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GLOBAL ASYMPTOTIC STABILITY OF BARNACLE-ALGAE INTERACTION MODEL WITH DELAY*

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Abstract This work investigates the dynamical behavior of a three-dimensional patch occupancy model that incorporates a discrete time delay into the barnacle-algae interaction. The model framework extends the non-delayed, periodically forced barnacle-algae-mussel system introduced by Benincà in [2]. In the absence of mussels ($M = 0$) and seasonal forcing, the system studied in [10] reduces to a three-dimensional system, for which global asymptotic stability was previously established. Under biologically motivated assumptions, this paper provides a complete characterization of the global dynamics for the corresponding three-dimensional delayed system. By leveraging the Poincaré-Bendixson Property applicable to type-K competitive systems with delay, we prove the global asymptotic stability of the positive equilibrium whenever it exists. Our analytical results demonstrate that the introduction of a time delay does not disrupt the global convergence behavior observed in the non-delayed system.

Keywords Global asymptotic stability, barnacle-algae interaction model, orbital stability, type-K competitive delayed system.

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1. Introduction and the model formulation

The incorporation of time delays into differential equation systems [13,16,34] is often motivated by the need for greater biological realism, reflecting processes such as maturation times, resource conversion lags, or spatial colonization periods in population dynamics. The time delay [6,28] in the given system of differential equations is motivated by biological realism in modeling the dynamics of interaction between different components here, likely representing populations, biochemical species or natural ecological systems.

In this study, we analyze the following delayed differential equation model describing barnacle-algae dynamics within intertidal rock-pool habitats:

$$\begin{cases} \dot{B}_0 = c_{BR}(B_0 + B_A)R - c_{AB}A(t-\tau)B_0 - m_B B_0 + m_A B_A, \\ \dot{B}_A = c_{AB}A(t-\tau)B_0 - m_B B_A - m_A B_A, \\ \dot{A} = c_{AR}A(t-\tau)R + c_{AB}A(t-\tau)B_0 - m_A A, \\ R = 1 - B_0 - A. \end{cases} \tag{1.1}$$

The model parameters carry specific ecological interpretations: c_{BR} quantifies the settlement rate of barnacles on unoccupied rock surfaces; c_{AB} represents the colonization rate of crustose

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algae on barnacles; c_{AR} denotes the colonization rate of crustose algae on bare rock; and m_B, m_A are the mortality rates of barnacles and crustose algae, respectively.

The delay τ in the model (1.1) represents the lag in algal overgrowth, i.e., the time required for algal biomass to accumulate to a level that subsequently influences colonization. Since bare rock availability $R(t)$ and barnacle density $B_0(t)$ reflect current environmental conditions and respond on a relatively fast timescale, they are evaluated at the present time t. In contrast, the colonizing algae originate from the historical overgrowth $A(t-\tau)$. This asymmetric formulation $A(t-\tau)R(t)B_0(t)$ is biologically justified by the distinct timescales of the source population (memory effect) and the target environmental conditions.

A typical model without delay assumes that interactions and responses occur instantaneously. For instance, the term $c_{AB}A(t)B_0(t)$ would imply that the effect of species overgrown on B by A is immediate. In this model, the delay is introduced specifically in the terms involving $A(t-\tau)$, meaning that the value of A from τ time delay in the influences of covering time lag between algae and barnacles.

This framework originates from Benincà et al. [2], who documented 20-year population time series of barnacles, crustose algae, and mussels in a rocky intertidal community at Cape Rodney-Okakari Point Marine Reserve, New Zealand. Their original model ([2], p.6391) incorporated mussel dynamics and seasonal forcing:

$$\begin{cases} \dot{B}_0 = c_{BR}(B_0 + B_A)R - c_{AB}AB_0 - c_M MB_0 - m_B B_0 + f(t)m_A B_A, \\ \dot{B}_A = c_{AB}AB_0 - c_M MB_A - m_B B_A - f(t)m_A B_A, \\ \dot{A} = c_{AR}AR + c_{AB}AB_0 - c_M MA - f(t)m_A A, \\ \dot{M} = c_M M(B_0 + A) - f(t)m_M M, \\ R = 1 - B_0 - A - M, \end{cases} \tag{1.2}$$

where

$$f(t) = 1 + \alpha \cdot (T_{\max} - T_{\text{mean}}) \cos(\frac{2\pi(t-32)}{365})$$

represents seasonal temperature fluctuations.

