

FAST ENERGY-NORM CONVERGENCE AND HIGH EFFICIENCY IN A MULTISCALE FINITE ELEMENT COMPUTATION FOR THE HELMHOLTZ EQUATION*

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Abstract This paper develops a fast-convergent and highly efficient multiscale finite element method for the Helmholtz equation. Through the construction of tailored multiscale basis functions, the proposed method is capable of capturing both local fine-scale characteristics and global macroscale behaviors of multiscale solutions. We further systematically elaborate on the associated data structure, in which a global mapping matrix intrinsically encodes rich microscopic information inside each macroscopic element. Rigorous error analysis is then carried out, combining approximation and interpolation error estimates to establish a second-order convergence result in the energy norm, which furnishes a solid theoretical foundation for the accuracy of the proposed scheme. Representative numerical experiments are presented to verify the performance of the method with regard to computational efficiency and solution stability.

Keywords Multiscale scheme, data structure, error analysis, energy norm, fast convergence.

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1. Introduction

The Helmholtz equation, a pivotal partial differential equation in applied mathematics, serves as a cornerstone across a wide range of scientific and engineering domains, underpinning both theoretical research and practical technology development. Its core value lies in describing wave propagation phenomena, such as those governing acoustic wave propagation and electromagnetic field oscillations. This makes it indispensable for modeling complex oscillatory physical processes.

In the field of acoustics, the Helmholtz equation provides a reliable mathematical framework for modeling sound wave propagation, a capability that not only supports the analysis of

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sound's interaction with different media but also enables the design and performance evaluation of sound-absorbing materials and noise control systems used in high-speed railways, aircraft cabins and other noise-sensitive scenarios. In electromagnetics, it plays an essential role in describing the behavior of electromagnetic waves across various frequency bands, from 5G communication radio waves to fiber-optic transmission optical waves, and this description helps optimize devices like communication antennas and radar systems by predicting how electromagnetic energy propagates through different structures.

Studies on the Helmholtz equation have a rich history, and conventional numerical techniques, most notably the finite element method (FEM), have long served as the primary tools for solving it. However, these approaches struggle with critical hurdles, particularly stemming from the equation's intrinsic multiscale character: Solutions often exhibit variations across vastly disparate length scales, with fine-scale oscillations coexisting alongside macroscale trends. To resolve both, standard FEM demands an extremely dense fine mesh, which triggers a sharp rise in degrees of freedom. This not only fuels exponential growth in computational expense but also heightens the risk of numerical instabilities, especially in problems involving large wave numbers. In such cases, abrupt material property variations magnify errors and produce non-physical results that undermine the reliability of the solution.

To address traditional methods' limitations like high computational costs and poor adaptability to small-scale features, the multiscale finite element method (MsFEM) has become a promising alternative for engineering and scientific multiscale problems. MsFEM is specifically designed to capture solutions' multiscale behavior without a globally fine mesh, which causes excessive memory use and long calculations in traditional approaches. It achieves this by constructing local multiscale basis functions tailored to the domain's local physical and geometric properties. These functions not only approximate solutions across scales but also adapt to small-scale variations like material heterogeneities. Compared to standard FEM, which requires dense mesh refinement to handle small-scale details, the basis functions of MsFEM incorporate small-scale information more efficiently, thereby reducing computational burden while preserving solution accuracy.

Foundational research on the Helmholtz equation dates back decades, examples of such work include pivotal contributions by Ihlenburg and Babuska [11], Oberai and Pinsky [22], Baiges and Codina [3], Ciarlet and Stohrer [8] and Peterseim [24]. Later among notable advances, in [5] a multiscale hybrid-mixed method was put forward specifically to address the Helmholtz equation in heterogeneous domains with complex material distributions. Ohlberger and Verfurth [23] developed a heterogeneous multiscale method (HMM) for the Helmholtz equation with high material contrast. However, the explicit stability estimate presented in Theorem 3.9 of that study led to a "worst case" resolution condition, where $k^5(H + h)$ had to be sufficiently small. This rendered the method computationally intensive, as it demanded extremely fine meshes (i.e., small values of H and h), particularly when dealing with high wave numbers k . that reduced the discrete system's dimension and computational time using multiscale basis functions in acoustic media, yielding convergent results under specific conditions. Meanwhile, Li [19] and Fu et al. [9] advanced fast solvers for the generalized multiscale finite element method (GMsFEM) in Helmholtz problems, while Jiang et al. [12, 13] extended balanced truncation (BT-GMsFEM) to parameter-dependent elliptic and parabolic systems, respectively. Kumar and Dubey [17] proposed a standard sixth-order compact finite difference scheme for the 2D Helmholtz equation and further presented a more accurate compact scheme for the same equation. This new scheme was less sensitive to the associated wave number than existing sixth-order schemes, with theoretical analysis provided and its high accuracy illustrated via comparisons of numerical solutions. Cao

and Wu [4] further demonstrated the notable efficacy of interior penalty methods (IPCDGM and MsIPDPGM) for solving large wave number Helmholtz problems in complex media. In [7, 26, 27], the authors applied the local discontinuous Galerkin (LDG) method to 1D and 2D singularly perturbed convection-diffusion and reaction-diffusion problems. They also presented improved pointwise error estimates, demonstrating that uniform pointwise convergence of order $O(N^{-k})$ and $O(N^{-(k-1)})$ was derived inside the boundary and corner layers, respectively, on a Shishkin mesh. Liu and Cheng [20] developed the LDG method to a fourth-order singularly perturbed problem of reaction-diffusion type, and obtained an $O((N^{-1} \ln N)^{k+\frac{1}{2}})$ convergence order in the energy norm on Shishkin and Bakhvalov-type meshes. Their studies on the performance of numerical results under the energy norm have provided us with useful inspiration.

Recent years have seen a surge of advances in Helmholtz problem solvers, including works in [6, 10, 14, 15, 18, 21]. For instance, Jiang et al. [14] developed a multiscale finite-difference frequency-domain (FDFD) method to solve the Helmholtz equation for acoustic wave modeling. Nevertheless, this method exhibited a limitation: Its solutions contained a small frequency shift, which resulted in increasingly larger errors as the source frequency increased. Kalachikova et al. [15] investigated the Helmholtz equation governing wave propagation, which was supplemented with absorbing boundary conditions. They used a GMsFEM-based multiscale approximation on a coarse grid, constructed multiscale spaces from local spectral problems, and studied two types of basis functions. Lerer et al. [18] proposed a deep learning-based iterative approach for solving the discrete heterogeneous Helmholtz equation at high wave numbers. By combining classical iterative multigrid solvers and convolutional neural networks (CNNs) through preconditioning, they obtained a faster learned neural solver that exhibited better scalability than standard multigrid solvers. These developments in fast solvers provide the impetus for our current work, motivating us to derive new results in this domain.

In recent years, numerous efforts have exploited machine learning and physics-informed neural networks (PINNs) to tackle the Helmholtz equation and high-frequency acoustic wave propagation in heterogeneous media [1, 2, 16, 25, 28]. Specifically, [28] proposes a two-scale PINN architecture embedding small parameters directly into neural network design to resolve steep gradients arising from small parameters, without relying on Fourier feature augmentation or exhaustive truncation parameter searching. For oscillatory high-frequency wavefield simulations, [1] devises a simplified Gabor-based PINN with a learnable coordinate mapping to capture localized oscillations, mitigating the low-frequency bias of standard PINNs and eliminating manual hyperparameter tuning. Targeting wave scattering problems, [16] reformulates the Helmholtz equation as decoupled first-order wave systems and develops a Lie-generator PINN that enforces transparent boundary conditions through hard constraints, yielding remarkable error reduction and faster training compared with conventional Helmholtz PINNs.

