Journal of Applied Analysis and Computation
Volume 17, Number 2, April 2027, 834–850
Website:http://www.jaac-online.com
DOI:10.11948/20260158

STABILITY AND SYNCHRONIZATION OF SET-VALUED SATURATED IMPULSIVE NONLINEAR SYSTEM*

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Abstract This paper investigates the criteria for the local exponential stability and synchronization of the set-valued saturated impulsive nonlinear system. Initially, the saturation nonlinearity of the system is addressed through the sector nonlinear model approach and the polyhedral representation approach, and the sufficient condition for the local exponential stability is derived by combining the generalized Cauchy-Schwarz inequality and the average impulsive interval method. Subsequently, by designing a saturated impulsive controller, the sufficient condition for the synchronization of the drive-response system is established, and the schematic diagram of the saturated impulsive input control is briefly plotted. Ultimately, the application example of the chaotic financial system is provided, and numerical simulations are conducted to verify the effectiveness of the theoretical results.

Keywords Local exponential stability, set-valued nonlinear system, saturated impulsive, synchronization, financial system.

MSC(2010) 34A06, 93C27, 93D23.

1. Introduction

Stability and synchronization analysis has long been one of the central topics in nonlinear dynamical systems [18,33,35]. In recent years, significant progress has been achieved in synchronization and stability analysis of neural networks with complex dynamics, including exponential synchronization of n-bidirectional chaotic neural networks with state-dependent delays [2], exponential synchronization and Ulam-Hyers-Rassias stability of n spiking neural networks with time delays [13], stability and synchronization challenges of neural networks with state-dependent and distributed delays [3], etc. Motivated by these developments, increasing attention has also been devoted to dynamical systems with uncertainty. Among them, set-valued differential equations (SDEs) represent a class of mathematical models that effectively address multi-valued mappings and uncertain problems, which provide more extensive and flexible modeling capabilities, and have multiple applications in fields including financial mathematics, engineering, and control. Since the 1970s, scholars have focused on investigating SDEs, see [4, 6, 20, 22, 23, 25, 32, 36]. In 2018, Martynyuk [24] summarized qualitative and stability results for solutions of SDEs.

In addition, impulses can dramatically affect the characteristics of the system solutions, such as the stability of the solutions [17], destabilization of the solutions [15], bifurcation of the solutions [21], controllability [1], chaos [5], etc. Therefore, many experts have started to pay

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attention to the set-valued impulsive differential equations (SIDEs). For example, Perestyuk and Skripnik [27] researched the averaging method of set-valued impulsive systems. The existence of a periodic solution for the impulsive differential inclusion was established by Kryszewski and Plaskacz [12]. Ho and Ngo [9] investigated the non-instantaneous impulsive value problem of interval differential equations. Wang et al. [31] discussed the stability of the set-valued system with non-instantaneous impulses:

$$\begin{cases} \mathcal{D}_H X(t) = \Gamma X(t) + \Phi(X(t)), \quad t \in (t_k, s_k], \\ X(t) = \Psi_k(t, X(s_k - 0)), \quad t \in (s_k, t_{k+1}], \\ X(t_0) = X_0. \end{cases}$$

However, in practice, any feasible system may be constrained by technological, physical, and other factors to the point where all control systems face the issue of actuator saturation during implementation. As a result, the maximum or minimum outputs are usually bounded and cannot be infinite. The saturation characteristic may lead to a reduction in the control performance of the system, where an actuator that could have stabilized the system globally is currently only locally stabilized. A series of problems arise from such situations, which could cause irreparable damage to the actual control system. Consequently, saturation exerts an essential influence on the dynamic characteristics and stability of the systems, which makes an in-depth study of saturated impulses systems of great significance. Nevertheless, the complexity and discontinuity of the impulses themselves, as well as the nonlinear properties induced by the actuator saturation, result in a complicated nonlinear characterization of the systems. In the last few years, several studies have focused on saturated impulses control systems and obtained some interesting results, such as, Wang et al. [30] studied the fractional order saturated impulsive system; Li et al. [14] established the nonautonomous delayed saturated impulsive systems; Jiang et al. [11] focused on the second-order multi-agent systems with input saturation constraint; Li and Zhu [16] derived the local exponential stabilization of the nonlinear system with saturated impulses:

$$\begin{cases} \dot{x}(t) = Ax(t) + Wf(x(t)), \quad t \neq t_k, t \geq t_0, \\ x(t_k^+) = x(t_k^-) + B\mathrm{sat}(u(t)), \quad t = t_k, k \in \mathcal{N}, \\ x(t_0) = x_0. \end{cases}$$

Wu et al. [34] considered the finite-time stabilization; In [29], the synchronization was examined; Ouyang et al. [26] based on the polytopic representation approach, the global asymptotic stability was given. In conclusion, incorporating the saturated impulsive into the investigation of SDEs becomes essential. To the best of our knowledge, no results have been found for the set-valued differential systems with the saturated impulsive. Therefore, in this paper, the following set-valued nonlinear system with saturated impulses is established:

$$\begin{cases} \mathcal{D}_H \Phi(t) = P\Phi(t) + F(\Phi(t)), \quad t \neq t_k, t \geq t_0, \\ \Phi(t_k^+) = \Phi(t_k^-) + \mathrm{sat}(U(t)), \quad t = t_k, k \in \mathcal{N}, \\ \Phi(t_0) = \Phi_0, \end{cases} \tag{1.1}$$

in which $\Phi(t) \in \mathcal{K}_c(\mathcal{R}^n)$, $F(\Phi(t)) : \mathcal{K}_c(\mathcal{R}^n) \to \mathcal{K}_c(\mathcal{R}^n)$, $P \in \mathcal{R}^{n\times n}$. $\mathrm{sat}(U(t)) : \mathcal{K}_c(\mathcal{R}^n) \to \mathcal{K}_c(\mathcal{R}^n)$, and $\mathrm{sat}(u(t)) \in \mathrm{sat}(U(t))$ is the saturated function, where $\mathrm{sat}(u(t)) = [\mathrm{sat}_1(u_1(t)), ..., \mathrm{sat}_n(u_n(t))]^T$ with the saturation level $\hat{u} = (\hat{u}_1, ..., \hat{u}_n) \in \mathcal{R}^n_+$, for any $\hat{u}_i \in \mathcal{R}$. $\mathrm{sat}_i(u_i(t)) = \mathrm{sign}(u_i(t))\min\{\hat{u}_i, |u_i(t)|\}, i = 1, 2, ..., n$. $\{t_k, k \in \mathcal{N}\}$ is a impulse sequence, which satisfying

$0 \leq t_0 < t_1 < ... < t_k, \lim_{k\to+\infty} t_k = +\infty$. $\Phi(t_k^+) = \lim_{t\to t_k^+} \Phi(t)$ and $\Phi(t_k^-) = \lim_{t\to t_k^-} \Phi(t)$. Assume that the solution of system (1.1) is right continuous on the interval $[t_{k-1}, t_k]$, then $\Phi(t_k^+) = \Phi(t_k), t = t_k, k \in \mathcal{N}$.