To understand the fundamental dynamical mechanisms of system (1.2), Hsu et al. [10] considered the crucial case with $\alpha = 0$ and absent mussels ($M = 0$), reducing system (1.1) to a three-dimensional system:

$$\begin{cases} \dot{B}_0 = c_{BR}(B_0 + B_A)R - c_{AB}AB_0 - m_B B_0 + m_A B_A, \\ \dot{B}_A = c_{AB}AB_0 - m_B B_A - m_A B_A, \\ \dot{A} = c_{AR}AR + c_{AB}AB_0 - m_A A, \\ R = 1 - B_0 - A. \end{cases} \tag{1.3}$$

The study of species interactions in ecological systems often involves mathematical models that incorporate temporal delays, reflecting processes such as maturation, resource conversion, or spatial colonization. In this paper, we extend the non-delayed barnacle-algae-mussel model introduced in [10] to include a discrete time delay $\tau > 0$, leading to delayed system (1.1). This delay accounts for the time required for crustose algae to overgrow barnacles, a key biological process in the interaction.

The non-delayed system (1.3) has been shown to exhibit rich dynamics [10], including the global stability of boundary equilibria and the coexistence of species under certain conditions.

However, the introduction of delay complicates the analysis due to the infinite-dimensional nature of the resulting dynamical system. Our work aims to bridge this gap by rigorously analyzing the delayed system (1.1) using tools from delay differential equations and Poincaré-Bendixson Property of type-K competitive system.

We first establish the positive invariance of the biologically meaningful domain Λ and analyze the stability of equilibria via characteristic equations and comparison principles. For the global stability of the positive equilibrium, we transform the system into a K-competitive form and apply Poincaré-Bendixson theory for monotone delay systems. The main results illustrate that the three-dimensional patch occupancy model of barnacle-algae interaction with delay preserves global convergence. This study not only extends results from the non-delayed case but also illuminates the ecological implications of time delays in species interactions, contributing to a deeper understanding of complex dynamics in rocky intertidal communities.

We organize this paper as follows: Section 2 presents notations, definitions, and preliminary results. Section 3 details the main results and their proofs. Section 4 provides discussion.

2. Preliminaries

Firstly, we give some notations and lemmas, which are useful to obtain the global asymptotical stability for system (1.1).

2.1. Some notations and lemmas

We mainly focus on the autonomous system (1.1) of Barnacle-Algae interactions. For this purpose, we rewrite (1.1) as the following new system

$$\begin{cases} \dot{B}_0(t) = B_0(t)[c_{BR}R(t) - m_B - c_{AB}A(t-\tau)] + B_A(t)(c_{BR}R(t) + m_A), \\ \dot{B}_A(t) = c_{AB}A(t-\tau)B_0(t) - (m_B + m_A)B_A(t), \\ \dot{A}(t) = A(t-\tau)[c_{AR}R(t) + c_{AB}B_0(t)] - m_A A(t), \end{cases} \tag{BA}$$

where $R(t) = 1 - B_0(t) - A(t)$.

Let $B = B_0 + B_A$ be the total barnacles. Then B satisfies

$$\dot{B} = B[c_{BR}(1 - B_0 - A) - m_B]. \tag{2.1}$$

Clearly, system (BA) is equivalent to system (1.1).

Due to B and A are the fractions of the patches occupied by barnacle and crustose algae in a rocky intertidal community, based on biological meaning, it is reasonable to require that $0 \leq B, A \leq 1$. Note that B_A represents the fraction occupied by barnacles overgrown with crustose algae, making it biologically reasonable to expect $B_A(t) \leq A(t)$ for all $t \geq 0$. Accordingly, we define in the present paper our working domain Σ as

$$\Lambda = \{(B_0, B_A, A) \in \mathcal{C}([-\tau, 0], \mathbb{R}^3_+) : B_0 + A \leq 1, \text{ and } B_A \leq A\}, \tag{2.2}$$

where $\mathbb{R}^3_+$ denotes the first orthant in $\mathbb{R}^3$, and $\mathcal{C}([-\tau, 0], \mathbb{R}^3_+) = \{\phi : [-\tau, 0] \to \mathbb{R}^3_+ \text{ continuous}\}$. In this present paper, for any $\theta \in [-\tau, 0)$, we also assume $(B_0, B_A, A)(\theta) \in \mathcal{C}([-\tau, 0], \mathbb{R}^3_+)$, and $B_0(0), B_A(0)$ and $A(0)$ are all positive.

We maintain the following fundamental assumption condition throughout our analysis:
(A1) $c_{AB} > c_{AR}$ and $c_{BR} > m_B$.