The key contributions of this work are summarized as follows. First, we propose a multiscale finite element scheme specially designed for the Helmholtz equation, together with a dedicated construction strategy for multiscale basis functions. The resulting basis functions can resolve local multiscale features with higher fidelity while maintaining accurate numerical approximations. Second, we improve the overall stability of the scheme via a carefully devised data structure, in which a global mapping matrix weakens the coupling between macroscale and microscale basis functions and cuts down the overall computational cost. Third, we carry out comprehensive error analysis to prove the scheme achieves second-order fast convergence in the energy norm, which renders our method a robust and efficient solver for complicated multiscale Helmholtz problems.

The paper is organized as follows. Section 2 presents the Helmholtz equation along with

its weak formulation. In Section 3, two numerical strategies are elaborated on, with particular emphasis on the mechanism and data structure of the multiscale strategy. Notably, the compact system of the reduced mapping matrix in this multiscale strategy plays a key role in substantially conserving critical computational resources. Of key importance, Section 4 carries out error analysis to confirm that the method achieves fast second-order convergence with respect to the energy norm. To validate the efficacy of the proposed MsFEM, Section 5 conducts numerical experiments and comparative assessments. Lastly, Section 6 concludes the paper with a summary.

2. Helmholtz equation

We consider a two-dimensional Helmholtz equation

$$\begin{cases} \mathcal{L}u := -\Delta u - k^2 u = f(\mathbf{x}), & \text{in } \Omega, \\ u = g(\mathbf{x}), & \text{on } \Gamma_1, \\ \nabla u \cdot \mathbf{n} + iku = p(\mathbf{x}), & \text{on } \Gamma_2, \end{cases} \quad (2.1)$$

where k stands for the variable wave number, $f : \Omega \rightarrow \mathbb{C}$ denotes the given source term and $g : \Gamma_1 \rightarrow \mathbb{C}$ specifies the prescribed Dirichlet boundary condition. For the remaining boundary segment $\Gamma_2 = \partial\Omega \setminus \Gamma_1$, a mixed boundary condition is imposed: $\mathbf{n}$ refers to the unit outward normal vector on Γ_2 , and $p : \Gamma_2 \rightarrow \mathbb{C}$ serves as the Neumann boundary condition. And $\mathbf{x}$ represents a spatial variable in the two-dimensional domain.

The weak formulation of (2.1) is to find $u \in U$ such that

$$a(u, v) + ik(u, v)_{\Gamma_2} = (f, v) + (p, v)_{\Gamma_2}, \quad \forall v \in V, \quad (2.2)$$

where the function spaces U and V are defined as

$$\begin{aligned} U &= \{u \mid u \in H^1(\Omega), \quad u = g \text{ on } \Gamma_1\}, \\ V &= \{v \mid v \in H^1(\Omega), \quad v = 0 \text{ on } \Gamma_1\}. \end{aligned}$$

The associated sesquilinear form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ is defined as

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v - k^2 uv) \, d\mathbf{x} + i \int_{\Gamma_2} kuv \, ds, \quad (2.3)$$

and the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\Gamma_2}$ are defined as

$$(f, v) = \int_{\Omega} f v \, d\mathbf{x}, \quad (2.4)$$

$$(p, v)_{\Gamma_2} = \int_{\Gamma_2} p v \, ds. \quad (2.5)$$

A key focus involves clarifying how scale decomposition mitigates scale-related challenges. In high wave number or media with spatial property variations problems, scale discrepancies between global wave fields and local features often force traditional methods to compromise accuracy for efficiency. We will analyze how the multiscale strategy works, specifically by constructing basis functions that capture local behaviors while solving the problem on global coarse

meshes. This approach promises deeper insights into how scale interactions affect wave propagation.

This study should substantially improve computational performance. Refining the multiscale strategy through optimized basis functions and mesh partitioning can reduce degrees of freedom, runtime, and memory requirements while maintaining good accuracy. Such advances will enable solving large-scale Helmholtz problems that challenge conventional methods.

3. Two types of numerical strategies

3.1. Finite element method

When working with the weak formulation given in (2.2), we are confronted with an infinite-dimensional problem. To overcome this issue, we must employ suitable numerical strategies that allow us to obtain a finite-dimensional approximation. For the standard finite element method, we use $\mathcal{T}^h$ to represent a fine-grained partition of the domain Ω into discrete elements. For each element $K \in \mathcal{T}^h$, the element size satisfies the condition $0 < h_K < 1$. We define h as the maximum value among all h_K , which is mathematically expressed as $h = \max_K h_K$. Within each fine-scale element K , we introduce a set of nodal basis functions. These functions are linked to the four nodes of a rectangular element in the discretization framework, ensuring consistency with the overall spatial discretization logic.

The finite element space is constructed by means of piecewise bilinear polynomials $\mathbb{Q}^1(K)$ and can be written as

$$V^h = \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{Q}^1(K), \forall K \in \mathcal{T}^h\}. \quad (3.1)$$

The Galerkin scheme aims to find $u_g \in V^h$ such that

$$a(u_g, v) + ik(u_g, v)_{\Gamma_2} = (f, v) + (p, v)_{\Gamma_2}, \quad \forall v \in V_0^h. \quad (3.2)$$

In this context, u_g stands for the Galerkin FEM solution, and it can be written as

$$u_g = \sum_i^{gb1} u_i \psi_i, \quad (3.3)$$

where ψ_i are the finite element basis functions, and $gb1$ denotes the number of global basis functions in the FEM.

In this method, NM denotes the number of fine-scale elements distributed along both the x -axis and y -axis. For two-dimensional problems, the total number of elements in the fine mesh is determined by multiplying the element numbers in these two directions, resulting in N^2M^2 elements in total.

This relationship between the number of fine-scale elements and computational cost is critical to understanding the efficiency of the standard finite element method. As a well-documented characteristic of the method, its computational complexity scales with the total number of fine elements. Specifically, the computational cost is known to be on the order of $\mathcal{O}(N^2M^2)$, meaning that when the mesh is refined to enhance accuracy (a refinement process that increases N and M accordingly), the computational burden grows quadratically with the number of elements along each direction. This scaling behavior underscores the method's reliance on fine mesh resolution for precision, which in turn directly impacts its practical feasibility for large-scale or high-resolution simulations.

3.2. Multiscale finite element method

In contrast to the finite element method, the multiscale finite element method initiates its process by solving local problems to derive the multiscale basis functions φ_i . Specifically, for two-dimensional scenarios, the domain Ω is first partitioned into N^2 coarse elements. Within the framework of the multiscale finite element method, this N^2 quantifies a coarse-grained discretization that divides the domain into distinct coarse-scale elements. Following this partitioning step, for each individual coarse element K , we proceed to solve

$$\begin{cases} \mathcal{L}\varphi := -\Delta\varphi_i - k^2\varphi_i = 0, & \text{in } K, \\ \varphi_i = \theta_i, & \text{on } \partial K_1, \\ \nabla\varphi_i \cdot \mathbf{n} + ik\varphi_i = 0, & \text{on } \partial K_2, \end{cases} \quad (3.4)$$

where $\partial K_1 \cup \partial K_2$ denotes the entire boundary of element K . Given the specified boundary conditions (e.g., θ_i), the local problem (3.4) can be solved via the FEM with a sub-element size of $h = H_K/M$, where H_K is the size of coarse element satisfying $h \ll H_K$, and M is the number of sub-partitions along each spatial direction.

When the multiscale basis functions are acquired, we construct the multiscale space U^H as follows

$$U^H = \text{span}\{\varphi_i : i = 1, \dots, 4, \forall K \in \mathcal{T}^H\}. \quad (3.5)$$

This construction mechanism endows the multiscale basis functions with a unique capability. They can effectively capture the microscopic details inherent in the macroscopic problem (2.1), such as those governing acoustic wave propagation and electromagnetic field oscillations. These microscopic features, often characterized by rapid spatial variations or local heterogeneities, are precisely the fine-scale behaviors that traditional numerical methods struggle to resolve without excessive mesh refinement. By embedding such small-scale information into the basis functions themselves, the multiscale framework integrates these details into the macroscopic solution. This eliminates the need for the solution to explicitly “resolve” every fine-scale fluctuation.