In general, there are two main approaches in dealing with saturated nonlinear systems: The sector nonlinear model approach (SNMA) [28] and the polyhedral representation approach (PRA) [10]. The SNMA allows the nonlinear properties of the system to be described by defining different sectors, which enables the model to be more intuitive and easy to understand. Meanwhile, the PRA is flexible and adaptable as it can approximate the nonlinear saturation properties of the system by defining multiple faces. He et al. [8] applied these two methods to study the exponential stability of continuous-time and discrete-time neural networks with the saturated impulses. In this study, we address the local exponential stability (LES) and synchronization by extending the two techniques to the set-valued sense. Main contributions are presented below.

(i) For the first time, we consider the saturated impulses for the set-valued nonlinear system, and the SNMA and PRA in the set-valued sense are used to handle the saturated term of the system.

(ii) Since the classical Cauchy-Schwarz inequality is not applicable owing to the uncertainty of the set-valued nonlinear system, a new inequality in the set-valued sense is provided. In addition, due to the multi-valued mapping property of the set-valued, the Lyapunov function is defined through the notion of minimum upper bound for the Cartesian product.

(iii) The set-valued financial system is established as the application, and the results are applied in an interval-valued sense to obtain the LES and synchronization by using the impulsive control theory. In contrast to earlier research, the set-valued saturated impulsive nonlinear system (SSINS) considers the financial system's randomness.

The rest of this paper is arranged as: Some definitions and lemmas are given in Section 2; the exponential stability of the set-valued saturated impulsive nonlinear system by the SNMA and the PRA are proposed in Section 3; the drive system and the response system are built, and the synchronization of the error system is obtained in Section 4; the numerical examples are provided in Section 5 and Section 6 concludes the paper.

2. Preliminaries

The convex, compact and nonempty subsets of $\mathcal{R}^n$ are denoted by $\mathcal{K}_c(\mathcal{R}^n)$. Define the Hausdorff metric

$$\mathcal{D}[F, G] = \max\{\sup_{g\in G} d(g, F), \sup_{f\in F} d(f, G)\},$$

where $F, G \in \mathcal{K}_c(\mathcal{R}^n)$, $d(g, F) = \inf\{d(g, f) : f \in F\}$. In particular,

$$\mathcal{D}[F, \theta] = \sup_{f\in F} d(f, \theta),$$

where $\theta = \{(0, 0, \ldots, 0)^T\} \in \mathcal{K}_c(\mathcal{R}^n)$.

Definition 2.1 ([7])**.** Let $F, G, H \in \mathcal{K}_c(\mathcal{R}^n)$. If the equality $F = G + H$ holds, then H is known as the Hukuhara difference, and which is signified by $F - G$. The sufficient condition for $F - G$ exists is $\mathrm{diam}(F) \geq \mathrm{diam}(G)$.

Definition 2.2 ([7]). The mapping $F : [0,T] \to \mathcal{K}_c(\mathcal{R}^n)$ has a Hukuhara derivative $\mathcal{D}_H F(t_0)$ at $t_0 \in [0,T]$, if

$$\mathcal{D}_H F(t_0) = \lim_{s \to 0^+} \frac{F(t_0 + s) - F(t_0)}{s} = \lim_{s \to 0^-} \frac{F(t_0) - F(t_0 - s)}{s},$$

where $s > 0$.

Definition 2.3 ([19]). The average impulsive interval (AII) of the impulsive sequence $\{t_k, k \in \mathcal{N}\}$ is equal to $\mathcal{T}$ satisfies the inequality

$$\frac{t - t_0}{\iota} - \mathcal{T} \le \mathcal{N}(t, t_0) \le \frac{t - t_0}{\iota} + \mathcal{T}, \quad \forall t \ge t_0,$$

where $\mathcal{N}(t, t_0)$ is the number of impulsive times, $\iota > 0$ is the AII and $\mathcal{T} \ge 0$ is the elasticity number.

Definition 2.4. Let $\mathcal{E}(\Upsilon, \delta) := \{\Phi(t) \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[\Phi^T(t)\Upsilon\Phi(t), \theta] \le \delta\}$ be the ellipsoid set, where $\delta > 0$ is a scalar. For any $\Phi(t_0) \in \mathcal{E}(\Upsilon, \delta)$, if there exist constants $\mathcal{M}, \gamma > 0$, for a given impulsive sequence $\{t_k, k \in \mathcal{N}\}$ satisfying

$$\mathcal{D}[\Phi(t), \theta] \le \mathcal{M} e^{-\gamma(t - t_0)} \mathcal{D}[\Phi_0, \theta], \quad \forall t \ge t_0,$$

the SSINS (1.1) is called LES.

Let $U(t) = A\Phi(t)$ is the state feedback, where $A \in \mathcal{R}^{n \times n}$ is a constant matrix. Define the polyhedral sets

$$\mathcal{O}_{AB} = \{\Phi \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[(A_{(i)} - B_{(i)})\Phi, \theta] \le \hat{u}_i, i = 1, 2, ..., n\},$$
$$\mathcal{O}_B = \{\Phi \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[B_{(i)}\Phi, \theta] \le \hat{u}_i, i = 1, 2, ..., n\},$$

where $A, B \in \mathcal{R}^{n \times n}$, and $A_{(i)}, B_{(i)} \in \mathcal{R}^{1 \times n}$.

Lemma 2.1. *For any real diagonal positive definite matrix $C \in \mathcal{R}^{n \times n}$, if $\Phi(t_k^-) \in \mathcal{O}_{AB}$, then the nonlinearity $G(\Phi(t_k^-))$ satisfying*

$$\mathcal{D}[G^T(\Phi(t_k^-))C(G(\Phi(t_k^-)) - B\Phi(t_k^-)), \theta] \le 0, \tag{2.1}$$

where

$$G(\Phi(t_k^-)) = \Phi(t_k^-) - \mathrm{sat}(\Phi(t_k^-)), k \in \mathcal{N}. \tag{2.2}$$

Lemma 2.2. *There exist $l_m \in (0,1), m = 1, 2, ..., 2^n$, if $\Phi(t_k^-) \in \mathcal{O}_B$, then the following holds*

$$\mathrm{sat}(A\Phi(t_k^-)) = \sum_{m=1}^{2^n} l_m (W_m A + W_m^- B)\Phi(t_k^-), \tag{2.3}$$

where $\sum_{m=1}^{2^n} l_m = 1, W_m^- = I - W_m, k \in \mathcal{N}$.

Lemma 2.3 ([31]). *Let $\Upsilon \in \mathcal{R}^{n \times n}$ is a real symmetrical matrix, for any $\Phi \in \mathcal{K}_c(\mathcal{R}^n)$, the following inequality holds*

$$\lambda_{\min}(\Upsilon) \sup_{\phi \in \Phi} \phi^T \phi \le \sup_{\phi \in \Phi} \phi^T \Upsilon \phi \le \lambda_{\max}(\Upsilon) \sup_{\phi \in \Phi} \phi^T \phi,$$

where $\lambda_{\max}(\Upsilon), \lambda_{\min}(\Upsilon)$ are the largest and smallest eigenvalues of Υ.

Lemma 2.4 ([31]). *Denote matrix $\Xi \in \mathcal{R}^{n\times n}$ is symmetric and positive definite, if $X, Y \in \mathcal{K}_c(\mathcal{R}^n)$ satisfy*

$$\mathcal{D}[X^T Y, \theta] \leq \varepsilon \mathcal{D}[X, \theta]\mathcal{D}[Y, \theta], \tag{2.4}$$

for a certain $\varepsilon \in [0, 1]$, then

$$(\mathcal{D}[X^T \Xi Y, \theta])^2 \leq \left(\frac{\lambda_{\max}(\Xi) - \omega(\varepsilon)\lambda_{\min}(\Xi)}{\lambda_{\max}(\Xi) + \omega(\varepsilon)\lambda_{\min}(\Xi)}\right)^2 \mathcal{D}[X^T \Xi X, \theta]\mathcal{D}[Y^T \Xi Y, \theta], \tag{2.5}$$

where $\omega(\varepsilon) = \frac{1-\varepsilon}{1+\varepsilon}$, $\lambda_{\min}(\Xi), \lambda_{\max}(\Xi)$ are the smallest and the largest eigenvalues of Ξ.

Lemma 2.5 ([24]). *Let $X, Y : I \to \mathcal{K}_c(\mathcal{R}^n)$ be Hukuhara differentiable set-valued mappings. Suppose the Hukuhara difference $F - G$ exists in the neighborhood $I^* \subseteq I$ of a point t_0, then the mapping $X - Y : I^* \to \mathcal{K}_c(\mathcal{R}^n)$ is differentiable at the point t and*

$$D_H(X(t) - Y(t)) = D_H X(t) - D_H Y(t).$$

3. Stability result

In this section, the SNMA and PRA in the set-valued sense are used to address the saturation term and to discuss the LES.

According to $G(U(t))$ and $U(t) = A\Phi(t)$, rewrite the SSINS (1.1) as

$$\begin{cases} \mathcal{D}_H \Phi(t) = P\Phi(t) + F(\Phi(t)), & t \neq t_k, t \geq t_0, \\ \Phi(t_k) = (I + A)\Phi(t_k^-) - G(A\Phi(t_k^-)), & t = t_k, k \in \mathcal{N}, \\ \Phi(t_0) = \Phi_0, \end{cases} \tag{3.1}$$

in which, $G(\cdot) \in \mathcal{K}_c(\mathcal{R}^n)$ is defined in (2.2).

Theorem 3.1. *Let matrices $A, B \in \mathcal{R}^{n\times n}$, matrix $\Upsilon \in \mathcal{R}^{n\times n}$ is symmetric and positive definite. Ellipsoid set $\mathcal{E}(\Upsilon, \delta) := \{\Phi(t) \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[\Phi^T(t)\Upsilon\Phi(t), \theta] \leq \delta\}$, where $\delta > 0$ is a scalar. λ is the largest eigenvalue of $\Upsilon^{-1}(\Upsilon P + P^T\Upsilon)$, $\lambda_{\max}(\Upsilon)$ and $\lambda_{\min}(\Upsilon)$ are the largest and the smallest eigenvalues of Υ. If there exists a real diagonal positive definite matrix $C \in \mathcal{R}^{n\times n}$, $\alpha, K, \iota, \mathcal{T} \in \mathcal{R}^+$ satisfying $\beta + \frac{\ln\alpha}{\iota} < 0$ and*

$$\mathcal{D}[\Phi^T F(\Phi), \theta] \leq \varepsilon \mathcal{D}[\Phi, \theta]\mathcal{D}[F(\Phi), \theta], \quad t \neq t_k, t \geq t_0, \tag{3.2}$$

for a certain $\varepsilon \in [0, 1]$, and

$$\mathcal{D}[F(\Phi), \theta] \leq K\mathcal{D}[\Phi, \theta], \quad t \neq t_k, t \geq t_0, \tag{3.3}$$

$$\begin{bmatrix} \Upsilon & (A_{(i)} - B_{(i)})^T \\ \star & \delta^{-1}\hat{u}_i^2 \alpha^{\operatorname{sgn}(1-\alpha)\mathcal{T}} \end{bmatrix} \geq 0, \quad i = 1, 2, ..., n, \tag{3.4}$$

$$\begin{bmatrix} -\alpha\Upsilon + (I + A)^T\Upsilon(I + A) & -(I + A)^T\Upsilon + B^T C \\ \star & \Upsilon - 2C \end{bmatrix} \leq 0, \tag{3.5}$$

where $\beta = \lambda + 2K\frac{\lambda_{\max}(\Upsilon) - \omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon) + \omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}$, $\omega(\varepsilon) = \frac{1-\varepsilon}{1+\varepsilon}$. Then, the SSINS (1.1) is LES, and convergence exponent is $\frac{1}{2}(\beta + \ln\alpha)$.

Proof. Let Lyapunov function $V(t)=\sup\limits_{\phi\in\Phi}\phi^T\Upsilon\phi$. For $t\in[t_k,t_{k+1}), k\in\mathcal{N}$, then

$$\dot{V}(t)=\sup_{\phi\in\Phi,f\in F}\{\phi^T(\Upsilon P+P^T\Upsilon)\phi+2\phi^T\Upsilon f\}. \tag{3.6}$$

We note that the matrix $\Upsilon^{-\frac{1}{2}}(\Upsilon P+P^T\Upsilon)\Upsilon^{-\frac{1}{2}}$ and the matrix $\Upsilon^{-1}(\Upsilon P+P^T\Upsilon)$ have the same eigenvalues, denote its largest eigenvalue by λ. From Lemma 2.3, it yields

$$\begin{aligned}\sup_{\phi\in\Phi}\{\phi^T(\Upsilon P+P^T\Upsilon)\phi\}&=\sup_{\phi\in\Phi}\{(\phi^T\Upsilon^{\frac{1}{2}})(\Upsilon^{-\frac{1}{2}}(\Upsilon P+P^T\Upsilon)\Upsilon^{-\frac{1}{2}})(\Upsilon^{\frac{1}{2}}\phi)\}\\&\leq\lambda\sup_{\phi\in\Phi}\{(\phi^T\Upsilon^{\frac{1}{2}})(\Upsilon^{\frac{1}{2}}\phi)\}\\&=\lambda V(t).\end{aligned}\tag{3.7}$$

Using Lemmas 2.3, 2.4 and the inequalities (3.2), (3.3), we have

$$\begin{aligned}&\sup_{\phi\in\Phi,f\in F}\{2\phi^T\Upsilon f\}\\&=2\mathcal{D}[\Phi^T\Upsilon F,\theta]\\&\leq 2\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\mathcal{D}[\Phi^T\Upsilon\Phi,\theta]\mathcal{D}[F^T\Upsilon F,\theta]}\\&\leq 2\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{\phi\in\Phi,f\in F}\{(\phi^T\Upsilon\phi)(f^Tf)\}}\\&\leq 2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{\phi\in\Phi}\{(\phi^T\Upsilon\phi)(\phi^T\phi)\}}\\&=2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{\phi\in\Phi}\{(\phi^T\Upsilon\phi)(\phi^T\Upsilon^{\frac{1}{2}}\Upsilon^{-1}\Upsilon^{\frac{1}{2}}\phi)\}}\\&\leq 2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}\sup_{\phi\in\Phi}\{(\phi^T\Upsilon\phi)(\phi^T\Upsilon^{\frac{1}{2}}\Upsilon^{\frac{1}{2}}\phi)\}}\\&=2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}V(t).\end{aligned}\tag{3.8}$$

Combining the inequalities (3.7) and (3.8), one can gain

$$\dot{V}(t)\leq\left(\lambda+2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}\right)V(t). \tag{3.9}$$

Denote $\beta=\lambda+2K\frac{\lambda_{\max}(\Upsilon)-\omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}$, thus

$$V(t)\leq e^{\beta(t-t_k)},\quad t\in[t_k,t_{k+1}),k\in\mathcal{N}. \tag{3.10}$$

For $\Phi(t_0)\in\mathcal{E}(\Upsilon,\delta)$, assume that $\Phi(t_k^-)\in\mathcal{O}_{AB}, k\in\mathcal{N}$. If not, then there are some impulsive times $t_{k_j}\in\{t_k,k\in\mathcal{N}\}, j=1,2,...,m$, satisfying $\Phi(t_{k_j}^-)\notin\mathcal{O}_{AB}$. Let $t_{k_j}=t_1$, then $\Phi(t_1^-)\notin\mathcal{O}_{AB}$.

According the AII and the inequality (3.10), when $t \in [t_0, t_1)$ one can deduce the subsequent inequality

$$\begin{aligned} V(t_1^-) &\leq e^{\beta(t_1-t_0)}V(t_0) \\ &\leq e^{-\frac{\ln\alpha}{\iota}(t_1-t_0)}V(t_0) \\ &\leq e^{-\ln\alpha(\mathcal{N}(t_1,t_0)+\mathrm{sgn}(1-\alpha)\mathcal{T})}V(t_0) \\ &\leq \alpha^{-\mathrm{sgn}(1-\alpha)\mathcal{T}}V(t_0). \end{aligned} \tag{3.11}$$

By the LMI (3.4), we get

$$\delta \hat{u}_i^{-2}\alpha^{-\mathrm{sgn}(1-\alpha)\mathcal{T}}(A_{(i)} - B_{(i)})^T(A_{(i)} - B_{(i)}) \leq \Upsilon, \tag{3.12}$$

it shows that for any $\Phi(t_0) \in \mathcal{E}(\Upsilon, \delta)$, then $\Phi(t_1^-) \in \mathcal{O}_{AB}$, which is in contradiction with $\Phi(t_1^-) \notin \mathcal{O}_{AB}$.

Denote that $t_s (s \geq 2)$ is the first impulsive time such that $\Phi(t_s^-) \notin \mathcal{O}_{AB}$, then $\Phi(t_k^-) \in \mathcal{O}_{AB}$ holds for $t = t_k^-, k = 1, 2, ..., s-1$. By Lemma 2.1, we have the following sector condition in the set-valued sense

$$\mathcal{D}[2G^T(A\Phi(t_k^-))C(G(A\Phi(t_k^-)) - B\Phi(t_k^-)), \theta] \leq 0, \tag{3.13}$$

where $k = 1, 2, ..., s-1$.

Combining (3.5) and (3.13), one derives

$$\begin{aligned} V(t_k) &= \sup_{\phi\in\Phi} \phi^T\Upsilon\phi \\ &= \sup_{\phi\in\Phi}\{((I+A)\phi(t_k^-) - G(A\phi(t_k^-)))^T\Upsilon((I+A)\phi(t_k^-) - G(A\phi(t_k^-)))\} \\ &\leq \sup_{\phi\in\Phi}\{((I+A)\phi(t_k^-) - G(A\phi(t_k^-)))^T\Upsilon((I+A)\phi(t_k^-) - G(A\phi(t_k^-))) \\ &\quad - 2G^T(A\phi(t_k^-))C(G(A\phi(t_k^-)) - B\phi(t_k^-))\} \\ &= \sup_{\phi\in\Phi}\left\{\begin{bmatrix}\phi^T(t_k^-) \; G^T(A\phi(t_k^-))\end{bmatrix} \times \Xi \times \begin{bmatrix}\phi(t_k^-) \\ G(A\phi(t_k^-))\end{bmatrix} + \alpha\phi^T(t_k^-)\Upsilon\phi(t_k^-)\right\} \\ &\leq \alpha V(t_k^-), \end{aligned} \tag{3.14}$$

where

$$\Xi = \begin{bmatrix} -\alpha\Upsilon + (I+A)^T\Upsilon(I+A) & -(I+A)^T\Upsilon + B^TC \\ \star & \Upsilon - 2C \end{bmatrix}.$$

For $t \in [t_{s-1}, t_s)$, considering (3.10) and (3.14), one yields

$$\begin{aligned} V(t) &\leq e^{\beta(t-t_{s-1})}V(t_{s-1}) \\ &\leq \alpha e^{\beta(t-t_{s-1})}V(t_{s-1}^-) \\ &\leq \alpha^2 e^{\beta(t-t_{s-2})}V(t_{s-2}^-) \\ &\leq ... \\ &\leq \alpha^{s-1}e^{\beta(t-t_0)}V(t_0), \end{aligned} \tag{3.15}$$

hence,

$$\begin{aligned}V(t_s^-) &\leq \alpha^{s-1}e^{\beta(t_s-t_0)}V(t_0)\\ &\leq \alpha^{s-1}e^{-\frac{\ln\alpha}{\iota}(t_s-t_0)}V(t_0)\\ &\leq \alpha^{s-1}e^{-\ln\alpha(\mathcal{N}(t_s-t_0)+\mathrm{sgn}(1-\alpha)\mathcal{T})}V(t_0)\\ &\leq \alpha^{-\mathrm{sgn}(1-\alpha)\mathcal{T}}V(t_0),\end{aligned} \tag{3.16}$$

similar to (3.12), we have

$$\delta\hat{u}_i^{-2}\alpha^{-\mathrm{sgn}(1-\alpha)\mathcal{T}}(A_{(i)}-B_{(i)})^T(A_{(i)}-B_{(i)}) \leq \Upsilon, \tag{3.17}$$

which means for any $\Phi(t_0) \in \mathcal{E}(\Upsilon,\delta)$, then $\Phi(t_s^-) \in \mathcal{O}_{AB}$. It is contradictory to $\Phi(t_s^-) \notin \mathcal{O}_{AB}$.

Following the condition (3.5) and the inequality (3.15), we have

$$V(t_k) \leq \alpha V(t_k^-), \quad k \in \mathcal{N}. \tag{3.18}$$

According to (3.10) and (3.18), it yields

$$V(t) \leq \alpha^k e^{\beta(t-t_0)}V(t_0), \tag{3.19}$$

for any $t \in [t_k, t_{k+1})$.

Hence, for any $t \geq t_0$, it follows that

$$\begin{aligned}V(t) &\leq \alpha^{\mathcal{N}(t,t_0)}e^{\beta(t-t_0)}V(t_0)\\ &\leq \alpha^{\frac{t-t_0}{\iota}-\mathrm{sgn}(1-\alpha)\mathcal{T}}e^{\beta(t-t_0)}V(t_0)\\ &= \alpha^{-\mathrm{sgn}(1-\alpha)\mathcal{T}}e^{(\beta+\ln\alpha)(t-t_0)}V(t_0),\end{aligned} \tag{3.20}$$

therefore, there exists a $\mathcal{L} = \max\{\alpha^{-\mathcal{T}}, 1, \alpha^{\mathcal{T}}\}$ satisfying

$$V(t) \leq \mathcal{L}e^{(\beta+\ln\alpha)(t-t_0)}V(t_0), \tag{3.21}$$

which is equal to

$$\mathcal{D}[\Phi(t),\theta] \leq \sqrt{\frac{\mathcal{L}\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}e^{\frac{1}{2}(\beta+\ln\alpha)(t-t_0)}\mathcal{D}[\Phi(t_0),\theta], \quad \forall t \geq t_0. \tag{3.22}$$

Further, base on Definition 2.3, the SSINS (1.1) is LES in ellipsoid set $\mathcal{E}(\Upsilon,\delta)$. □

Next, we use the PRA to derive the LES of the SSINS (1.1).

Following Lemma 2.2, if $\Phi(t_k^-) \in \mathcal{O}_B, k \in \mathcal{N}$, then the SSINS (1.1) is reformulated as

$$\begin{cases}\mathcal{D}_H\Phi(t) = P\Phi(t) + F(\Phi(t)), \quad t \neq t_k, t \geq t_0,\\ \Phi(t_k) = \sum_{m=1}^{2^n} l_m(I + W_mB + W_m^-B)\Phi(t_k^-), \quad t = t_k, k \in \mathcal{N},\\ \Phi(t_0) = \Phi_0,\end{cases} \tag{3.23}$$

where $\sum_{m=1}^{2^n} l_m = 1, W_m^- = I - W_m$.

Theorem 3.2. *Let matrices $A, B \in \mathcal{R}^{n\times n}$, matrix $\Upsilon \in \mathcal{R}^{n\times n}$ is symmetric and positive definite. Ellipsoid set $\mathcal{E}(\Upsilon, \delta) := \{\Phi(t) \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[\Phi^T(t)\Upsilon\Phi(t), \theta] \leq \delta\}$, where $\delta > 0$ is a scalar. If $\alpha, K, \iota, \mathcal{T} \in \mathcal{R}^+$ satisfying $\beta + \frac{\ln\alpha}{\iota} < 0$ and*

$$\mathcal{D}[\Phi^T F(\Phi), \theta] \leq \varepsilon \mathcal{D}[\Phi, \theta]\mathcal{D}[F(\Phi), \theta], \quad t \neq t_k, t \geq t_0, \tag{3.24}$$

for a certain $\varepsilon \in [0, 1]$, and

$$\mathcal{D}[F(\Phi), \theta] \leq K\mathcal{D}[\Phi, \theta], \quad t \neq t_k, t \geq t_0, \tag{3.25}$$

$$\begin{bmatrix} \Upsilon & B^T_{(i)} \\ \star & \delta^{-1}\hat{u}_i^2 \alpha^{\mathrm{sgn}(1-\alpha)\mathcal{T}} \end{bmatrix} \geq 0, \quad i = 1, 2, ..., n, \tag{3.26}$$

$$\begin{bmatrix} -\alpha\Upsilon & (I + W_m A + W_m^- B)^T \Upsilon \\ \star & -\Upsilon \end{bmatrix} \leq 0, \tag{3.27}$$

where $\beta = \lambda + 2K \frac{\lambda_{\max}(\Upsilon) - \omega(\varepsilon)\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\varepsilon)\lambda_{\min}(\Upsilon)} \sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}$, $\omega(\varepsilon) = \frac{1-\varepsilon}{1+\varepsilon}$. Then, the system (1.1) is LES, and the convergence exponent is $\frac{1}{2}(\beta + \ln\alpha)$.

Proof. The proof is similar to Theorem 3.1, we omitted here. □

4. Synchronization result

The synchronization of SSINS is investigated in this section. Consider the drive system (DS) below

$$\mathcal{D}_H \Phi(t) = P\Phi(t) + F(\Phi(t)), \tag{4.1}$$

and the response system (RS) is described as

$$\mathcal{D}_H X(t) = PX(t) + F(X(t)) + \tilde{U}(t). \tag{4.2}$$

Assume that the Hukuhara difference $X - \Phi = Y$ exists, Y represents the error between the DS and the RS, and $\mathrm{diam}(F(X)) \geq \mathrm{diam}(F(\Phi))$. By Lemma 2.5, we have

$$\mathcal{D}_H Y(t) = PY(t) + \tilde{F}(Y(t)) + \tilde{U}(t), \tag{4.3}$$

where $\tilde{F}(Y(t)) = F(X(t)) - F(\Phi(t))$, and the control input $\tilde{U}(t)$ is designed as

$$\tilde{U}(t) = \sum_{k=1}^{\infty} \mathrm{sat}(AY(t_k))\Delta(t - t_k), \quad k \in \mathcal{N}, \tag{4.4}$$

where Δ is the Dirac function set.

Combining (4.3) and (4.4), the synchronization error set-valued system can be denoted as

$$\begin{cases} \mathcal{D}_H Y(t) = PY(t) + \tilde{F}(Y(t)), \quad t \neq t_k, t \geq t_0, \\ Y(t_k^+) = Y(t_k^-) + \mathrm{sat}(AY(t_k^-)), \quad t = t_k, k \in \mathcal{N}. \end{cases} \tag{4.5}$$

Theorem 4.1. *Let matrices $A, B \in \mathcal{R}^{n\times n}$, matrix $\Upsilon \in \mathcal{R}^{n\times n}$ is symmetric and positive definite. Ellipsoid set $\mathcal{E}(\Upsilon,\delta) := \{Y(t) \in \mathcal{K}_c(\mathcal{R}^n) : \mathcal{D}[Y^T(t)\Upsilon Y(t),\theta] \leq \delta\}$, where $\delta > 0$ is a scalar. If there exists a real diagonal positive definite matrix $\tilde{C} \in \mathcal{R}^{n\times n}$, $\tilde{\alpha}, \tilde{K}, \iota, \mathcal{T} \in \mathcal{R}^+$ satisfying $\tilde{\beta} + \frac{\ln\tilde{\alpha}}{\iota} < 0$ and*

$$\mathcal{D}[Y^T\tilde{F}(Y),\theta] \leq \tilde{\varepsilon}\mathcal{D}[Y,\theta]\mathcal{D}[\tilde{F}(Y),\theta], \quad t \neq t_k, t \geq t_0, \tag{4.6}$$

for a certain $\tilde{\varepsilon} \in [0,1]$, and

$$\mathcal{D}[\tilde{F}(Y),\theta] \leq \tilde{K}\mathcal{D}[Y,\theta], \quad t \neq t_k, t \geq t_0, \tag{4.7}$$

$$\begin{bmatrix} \Upsilon & (A_{(i)} - B_{(i)})^T \\ \star & \delta^{-1}\hat{u}_i^2\tilde{\alpha}^{\mathrm{sgn}(1-\tilde{\alpha})\mathcal{T}} \end{bmatrix} \geq 0, \quad i = 1,2,...,n, \tag{4.8}$$

$$\begin{bmatrix} -\tilde{\alpha}\Upsilon + (I+A)^T\Upsilon(I+A) & -(I+A)^T\Upsilon + B^T\tilde{C} \\ \star & \Upsilon - 2\tilde{C} \end{bmatrix} \leq 0, \tag{4.9}$$

where $\tilde{\beta} = \lambda + 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}$, $\omega(\tilde{\varepsilon}) = \frac{1-\tilde{\varepsilon}}{1+\tilde{\varepsilon}}$. Then, the error system (4.5) is LES, which implies that the DS (4.1) and the RS (4.2) achieved local exponential synchronization under the saturated impulses controller (4.4).

Proof. Let Lyapunov function $V(t) = \sup\limits_{y\in Y} y^T\Upsilon y$. For $t \in [t_k, t_{k+1}), k \in \mathcal{N}$, we have

$$\dot{V}(t) = \sup_{y\in Y,\tilde{f}\in\tilde{F}}\{y^T(\Upsilon P + P^T\Upsilon) + 2y^T\Upsilon\tilde{f}\}. \tag{4.10}$$

By Lemma 2.3, we obtain

$$\begin{aligned} \sup_{y\in Y}\{y^T(\Upsilon P + P^T\Upsilon)y\} &= \sup_{y\in Y}\{(y^T(t)\Upsilon^{\frac{1}{2}})(\Upsilon^{-\frac{1}{2}}(\Upsilon P + P^T\Upsilon)\Upsilon^{-\frac{1}{2}})(\Upsilon^{\frac{1}{2}}y)\} \\ &\leq \lambda\sup_{y\in Y}\{(y^T\Upsilon^{\frac{1}{2}})(\Upsilon^{\frac{1}{2}}y)\} \\ &= \lambda V(t). \end{aligned} \tag{4.11}$$

Using Lemmas 2.3, 2.4 and the inequalities (4.6), (4.7), we have

$$\begin{aligned} &\sup_{y\in Y,\tilde{f}\in\tilde{F}}\{2y^T\Upsilon\tilde{f}\} \\ &= 2\mathcal{D}[Y^T\Upsilon\tilde{F},\theta] \\ &\leq 2\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\mathcal{D}[Y^T\Upsilon Y,\theta]\mathcal{D}[\tilde{F}^T\Upsilon\tilde{F},\theta]} \\ &\leq 2\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{y\in Y,\tilde{f}\in\tilde{F}}\{(y^T\Upsilon y)(\tilde{f}^T\tilde{f})\}} \\ &\leq 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{y\in Y}\{(y^T\Upsilon y)(y^Ty)\}} \\ &= 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\lambda_{\max}(\Upsilon)\sup_{y\in Y}\{(y^T\Upsilon y)(y^T\Upsilon^{\frac{1}{2}}\Upsilon^{-1}\Upsilon^{\frac{1}{2}}y)\}} \end{aligned}$$

$$\leq 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}\sup_{y\in Y}\{(y^T\Upsilon y)(y^T\Upsilon^{\frac{1}{2}}\Upsilon^{\frac{1}{2}}y)\}}$$

$$= 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}V(t). \tag{4.12}$$

Combining the inequalities (4.11) and (4.12), we have

$$\dot{V}(t) \leq \left(\lambda + 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}\right)V(t). \tag{4.13}$$

Denote $\tilde{\beta} = \lambda + 2\tilde{K}\frac{\lambda_{\max}(\Upsilon)-\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}{\lambda_{\max}(\Upsilon)+\omega(\tilde{\varepsilon})\lambda_{\min}(\Upsilon)}\sqrt{\frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(\Upsilon)}}$, thus

$$V(t) \leq e^{\tilde{\beta}(t-t_k)}, \quad t \in [t_k, t_{k+1}), k \in \mathcal{N}. \tag{4.14}$$

The remaining part of the proof is similar to Theorem 3.1, omitted here. □

Remark 4.1. Similarly, we can also use the PRA to realize the synchronization under the saturated impulses controller. The proof process is identical to Theorem 3.2 and is not repeated here.

5. Application

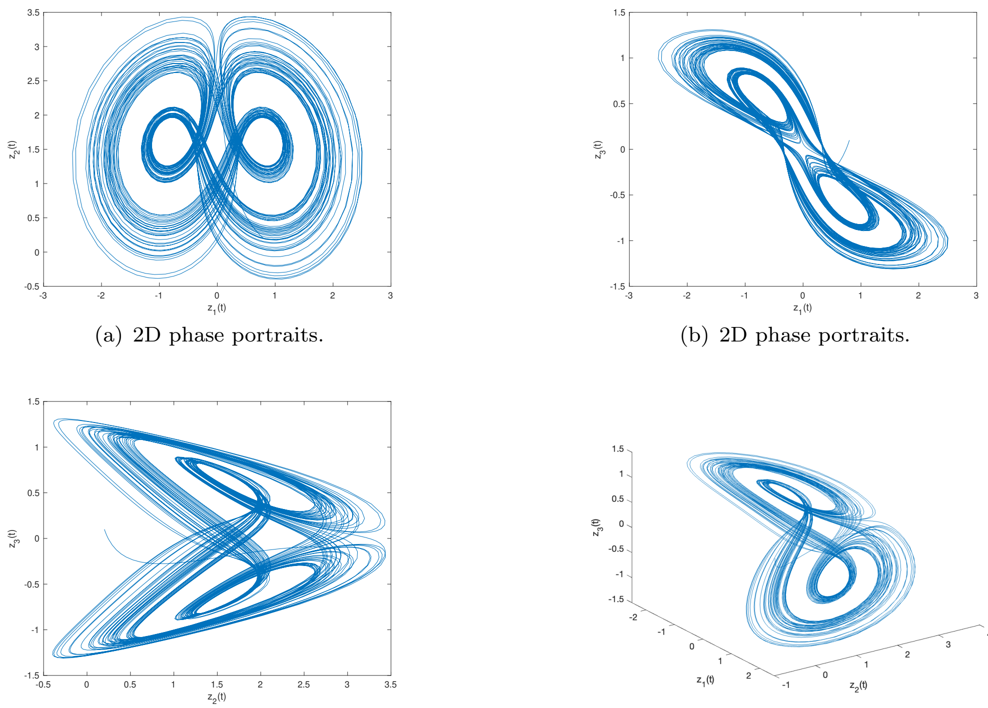
Information about the financial system is uncertain because it can be affected by international situations, political policies, extreme weather, and other factors. Therefore, it is suitable to establish the set-value financial system. In addition, the state variables in real systems are often influenced by the control system itself or the external environment, and thus limited to a certain range, thereby we consider the following set-valued financial system with the saturated impulses.

$$\begin{cases} \mathcal{D}_H\Phi(t) = P\Phi(t) + F(\Phi(t)), \quad t \neq t_k, t \geq t_0, \\ \Phi(t_k^+) = \Phi(t_k^-) + \mathrm{sat}(A\Phi(t_k^-)), \quad t = t_k, k \in \mathcal{N}, \\ \Phi(t_0) = \Phi_0, \end{cases} \tag{5.1}$$

where

$$P = \begin{bmatrix} -a_1 & 0 & 1 \\ 0 & -a_2 & 0 \\ -1 & 0 & -a_3 \end{bmatrix}, \phi(t) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, f(\phi(t)) = \begin{bmatrix} \phi_1\phi_2 \\ 1-\phi_2^2 \\ 0 \end{bmatrix},$$

$\phi(t) \in \Phi(t), f(\phi(t)) \in F(\Phi(t))$. ϕ_1, ϕ_2, ϕ_3 are the state variables, which represent the interest rate, the investment demand and the price index. a_1, a_2, a_3 are positive constants and describe the amount saved, the cost per investment and the elasticity demand involved in the commercial market.



(a) 2D phase portraits. (b) 2D phase portraits.

(c) 2D phase portraits. (d) 3D phase portraits.

Figure 1. Chaos trajectories of the financial system (5.1).

Assume the above set-valued financial system (5.1) with the initial condition $\phi_0 \in \Phi_0 = ([0,1],[0,1],[0,1])^T$, and $a_1 = 0.9, a_2 = 0.2, a_3 = 1.2$. From Figure 1, it can be found that the system (5.1) is chaotic, and $\max\limits_{\phi\in\Phi, i=1,2,3}\{\mathcal{D}[\phi_i,\theta]\} \le 4$.

Then,

$$\begin{aligned}\mathcal{D}[F(\Phi),\theta] &= \sup_{\phi\in\Phi}\left\{\sqrt{\phi_1^2\phi_2^2 + (1-\phi_2^2)^2}\right\}\\ &\le \max_{\phi\in\Phi,i=1,2,3}\{\mathcal{D}[\phi_i,\theta]\}\mathcal{D}[\Phi(t),\theta]\\ &\le 4\mathcal{D}[\Phi(t),\theta],\end{aligned}$$

therefore, we choose $K = 4$.

$$\begin{aligned}\mathcal{D}[\Phi^T F(\Phi),\theta]^2 &= \sup_{\phi\in\Phi}\left\{\phi_1^4\phi_2^2 + \phi_2^2(1-\phi_2^2)^2\right\}\\ &\le \frac{1}{3}\sup_{\phi\in\Phi}\left\{(\phi_1^2+\phi_2^2+\phi_3^2)(\phi_1^2\phi_2^2+(1-\phi_2^2)^2)\right\},\end{aligned}$$

namely,

$$\begin{aligned}\mathcal{D}[\Phi^T F(\Phi),\theta] &\le \frac{\sqrt{3}}{3}\sup_{\phi\in\Phi}\left\{\sqrt{\phi_1^2+\phi_2^2+\phi_3^2}\sqrt{\phi_1^2\phi_2^2+(1-\phi_2^2)^2}\right\}\\ &= \frac{\sqrt{3}}{3}\mathcal{D}[\Phi,\theta]\mathcal{D}[F(\Phi),\theta].\end{aligned}$$

Let $\varepsilon = \frac{\sqrt{3}}{3}$ and $\Upsilon = I$.

Denote the impulsive sequence $\{t_k, k \in \mathcal{N}\} = 0.5k$, and the impulsive gain matrix $A = -0.5I$, $\delta = 1, \alpha = 0.3, \hat{u}_0 = 1, \iota = 0.5, B = 0.05I$. By calculation, it is known that $\lambda = -0.4, \beta = 1.7435, \alpha < e^{-\beta\iota} = 0.4182$. From the linear matrix inequality (LMI) toolbox in Matlab, it can be seen that there exists a feasible solution for LMIs (3.4)-(3.5)

$$C = \begin{bmatrix} 2.5929 & 0 & 0 \\ 0 & 2.5929 & 0 \\ 0 & 0 & 2.5929 \end{bmatrix}.$$

By Theorem 3.1, the financial chaotic system (5.1) can eliminate chaos with the appropriate saturated impulses control input. Figure 2 shows the state trajectories of the system (5.1) with the saturated impulses eventually converging to the equilibrium state, where the red dots are the impulsive times, and the convergence exponent is $\frac{1}{2}(\beta + \ln\alpha) = 0.2698$. The comparison in Figure 3 shows that the saturated impulses control strategy is effective in eliminating chaotic behavior in the financial market and achieving stability in the market state.

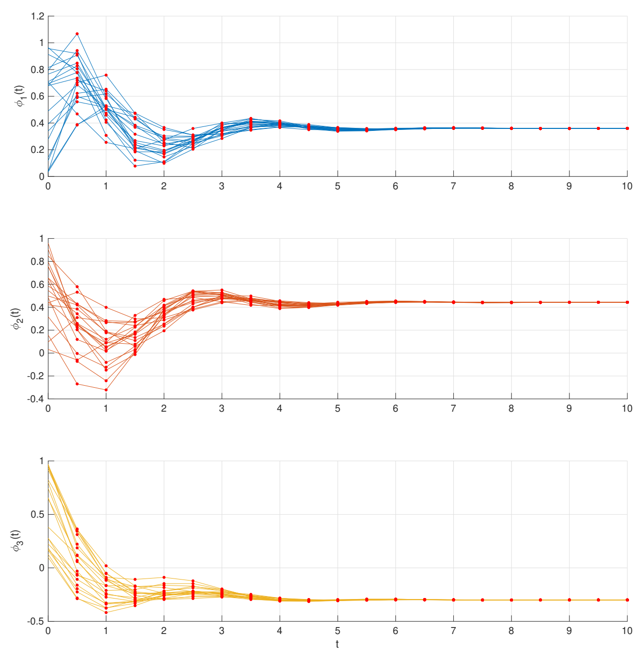
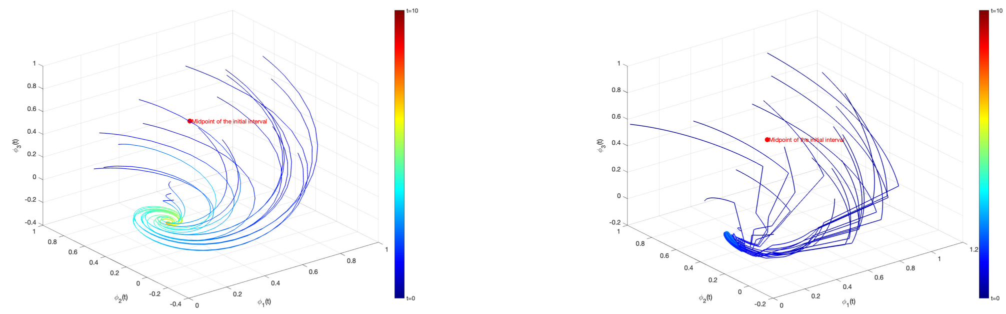


Figure 2. The state trajectories of the system (5.1).

To achieve synchronization of the system (5.1), consider the system (5.1) as a DS, and the corresponding RS is described as

$$\mathcal{D}_H X(t) = PX(t) + F(X(t)) + \sum_{k=1}^{\infty} \mathrm{sat}(AY(t_k))\Delta(t - t_k), \tag{5.2}$$

where $Y(t) = X(t) - \Phi(t)$ is the synchronization error.



(a) The phase space of the system (5.1) without the saturated impulses.

(b) The phase space of the system (5.1) with the saturated impulses.

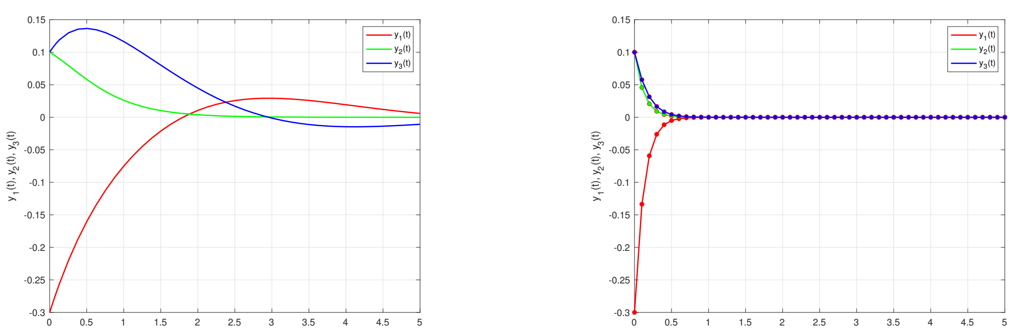
Figure 3. The phase space of the system (5.1).

Then, the error system is described below

$$\begin{cases} \mathcal{D}_H Y(t) = PY(t) + \tilde{F}(Y(t)), & t \neq t_k, t \geq t_0, \\ Y(t_k^+) = Y(t_k^-) + \mathrm{sat}(AY(t_k^-)), & t = t_k, k \in \mathcal{N}. \end{cases} \tag{5.3}$$

Let the RS (5.2) with the initial value $x_0 = ([0,1],[0,1],[0,1])^T$. The state trajectories of the synchronization error for the DS (5.1) and the RS (5.2) are shown in Figure 4(a). It implies that without the saturated impulses control input, the two financial systems with different initial states are unable to synchronize autonomously.

Figure 4(b) illustrates that the error system (5.3) achieves the exponential synchronization under the saturated impulses control and converges to zero with the impulsive sequence $\{t_k, k \in \mathcal{N}\} = 0.1k$, and the impulsive gain matrix $A = -0.5I$, $\delta = 1, \tilde{\alpha} = 0.3, \hat{u}_0 = 1, \iota = 0.5, B = 0.2I$.



(a) The state trajectories of the error system (5.3) without the saturated impulses.

(b) The state trajectories of the error system (5.3) with the saturated impulses.

Figure 4. The state trajectories of the error system (5.3).

6. Conclusion

In this paper, the LES and synchronization problems of the set-valued nonlinear system under the actuator saturation are studied. Firstly, for the saturated nonlinear issue of the system at the impulsive time, the sufficient condition for the LES is obtained by using the SNMA and PRA, combining the generalized Cauchy-Schwarz inequality and the Lyapunov method. Subsequently, we derive the LES criteria of the error system under the synchronization concept of the drive-response system with the saturated impulses controller. Finally, the above results are applied to a chaotic financial system, and numerical simulations demonstrate the effectiveness of the saturated impulses control in achieving the stability and chaotic synchronization for the set-valued financial system. It is worth noting that, for the first time, we consider the actuator saturation constraint in the impulsive control of the set-valued financial system and discuss the local properties of the system, which aligns with the limitations of macro-regulation policies in real-world applications.

Acknowledgements

This paper is supported by the National Natural Science Foundation of China (12471505, 12171135).

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Received May 2026; Accepted September 2026; Available online September 2026.