Fundamental Assumption **(A1)**: The colonization efficiency of crustose algae is greater on barnacle surfaces than on bare rock ($c_{AB} > c_{AR}$), while barnacle colonization exceeds their mortality rate ($c_{BR} > m_B$). These conditions align with empirical observations from rocky intertidal communities.

From the right forms of delayed system (BA), we directly calculate the boundary equilibria:

$$E_0 = (0,0,0),\ E_B = (B_0^*,0,0),\ E_A = (0,0,A^*),$$

on $\Lambda \cap \partial\mathbb{R}^3_+$, with $B_0^* = 1 - \frac{m_B}{c_{BR}},\ A^* = 1 - \frac{m_A}{c_{AR}}$, and by the positive equilibrium

$$E_c = (\bar{B}_0, \bar{B}_A, \bar{A}) \in \Lambda \cap \mathrm{Int}\mathbb{R}^3_+,$$

with $(\bar{B}_0, \bar{B}_A, \bar{A})$ satisfies $\bar{B}_0 = \frac{c_{AR}}{c_{AB}} \cdot (\frac{m_A}{c_{AR}} - \frac{m_B}{c_{BR}})$, $\bar{B}_A = \frac{c_{AB}}{m_A+m_B} \cdot \bar{A}\bar{B}_0$ and $c_{AB}\bar{A} = (c_{AR} - m_A) + (c_{AB} - c_{AR})B_0^*$, where $\partial\mathbb{R}^3_+$ and $\mathrm{Int}\mathbb{R}^3_+$ mean the boundary and interior of $\mathbb{R}^3_+$. For convenience, let

$$\gamma = (c_{AR} - m_A) + B_0^*(c_{AB} - c_{AR}), \tag{2.3}$$
$$\delta = c_{BR}A^* - (c_{BR} - m_B). \tag{2.4}$$

We maintain the following standing assumption throughout this paper.
(A2) The following parameter conditions hold at the positive equilibrium $E_c = (\bar{B}_0, \bar{B}_A, \bar{A})$:

$$\text{(i)}\quad C_1 > 0, \qquad \text{(ii)}\quad C_2 > 0, \qquad \text{(iii)}\quad m_A\Delta_1 > \Delta_0, \tag{2.5}$$

where

$$\Delta_1 := (c_{AB} - c_{AR})\bar{A}\big(c_{AB}\bar{B}_0 + c_{BR}(\bar{B}_0 + \bar{B}_A)\big),$$
$$\Delta_0 := (m_B + m_A)\bar{A}c_{AB}c_{BR}(\bar{B}_0 + \bar{B}_A),$$

and

$$\begin{aligned}
C_1 :=\ & m_A^2 + 2m_Ac_{AB}\bar{A} - 2m_Bc_{AR}\bar{A} \\
& - 2c_{BR}(\bar{B}_0 + \bar{B}_A)(m_B + m_A + c_{AB}\bar{A}) \\
& - 2c_{AR}c_{AB}\bar{A}^2 - 2(c_{AB} - c_{AR})c_{AB}\bar{A}\bar{B}_0, \\
C_2 :=\ & 2\Big[(m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) + c_{AR}\bar{A}\big(m_B + m_A + c_{BR}(\bar{B}_0 + \bar{B}_A) + c_{AB}\bar{A}\big) \\
& + m_A\big(m_B + m_A + c_{BR}(\bar{B}_0 + \bar{B}_A)\big)\Big] \\
& \times (c_{AB} - c_{AR})\bar{A}\big(c_{AB}\bar{B}_0 + c_{BR}(\bar{B}_0 + \bar{B}_A)\big) \\
& + (c_{AB} - c_{AR})^2\bar{A}^2\big(c_{AB}\bar{B}_0 + c_{BR}(\bar{B}_0 + \bar{B}_A)\big)^2 \\
& - 2m_A^2(m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) \\
& - 2\big(m_B + m_A + c_{BR}(\bar{B}_0 + \bar{B}_A) + c_{AR}\bar{A} + c_{AB}\bar{A}\big) \\
& \times (m_B + m_A)\bar{A}c_{AB}c_{BR}(\bar{B}_0 + \bar{B}_A) \\
& - 2m_A(m_B + m_A)\bar{A}c_{AB}c_{BR}(\bar{B}_0 + \bar{B}_A).
\end{aligned}$$

Remark 2.1. Assumption (A2) is a parametric restriction that guarantees the local asymptotic stability of E_c in the delayed system. It is satisfied for biologically meaningful parameter ranges,

including those reported in Benincà et al. [2] and Hsu et al. [10]. Biologically, (A2)(iii) ensures that algal mortality is sufficiently large to prevent algal overgrowth, while (A2)(i)–(ii) ensure that the system does not exhibit oscillatory instabilities.

From the assumption (A1), the equilibria E_0, E_B always exist, and E_A exists if and only if $\frac{m_A}{c_{AR}} < 1$. Note that $\bar{B}_0 = -\frac{c_{AR}}{c_{AB}} \cdot c_{BR} \cdot \delta$ and $\bar{A} = \frac{1}{c_{AB}}\gamma$, due to the form of the positive equilibrium E_c, it is easy to see that E_c exists if and only if $\gamma > 0$ and ¡0, moreover, E_c is unique as long as it exists.

To improve the readability of the paper, we now introduce the following two frequently used lemmas.

Lemma 2.1. (Hayes, see Bellman and Cooke [1], p.444) *Let $A, B \in \mathbb{R}$, then all roots ω of the equation*

$$Ae^{\omega} + B - \omega e^{\omega} = 0$$

have negative real parts if and only if the following conditions hold: $A < 1$ and $A < -B < \sqrt{\alpha^2 + A^2}$, where $\alpha$is the root of $\alpha = A\tan\alpha$ satisfying $0 < \alpha < \pi$ (If $A = 0$, set $\alpha = \pi/2$.)

Lemma 2.2. ([17], p.159) *If $a < b$, then the solution of the equation*

$$u'(t) = au(t-\tau) - bu(t),$$

where $a, b, \tau > 0$, and $u(t) > 0$ for $-\tau \le t \le 0$, satisfies $\lim_{t\to\infty} u(t) = 0$.

2.2. Invariant domain

The following proposition confirms the validity of the choice of Λ from mathematical point of view.

Proposition 2.1. *$\Lambda = \{(B_0, B_A, A) \in \mathcal{C}([-\tau, 0], \mathbb{R}^3_+) : B_0 + A \le 1,$ and $B_A \le A\}$ is positively invariant with respect to delayed system* (BA). *Moreover, let $(B_0(t), B_A(t), A(t))$ be a solution of* (BA) *in Λ. If $B(0) = B_0(0) + B_A(0) > 0$ and $A(s) \in \mathcal{C}$ on $[-\tau, 0]$, then $B_0(t), B_A(t)$ and $A(t)$ are all positive for all $t > 0$.*

Proof. Let $(B_0(t), B_A(t), A(t))$ be any solution of the delayed system (BA) with $B_0(0) + B_A(0) > 0$, $A(s) = \phi(s) > 0$ on $[-\tau, 0]$, $B_A(0) < A(0)$, and $(B_0 + A)(0) < 1$.

From (2.1), we have

$$B(t) = B(0)\exp\left\{\int_0^t [c_{BR}(1 - B_0(s) - A(s)) - m_B]\, ds\right\} \ge 0,$$

so $B(t) \ge 0$ for all $t \ge 0$.

We first prove that $A(t) > 0$ for all $t \ge 0$. Suppose, for contradiction, that there exists $t_0 > 0$ such that $A(t_0) = 0$, and let $t_0 = \inf\{t > 0 : A(t) = 0\}$. Then $A(t) > 0$ for $0 \le t < t_0$. From the A-equation in (BA),

$$\dot{A}(t_0) = A(t_0 - \tau)[c_{AR}R(t_0) + c_{AB}B_0(t_0)] - m_A A(t_0) = A(t_0 - \tau)[c_{AR} + (c_{AB} - c_{AR})B_0(t_0)].$$

If $t_0 \le \tau$, then $A(t_0 - \tau) > 0$ by the initial condition. If $t_0 > \tau$, then by the minimality of t_0, we also have $A(t_0 - \tau) > 0$. Since $c_{AB} > c_{AR}$ by (A1) and $B_0(t_0) \ge 0$, the bracket is positive. Thus $\dot{A}(t_0) > 0$, contradicting the definition of t_0. Hence $A(t) > 0$ for all $t \ge 0$.

Now define

$$S(t) := A(t) + B_0(t), \qquad D(t) := A(t) - B_A(t).$$

By the initial conditions, $S(0) < 1$ and $D(0) > 0$. A direct calculation from (BA) yields

$$\dot{S} = c_{AR}A(t-\tau)R + c_{BR}BR - m_B B_0 - m_A D, \tag{2.6a}$$

$$\dot{D} = c_{AR}A(t-\tau)R + m_B B_A - m_A D, \tag{2.6b}$$

where $R = 1 - S$. We claim that $S(t) < 1$ and $D(t) > 0$ for all $t \geq 0$.

Define

$$t_1 := \sup\{t > 0 : S(t) < 1\}, \qquad t_2 := \sup\{t > 0 : D(t) > 0\},$$

and let $t_* := \min\{t_1, t_2\}$. Clearly $t_* > 0$. Suppose, for contradiction, that $t_* < +\infty$. Then one of the following three cases must occur.

Case 1. $t_* = t_1 < t_2$**.**

At $t = t_*$, we have $S(t_*) = 1$ (so $R(t_*) = 0$) and $D(t_*) > 0$. From (2.6a),

$$\dot{S}(t_*) = -m_B B_0(t_*) - m_A D(t_*).$$

Since $S(t_*) = 1$, we have $B_0(t_*) = 1 - A(t_*)$. Because $A(t_*) > 0$, and $S(t_*) = 1$ with $S(t) < 1$ for $t < t_*$, by continuity we have $A(t_*) \leq S(t_*) = 1$. Hence $B_0(t_*) \geq 0$. Therefore $\dot{S}(t_*) < 0$, which implies that $S(t) > 1$ for $t < t_*$ sufficiently close to t_*, contradicting the definition of t_1.

Case 2. $t_* = t_2 < t_1$**.**

At $t = t_*$, we have $D(t_*) = 0$ and $S(t_*) < 1$ (so $R(t_*) > 0$). From (2.6b),

$$\dot{D}(t_*) = c_{AR}A(t_* - \tau)R(t_*) + m_B B_A(t_*).$$

We have $A(t_* - \tau) > 0$ from above, $R(t_*) > 0$, and $B_A(t_*) = A(t_*) \geq 0$. Hence $\dot{D}(t_*) > 0$, which implies that $D(t) < 0$ for $t < t_*$ sufficiently close to t_*, contradicting the definition of t_2.

Case 3. $t_* = t_1 = t_2$**.**

At $t = t_*$, we have $S(t_*) = 1$ (so $R = 0$) and $D(t_*) = 0$. From (2.6a) and (2.6b),

$$\dot{S}(t_*) = -m_B B_0(t_*), \qquad \dot{D}(t_*) = m_B A(t_*).$$

Since $A(t_*) > 0$ and $m_B > 0$, we have $\dot{D}(t_*) > 0$, which again implies that $D(t) < 0$ for $t < t_*$ sufficiently close to t_*, contradicting the definition of t_2.

Thus t_* cannot be finite, so $t_* = +\infty$. Hence

$$S(t) < 1, \qquad D(t) > 0, \qquad \forall t \geq 0,$$

that is,

$$B_0(t) + A(t) < 1, \qquad B_A(t) < A(t), \qquad \forall t \geq 0.$$

It remains to prove the strict positivity of B_0 and B_A. Suppose there exists $t_B > 0$ such that $B_A(t_B) = 0$, and let t_B be the first such time. Then $B_A(t_B) = 0$ and $B_A(t) > 0$ for $0 \leq t < t_B$. From the B_A-equation in (BA),

$$\dot{B}_A(t_B) = c_{AB}A(t_B)B_0(t_B) - (m_B + m_A)B_A(t_B) = c_{AB}A(t_B)B_0(t_B).$$

Since $B(t_B) = B_0(t_B)+B_A(t_B) = B_0(t_B) > 0$ and $A(t_B) > 0$, we have $\dot{B}_A(t_B) > 0$, contradicting the assumption that t_B is the first zero time. Hence $B_A(t) > 0$ for all $t > 0$.

Similarly, suppose there exists $t_0 > 0$ such that $B_0(t_0) = 0$, and let t_0 be the first such time. From the B_0-equation in (BA),

$$\dot{B}_0(t_0) = B_A(t_0)(c_{BR}R(t_0) + m_A).$$

We have $B_A(t_0) > 0$ from the previous step, and $R(t_0) = 1 - S(t_0) > 0$, so $c_{BR}R(t_0) + m_A > 0$. Hence $\dot{B}_0(t_0) > 0$, contradicting the assumption that t_0 is the first zero time. Therefore $B_0(t) > 0$ for all $t > 0$.

Since system is a delay differential equation, the initial condition is given by a continuous history function on the interval $[-\tau, 0]$. Let $\Phi = (\phi_A, \phi_R, \phi_B) \in C([-\tau, 0], \mathbb{R}^3_+)$ denote the initial history. We define the invariant domain in the phase space $C([-\tau, 0], \mathbb{R}^3_+)$. The constraint $\phi_{B_A}(\theta) \le \phi_A(\theta)$ is interpreted as a uniform condition on the initial history, meaning that for every $\theta \in [-\tau, 0]$, the bound holds continuously. By the method of steps, the solution of a delay differential equation preserves continuity on each interval $[k\tau, (k+1)\tau]$. Thus, if the initial history satisfies the bound at all $\theta \in [-\tau, 0]$, then the same bound propagates to the subsequent intervals, provided that the rightmost derivative condition is satisfied. To verify invariance, consider any initial history $\Phi \in \Lambda$. For the corresponding solution $(A(t), R(t), B_0(t))$, on the boundary $\phi_A(\theta) = 0$, by the comparison principle, $\dot{A}(t) \ge 0$ for all t where the history lies on the boundary; on $\phi_A(\theta) = A_{\max}$, we have $\dot{A}(t) \le rA_{\max} - dA_{\max} = 0$; on the boundaries $\phi_R(\theta) = 0$ and $\phi_R(\theta) = 1$, we have $\dot{R}(t) \ge 0$ and $\dot{R}(t) \le 0$, respectively. on the boundaries $\phi_B(\theta) = 0$ and $\phi_B(\theta) = B_{\max}$, we have $\dot{B}_0(t) \ge 0$ and $\dot{B}_0(t) \le 0$ under the parameter condition.

The constraint $\phi_{B_A}(\theta) \le \phi_A(\theta)$ is preserved because the difference $A(t) - B_A(t)$ satisfies a linear delay inequality whose right-hand side is nonnegative whenever the history satisfies the bound, then $A(t) \ge B_A(t)$ holds. Thus, Λ is positively invariant. The proof is complete. □

3. Main results

Firstly, we give the characteristic equation of the Jacobian of the vector field for (BA) as

$$\begin{vmatrix} \lambda - L_1 & -c_{BR}(1 - B_0 - A) - m_A\, c_{AB}B_0e^{-\lambda\tau} + c_{BR}(B_0 + B_A) \\ -c_{AB}A & \lambda + (m_B + m_A) & -c_{AB}B_0e^{-\lambda\tau} \\ -(c_{AB} - c_{AR})A & 0 & \lambda - L_2 \end{vmatrix} = 0, \tag{3.1}$$

where

$$L_1 = c_{BR} - m_B - 2c_{BR}B_0 - (c_{BR} + c_{AB})A - c_{BR}B_A$$

and

$$L_2 = -(c_{AR}A + m_A) + [c_{AR}(1 - B_0 - A) + c_{AB}B_0]e^{-\lambda\tau}.$$

The following two lemmas give a complete classification of the stability of the equilibria of (BA) according to the sign of λ_* and Δ_1.

Lemma 3.1. *If $c_{BR} < m_B$ and $c_{AR} < m_A$, then E_0 is globally asymptotically stable (G.A.S.).*

Proof. From (3.1), the characteristic equation at E_0 becomes

$$H_0(\lambda) = (\lambda - (c_{BR} - m_B))(\lambda + m_B + m_A)(\lambda + m_A + e^{-\lambda\tau}c_{AR}) = 0.$$

Thus, the real parts of the roots for the equation $H_0(\lambda) = 0$ are all negative if and only if $c_{BR} < m_B$ and the roots of the equation $\lambda + m_A + e^{-\lambda\tau}c_{AR} = 0$ just exist negative real parts. Now we show that the real parts of the roots for the equation $\lambda + m_A + e^{-\lambda\tau}c_{AR} = 0$ are all negative. Let $\omega = \lambda\tau$, then the equation becomes

$$-\omega e^{\omega} - \tau c_{AR} - \tau m_A e^{\omega} = 0.$$

Then, $-\tau m_A < 1$ and $-\tau m_A < 0 < \tau c_{AR}$ automatically hold. And, note that $\tau c_{AR} < \sqrt{\alpha^2 + (\tau m_A)^2}$ if $c_{AR} < m_A$, where $\alpha \in (0, \pi)$ is the root of $\alpha = \tau m_A \tan\alpha$. By using Lemma 2.1, the real parts of the roots for the equation

$$\lambda + m_A + e^{-\lambda\tau}c_{AR} = 0$$

are all negative if $c_{AR} < m_A$. Therefore, if $c_{BR} < m_B$ and $c_{AR} < m_A$, one has E_0 is locally asymptotically stable (L.A.S.). From (3.2) and Proposition 2.1, we have

$$\frac{\dot{B}(t)}{B(t)} = c_{BR}(1 - B_0(t) - A(t)) - m_B < c_{BR} - m_B < 0, \tag{3.2}$$

which entails that $B(t) \to 0$ as $t \to +\infty$, then $B_0(t) \to 0$ and $B_A(t) \to 0$ as $t \to +\infty$. And due to $B(t) \to +\infty 0$, again by Proposition 2.1, then the equation (A) in (BA) becomes into

$$\begin{aligned}\dot{A}(t) &= c_{AR}A(t-\tau)R(t) - m_A A(t)\\ &< c_{AR}A(t-\tau) - m_A A(t).\end{aligned}$$

Since $c_{AR} < m_A$, by virtue of *Theorem 4.9.4* ([17], p.159) and comparison principle of delay differential equations, we have $A(t) \to 0$ as $t \to +\infty$. We have completed the proof. □

Remark 3.1. No matter how the time delay changes, from Lemma 3.1, when $c_{BR} > m_B$, one has the equilibrium E_0 is always unstable; and Lemma 3.1 says that barnacles and algae will ultimately be extinct if the mortality rates of barnacles and algae are larger than the colonization rate of barnacles and algae on bare rock.

Lemma 3.2. *Assume (A1) holds.*

(i) *If $\gamma < 0$, then E_B is globally asymptotically stable (G.A.S.) for $\Lambda \setminus E_0$;*

(ii) *If $\gamma > 0$ and $\delta > 0$, then E_A exists and is G.A.S. for $L \setminus \{A = 0\}$.*

Proof. (i) Together with (A1) and $\gamma < 0$, it yields to E_B exists, neither E_c nor E_A exists. Again from (3.1), the characteristic equation at E_B becomes

$$H_B(\lambda) = [\lambda - (m_B - c_{BR})](\lambda + m_B + m_A)(\lambda - h_\lambda) = 0,$$

where $h_\lambda = c_{AR} - m_A + e^{-\lambda\tau}(c_{AB} - c_{AR})B_0^*$. E_B is locally asymptotically stable as long as the real parts of the roots for equation $\lambda - h_\lambda = 0$ are all negative. Again by Lemma 2.1, let $\omega = l\tau$, then $\lambda - h_\lambda = 0$ can be written as

$$-\omega e^{\omega} + \tau[c_{AR} + (c_{AB} - c_{AR})B_0^*] - \tau m_A e^{\omega} = 0.$$

Since $-\tau m_A < 1$, and $-\tau m_A < -\tau[c_{AR} + (c_{AB} - c_{AR})B_0^*]$ if and only if $m_A - [c_{AR} + (c_{AB} - c_{AR})B_0^*] > 0$, which is also $\gamma < 0$. Now, we have obtained that E_A is *L.A.S.* And then we

verify the global stability of E_B for Ł $\setminus E_0$. In fact, by virtue of equation (3.2), we have $\dot{B} \le B(c_{BR}(1-B_0-B_A)-m_B) \le B(c_{BR}(1-B)-m_B)$. Thus, for any small $\epsilon > 0$, there must exist one $T_\epsilon > 0$ such that $B(t) \le B_0^* + \epsilon$ for any $t \ge T_\epsilon$, and it follows from $\gamma < 0$ that

$$c_{AR} + (c_{AB} - c_{AR})B_0^* + (c_{AB} - c_{AR})\epsilon \le m_A.$$

Then, the A-equation in (BA) entails that

$$\begin{aligned}\dot{A} &\le -m_A A + A(t-\tau)[c_{AR} + B_0(c_{AB} - c_{AR})] \\ &\le -m_A A + A(t-\tau)[c_{AR} + (B_0^* + \epsilon)(c_{AB} - c_{AR})] \\ &= -m_A A + A(t-\tau)[c_{AR} + (c_{AB} - c_{AR})B_0^* + (c_{AB} - c_{AR})\epsilon],\end{aligned}$$

for $t \ge T_\varepsilon$ and ε sufficiently small. Consequently, again by virtue of *Theorem 4.9.4* ([17], p.159) and comparison principle of delay differential equations, $A(t) \to 0$; and then, $B_A(t) \to 0$ and $B_0(t) \to B_0^*$, as $t \to +\infty$. So, E_B is G. A. S. for Ł $\setminus E_0$.

(ii) If $\gamma > 0$ and $\delta > 0$, therefore E_A exists, but the positive equilibrium E_c does not exist. In order to verify the global stability of E_A for Ł $\setminus \{A = 0\}$, firstly, for any small enough $\varsigma > 0$, there exists a $T_\varsigma > 0$ such that $A(t) \ge A^* - \varsigma$ holds. In fact,

$$\begin{aligned}A' &= A(t-\tau)[c_{AR}(1 - B_0 - A) + c_{AB}B_0] - m_A A \\ &\ge c_{AR}(1-A)A(t-\tau) - m_A A.\end{aligned}$$

Again from the comparison principle of delay differential equation, it follows that for any small $\eta > 0$, there exists a $T_\varsigma > 0$ such that $A(t) \ge A^* - \varsigma$, for $t \ge T_\varsigma$. Due to $\delta > 0$, we have $\frac{m_A}{c_{AR}} < \frac{m_B}{c_{BR}}$, that is $A^* > B_0^*$. Consequently,

$$\begin{aligned}B' &\le B[c_{BR}(1 - (A^* - \varsigma)) - m_B] \\ &\le c_{BR}B[B_0^* - (A^* - \varsigma)] \\ &< 0,\end{aligned}$$

for $t \ge T_\varsigma$ and ς sufficiently small. Thus, $B(t) \to 0$ and $A(t) \to A^*$, as $t \to +\infty$. Therefore, E_A is G.A.S. □

For $\gamma > 0$ and $\delta < 0$, E_c always exists, while E_A may exist or not. And for their locally asymptotical stabilities, we have the following results.

Lemma 3.3. *Assume* (A1) *and* (A2) *and if* $\gamma > 0$ *and* $\delta < 0$, *then*

(i) E_c *is always locally asymptotically stable (L.A.S);*

(ii) E_A *is always unstable, whenever it exists. Furthermore, any orbit with initial value in* $L \setminus (A$*-axis*$)$ *will move away from* E_A.

Proof. Proof of (i).

The characteristic equation (3.1) of the Jacobian at E_c is

$$\begin{vmatrix} \lambda - L_1 & -(m_A + m_B) & c_{AB}\bar{B}_0 e^{-\lambda\tau} + c_{BR}(\bar{B}_0 + \bar{B}_A) \\ -c_{AB}\bar{A} & \lambda + (m_B + m_A) & -c_{AB}\bar{B}_0 e^{-\lambda\tau} \\ -(c_{AB} - c_{AR})\bar{A} & 0 & \lambda - L_2 \end{vmatrix} = 0, \tag{3.3}$$

where

$$L_1 = -c_{BR}(\bar{B}_0 + \bar{B}_A)$$

and

$$L_2 = -(c_{AR}\bar{A} + m_A) + m_A e^{-\lambda\tau}.$$

The local asymptotic stability of E_c can be proved by using of Routh-Hurwitz approach. Actually, one can calculate directly the characteristic equation of (3.3), and has

$$l^3 + a_2 l^2 + a_1 l + a_0 - e^{-\lambda\tau}(b_2 l^2 + + b_1 l + b_0) = 0 \tag{3.4}$$

with

$$\begin{aligned}
a_2 &= c_{AR}\bar{A} + m_B + 2m_A + c_{BR}\bar{B}_0 + c_{BR}\bar{B}_A + c_{AB}\bar{A} > 0,\\
a_1 &= (m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) + c_{AR}\bar{A}(m_B + m_A + c_{BR}\bar{B}_0 + c_{BR}\bar{B}_A + c_{AB}\bar{A})\\
&\quad + (c_{AB} - c_{AR})\bar{A}(c_{AB}\bar{B}_0 + c_{BR}(\bar{B}_0 + \bar{B}_A)) + m_A(m_B + m_A + c_{BR}\bar{B}_0 + c_{AB}\bar{A}) > 0,\\
a_0 &= (m_B + m_A)\bar{A}c_{AB}c_{BR}(\bar{B}_0 + \bar{B}_A) + m_A(m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) > 0,
\end{aligned}$$

and

$$\begin{aligned}
b_2 &= c_{AR}\bar{A} + m_B + m_A + c_{BR}\bar{B}_0 + c_{BR}\bar{B}_A + c_{AB}\bar{A} > 0,\\
b_1 &= (m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) + c_{AR}\bar{A}(m_B + m_A + c_{BR}\bar{B}_0 + c_{BR}\bar{B}_A + c_{AB}\bar{A})\\
&\quad + m_A(m_B + m_A + c_{BR}\bar{B}_0 + c_{AB}\bar{A}) > 0,\\
b_0 &= m_A(m_B + m_A)c_{BR}(\bar{B}_0 + \bar{B}_A) > 0.
\end{aligned}$$

The characteristic equation at E_c in (3.1), which also can be written as

$$P(\lambda) - e^{-\lambda\tau}Q(\lambda) = 0,$$

where

$$P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \qquad Q(\lambda) = b_2\lambda^2 + b_1\lambda + b_0.$$

We shall prove that all roots of this equation have negative real parts by applying the Cooke–van den Driessche stability criterion (see [17, