi-definite solutions quaternion matrix equation (1.1).

3.1. Conditions for the existence of positive semi-definite solutions

Theorem 3.1. *Let $B \in \mathcal{H}(\mathbb{H})$, the matrix equation (1.1) has a positive semi-definite solution X only if the coefficient matrix $B \leq 0$.*

Proof. If equation (1.1) has a positive semi-definite solution X , then

$$XB^*X + BX = -X^2 - Q \leq 0.\tag{3.1}$$

Since $B \in \mathcal{H}(\mathbb{H})$, $Q \geq 0$, it follows that B is unitarily diagonalizable, that is, $B = UD_1U^*$, where $D_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and U is a unitary matrix. Thus, from (3.1), we obtain

$$XUD_1U^*X + UD_1U^*X = -X^2 - Q,$$

which gives

$$(U^*XU)D_1(U^*XU) + D_1(U^*XU) = U^*(-X^2 - Q)U \leq 0.$$

Write $U^*XU = Z$. Then $Z \geq 0$ as $X \geq 0$ and

$$ZD_1Z + D_1Z = (Z + I)D_1Z \leq 0.\tag{3.2}$$

Furthermore, by writing $Z = (x_{ij})_{n \times n}$ on the LHS of inequality (3.2) and performing matrix multiplication, we obtain

$$\begin{pmatrix} \lambda_1(x_{11} + 1)x_{11} & & * \\ & \ddots & \\ * & & \lambda_n(x_{nn} + 1)x_{nn} \end{pmatrix} \leq 0.$$

Since $x_{ii} \geq 0$, it follows that $\lambda_i \leq 0$, for all $i = 1, 2, \dots, n$ and so $D_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \leq 0$. Hence $B = UD_1U^* \leq 0$. This completes the proof. $\square$

Theorem 3.2. *Suppose that the matrix B in equation (1.1) is negative semi-definite and there exists a constant $\delta_1 > 1$ satisfying*

$$2B + I \leq -Q \leq \delta_1(\delta_1 + 1)B + \delta_1^2 I.\tag{3.3}$$

Then the quaternion matrix equation (1.1) has a positive definite solution X where $I \leq X < \delta_1 I$.

Proof. Let S_1 be set of positive definite matrices given by

$$S_1 = \{X \in \mathcal{H}(\mathbb{H}) : I \leq X \leq \delta_1 I, \delta_1 > 1\}.$$

Write $M = -B$ and define $\varphi(X) = MX + XMX$. Then for all $X \in S_1$,

$$\begin{aligned} \lambda(MX) &= \lambda[M^{\frac{1}{2}}(M^{\frac{1}{2}}XM^{\frac{1}{2}})M^{-\frac{1}{2}}] \\ &= \lambda(M^{\frac{1}{2}})\lambda(M^{\frac{1}{2}}XM^{\frac{1}{2}})\lambda(M^{-\frac{1}{2}}) \\ &= \lambda(M^{\frac{1}{2}}XM^{\frac{1}{2}}) \\ &> 0, \end{aligned}$$

where $\lambda(A)$ denote the eigenvalue of a matrix A . Since $MX + XMX \in \mathcal{H}(\mathbb{H})$ and $XMX > 0$, it follows from Weyl's inequality that

$$0 < \lambda(MX) + \lambda_{\min}(XMX) \leq \lambda(MX + XMX),$$

where $\lambda_{\min}(A)$ denote the smallest eigenvalue of a matrix A . Hence $\varphi(X) > 0$. Furthermore, $\varphi'(X) = M + MX + XM > 0$ for all $X \in S_1$ which implies that $\varphi(X)$ is strictly increasing on S_1 . Note that the matrix equation (1.1) is equivalent to

$$X = (MX + XMX - Q)^{\frac{1}{2}}. \quad (3.4)$$

Now, consider the function h on S_1 defined by

$$h(X) = (MX + XMX - Q)^{\frac{1}{2}}, \quad X \in S_1. \quad (3.5)$$

For all $X \in S_1$, the monotonicity of $\varphi(X)$ gives

$$\begin{aligned} h(X) &\geq h(I) = (2M - Q)^{1/2} \geq I, \quad \text{and} \\ h(X) &\leq h(\delta_1 I) = (\delta_1 M + \delta_1^2 M - Q)^{1/2} < (\delta_1^2 I)^{1/2} = \delta_1 I. \end{aligned}$$

Thus, $h(S_1) \subseteq S_1$. Hence $h(X)$ must have an fixed point X on the compact convex set S_1 , that is, there exists a positive definite solution $X \in S_1$ to the quaternion matrix equation (1.1). This completes the proof. $\square$

Theorem 3.3. Suppose that the matrix B in equation (1.1) is negative semi-definite with $I + B > 0$ and there exists a constant $\delta_2 > 0$ such that

$$0 \leq Q \leq -\delta_2 B - \delta_2^2(I + B)^*. \quad (3.6)$$

Then the quaternion matrix equation (1.1) has a positive semi-definite solution X where $0 \leq X \leq \delta_2 I$.

Proof. Let S_2 be the set of positive semi-definite matrices given by

$$S_2 = \{X \in \mathcal{H}(\mathbb{H}) : 0 \leq X \leq \delta_2 I, \delta_2 > 0\}.$$

Since the matrix equation (1.1) is equivalent to

$$X = X(B + I)^*X + (B + I)X + Q, \quad (3.7)$$

we define a function f on S_2 by

$$f(X) = X(B + I)^*X + (B + I)X + Q, \quad \text{for all } X \in S_2. \quad (3.8)$$

Then the matrix equation (1.1) is equivalent to $X = f(X)$ on S_2 . Since $B < -\delta_2(I + B)^*$, it follows that

$$f'(X) = (B + I)^*X + X(B + I)^* + (B + I) > 0$$

for all $X \in S_2$ as $I + B > 0$. It then follows from (3.6) that

$$\begin{aligned} f(X) &\geq f(0) = Q \geq 0 \quad \text{and} \\ f(X) &\leq f(\delta_2 I) = \delta_2^2(B + I)^* + \delta_2(B + I) + Q \leq \delta_2 I. \end{aligned}$$

Hence $f(S_2) \subseteq S_2$, $f(X)$ must have a fixed point X on the compact convex set S_2 , i.e., there exists a positive semi-definite solution X to quaternion matrix equation (1.1) with $0 \leq X \leq \delta_2 I$. This completes the proof. $\square$

Theorem 3.4. *Suppose that the matrix B in equation (1.1) is negative semi-definite with $(I + B)^* = I + B^* < 0$ and there exists a positive number δ_3 such that*

$$Q + \delta_3 B + \delta_3^2(I + B^*) \leq 0. \quad (3.9)$$

Then the quaternion matrix equation (1.1) has a positive semi-definite solution X where $0 \leq X \leq \delta_3 I$.

Proof. Let S_3 the set of positive semi-definite matrices given by

$$S_3 = \{X \in \mathcal{H}(\mathbb{H}) : 0 \leq X \leq \delta_3 I, \delta_3 > 0\}.$$

Also, let α be a quaternion such that the matrix $\alpha I - B$ is invertible. Since

$$(\alpha I - B)X(\alpha I - B)^* = BXB^* + \alpha^2 X - \alpha XB^* - \alpha BX$$

it follows that equation (1.1) can be written equivalently as

$$X = (\alpha I - B)^{-1}(BXB^* + \alpha X^2 + \alpha^2 X + \alpha XB^*X - \alpha XB^* + \alpha Q)[(\alpha I - B)^{-1}]^*. \quad (3.10)$$

Now, define

$$g(X) = (\alpha I - B)^{-1}(BXB^* + \alpha X^2 + \alpha^2 X + \alpha XB^*X - \alpha XB^* + \alpha Q)[(\alpha I - B)^{-1}]^*, \quad (3.11)$$

on S_3 . Then the matrix equation (1.1) is equivalent to $X = g(X)$. For all $X \in S_3$, the eigenvalues of matrix BXB^* are satisfied that:

$$\lambda(BXB^*) = \lambda(B)\lambda(X)\lambda(B^*) > 0.$$

Hence $BXB^* > 0$, by using the condition $I + B^* < 0$ and the inequality (3.9), we get $0 < g(X)$ and

$$\begin{aligned} g(X) &\leq (\alpha I - B)^{-1} (\delta_3 BB^* + \delta_3^2 \alpha I + \delta_3 \alpha^2 I + \alpha \delta_3^2 B^* - \alpha \delta_3 B^* + \alpha Q) [(\alpha I - B)^{-1}]^* \\ &= (\alpha I - B)^{-1} [\delta_3 (\alpha I - B)(\alpha I - B)^* + \delta_3^2 \alpha (I + B^*) + \delta_3 \alpha B + \alpha Q] [(\alpha I - B)^{-1}]^* \\ &= \delta_3 I + \alpha (\alpha I - B)^{-1} [\delta_3^2 (I + B^*) + \delta_3 B + Q] [(\alpha I - B)^{-1}]^* \leq \delta_3 I. \end{aligned}$$

Thus $g(S_3) \subseteq S_3$, and hence $g(X)$ must have an fixed point X on the compact convex set S_3 , i.e., there exists a positive semi-definite solution X to equation (1.1) and $0 \leq X \leq \delta_3 I$. This completes the proof. $\square$

In the following, we discuss how to estimate the value of the parameter δ_i ($i = 1, 2, 3$) that makes inequalities (3.3), (3.6), and (3.9) valid when the matrices B, Q are given.

i) Estimating δ_1 : Since $\delta_1 > 1$, from inequality (3.3), we have

$$Q + I \leq 2M < \delta_1 (\delta_1 + 1) M \leq Q + \delta_1^2 I \quad \text{and}$$

where $M = -B \geq 0$ is positive semi-definite. For $2M < Q + \delta_1^2 I$, we easy to known that

$$\lambda(\delta_1^2 I + Q - 2M) > 0.$$

Then by the eigenvalue inequality of the quaternion Hermitian matrix [15, Theorem 2.2],

$$\begin{aligned} \lambda(\delta_1^2 I + Q - 2M) &\leq \lambda_{\max}(\delta_1^2 I + Q - 2M) \\ &\leq \lambda_{\max}(\delta_1^2 I) + \lambda_{\max}(Q - 2M) \\ &= \delta_1^2 + \lambda_{\max}(Q - 2M), \end{aligned}$$

which gives

$$\delta_1^2 + \lambda_{\max}(Q - 2M) > 0.$$

From inequality (3.3), we get $2M \geq Q$, and then one can choose

$$\delta_1 = \sqrt{\lambda_{\max}(2M - Q)}. \quad (3.12)$$

ii) Estimating δ_2 : By inequality (3.6), we have $\delta_2(-B) \geq Q + \delta_2^2 P$, where $P = (I + B)^* > 0$. For $\delta_2^2 P + \delta_2 B + Q \leq 0$, we can obtain that

$$\begin{aligned} \delta_2^2 I + \delta_2 B P^{-1} + Q P^{-1} &\leq 0, \\ \lambda(\delta_2^2 I + \delta_2 B P^{-1} + Q P^{-1}) &\leq 0. \end{aligned}$$

It then follows from [15, Theorem 2.2] that

$$\begin{aligned} \lambda_{\min}(\delta_2^2 I) + \lambda_{\min}(\delta_2 B P^{-1}) + \lambda_{\min}(Q P^{-1}) &= \delta_2^2 + \lambda_{\min}(\delta_2 B P^{-1}) + \lambda_{\min}(Q P^{-1}) \\ &\leq \lambda_{\min}(\delta_2^2 I + \delta_2 B P^{-1} + Q P^{-1}) \\ &\leq \lambda(\delta_2^2 I + \delta_2 B P^{-1} + Q P^{-1}), \end{aligned}$$

which gives

$$\delta_2^2 + \delta_2 \lambda_{\min}(B P^{-1}) + \lambda_{\min}(Q P^{-1}) \leq 0,$$

and then one can choose

$$\delta_2 = \frac{1}{2} \left[-\lambda_{\min}(B P^{-1}) + \sqrt{|\lambda_{\min}^2(B P^{-1}) - 4\lambda_{\min}(Q P^{-1})|} \right]. \quad (3.13)$$

iii) Estimating δ_3 : Remember that $R = -(I + B^*) > 0$, then from inequality (3.9), we have

$$\delta_3 B + Q \leq \delta_3^2 R,$$

which implies

$$\begin{aligned} \delta_3^2 I - \delta_3 B R^{-1} - Q R^{-1} &\geq 0, \\ \lambda(\delta_3^2 I - \delta_3 B R^{-1} - Q R^{-1}) &\geq 0. \end{aligned}$$

Therefore, from the [15, Theorem 2.2], such that

$$\begin{aligned} \lambda(\delta_3^2 I - \delta_3 B R^{-1} - Q R^{-1}) &\leq \lambda_{\max}(\delta_3^2 I - \delta_3 B R^{-1} - Q R^{-1}) \\ &\leq \lambda_{\max}(\delta_3^2 I) - \lambda_{\max}(\delta_3 B R^{-1}) + \lambda_{\max}(-Q R^{-1}) \\ &= \delta_3^2 - \delta_3 \lambda_{\max}(B R^{-1}) + \lambda_{\max}(-Q R^{-1}), \end{aligned}$$

which gives

$$\delta_3^2 - \delta_3 \lambda_{\max}(B R^{-1}) + \lambda_{\max}(-Q R^{-1}) \geq 0.$$

Since $Q R^{-1} > 0$, $\lambda_{\max}(Q R^{-1}) > 0$. And then one can choose

$$\delta_3 = \frac{1}{2} \left[\lambda_{\max}(B R^{-1}) + \sqrt{\lambda_{\max}^2(B R^{-1}) + 4\lambda_{\max}(Q R^{-1})} \right]. \quad (3.14)$$

3.2. Construction of the iterative method

According to the results of Theorems 3.1 to 3.4, when the coefficient matrices B, Q of equation (1.1) satisfy the given conditions, we can construct the iterative method to compute the positive semi-definite solution of equation (1.1). For the convenience of discussion, we set $\tilde{X}$ as a root of equation (1.1), which is the fixed point of the following three iterative methods.

a) When $M = -B \geq 0$, the proposed Newtonian iteration method is established under the conditions of Theorem 3.2 as follows:

$$\begin{cases} X_{k+1} = X_k - (2X_k - 2MX_k - M)^{-1} (X_k^2 - X_k M X_k - M X_k + Q), \\ X_0 = I_n, \quad k \in \mathbb{Z}^+. \end{cases} \quad (3.15)$$

Its equivalent complex representation is given by

$$\begin{cases} X_{k+1}^\sigma = X_k^\sigma - (2X_k^\sigma - 2M^\sigma X_k^\sigma - M^\sigma)^{-1} \left[(X_k^\sigma)^2 - X_k^\sigma M^\sigma X_k^\sigma - M^\sigma X_k^\sigma + Q^\sigma \right], \\ X_0 = I_{2n}, \quad k \in \mathbb{Z}^+, \end{cases} \quad (3.16)$$

where A^σ denotes the complex operator of the quaternion matrix A . In practice, due to the non-commutative reason of quaternion multiplication, we run MATLAB software for the iterative method of (3.16) above, and express the k -th approximate solution X_k as $X_k = X_1 + X_2 j$, which is an approximate solution of equation (1.1) (see **Algorithm 1** below). Based on the relationship between a quaternion matrix and the Frobenius norm of its complex representation matrix, the k -th approximate solution of equation (1.1) has the residual term norm of

$$R_k = \|X_k^2 + X_k B^* X_k + B X_k + Q\|_F = \frac{1}{\sqrt{2}} \| (X_k^2)^\sigma + X_k^\sigma (B^*)^\sigma X_k^\sigma + B X_k^\sigma + Q^\sigma \|_F.$$

Algorithm 1 For the Iterative Method (a), (Newtonian iteration)

- (1). **Input:** k, n , iterative function $h(X) = \text{Iterative Method}(a)$, $X_0 = I_{2n}$, T, α, B, Q .
 $(X_0 \in \mathbb{H}^{2n \times 2n}, B \in \mathbb{H}^{2n \times 2n}, Q \in \mathbb{H}^{2n \times 2n}, \text{ and } T \in \mathbb{R}, k \in \mathbb{Z}^+, n \in \mathbb{Z}^+, \alpha \in \mathbb{R}^+)$.
 - (2). Set $M = -B$, calculate $X_k = \text{Iterative Method}(a)$, then let $X^\sigma = X_k$.
 - (3). Matrix X^σ can be decomposed to $X^\sigma = \begin{pmatrix} X_1 - X_2 \\ \overline{X_2} \quad \overline{X_1} \end{pmatrix}$, $(X_1 \in \mathbb{H}^{n \times n}, X_2 \in \mathbb{H}^{n \times n})$.
 - (4). Let $X_{k+1} = X_1 + X_2 j$.
 - (5). Compute the norm $\|X_{k+1} - X_k\|_F$.
 - (6). If norm $\|X_{k+1} - X_k\|_F < T$, stop. Otherwise circle the above step (2)-(6).
 - (7). Let $X_{k+1} = X$.
 - (8). Compute the residual term norm $R_k = \|X_k^2 + X_k B^* X_k + B X_k + Q\|_F$.
 - (9). **Output:** The solution X, R_k .
-

Remark 3.1 (Convergence analysis of Newtonian Iterative Method (a)). Under the assumption of Theorem 3.2, we define a function

$$\Phi(X_k) = (X_k^2 - X_k M X_k - M X_k + Q),$$

and the Newtonian Iterative (a) can be expressed that

$$\varphi_1(X_{k+1}) = X_k - \frac{\Phi(X_k)}{\Phi'(X_k)}.$$

We take the derivative of both sides of the equation with respect to X_k ,

$$\varphi'_1(X_{k+1}) = \left[X_k - \frac{\Phi(X_k)}{\Phi'(X_k)} \right]' = \frac{\Phi(X_k)\Phi''(X_k)}{[\Phi'(X_k)]^2}.$$

For $\tilde{X}$ is a root of equation (1.1), that is, $\Phi(\tilde{X}) = 0$ and $\Phi'(\tilde{X}) \neq 0$, then from the above equation it can be concluded that $\varphi'_1(\tilde{X}) = 0$ and $\varphi''_1(\tilde{X}) = \frac{\Phi''(\tilde{X})}{\Phi'(\tilde{X})} \neq 0$. Therefore, according to Theorem 2.4, we can get that Newtonian method (a) is quadratic order convergent in the vicinity of $\tilde{X}$. Then perform a Taylor expansion of $\varphi_1(X_k)$ at point $\tilde{X}$, and using condition equation (2.1), we have

$$\varphi_1(X_k) = \varphi_1(\tilde{X}) + \frac{\varphi''_1(\xi)}{2!}(X_k - \tilde{X}), \quad X_k \leq \xi \leq \tilde{X}.$$

We noted that $\varphi_1(X_k) = X_{k+1}$, $\varphi_1(\tilde{X}) = \tilde{X}$, and from the above equation,

$$X_{k+1} - \tilde{X} = \frac{1}{2}\varphi''_1(\xi)(X_k - \tilde{X}) = \frac{\Phi''(\xi)}{2\Phi'(\xi)}(X_k - \tilde{X}), \quad X_k \leq \xi \leq \tilde{X}.$$

Therefore, regarding the iterative error e_k , when $k \rightarrow \infty$,

$$\frac{e_{k+1}}{e_k^p} \rightarrow \frac{\Phi''(\tilde{X})}{2\Phi'(\tilde{X})}.$$

Set $C_1 = \frac{\Phi''(\tilde{X})}{2\Phi'(\tilde{X})}$, based on the above analysis, we can get Newtonian method (a) is quadratic order convergent within neighborhood $U(\tilde{X}, e_k C_1) \cap U(I, \delta_1 I)$.

- b) When $B < -\delta_2(I+B)^*$, $I+B > 0$, the iteration format is established under the conditions of Theorem 3.3 as follows:

$$\begin{cases} X_{k+1} = X_k(B+I)^*X_k + (B+I)X_k + Q, \\ X_0 = 0.001I_n, k \in \mathbb{Z}^+. \end{cases} \quad (3.17)$$

Its equivalent complex representation format is:

$$\begin{cases} X_{k+1}^\sigma = X_k^\sigma (B^\sigma + I_{2n})^* X_k^\sigma + (B^\sigma + I_{2n}) X_k^\sigma + Q^\sigma, \\ X_0 = 0.001I_{2n}, k \in \mathbb{Z}^+. \end{cases} \quad (3.18)$$

The details of (3.18) iterative procedures in MATLAB is shown in **Algorithm 2**.

Algorithm 2 For the Iterative Method (b)

- (1). **Input:** k, n , iterative function $f(X) = \text{Iterative Method}(b)$, $X_0 = 0.001I_{2n}$, T, α, B, Q . ($X_0 \in \mathbb{H}^{2n \times 2n}, B \in \mathbb{H}^{2n \times 2n}, Q \in \mathbb{H}^{2n \times 2n}$, and $T \in \mathbb{R}, k \in \mathbb{Z}^+, n \in \mathbb{Z}^+, \alpha \in \mathbb{R}^+$).
 - (2). Calculate $X_k = \text{Iterative Method}(b)$, then let $X^\sigma = X_k$.
 - (3). Matrix X^σ can be decomposed to $X^\sigma = \begin{pmatrix} X_1 - X_2 \\ \overline{X_2} \quad \overline{X_1} \end{pmatrix}$, ($X_1 \in \mathbb{H}^{n \times n}, X_2 \in \mathbb{H}^{n \times n}$).
 - (4). Let $X_{k+1} = X_1 + X_2 j$.
 - (5). Compute the norm $\|X_{k+1} - X_k\|_F$.
 - (6). If norm $\|X_{k+1} - X_k\|_F < T$, stop. Otherwise circle the above step (2)-(6).
 - (7). Let $X_{k+1} = X$.
 - (8). Compute the residual term norm $R_k = \|X_k^2 + X_k B^* X_k + B X_k + Q\|_F$.
 - (9). **Output:** The solution X, R_k .
-

Remark 3.2 (Convergence analysis of Iterative Method (b)). Under the assumption of Theorems 3.1 and 3.3, we define a function

$$\varphi_2(X_k) = X_k(B+I)^*X_k + (B+I)X_k + Q,$$

and we take the derivative of both sides of the equation with respect to X_k ,

$$\varphi_2'(X_k) = (B+I)^*X_k + X_k(B+I)^* + (B+I).$$

By observation, it can be seen that when $Q = 0$ and $\tilde{X} = 0$, $\varphi_2(\tilde{X}) = \varphi_2(0) = 0 = \tilde{X}$. Therefore, $\tilde{X} = 0$ be a special fixed point of the iterative function $\varphi_2(X_k)$, we can obtain

$$0 < \varphi_2'(\tilde{X}) = \varphi_2'(0) = B + I < I,$$

which imply that $0 < \rho[\varphi_2'(\tilde{X})] = \rho[\varphi_2'(0)] < 1$. According to the Theorem 2.3, iterative method (b) converges within neighborhood $U(0, \delta_2 I)$ if the matrix Q approaches the zero matrix.

- c) When $I + B^* = (I + B)^* < 0$, the iteration format is established under the conditions of Theorem 3.4 as follows:

$$\begin{cases} X_{k+1} = (\alpha I - B)^{-1} \begin{bmatrix} BX_k B^* + \alpha X_k^2 + \alpha^2 X_k + \alpha X_k B^* X_k \\ -\alpha X_k B^* + \alpha Q \end{bmatrix} [(\alpha I - B)^{-1}]^*, \\ X_0 = 0.001I_n, k \in \mathbb{Z}^+. \end{cases} \quad (3.19)$$

Its equivalent complex representation format is:

$$\begin{cases} X_{k+1}^\sigma = (\alpha I_{2n} - B^\sigma)^{-1} \begin{bmatrix} B^\sigma X_k^\sigma (B^\sigma)^* + \alpha (X_k^\sigma)^2 + \alpha^2 X_k^\sigma + \\ \alpha X_k^\sigma (B^\sigma)^* X_k^\sigma - \alpha X_k^\sigma (B^\sigma)^* + \alpha Q^\sigma \end{bmatrix} [(\alpha I_{2n} - B^\sigma)^{-1}]^*, \\ X_0 = 0.001I_{2n}, k \in \mathbb{Z}^+. \end{cases} \quad (3.20)$$

The detail of (3.20) iterative procedures in MATLAB is shown in **Algorithm 3**.

Algorithm 3 For the Iterative Method (c)

- (1). **Input:** k, n , iterative function $g(X) = \text{Iterative Method}(c)$, $X_0 = 0.001I_{2n}$, T, α, B, Q . ($X_0 \in \mathbb{H}^{2n \times 2n}, B \in \mathbb{H}^{2n \times 2n}, Q \in \mathbb{H}^{2n \times 2n}$, and $T \in \mathbb{R}, k \in \mathbb{Z}^+, n \in \mathbb{Z}^+, \alpha \in \mathbb{R}^+$).
 - (2). Calculate $X_k = \text{Iterative Method}(c)$, then let $X^\sigma = X_k$.
 - (3). Matrix X^σ can be decomposed to $X^\sigma = \begin{pmatrix} X_1 - X_2 \\ \overline{X_2} \quad \overline{X_1} \end{pmatrix}$, ($X_1 \in \mathbb{H}^{n \times n}, X_2 \in \mathbb{H}^{n \times n}$).
 - (4). Let $X_{k+1} = X_1 + X_2 j$.
 - (5). Compute the norm $\|X_{k+1} - X_k\|_F$.
 - (6). If norm $\|X_{k+1} - X_k\|_F < T$, stop. Otherwise circle the above step (2)-(6).
 - (7). Let $X_{k+1} = X$.
 - (8). Compute the residual term norm $R_k = \|X_k^2 + X_k B^* X_k + B X_k + Q\|_F$.
 - (9). **Output:** The solution X, R_k .
-

Remark 3.3 (Convergence analysis of Iterative Method (c)). Under the assumption of Theorems 3.1 and 3.4, we define a function

$$\varphi_3(X_k) = (\alpha I - B)^{-1} \begin{bmatrix} BX_k B^* + \alpha X_k^2 + \alpha^2 X_k + \alpha X_k B^* X_k \\ -\alpha X_k B^* + \alpha Q \end{bmatrix} [(\alpha I - B)^{-1}]^*,$$

and we take the derivative of both sides of the equation with respect to X_k . And in order to ensure the invertibility of matrix $\alpha I - B$, the value of $|\alpha|$ should be sufficiently small. By using the condition $I + B^* = (I + B)^* < 0$ and $B \leq 0$,

$$\begin{aligned} & \varphi_3'(X_k) \\ &= (\alpha I - B)^{-1} (BB^* + 2\alpha X_k + \alpha^2 I + \alpha B^* X_k + \alpha X_k B^* - \alpha B^*) [(\alpha I - B)^{-1}]^* \\ &= (\alpha I - B)^{-1} [\alpha X_k B^* + \alpha(2I + B^*)X_k + \alpha^2 I - (\alpha I - B)B^*] [(\alpha I - B)^{-1}]^* \end{aligned}$$

$$\begin{aligned}
&= (\alpha I - B)^{-1} [\alpha X_k (B^* + I) + \alpha (I + B) X_k + \alpha^2 I] [(\alpha I - B)^{-1}]^* - B^* [(\alpha I - B)^{-1}]^* \\
&\leq \alpha^2 (\alpha I - B)^{-1} [(\alpha I - B)^{-1}]^* - B^* [(\alpha I - B)^{-1}]^* \\
&= \alpha^2 [(\alpha I - B)^{-1} - B^*] [(\alpha I - B)^{-1}]^* \\
&\leq \alpha^2 I,
\end{aligned}$$

From above inequality, we can get $0 < \varphi'_3(X_k) \leq \alpha^2 I$, which imply that

$$0 \leq \rho [\varphi'_3(\tilde{X})] \leq \rho(\alpha^2 I) = \alpha^2.$$

According to the Theorem 2.3, iterative method (c) converges within neighborhood $U(0, \delta_3 I)$ if the value of $|\alpha|$ small enough.

3.3. Perturbation analysis of positive semi-definite solutions

Let X be a Hermitian positive semi-definite solution of equation (1.1), and for a given matrix of coefficients B, Q , suppose that we introduce perturbations $\Delta B, \Delta Q$, respectively, and that the solution of the equation after the perturbation is $X + \Delta X$, then the equation after the perturbation of equation (1.1) is

$$\begin{aligned}
&(X + \Delta X)(X + \Delta X) + (X + \Delta X)(B + \Delta B)^*(X + \Delta X) \\
&+ (B + \Delta B)(X + \Delta X) + (Q + \Delta Q) = 0.
\end{aligned} \tag{3.21}$$

Expanding equation (3.21) and using equation (1.1) yields

$$\begin{aligned}
&(\Delta X)^2 + (X + B + \Delta B + XB^* + X\Delta B^*)\Delta X \\
&+ \Delta X[(I + B + \Delta B)^*X] + \Delta X(B + \Delta B)^*\Delta X \\
&= -(\Delta B \cdot X + X\Delta B^*X + \Delta Q).
\end{aligned} \tag{3.22}$$

The following theorem gives an upper bound on the perturbation error ΔX .

Theorem 3.5. *Let X be a Hermitian positive semi-definite solution of equation (1.1). $\Delta B, \Delta Q$ are perturbations of the coefficient matrices B, Q respectively and $X + \Delta X$ is a solution of equation (1.1) after the perturbations. If $0 < \|B + \Delta X\|_F \leq 1$, then we have*

$$\|\Delta X\|_F \leq -\frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 2\|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F}, \tag{3.23}$$

where $\lambda_n = \lambda_{\min}(X)$, $\lambda_1 = \lambda_{\max}(X)$, $r = \sqrt{n}\lambda_1 - \lambda_n + 1 > 0$ and $2\|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F < r^2$.

Proof. From equation (3.22) and the trigonometric inequality for the Frobenius norm, we obtain that

$$\begin{aligned}
&\|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \\
&= \left\| \begin{aligned} &(\Delta X)^2 + (X + B + \Delta B + XB^* + X\Delta B^*)\Delta X \\ &+ \Delta X(B + \Delta B)^*\Delta X + \Delta X(I + B + \Delta B)^*X \end{aligned} \right\|_F
\end{aligned}$$

$$\begin{aligned}
&\geq \left\| \begin{aligned} &X\Delta X + \Delta X \cdot X + (B + \Delta B)\Delta X + X(B + \Delta B)^*\Delta X \\ &+ \Delta X(B + \Delta B)^*X + \Delta X(B + \Delta B)^*\Delta X \end{aligned} \right\|_F - \|\Delta X\|_F^2 \\
&\geq \|X\Delta X + \Delta X \cdot X + (B + \Delta B)\Delta X + \Delta X(B + \Delta B)^*\Delta X\|_F \\
&\quad - 2\|X\|_F \|(B + \Delta B)^*\|_F \|\Delta X\|_F - \|\Delta X\|_F^2 \\
&\geq \|X\Delta X + \Delta X \cdot X\|_F - \|B + \Delta B\|_F \|\Delta X\|_F - \|B + \Delta B\|_F \|\Delta X\|_F^2 \\
&\quad - 2\|X\|_F \|B + \Delta B\|_F \|\Delta X\|_F - \|\Delta X\|_F^2 \\
&\geq \|X\Delta X + \Delta X \cdot X\|_F - (2\|X\|_F + 1) \|B + \Delta B\|_F \|\Delta X\|_F \\
&\quad - (\|B + \Delta B\|_F + 1) \|\Delta X\|_F^2.
\end{aligned} \tag{3.24}$$

Since X is Hermitian positive semi-definite, it follows that X can be unitary diagonally decomposed as

$$X = UD_2U^*, \tag{3.25}$$

where $D_2 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is a unitary matrix, $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$. Write $U^*(\Delta X)U = (\Delta x_{ij})_{n \times n}$. By the unitarily invariance of Frobenius norm and (3.25), we can achieve:

$$\begin{aligned}
\|X\Delta X + \Delta X \cdot X\|_F &= \|UD_2U^*\Delta X + \Delta XU D_2U^*\|_F \\
&= \|U^*(UD_2U^*\Delta X + \Delta XU D_2U^*)U\|_F \\
&= \|D_2U^*\Delta XU + U^*\Delta XU D_2\|_F \\
&= \left(\sum_{i,j=1}^n |(\lambda_i + \lambda_j) \Delta x_{ij}|^2 \right)^{1/2} \\
&= 2\lambda_n \left(\sum_{i,j=1}^n |\Delta x_{ij}|^2 \right)^{1/2} \\
&= 2\lambda_n \|U^*\Delta XU\|_F \\
&= 2\lambda_n \|\Delta X\|_F.
\end{aligned} \tag{3.26}$$

Also

$$\begin{aligned}
\|X\|_F &= \|UD_2U^*\|_F \leq \|U\|_F \|D_2\|_F \|U^*\|_F = \|D_2\|_F \\
&= \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} \leq (n\lambda_1^2)^{1/2} = \sqrt{n}\lambda_1.
\end{aligned} \tag{3.27}$$

From inequalities (3.24), (3.27), equality (3.26) and the condition $0 < \|B + \Delta X\|_F \leq 1$, we get

$$\begin{aligned}
&\|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \\
&\geq 2\lambda_n \|\Delta X\|_F - (2\|X\|_F + 1) \|\Delta X\|_F - 2\|\Delta X\|_F^2 \\
&\geq 2\lambda_n \|\Delta X\|_F - (2\sqrt{n}\lambda_1 + 1) \|\Delta X\|_F - 2\|\Delta X\|_F^2 \\
&\geq 2(\lambda_n - \sqrt{n}\lambda_1 - 1) \|\Delta X\|_F - 2\|\Delta X\|_F^2.
\end{aligned}$$

So

$$\|\Delta X\|_F^2 + (\sqrt{n}\lambda_1 - \lambda_n + 1) \|\Delta X\|_F + \frac{1}{2} \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \geq 0. \tag{3.28}$$

Write $r = \sqrt{n}\lambda_1 - \lambda_n + 1 > 0$, for $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$. Note that $r > 0$.

Taking the perturbation matrices $\Delta B, \Delta Q$ with sufficiently small Frobenius norm,

$$r > \sqrt{2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F}. \quad (3.29)$$

Thus, from inequality (3.28), we have

$$\|\Delta X\|_F \leq -\frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F}.$$

This completes the proof. $\square$

Theorem 3.5 discusses the perturbation error bounds for Hermitian positive semi-definite solutions of equation (1.1) when B is a general quaternion matrix. When B is a self-conjugate matrix, i.e., the matrix B satisfies $B^* = B$, discovering that inequality (3.28) has changed. Next we analyze the perturbation of its positive semi-definite solution when B is a Hermitian matrix. Based on the process of derivation of the inequation (3.24), we can deduce that

$$\begin{aligned} & \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \\ = & \left\| \begin{aligned} & (\Delta X)^2 + X(I+B)\Delta X + (B+\Delta B)\Delta X + X\Delta B^*\Delta X \\ & + \Delta X\Delta B^*X + \Delta X(I+B)X + \Delta X(B+\Delta B^*)\Delta X \end{aligned} \right\|_F \\ \geq & \|X(I+B)\Delta X + \Delta X(I+B)X\|_F - \|X\Delta B^*\Delta X + \Delta X\Delta B^*X\|_F \\ & - \|(B+\Delta B)\Delta X\|_F - \|\Delta X(B+\Delta B^*)\Delta X\|_F - \|\Delta X\|_F^2 \\ \geq & \|X(I+B)\Delta X + \Delta X(I+B)X\|_F - 2\|X\|_F\|\Delta B\|_F\|\Delta X\|_F \\ & - \|B+\Delta B^*\|_F\|\Delta X\|_F^2 - \|B+\Delta B\|_F\|\Delta X\|_F - \|\Delta X\|_F^2. \end{aligned} \quad (3.30)$$

Theorem 3.6. Let X be a Hermitian positive semi-definite solution of equation (1.1). $\Delta B, \Delta Q$ are perturbations of the coefficient matrices B, Q respectively, B is a Hermitian matrix, and $X + \Delta X$ is a solution of equation (1.1) after the perturbation. If $I + B > 0$, $\|B + \Delta B\| < 1$, and $\beta = \lambda_n\mu_n - \lambda_1\|\Delta B\|_F - 1 > 0$, then we have

$$\|\Delta X\|_F \leq \frac{1}{2} \left[\sqrt{n}\beta - \sqrt{n\beta^2 - 2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F} \right], (n \in \mathbb{Z}^+). \quad (3.31)$$

Where

$$\begin{aligned} \mu_n &= \lambda_{\min}(I+B), \lambda_n = \lambda_{\min}(X), \lambda_1 = \lambda_{\max}(X), \\ 2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F &< n\beta^2. \end{aligned}$$

Proof. When $I + B > 0$, a similar procedure to the proof of Theorem 3.5 leads to

$$\|X(I+B)\Delta X + \Delta X(I+B)X\|_F \geq 2\sqrt{n}\lambda_n\mu_n\|\Delta X\|_F. \quad (3.32)$$

Where $\mu_n = \lambda_{\min}(I+B)$, $\lambda_n = \lambda_{\min}(X)$, from inequality (3.30), inequality (3.32) and condition $\|B + \Delta B\|_F < 1$, we get

$$\begin{aligned} & \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \\ \geq & 2\sqrt{n}\lambda_n\mu_n\|\Delta X\|_F - \|\Delta X\|_F - 2\sqrt{n}\|\Delta B\|_F\|\Delta X\|_F - 2\|\Delta X\|_F^2 \end{aligned}$$

$$\begin{aligned}
&= (2\sqrt{n}\lambda_n\mu_n - 2\sqrt{n}\lambda_1\|\Delta B\|_F - 1) \|\Delta X\|_F - 2\|\Delta X\|_F^2 \\
&\geq 2\sqrt{n}(\lambda_n\mu_n - \lambda_1\|\Delta B\|_F - 1) \|\Delta X\|_F - 2\|\Delta X\|_F^2.
\end{aligned} \tag{3.33}$$

i.e.

$$\|\Delta X\|_F^2 - \sqrt{n}(\lambda_n\mu_n - \lambda_1\|\Delta B\|_F - 1) \|\Delta X\|_F + \frac{1}{2} \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F \geq 0. \tag{3.34}$$

When $\beta = \lambda_n\mu_n - \lambda_1\|\Delta B\|_F - 1 > 0$, taking the perturbation matrices $\Delta B, \Delta Q$ with sufficiently small Frobenius norms, such that

$$2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F < n\beta^2.$$

In this case, from inequality (3.34), we known

$$\|\Delta X\|_F \leq \frac{1}{2} \left[\sqrt{n}\beta - \sqrt{n\beta^2 - 2 \|\Delta B \cdot X + X\Delta B^*X + \Delta Q\|_F} \right], (n \in \mathbb{Z}^+).$$

This completes the proof. $\square$

Note that under the conditions of Theorem 3.5 and Theorem 3.6, from the perturbation error inequalities (3.23) and (3.31), when the perturbation factor $(\Delta B, \Delta Q) \rightarrow 0$, there is $\Delta X \rightarrow 0$.

4. Numerical experiments

Three examples of a positive semi-definite solution to equation (1.1) are given below. Here, we mainly focus on the characteristics of the system matrix of the equation (1.1) and use appropriate iterations to calculate according to the results of Theorem 3.2, Theorem 3.3 and Theorem 3.4.

To facilitate the comparison of the advantages and disadvantages of the three iterative methods to be carried out later, in conjunction with inequalities (3.3), (3.6), (3.9), we give the matrix Q that simultaneously satisfies these three inequalities:

$$Q = \begin{pmatrix} 0.3 & 0.06i + 0.06k & & \\ -0.06i - 0.06k & 0.3 & \ddots & \\ & \ddots & \ddots & 0.06i + 0.06k \\ & & -0.06i - 0.06k & 0.3 \end{pmatrix}.$$

According to the decomposition property of quaternion matrices, we get $Q = Q_1 + Q_2j$, where

$$Q_1 = \begin{pmatrix} 0.3 & 0.06i & & \\ -0.06i & 0.3 & \ddots & \\ & \ddots & \ddots & 0.06i \\ & & -0.06i & 0.3 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0.06i & & \\ -0.06i & 0 & \ddots & \\ & \ddots & \ddots & 0.06i \\ & & -0.06i & 0 \end{pmatrix}.$$

The following three examples all use the same matrix Q, Q_1 and Q_2 .

Example 4.1. Given 2 n-order quaternion matrices:

$$M = \begin{pmatrix} 0.7 & 0.03i + 0.03k & & & \\ -0.03i - 0.03k & 0.7 & & \ddots & \\ & \ddots & \ddots & \ddots & 0.03i + 0.03k \\ & & -0.03i - 0.03k & 0.7 & \end{pmatrix} = -B.$$

Try to establish an appropriate iterative format to find a positive definite solution to equation (1.1).

From the decomposition equation $M = M_1 + M_2j$, we can achieve that

$$M_1 = \begin{pmatrix} 0.7 & 0.03i & & & \\ -0.03i & 0.7 & & \ddots & \\ & \ddots & \ddots & \ddots & 0.03i \\ & & -0.03i & 0.7 & \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0.03i & & & \\ -0.03i & 0 & & \ddots & \\ & \ddots & \ddots & \ddots & 0.03i \\ & & -0.03i & 0 & \end{pmatrix}.$$

By direct calculation it is clear that

$$M = -B > Q, \quad \lambda_{\max}(2M - Q) \doteq 1.1.$$

Selected according to the estimated equation (3.12),

$$\delta_1 = \sqrt{\lambda_{\max}(2M - Q)} \doteq 1.0488.$$

That the condition (3.3) holds. Therefore, according to Theorem 3.2, there exists a positive definite solution to equation (1.1) in the interval $I \leq X \leq 1.0488I$. We use the proposed Newtonian iteration (3.15) for the solution. Taking $X_0 = I$ when $n = 5$, the calculation yields:

$$X_1 = \begin{pmatrix} 0.91 & -0.2114i & -0.1644 & -0.008i & -0.1799 \\ 0.2114i & 1.0743 & -0.1314i & 0.0155 & -0.008i \\ -0.1644 & 0.1314i & 0.8945 & -0.1314i & -0.1644 \\ 0.008i & 0.0155 & 0.1314i & 1.0743 & -0.2114i \\ -0.1799 & 0.008i & -0.1644 & 0.2114i & 0.91 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & -0.2114i & 0 & -0.008i & 0 \\ 0.2114i & 0 & -0.1314i & 0 & -0.008i \\ 0 & 0.1314i & 0 & -0.1314i & 0 \\ 0.008i & 0 & 0.1314i & 0 & -0.2114i \\ 0 & 0.008i & 0 & 0.2114i & 0 \end{pmatrix},$$

and we get the solution $\tilde{X} = X_1 + X_2 j$.

Through the discussion of Remark 3.1, we work out that $e_k C_1 = e_k C_{11} + e_k C_{12} \mathbf{j}$, where

$$e_k C_{11} = \begin{pmatrix} 0.82 & -0.1224i & -0.1846 & -0.007i & -0.1679 \\ 0.2114i & 1.0524 & -0.1255i & 0.0157 & -0.007i \\ -0.1623 & 0.1255i & 0.8722 & -0.1255i & -0.1644 \\ 0.007i & 0.0157 & 0.1255i & 1.0211 & -0.1224i \\ -0.1623 & 0.007i & -0.1846 & 0.1224i & 0.82 \end{pmatrix},$$

$$e_k C_{12} = \begin{pmatrix} 0 & -0.1224i & 0 & -0.007i & 0 \\ 0.2114i & 0 & -0.1255i & 0 & -0.007i \\ 0 & 0.1255i & 0 & -0.1255i & 0 \\ 0.007i & 0 & 0.1255i & 0 & -0.1224i \\ 0 & 0.007i & 0 & 0.1224i & 0 \end{pmatrix}.$$

Then, the Newtonian iterative method (a) converges within the neighborhood of $\tilde{X}$ belonging to $[0.8761I, 1.0445I] \cap [I, 1.0488I] = [I, 1.0445I]$.

When $n = 100, 500, 1000$, the iteration results are shown in Table 1.

Table 1. Newtonian iteration.

Matrix order n	iterative	R_k	t/s
100	30	2.5210e-15	0.165
	40	2.2084e-15	0.272
	50	2.2165e-15	0.421
500	30	4.8683e-07	6.544
	40	6.7518e-08	7.972
	50	1.4899e-08	10.252
1000	30	2.7579e-09	29.964
	40	8.2364e-10	32.742
	50	2.5507e-10	35.177

In the Table 1, we see that if the order of the matrix is small, then only 30 iterations are required by the MATLAB which runs less than 1 second, so that the residual term norm R_k reach an algebraic accuracy of roughly e-15. However, as the order of the matrix increases, its residual term norm R_k accuracy decreases. When the order of the matrix reaches 1000, it takes about 30 seconds of run time to get the residual term norm R_k accuracy to be roughly e-09.

Example 4.2. Given 2 quaternion matrices of order n :

$$B = \begin{pmatrix} -0.7 & 0.04i - 0.03k & & & \\ -0.04i + 0.03k & -0.7 & & \ddots & \\ & & \ddots & \ddots & 0.04i - 0.03k \\ & & & -0.04i + 0.03k & -0.7 \end{pmatrix}.$$

Try to establish an appropriate iterative format to find a positive semi-definite solution to equation (1.1)

From the decomposition equation $B = B_1 + B_2j$, we can get that

$$B_1 = \begin{pmatrix} -0.7 & 0.04i & & & \\ -0.04i & -0.7 & & \ddots & \\ & & \ddots & \ddots & 0.04i \\ & & & -0.04i & -0.7 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & -0.03i & & & \\ 0.03i & 0 & & \ddots & \\ & & \ddots & \ddots & -0.03i \\ & & & 0.03i & 0 \end{pmatrix}.$$

Direct calculation shows that

$$P = (I + B)^* > Q, \quad \lambda_{\min}(BP^{-1}) \doteq -1.6316, \quad \lambda_{\min}(QP^{-1}) \doteq 0.4266.$$

According to the estimation equation (3.13) select

$$\delta_2 = \frac{1}{2} \left[-\lambda_{\min}(BP^{-1}) + \sqrt{|\lambda_{\min}^2(BP^{-1}) - 4\lambda_{\min}(QP^{-1})|} \right] \doteq 1.3047.$$

That the condition (3.6) holds. Therefore, according to Theorem 3.3, there exists a positive semi-definite solution to equation (1.1) in the interval $0 \leq X \leq 1.3047I$. The solution can be obtained by iterating (3.17). Taking $X_0 = 0.001I$, when $n = 5$, the calculation can be obtained as follows:

$$X_1 = \begin{pmatrix} 0.4392 & 0.222i & -0.018 & 0i & 0 \\ -0.222i & 0.4428 & 0.0228i & -0.018 & 0 \\ -0.018 & -0.0228i & 0.4428 & 0.0228i & -0.018 \\ 0 & -0.018 & -0.0228i & 0.4452 & 0.222i \\ 0 & 0 & -0.018 & -0.222i & 0.4392 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & -0.09i & 0 & 0 & 0 \\ 0.12i & 0 & -0.09i & 0 & 0 \\ 0 & 0.12i & 0 & -0.09i & 0 \\ 0 & 0 & 0.12i & 0 & -0.09i \\ 0 & 0 & 0 & 0.12i & 0 \end{pmatrix},$$

and we get the solution $\tilde{X} = X_1 + X_2j$.

Through the result of Remark 3.2, we learned that the iterative method (b) converges within the neighborhood $U(0, 1.3047I)$ when the matrix Q approaches to the zero matrix.

When $n = 100, 500, 1000$, the iteration results are shown in Table 2.

Table 2

Matrix order n	iterative	R_k	t/s
100	20	4.0782e-09	0.097
	30	9.2013e-13	0.155
	40	2.1714e-16	0.178
500	20	9.2363e-09	3.323
	30	2.0919e-12	3.889
	40	4.8226e-16	4.471
1000	20	1.3083e-08	20.421
	30	2.9645e-12	22.732
	40	6.8325e-16	24.117

In the Table 2, we can see that when the order of the matrix reaches 100, with only 20 iterations, MATLAB obtain a numerical solution of the equation (1.1) in 1 second, while the residual term norm R_k reach an algebraic accuracy of e-09. Then the algebraic accuracy of the residual term norm R_k is getting higher with the increase of the number of iterations. When the matrix order reaches 1000, its solution time can also be well controlled at about 20 seconds, with good iteration speed.

Example 4.3. Given $2n$ -order quaternion matrices:

$$B = \begin{pmatrix} -0.75 - 0.06j - 0.03i + 0.03k & & & \\ 0.03i - 0.03k & -0.75 - 0.06j & & \ddots \\ & \ddots & \ddots & -0.03i + 0.03k \\ & & 0.03i - 0.03k & -0.75 - 0.06j \end{pmatrix}.$$

Try to establish an appropriate iterative format to find a positive semi-definite solution to equation (1.1).

From the decomposition equation $B = B_1 + B_2j$, we can get that

$$B_1 = \begin{pmatrix} -0.75 - 0.03i & & & \\ 0.03i & -0.75 & & \ddots \\ & \ddots & \ddots & -0.03i \\ & & 0.03i & -0.75 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.06 & 0.03i & & \\ -0.03i & -0.06 & & \ddots \\ & \ddots & \ddots & 0.03i \\ & & -0.03i & -0.06 \end{pmatrix}.$$

Direct calculations give

$$B > -2Q, \quad \lambda_{\max}(BR^{-1}) \doteq 2.3225, \quad \lambda_{\max}(-QR^{-1}) \doteq 0.516.$$

According to the estimation equation (3.14) choose

$$\delta_3 = \frac{1}{2} \left[\lambda_{\max}(BR^{-1}) + \sqrt{\lambda_{\max}^2(BR^{-1}) + 4\lambda_{\max}(QR^{-1})} \right] \doteq 2.5267.$$

That the condition (3.9) holds. Therefore, according to Theorem 3.4, equation (1.1) has a positive semi-definite solution in the interval $0 \leq X \leq 2.5267I$. The solution can be obtained by iterating (3.19). For the result of Remark 3, the value of $|\alpha|$ should be sufficiently small to ensure the invertibility of matrix $\alpha I - B$. So we can taking $X_0 = 0.001I$ and $\alpha = 0.95$, when $n = 5$, the calculation yields

$$X_1 = \begin{pmatrix} 0.086 & 0.0798i & -0.002 & 0 & 0 \\ -0.0798i & 0.388 & 0.0798i & -0.002 & 0 \\ -0.002 & -0.0798i & 0.388 & 0.0798i & -0.002 \\ 0 & -0.002 & -0.0798i & 0.388 & 0.0798i \\ 0 & 0 & -0.002 & -0.0798i & 0.386 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0.0732i & 0 & 0 & 0 \\ -0.0732i & 0 & 0.0732i & 0 & 0 \\ 0 & -0.0732i & 0 & 0.0732i & 0 \\ 0 & 0 & -0.0732i & 0 & 0.0732i \\ 0 & 0 & 0 & -0.0732i & 0 \end{pmatrix},$$

and we get the solution $\tilde{X} = X_1 + X_2j$.

The iterative method (c) converges within the neighborhood $U(0, 2.5267I)$ according to the Remark 3.3.

When $n = 100, 500, 1000$, the iteration results are shown in Table 3.

In the Table 3, we observe that when the order of the matrix reaches 100, it takes 60 iterations to bring the residual term norm R_k of equation (1.1) to an algebraic accuracy of e-08, and as the number of iterations increases, the algebraic accuracy of the residual term norm R_k stabilizes at e-09. When the order of the matrix reaches 1000, it takes about 1 minute of running time for the algebraic accuracy of the residual term norm R_k to stabilize at e-09.

5. Conclusion

In this work, firstly, we present some necessary and sufficient conditions for the existence of Hermitian positive definite or positive semi-definite solutions of the quadratic matrix equation (1.1) as well as iterative solution methods. Under the condition that the matrix Q is positive semi-definite, three different iterative methods are given for different cases of the coefficient matrix B . And convergent iterative formulas (3.15), (3.19) and (3.22) are constructed.

Table 3

Matrix order n	iterative	R_k	t/s
100	60	4.9062e-08	0.124
	70	3.9368e-09	0.232
	80	2.4531e-09	0.783
500	60	1.1066e-07	10.972
	70	8.8914e-09	13.223
	80	8.8244e-09	15.471
1000	60	1.5667e-07	55.496
	70	1.2590e-09	60.922
	80	1.8372e-09	61.227

In addition, the solution of equation (1.1) is analyzed by perturbation analysis and two upper bounds for the perturbation error ΔX are given.

Finally, by comparing and analyzing the three numerical algorithms, when the positive semi-definite solution of equation (1.1) exists, the iterative method using Theorem 3.3 has a high convergence speed, which can quickly converge to the algebraic accuracy of the residual term norm R_k of e-09 with a small number of iterations, and improve its accuracy requirements with the increase of the number of iterations. However, when the order of the matrix equation is small, the Newton iterative method of Theorem 3.2 can be well used to solve the matrix equation with an algebraic accuracy of e-15 rapidly. The iterative solution of Theorem 3.4 requires a higher number of iterations to reach a stable value R_k , which the algebraic accuracy of e-09.

This paper on the construction of parametric matrix inequalities and equation can be generalized to the solution of other types of nonlinear quaternion matrix equations with positive semi-definite solutions.

Acknowledgement

The authors are grateful to the anonymous reviewers for their valuable feedback.

References

- [1] M. S. I. Adam, J. Ding, Q. Huang and L. Zhu, *Solving a class of quadratic matrix equations*, Appl. Math. Lett., 2018, 82, 58–63.
- [2] S. C. Chacha, *Fixed point theorems and sensitivity analysis in solving nonlinear matrix equations: A study involving partially ordered sets*, Tanzan. J. Sci., 2024, 50(4), 813–820.
- [3] C. Chen, R.-C. Li and C. Ma, *Highly accurate doubling algorithm for quadratic matrix equation from quasi-birth-and-death process*, Linear Algebra Appl., 2019, 583, 1–45.
- [4] L. Chen, N. Yang, Y. Zhao and Z. Ma, *Fixed point theorems for set-valued G -contractions in a graphical convex metric space with applications*, J. Fixed Point Theory Appl., 2020, 22(4), Paper No. 88, 23.

- [5] X. Ding, L. Guo and P. M. Frank, *Parameterization of linear observers and its application to observer design*, IEEE Trans. Automat. Control, 1994, 39(8), 1648–1652.
- [6] X. Duan, Z. Jiang and A. Liao, *Low rank approximation solution of a class of generalized Lyapunov equation*, J. Comput. Math., 2016, 34(4), 407–420.
- [7] R. Erfanifar, K. Sayevand and M. Hajarian, *Solving system of nonlinear matrix equations over Hermitian positive definite matrices*, Linear Multilinear Algebra, 2023, 71(4), 597–630.
- [8] J. Guan, R. Song and A. Zubair, *New iteration method for a quadratic matrix equation associated with an M-matrix*, Math. Appl. (Wuhan), 2021, 34(1), 1–7.
- [9] B. Huang and C. Ma, *Some iterative algorithms for positive definite solution to nonlinear matrix equations*, J. Appl. Anal. Comput., 2019, 9(2), 526–546.
- [10] J. Huang, S. Zhang and H. Xiong, *Iterative methods of the positive definite solution to a quadratic quaternion system*, Math. Appl. (Wuhan), 2021, 34(2), 357–364.
- [11] J. P. Huang, *Parameter iterative method for solving a mixed-type Lyapunov matrix equation*, Math. Numer. Sin., 2007, 29(3), 285–292.
- [12] J. P. Huang and Y. Yu, *Complex representation of quaternion matrix equations and a structure-preserving algorithm*, Pure Appl. Math. (Xi'an), 2008, 24(2), 321–326.
- [13] M. Jleli and B. Samet, *Lyapunov-type inequalities for fractional boundary-value problems*, Electron. J. Differential Equations, 2015, No. 88, 11.
- [14] A. Kassymov and B. T. Torebek, *Lyapunov-type inequalities for a nonlinear fractional boundary value problem*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, 115(1), Paper No. 15, 10.
- [15] W. L. Li, *Quaternion Matrix*, National University of Defense Technology Press, Changsha, 2002.
- [16] J. Liu, J. Zhang and Q. Li, *Upper and lower eigenvalue summation bounds of the Lyapunov matrix differential equation and the application in a class time-varying nonlinear system*, Internat. J. Control, 2020, 93(5), 1115–1126.
- [17] J. Liu, J. Zhang and Q. Li, *Upper and lower eigenvalue summation bounds of the Lyapunov matrix differential equation and the application in a class time-varying nonlinear system*, Internat. J. Control, 2020, 93(5), 1115–1126.
- [18] L. Lu, Z. Ahmed and J. Guan, *Numerical methods for a quadratic matrix equation with a nonsingular M-matrix*, Appl. Math. Lett., 2016, 52, 46–52.
- [19] M. Masoudi and A. Salemi, *On Hermitian positive definite solutions of a nonlinear matrix equation*, J. Fixed Point Theory Appl., 2021, 23(2), Paper No. 30, 19.
- [20] M. Masoudi and A. Salemi, *On Hermitian positive definite solutions of a nonlinear matrix equation*, J. Fixed Point Theory Appl., 2021, 23(2), Paper No. 30, 19.
- [21] V. Simoncini, *Computational methods for linear matrix equations*, SIAM Rev., 2016, 58(3), 377–441.
- [22] Y. O. Vorontsov and K. D. Ikramov, *Numerical algorithm for solving quadratic matrix equations of a certain class*, Comput. Math. Math. Phys., 2014, 54(11), 1643–1646.
- [23] L. W. Yi, *Numeric Analysis*, Tsinghua University Press, Peking, 2012.

- [24] S. Zhang and L. Hu, *The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order*, AIMS Math., 2020, 5(4), 2923–2943.

Received June 2025; Accepted August 2026; Available online September 2026.