This integration is far from trivial. The flexibility of the multiscale basis functions lies in their ability to adaptively incorporate sufficient microscopic information based on the problem’s intrinsic scale characteristics. Examples include the wavelength of acoustic waves or the frequency-dependent variations in electromagnetic fields. In doing so, they bridge the gap between the macroscopic solution’s global structure and the local fine-scale dynamics. This ensures that the overall behavior of the problem is accurately represented across all relevant scales. This dual capacity to capture both large-scale trends and small-scale nuances underscores the critical role of multiscale basis functions in enhancing the method’s accuracy and efficiency for multiphysics problems involving multiple scales in the Helmholtz equation.

The weak formulation underpinning the multiscale strategy is designed to restrict the solution to equation (2.2) within a finite-dimensional multiscale subspace U^H , with U^H being a subset of $H^1(\Omega)$. More precisely, our goal is to determine a function $u_H \in U^H$ that meets the following

$$a(u_H, v) + ik(u_H, v)_{\Gamma_2} = (f, v) + (p, v)_{\Gamma_2}, \quad \forall v \in U^H. \quad (3.6)$$

u_H is the multiscale solution, and it can be represented as

$$u_H = \sum_i^{gb2} u_i \varphi_i. \quad (3.7)$$

Here, $gb2$ denotes the count of global multiscale basis functions in the MsFEM, as opposed to the earlier mentioned $gb1$ in the FEM (see (3.3)). It is important to note that $\mathcal{O}(N^2) = gb2 \ll gb1 = \mathcal{O}(N^2 M^2)$. This relationship indicates that the multiscale scheme demands far fewer degrees of freedom (DOFs) than the standard finite element scheme.

Such a substantial reduction in complexity directly leads to notable savings in computational costs. This confirms that the multiscale scheme excels over the finite element scheme in both DOF efficiency and computational cost-effectiveness. A detailed breakdown of the data structure will be presented in the following sections.

To analyze the well-posedness of the proposed discrete scheme, we employ the framework of continuity and coercivity for sesquilinear forms. For the Helmholtz problem equipped with Robin absorbing boundary conditions, we first establish the boundedness property

$$|a(v_H, w_H)| \leq C(k) \|v_H\|_{1,H} \|w_H\|_{1,H}, \quad \forall v_H, w_H \in U^H, \quad (3.8)$$

where $\|\cdot\|_{1,H}$ stands for a mesh-dependent discrete H^1 -type norm, and the constant $C(k) > 0$ depends explicitly on the wave number k .

As we know that the Helmholtz sesquilinear form is indefinite over standard $H^1(\Omega)$. Nevertheless, under suitable resolution conditions between mesh scale H and wave number k , the real part satisfies

$$\operatorname{Re} a(v_H, v_H) \geq \frac{1}{2} \|\nabla v_H\|_{L^2(\Omega)}^2, \quad \forall v_H \in U^H. \quad (3.9)$$

This estimate yields a mesh-dependent coercivity property on the discrete multiscale space U^H , which lays the foundation for subsequent stability and error analysis.

3.3. Multiscale data structure

To provide a detailed clarification of the data structure underlying the multiscale strategy, we begin by examining a specific coarse element denoted as $K_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, which represents a rectangular subdomain in the partitioned computational grid. This choice of starting with a single coarse element allows us to systematically unpack the structural components that form the basis of the multiscale framework.

Before introducing the fine subgrid partition within each coarse element, we clarify the choice of subgrid number M adopted for solving local problem (3.4). Small values such as $M = 2$ fail to provide sufficient spatial resolution to resolve fine-scale oscillations inside coarse elements, which inevitably degrades the approximation quality of the constructed multiscale basis functions. Conversely, considerably larger choices including $M = 10$ or $M = 16$ will dramatically expand the dimension of each local boundary value problem, increasing memory consumption and runtime overhead during basis assembly. Throughout this work, we adopt $M = 4$ or $M = 5$ as a balanced configuration: It delivers enough local resolution to capture microscale features while avoiding excessive computational burden from over-refined subgrids. Consequently, a proper choice of M offers a reasonable balance between numerical accuracy and computational cost.

In the x -direction, the number of fine mesh nodes is $M + 1$, expressed as $x_i = x_{i,1} < x_{i,2} < \dots < x_{i,M+1} = x_{i+1}$. The distance between consecutive nodes satisfies $x_{i,k+1} - x_{i,k} = \frac{x_{i+1} - x_i}{M}$, $k = 1, \dots, M$. In a similar way, the fine mesh nodes $y_{j,k}$ in the y -direction adhere to an analogous pattern. Following this, the multiscale basis functions denoted as φ_i can be expressed

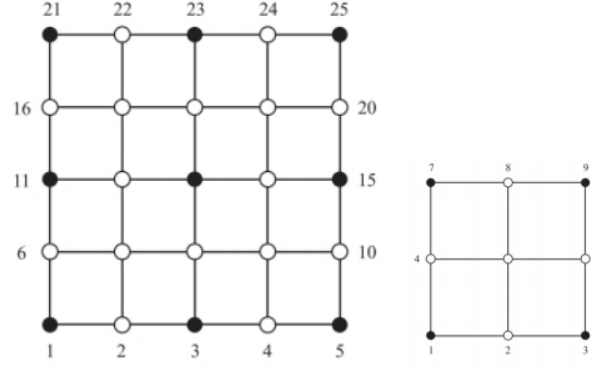


Figure 1. For $N = 2, M = 2$, fine grid nodes (total $25 = (NM + 1)^2$) with solid points denoting coarse grid nodes ($9 = (N + 1)^2$). Right part is a local zoom.

in a specific mathematical form, which encapsulates the essential features of both macroscopic trends and microscopic fluctuations inherent to the problem. This representation serves as a critical link between the coarse-scale structure and fine-scale details in the expression

$$\varphi_i = \sum_{l=1}^{(M+1)^2} r_{il} \psi_l, \quad \varphi_j = \sum_{n=1}^{(M+1)^2} r_{jn} \psi_n. \quad (3.10)$$

Here, ψ_l and ψ_n refer to the standard finite element basis functions commonly used in traditional finite element methods. The coefficients r_{il} and r_{jn} stand for the combination coefficients that bridge these two distinct groups of basis functions, quantifying their interrelationship within the multiscale framework. These coefficients are specifically stored as entries in a global mapping matrix, which serves as a key data structure for coordinating scale interactions.

For example, the entire domain is partitioned with $N = 2$ and $M = 2$, as illustrated in Figure 1(a). For the two-dimensional case, all coarse and fine grid nodes are numbered from bottom to top and left to right. The number of solid coarse grid nodes corresponds to 9 multiscale basis functions, and the total count of fine grid nodes (represented by both hollow and solid points, with solid points denoting coarse grid nodes) corresponds to 25 finite element basis functions. Together, these basis functions form the matrix subblocks of the corresponding system of equations, which collectively capture the coupling relationships between different scales in the computational framework as

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \vdots \\ \varphi_8 \\ \varphi_9 \end{bmatrix} = \begin{bmatrix} r_{1,1} & r_{1,2} & & & r_{1,6} & r_{1,7} & & & \\ & r_{2,2} & r_{2,3} & r_{2,4} & & r_{2,7} & r_{2,8} & r_{2,9} & \\ & & r_{3,4} & r_{3,5} & & r_{3,9} & r_{3,10} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & r_{8,17} & r_{8,18} & r_{8,19} & & r_{8,22} & r_{8,23} & r_{8,24} \\ & & & & & & r_{9,19} & r_{9,20} & & r_{9,24} & r_{9,25} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \vdots \\ \psi_{24} \\ \psi_{25} \end{bmatrix}.$$

In this way, (3.10) can be rewritten in the corresponding matrix block form. Let $R = (r_{ij})$

(where $i = 1, 2, \dots, 9$ and $j = 1, 2, \dots, 25$) denote the global mapping matrix, as exemplified in Figure 1(a).

As for the general case, the global mapping matrix $R = (r_{ij})$ has indices ranging from $i = 1, 2, \dots, N_c$ to $j = 1, 2, \dots, N_f$. In this multiscale context, $N_c = (N_{\text{MsFEM}} + 1)^2 = (N + 1)^2$ and $N_f = (N_{\text{FEM}} + 1)^2 = (N \cdot M + 1)^2$. For the two-dimensional Helmholtz problem, it is obvious that $N_c \ll N_f$, which indicates that the fine grid possesses a far higher degree of discretization than the coarse grid. This disparity directly reflects the hierarchical resolution design inherent to the multiscale framework, where the fine grid captures detailed local features while the coarse grid provides a macroscopic structural framework.

In the context of the multiscale scheme, the global stiffness matrix is denoted as $A^c = (a_{ij}^c) = (a(\varphi_i, \varphi_j))$ as defined in (2.3), where the superscript c explicitly indicates that this matrix corresponds to the coarse-scale problem. Building on the representation of multiscale basis functions provided in (3.10), the elements of the global stiffness matrix are assembled using the following expression

$$a_{ij}^c = \sum_{l=1}^{(M+1)^2} \sum_{n=1}^{(M+1)^2} r_{il} r_{jn} \int_K (\nabla \psi_l \cdot \nabla \psi_n - k^2 \psi_l \psi_n) \, dK \\ + i \sum_{l=1}^{(M+1)^2} \sum_{n=1}^{(M+1)^2} r_{il} r_{jn} \int_{\partial K_2} k \psi_l \psi_n \, ds,$$

where both the indices $i, j = 1, 2, \dots, (N_{\text{MsFEM}} + 1)^2$. This double-sum formulation directly incorporates the coupling between multiscale basis functions (via the coefficients r_{il}, r_{jn} from the global mapping matrix) and the local finite element basis functions ψ_l, ψ_n , ensuring that both the interior domain contributions (from the integral over K) and boundary effects (from the integral over ∂K_2) are properly accounted for in the coarse-scale matrix construction.

Taking an arbitrary coarse element $K_{i,j}$ as an example, a clearer explanation can be provided, as shown in the local zoom of Figure 1(b). As can be seen from the matrix operations between the multiscale basis functions and the finite element basis functions,

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & r_{17} & r_{18} & r_{19} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} & r_{27} & r_{28} & r_{29} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & r_{37} & r_{38} & r_{39} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & r_{47} & r_{48} & r_{49} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{bmatrix} = R_{\text{loc}} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{bmatrix}.$$

Herein, R_{loc} denotes the local mapping matrix. Therefore, the element stiffness matrix of the

multiscale finite element formulation is given by

$$\begin{aligned}
 & \begin{bmatrix} a(\varphi_1, \varphi_1) & a(\varphi_1, \varphi_2) & a(\varphi_1, \varphi_3) & a(\varphi_1, \varphi_4) \\ a(\varphi_2, \varphi_1) & a(\varphi_2, \varphi_2) & a(\varphi_2, \varphi_3) & a(\varphi_2, \varphi_4) \\ a(\varphi_3, \varphi_1) & a(\varphi_3, \varphi_2) & a(\varphi_3, \varphi_3) & a(\varphi_3, \varphi_4) \\ a(\varphi_4, \varphi_1) & a(\varphi_4, \varphi_2) & a(\varphi_4, \varphi_3) & a(\varphi_4, \varphi_4) \end{bmatrix} \\
 &= R_{\text{loc}} \begin{bmatrix} a(\psi_1, \psi_1) & a(\psi_1, \psi_2) & & a(\psi_1, \psi_4) & a(\psi_1, \psi_5) & & & & \\ a(\psi_2, \psi_1) & a(\psi_2, \psi_2) & a(\psi_2, \psi_3) & a(\psi_2, \psi_4) & a(\psi_2, \psi_5) & a(\psi_2, \psi_6) & & & \\ & a(\psi_3, \psi_2) & a(\psi_3, \psi_3) & & a(\psi_3, \psi_5) & a(\psi_3, \psi_6) & & & \\ a(\psi_4, \psi_1) & a(\psi_4, \psi_2) & & a(\psi_4, \psi_4) & a(\psi_4, \psi_5) & & a(\psi_4, \psi_7) & a(\psi_4, \psi_8) & \\ a(\psi_5, \psi_1) & a(\psi_5, \psi_2) & a(\psi_5, \psi_3) & a(\psi_5, \psi_4) & a(\psi_5, \psi_5) & a(\psi_5, \psi_6) & a(\psi_5, \psi_7) & a(\psi_5, \psi_8) & a(\psi_5, \psi_9) \\ & a(\psi_6, \psi_2) & a(\psi_6, \psi_3) & & a(\psi_6, \psi_5) & a(\psi_6, \psi_6) & & a(\psi_6, \psi_8) & a(\psi_6, \psi_9) \\ & & & a(\psi_7, \psi_4) & a(\psi_7, \psi_5) & & a(\psi_7, \psi_7) & a(\psi_7, \psi_8) & \\ & & & a(\psi_8, \psi_4) & a(\psi_8, \psi_5) & a(\psi_8, \psi_6) & a(\psi_8, \psi_7) & a(\psi_8, \psi_8) & a(\psi_8, \psi_9) \\ & & & & a(\psi_9, \psi_5) & a(\psi_9, \psi_6) & & a(\psi_9, \psi_8) & a(\psi_9, \psi_9) \end{bmatrix} R_{\text{loc}}^T.
 \end{aligned}$$

As a result, the coarse-scale global stiffness matrix on the left-hand side of the equation achieves an effective reduction in order. This reduction directly alleviates the computational burden caused by large-scale matrix operations, laying a solid foundation for efficient numerical calculations in the multiscale scheme.

Similarly, the right-hand side term at the coarse-scale defined in (2.4) is given by

$$b_i^c = (f, \varphi_i) = \sum_{l=1}^{(M+1)^2} r_{il} \int_K f \psi_l d\mathbf{x}.$$

This formulation links coarse-scale components to fine-scale integrals via the mapping coefficients r_{il} , ensuring consistency between the two scales in the numerical framework.

Building on the scale-correlation logic discussed previously, a key transformation worth emphasizing is that formula (3.10) can be converted into an equivalent matrix form. This form not only simplifies the expression of inter-scale relationships but also directly supports the subsequent construction of the coarse-scale stiffness matrix

$$\Phi = R\Psi. \quad (3.11)$$

Here, $\Phi \in \mathbb{C}^{N_c \times 1}$ serves as the coarse-scale vector. Correspondingly, $\Psi \in \mathbb{C}^{N_f \times 1}$ specifically denotes the fine-scale component at all fine-grid nodes.

The matrix R , which belongs to the set of $N_c \times N_f$ complex matrices ($R \in \mathbb{C}^{N_c \times N_f}$), is the global mapping matrix that was formally defined in the preceding sections. Each element within matrix R encodes the linear interconnection between the basis functions of the fine scale and those of the coarse scale. This linear coupling facilitates a critical computational step: Through the matrix-vector multiplication operation shown in (3.11), the coarse-scale vector Φ

can be derived directly from the fine-scale vector Ψ , effectively realizing the transformation of information between the two scales.

In comparison to the multiscale scheme, the traditional finite element method relies on a fine-scale global stiffness matrix denoted as $A^f = (a_{ij}^f) = (a(\psi_i, \psi_j))$, where the superscript f indicates its association with fine-scale computations. The indices i, j both range over $1, 2, \dots, (N_{\text{FEM}} + 1)^2$, and since $N_{\text{FEM}} = N \cdot M$, 2D problems using traditional FEM exhibit quadratically increasing computational costs, often becoming prohibitively high.

We define the fine-scale right-hand side term as $b^f = (b_i^f) = ((f, \psi_i))$, where the inner product (f, ψ_i) quantifies the contribution of each fine-scale node to the system's right-hand side. Building on the definition, the coarse-scale right-hand side term b^c satisfies the relationship $b^c = Rb^f$. This expression intuitively illustrates how b^c inherits and aggregates information from its fine-scale counterpart b^f via the mapping matrix R , thereby ensuring scale consistency within the multiscale finite element framework.

The connection between these two strategies is illustrated via

$$A^c = RA^f R^T. \quad (3.12)$$

Within the multiscale framework, the global system for solving u_H takes the form

$$A^c u_H = b^c. \quad (3.13)$$

By contrast, applying the finite element method to an extremely fine mesh leads to a drastically expanded global system, with the specific form

$$A^f u_g = b^f. \quad (3.14)$$

From a multiscale perspective, the sparse mapping matrix R acts as a scale information transfer operator. It projects fine-scale information, including oscillatory features captured by fine-grid basis functions, onto the low-dimensional coarse space spanned by multiscale basis functions. The transformation $\Phi = R\Psi$ establishes the mapping between fine-scale and coarse-scale coefficient vectors, while the relations $A^c = RA^f R^T$ and $b^c = Rb^f$ perform consistent variational compression of the fine-scale discrete system. This algebraic construction naturally retains the essential oscillatory information required for accurate coarse-grid computation, which constitutes the core merit of the proposed multiscale data structure.

In summary, this section systematically contrasts the multiscale finite element scheme with the traditional finite element method while clarifying their mathematical connections. The traditional FEM depends on a fine-scale global stiffness matrix and a corresponding right-hand side vector. However, the number of fine-scale element partitions needs to be large to capture small-scale features of the problem. This leads to quadratically increasing computational costs for two-dimensional problems, which often become too high to be practical.

Hence, the multiscale scheme makes use of a global mapping matrix. This mapping matrix inherently contains sufficient microscopic information within each macroscopic element. Using this mapping matrix, the multiscale method derives a reduced coarse-scale stiffness matrix and a reduced coarse-scale right-hand side vector. These reduced components form a compact coarse-scale system for solving the coarse-scale solution u_H . This compact system helps significantly conserve key computational resources such as memory and processing time, which clearly highlights the efficiency advantage of the multiscale method relative to traditional FEM.

4. Superconvergence analysis of energy norm

To assess the errors corresponding to numerical solutions, we will conduct norm-based tests in subsequent sections to support error analyses and numerical experiments. For quantitative evaluation, two key norms are employed: The L^2 norm and the energy norm. Their specific definitions are given by

$$\|u - u_H\|_{L^2} = \left(\int_{\Omega} (u - u_H)^2 \, d\mathbf{x} \right)^{1/2}, \quad (4.1)$$

$$\|u - u_H\|_{En} = (\|\nabla(u - u_H)\|_{L^2}^2 + k^2 \|u - u_H\|_{L^2}^2)^{1/2}. \quad (4.2)$$

Note that the MsFEM solution u_H can be replaced by the FEM solution u_g when conducting comparisons between the two methods, and k is the wave number defined in the preceding Helmholtz equation.

Let $\kappa(\mathbf{x})$ denote the medium heterogeneity with $\kappa_{\min} \leq \kappa(\mathbf{x}) \leq \kappa_{\max}$. Let $\mathcal{T}^H$ be a coarse-grained partition of Ω with mesh size H , and let Π_H denote the projection operator that maps functions from the Sobolev space $H_0^1(\Omega)$ to the multiscale space U^H . Next, let $u_I \in V^h$ be the standard piecewise linear nodal interpolant of the exact solution u . We make the assumption that the exact solution has sufficient regularity, i.e., $u \in H^3(\Omega) \cap H_0^1(\Omega)$. Additionally, let δ stand for the small parameter that quantifies the local error arising from solving the fine-scale local problems. This local error can be controlled, such that δ scales with the square of the fine-grid mesh size h (formally expressed as $\delta = \mathcal{O}(h^2)$). Finally, C_i are generic positive constants.

Lemma 4.1. *The error between the multiscale interpolant $\Pi_H u$ and the standard linear interpolant u_I in the energy norm is of second order,*

$$\|\Pi_H u - u_I\|_{En} \leq C(k, \kappa) H^2 |u|_{H^3(\Omega)}. \quad (4.3)$$

Proof. The core of superconvergence lies in the local properties of the multiscale basis functions φ_i . In the above sections, ψ_i are the standard linear hat functions. Define the correction function $\mu_i = \varphi_i - \psi_i$ on a coarse element K . By construction (3.4), $\mathcal{L}\varphi_i = (-\Delta - k^2)\varphi_i = 0$ in K , while $\mathcal{L}\psi_i$ is a linear function (since $\Delta\psi_i = 0$). Thus, μ_i satisfies

$$\mathcal{L}\mu_i = \mathcal{L}(\varphi_i - \psi_i) = k^2\psi_i \quad \text{in } K, \quad (4.4)$$

subject to appropriate boundary conditions. The right-hand side, $k^2\psi_i$, is linear. A detailed local analysis, involving Taylor expansion of u about element nodes and the application of the Bramble-Hilbert lemma to the linear functional defined by the nodal values acting on μ_i , shows that the leading $\mathcal{O}(H)$ error terms cancel out. This yields

$$\|\Pi_H u - u_I\|_{H^1(K)} \leq C(k, \kappa) H^2 |u|_{H^3(K)}. \quad (4.5)$$

Given the equivalence between $\|\cdot\|_{En}$ and $\|\cdot\|_{H^1}$ (with k -dependent constants), the result follows by summing over all coarse elements $K \in \mathcal{T}^H$. $\square$

Lemma 4.2. *The error between the multiscale interpolant and the discrete solution is bounded by*

$$\|\Pi_H u - u_H\|_{En} \leq \frac{L}{\alpha} \|\Pi_H u - u\|_{En}. \quad (4.6)$$

Here, L and α are the continuity and coercivity constants of the sesquilinear form $a(\cdot, \cdot)$ with respect to the energy norm.

Proof. The Galerkin orthogonality condition holds: $a(u - u_H, v) = 0$ for all $v \in U^H$. Choosing $v = \Pi_H u - u_H \in U^H$, we have

$$a(\Pi_H u - u_H, \Pi_H u - u_H) = a(\Pi_H u - u, \Pi_H u - u_H). \quad (4.7)$$

By the coercivity $\alpha|v|^2 \leq |a(v, v)|$ and continuity $|a(u, v)| \leq L|u||v|$ of the sesquilinear form in the energy norm, we obtain

$$\alpha \|\Pi_H u - u_H\|_{E_n}^2 \leq L \|\Pi_H u - u\|_{E_n} \cdot \|\Pi_H u - u_H\|_{E_n}. \quad (4.8)$$

Dividing both sides by $\|\Pi_H u - u_H\|_{E_n}$ yields the desired result. Note that for the Helmholtz equation, coercivity holds in the energy norm for sufficiently fine meshes or under certain conditions on k . $\square$

Lemma 4.3. *The error of the multiscale interpolation in the energy norm is of second order,*

$$\|\Pi_H u - u\|_{E_n} \leq C(k, \kappa) H^2 (|u|_{H^3(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (4.9)$$

Proof. Applying the triangle inequality

$$\|\Pi_H u - u\|_{E_n} \leq \|\Pi_H u - u_I\|_{E_n} + \|u_I - u\|_{E_n}. \quad (4.10)$$

The first term is bounded by Lemma 4.1. For the second term, the standard interpolation error estimate in H^1 gives $|u_I - u|_{H^1(\Omega)} \leq C(k, \kappa) H^2 |u|_{H^3(\Omega)}$. By the equivalence of norms, $|u_I - u|_{E_n} \leq C(k, \kappa) H^2 |u|_{H^3(\Omega)}$. The dependence on $\|f\|_{L^2}$ enters due to the potential coupling of errors across coarse element boundaries if the source term f has fine-scale structure, which can be rigorously shown via a duality argument. Combining these estimates proves the lemma. $\square$

Theorem 4.1. *Assume the multiscale basis functions satisfy the local homogeneous Helmholtz equation, and the exact solution u is sufficiently regular. Then, the error between the exact solution u and the multiscale solution u_H in the energy norm satisfies*

$$\|u - u_H\|_{E_n} \leq C(k, \kappa) H^2 (|u|_{H^3(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (4.11)$$

Proof. We now combine the lemmas to prove the main theorem. Using the triangle inequality

$$\|u - u_H\|_{E_n} \leq \|u - \Pi_H u\|_{E_n} + \|\Pi_H u - u_H\|_{E_n}. \quad (4.12)$$

Applying Lemma 4.3 to the first term and Lemmas 4.2 and 4.3 to the second term yields

$$\|u - u_H\|_{E_n} \leq C_1(k, \kappa) H^2 (|u|_{H^3(\Omega)} + \|f\|_{L^2(\Omega)}) + \frac{L}{\alpha} C_2 H^2 (|u|_{H^3(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (4.13)$$

Combining the constants concludes the proof

$$\|u - u_H\|_{E_n} \leq C(k, \kappa) H^2 (|u|_{H^3(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (4.14)$$

$\square$

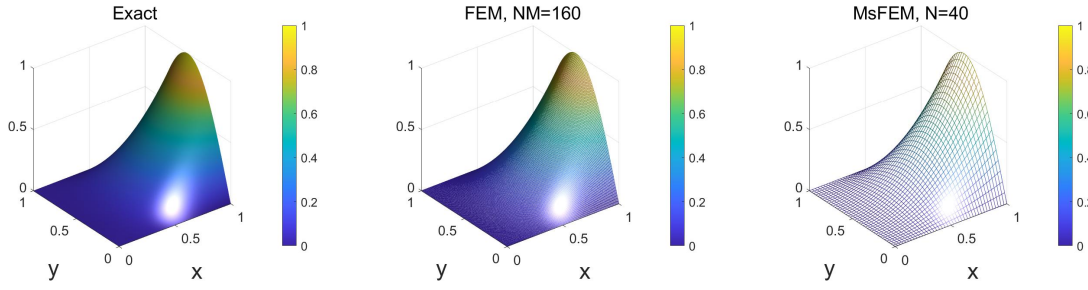


Figure 2. For Example 5.1, exact, FEM (fine mesh, $NM = 160$), and MsFEM (coarse mesh, $N = 40$) solutions.

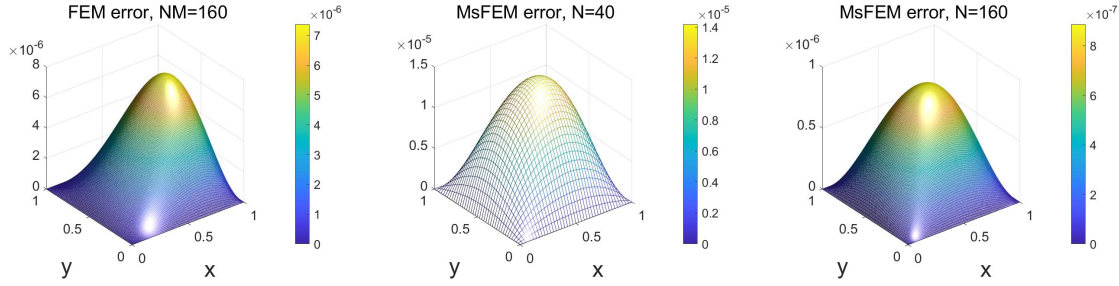


Figure 3. For Example 5.1, errors of FEM (fine mesh, $NM = 160$), MsFEM (coarse mesh, $N = 40$), and MsFEM (fine mesh, $N = 160$) solutions.

5. Numerical experiments

This section is dedicated to demonstrating how well the newly put forward multiscale scheme works when tackling the Helmholtz equation. To carry out the relevant computations, all the necessary codes are written in MATLAB 2020b. These codes are run on a desktop computer, which is outfitted with an Intel Core i9-10900K CPU @3.70GHz. It is equipped with 32GB of RAM, providing sufficient storage capacity for handling the numerical tasks.

Example 5.1. In the Helmholtz equation (2.1), we take an exact solution as

$$u(x, y) = x^2 \sin(\pi y). \quad (5.1)$$

This solution is a smooth function with explicit partial derivatives both in x and y directions, making it straightforward to compute the corresponding source term f by substituting into the Helmholtz equation. Additionally, its dependence on both spatial variables x and y allows us to test the method's ability to capture two-dimensional variations, while the trigonometric component $\sin(\pi y)$ introduces oscillatory behavior.

And we choose a variable wave number $k(x, y) = \exp(xy)$. The exponential function $\exp(xy)$ generates a smoothly varying wave number that increases with x and y , mimicking scenarios where wave propagation properties change gradually across the domain. This type of scenario is common in applications such as acoustic wave propagation. Then the source term is derived as

$$f = -\sin(\pi y)[x^2 \exp(2xy) - \pi^2 x^2 + 2].$$

Consistent with the setting introduced in Section 3.3, we take $M = 4$ for local subgrid refinement in local problem (3.4). We find that in Figure 2, both the FEM solution, computed on a fine mesh with partition number $NM = 160$, and the MsFEM solution, obtained on a

Table 1. Errors and convergence history of FEM/MsFEM for Example 5.1.

FEM					MsFEM				
NM	L^2 norm	order	energy norm	order	N	L^2 norm	order	energy norm	order
40	4.369e-5	-	1.659e-4	-	10	2.385e-4	-	1.811e-3	-
80	1.093e-5	2.07	4.153e-5	2.00	20	5.996e-5	1.99	4.547e-4	1.99
160	2.734e-6	2.00	1.038e-5	2.00	40	1.501e-5	2.00	1.138e-4	2.00
320	6.837e-7	2.00	2.596e-6	2.00	80	3.754e-6	2.00	2.846e-5	2.00
640	1.709e-7	2.00	6.491e-7	2.00	160	9.387e-7	2.00	7.116e-6	2.00
1280	-		-		320	2.346e-7	2.00	1.779e-6	2.00
2560	-		-		640	5.850e-8	2.00	4.446e-7	2.00

coarse mesh with partition number $N = 40$, both show good agreement with the exact solution. The corresponding discrete errors are visualized in Figure 3(a) and Figure 3(b). It should be admitted that while the two numerical solutions show qualitative consistency, the error of the FEM solution on the fine mesh is actually one order of magnitude smaller than that of the MsFEM solution on the coarse mesh. That said, this higher accuracy of FEM comes at the cost of significantly heavier computational loads, including longer simulation time and higher memory consumption. This contrast underscores the difference in computational cost versus solution accuracy between the two methods. For further clarification, Figure 3(c) displays the MsFEM errors calculated using the same fine mesh with $N = 160$, and these MsFEM errors are notably smaller than the FEM errors shown in Figure 3(a).

Table 1 is dedicated to analyzing the numerical errors and convergence performance of the FEM and MsFEM in the L^2 norm and energy norm for Example 5.1. Both methods exhibit normal second-order convergence in the L^2 norm, confirming their consistency in basic convergence behavior. It is worth noting that within the same row of the table the FEM, implemented on a fine mesh such as $NM = 320$, achieves higher accuracy than the MsFEM on a coarse mesh with $N = 80$. However, when both methods are applied to the same fine mesh with 320, the MsFEM outperforms the FEM in precision. Additionally, the FEM becomes computationally unfeasible on the desktop computer for extremely fine meshes like $NM = 1280$ or $NM = 2560$, whereas the MsFEM avoids this limitation. A key advantage of the MsFEM lies in its exceptional efficiency: It maintains fast convergence even when using much coarser meshes, significantly reducing computational burdens. Importantly, the results validate the superconvergence of the MsFEM in the energy norm, and notably, the MsFEM achieves this second-order superconvergence in the energy norm even with linear multiscale basis functions only. This observation aligns perfectly with the conclusion of Theorem 4.1.

Example 5.2. To further test the performance of the proposed method in capturing solutions with localized features (a common challenge in Helmholtz equation simulations, especially for problems involving boundary layers or inhomogeneous wave propagation), we introduce another exact solution for the Helmholtz equation. Specifically, the exact solution is defined as

$$u(x, y) = \exp\left(\frac{2x(x-1)}{\varepsilon}\right) \cdot \exp\left(\frac{2y(y-1)}{\varepsilon}\right), \quad (5.2)$$

where $\varepsilon = 0.1$ is a small parameter that controls the solution's spatial variation, and the variable

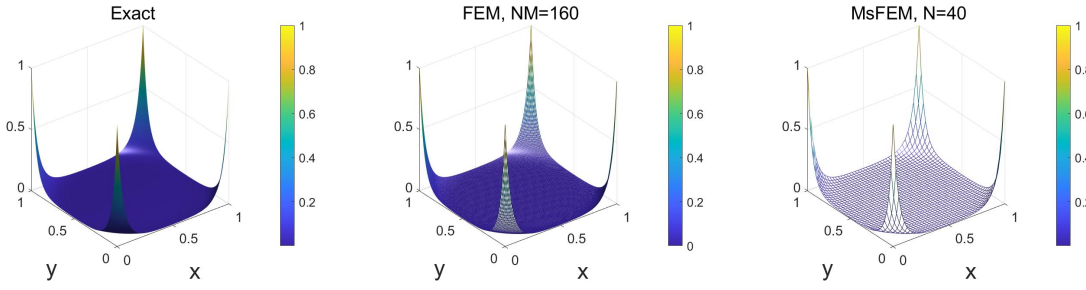


Figure 4. For Example 5.2, exact, FEM (fine mesh, $NM = 160$), and MsFEM (coarse mesh, $N = 40$) solutions.

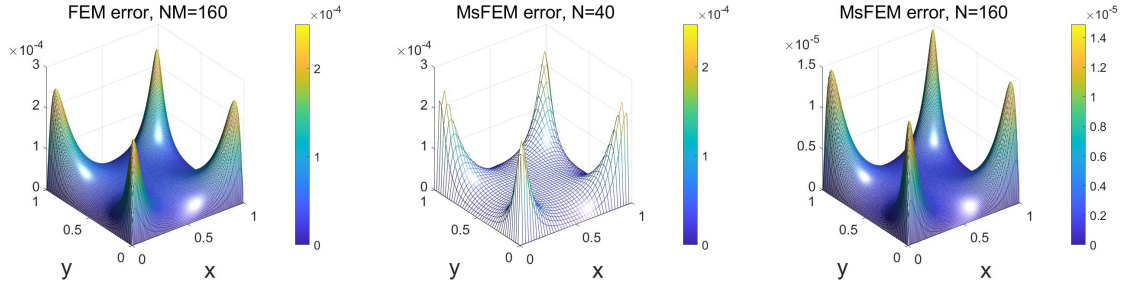


Figure 5. For Example 5.2, errors of FEM (fine mesh, $NM = 160$), MsFEM (coarse mesh, $N = 40$), and MsFEM (fine mesh, $N = 160$) solutions.

wave number $k(x, y) = \tan(xy)$. Consequently, the source term is calculated as

$$f = \frac{1}{\varepsilon^2} \left[\exp\left(\frac{2(x^2 - x + y^2 - y)}{\varepsilon}\right) (-8\varepsilon + 16x + 16y - \varepsilon^2 k^2 - 16x^2 - 16y^2 - 8) \right].$$

This source term retains the exponential structure of the exact solution, ensuring consistency between the forcing function and the solution's spatial distribution, which is a key requirement for generating a physically meaningful test case to validate the numerical method.

In Figure 4, the boundary layer phenomenon induced by the small parameter $\varepsilon = 0.1$ is clearly visualized. Boundary layers, as typical features of partial differential equation solutions with small parameters, bring challenges to accurate numerical approximation due to their rapid spatial variations. We remark that $\varepsilon = 0.1$ is not an extremely small perturbation parameter

Table 2. Errors and convergence history of FEM/MsFEM for Example 5.2.

FEM					MsFEM				
NM	L^2 norm	order	energy norm	order	N	L^2 norm	order	energy norm	order
40	4.139e-4	-	1.768e-2	-	10	2.921e-3	-	9.275e-2	-
80	1.107e-4	1.90	4.609e-3	1.94	20	8.551e-4	1.77	3.092e-2	1.58
160	2.819e-5	1.97	1.165e-3	1.98	40	2.266e-4	1.92	7.995e-3	1.95
320	7.080e-6	1.99	2.922e-4	2.00	80	5.764e-5	1.98	1.990e-3	2.01
640	1.772e-6	2.00	7.312e-5	2.00	160	1.447e-5	1.99	4.962e-4	2.00
1280	-	-	-	-	320	3.618e-6	2.00	1.240e-4	2.00
2560	-	-	-	-	640	9.044e-7	2.00	3.098e-5	2.00

in this example, so we do not adopt layer-adapted non-uniform meshes specially designed for boundary-layer capturing. Instead, uniform meshes are uniformly utilized across all test cases to objectively compare the numerical performance of the proposed scheme. Both the FEM solution on a fine mesh and the MsFEM solution on a coarse mesh in Figure 4 match the exact solution well. This demonstrates that the MsFEM can still maintain reliable accuracy for boundary layer problems without excessive mesh refinement.

Figure 5 provides quantitative error analysis, and among all solutions presented, the MsFEM solution on the fine mesh with $N = 160$ has the smallest error. Its maximum discrete error is as low as $1.5\text{e-}5$. This result highlights the MsFEM's flexibility: It balances efficiency and accuracy on coarse meshes, and can also deliver high-precision results when paired with fine meshes.

In Table 2, performance differences between the FEM and the MsFEM are observed across key evaluation metrics, providing clear insights into the strengths of each method. A particular comparison emerges when analyzing solutions on meshes of equivalent resolution: For example, the MsFEM with $N = 640$ outperforms the FEM with $NM = 640$ in the L^2 norm (with respective errors of $9.044\text{e-}7$ versus $1.772\text{e-}6$), and in the energy norm (with respective errors of $3.098\text{e-}5$ versus $7.312\text{e-}5$). This error gap underscores the MsFEM's unique advantage—even when operating on the same mesh scale, it consistently delivers higher precision than the FEM. Notably, the MsFEM demonstrates remarkable potential for extension to finer mesh partitions, a capability that further amplifies its practical value. Unlike the FEM, which often suffers from diminishing returns in efficiency (e.g., sharply increased computational cost) or compromised stability as meshes are refined, the MsFEM retains its performance advantages: It achieves highly accurate results even with relatively sparse meshes, and this efficiency-accuracy balance remains robust when applied to finer grids. In turn, this makes the MsFEM's precision and efficiency both more scientifically extensible and practically applicable than the FEM. Importantly, across all tested mesh configurations, the MsFEM maintains stable second-order convergence in both the L^2 -norm and, particularly, the energy norm. This observation aligns well with the theoretical conclusions of Theorem 4.1, further validating the reliability of the MsFEM's design and its consistency with established theoretical results.

Example 5.3. We consider an additional solution given by

$$u(x, y) = \sin(\pi x) \sin(\pi y), \quad (5.3)$$

and set a large wave number $k^2 = 10000$, then

$$f = (2\pi^2 - k^2) \sin(\pi x) \sin(\pi y).$$

Specifically, Example 5.3 adopts the Neumann boundary condition as

$$u_y(x, 0) = \pi \sin(\pi x),$$

which differs from the Dirichlet boundary condition used in above examples. This choice is intended to broaden the scope of numerical validation for our methods. Neumann conditions prescribe the solution's normal derivative (instead of its value) on domain boundaries, a common setup in practical Helmholtz applications like acoustic wave propagation where boundary energy flux is known. Testing under this condition helps evaluate the methods' robustness across different constraints, thereby ensuring their reliable performance in real-world high wave number scenarios.

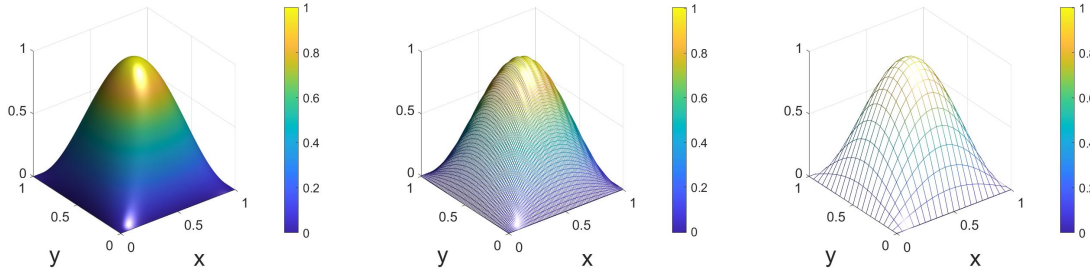


Figure 6. For Example 5.3, exact, FEM (fine mesh, $NM = 100$), and MsFEM (coarse mesh, $N = 20$) solutions.

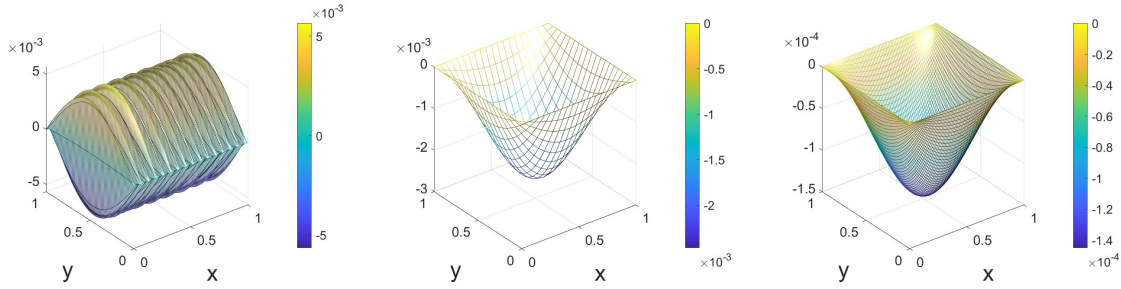


Figure 7. For Example 5.3, errors of FEM (fine mesh, $NM = 100$), MsFEM (coarse mesh, $N = 20$), and MsFEM (fine mesh, $N = 80$) solutions.

To explore the influence of subgrid resolution and conduct parameter-sensitivity tests, we deliberately set $M = 5$ for local subgrid refinement in this example, which differs from the reference value $M = 4$ given in above examples.

In Figure 6, the exact solution, the FEM solution computed on a fine mesh with $NM = 100$, and the MsFEM solution obtained using a coarse mesh with $N = 20$ are plotted respectively. A clear observation is that the FEM solution on the fine mesh ($NM = 100$) exhibits noticeable non-smooth features in the y direction, which are caused by the Neumann boundary condition adopted in this case. In contrast, such non-smoothness is effectively repaired by the MsFEM solution even when computed on the coarse mesh ($N = 20$). This visual result further reflects the advantage of the MsFEM in handling boundary condition-induced solution irregularities, even with relatively sparse meshing.

In Figure 7, we plot the discrete errors corresponding to three cases: The error of the FEM solution computed on a fine mesh with $NM = 100$, the error of the MsFEM solution obtained using a coarse mesh with $N = 20$, and the error of the MsFEM solution calculated on a fine mesh with $N = 80$. A prominent feature is that the FEM solution on the fine mesh ($NM = 100$) not only exhibits significantly large discrete errors but also suffers from obvious numerical oscillations. These issues undermine the reliability of its approximation results. Meanwhile, the MsFEM solution on the coarse mesh ($N = 20$) also maintains relatively stable error performance. Further in sharp contrast, the MsFEM solution on the fine mesh ($N = 80$) performs far better. The error in Figure 7(c) is significantly smoother and smaller than that of the FEM with $NM = 100$ in Figure 7(a), and it avoids the numerical oscillation that plagues the FEM. This further highlights the superior numerical stability and accuracy of the MsFEM compared to the FEM under the current problem settings.

For Example 5.3 with the Neumann boundary condition, Table 3 shows that the FEM solution exhibits spurious numerical oscillations and unstable high-order convergence. These

Table 3. Errors and convergence history of FEM/MsFEM for Example 5.3.

FEM					MsFEM				
NM	L^2 norm	order	energy norm	order	N	L^2 norm	order	energy norm	order
50	8.145e-4	-	1.368e+0	-	10	3.643e-4	-	8.686e-4	-
100	8.600e-6	6.56	1.808e-2	6.24	20	7.393e-5	2.30	1.360e-4	2.68
200	3.283e-7	1.39	6.608e-4	4.77	40	2.011e-5	1.88	4.155e-5	1.71
400	3.378e-8	3.28	6.152e-5	3.43	80	5.084e-6	1.98	1.064e-5	1.97
800	4.555e-9	2.89	7.051e-6	3.13	160	1.274e-6	2.00	2.676e-6	1.99
1600	-		-		320	3.185e-7	2.00	6.690e-7	2.00

issues persist even on the fine mesh $NM = 100$, indicating that mesh refinement fails to alleviate the instability induced by the boundary condition and large wave number. In contrast, the MsFEM delivers notably stable and accurate results. On the coarse mesh $N = 20$, it avoids the severe oscillations observed in the FEM and maintains a smooth solution that aligns with the exact solution. Notably, the MsFEM with $N = 160$ is more accurate than the FEM with $NM = 100$, particularly in the energy norm (2.676e-6 vs. 1.808e-2). It achieves this with a much sparser mesh, highlighting its superior efficiency and accuracy. This confirms the MsFEM's robustness for Neumann boundary conditions and high wave numbers, where FEM struggles with stability and precision.

6. Conclusion

In this work, we propose a fast-convergent multiscale finite element scheme for efficiently capturing the multiscale solution of the Helmholtz equation. We systematically expound the underlying multiscale data structure, where a global mapping matrix inherently embeds sufficient microscopic information within each macroscopic element. This characteristic not only highlights its superior efficiency but also reinforces its edge in balancing computational cost and solution fidelity. Meanwhile, rigorous error analyses yield a second-order convergence theorem with respect to the energy norm. Finally, numerical experiments validate that the proposed MsFEM produces robust solutions, which exhibit consistent stability and enhanced reliability. These advantages solidify its suitability for complex multiscale problems where traditional methods may falter, such as acoustic wave propagation in heterogeneous media and seismic modeling with fine-scale geological variations.

We also note several limitations of the present study. The current implementation and theoretical analysis are restricted to two-dimensional uniform coarse meshes. Extension to unstructured coarse meshes is not addressed and remains an open topic for further investigation. Future work will also include generalizing the approach to three-dimensional problems and exploring improved preconditioning strategies for high-wavenumber Helmholtz multiscale computations.

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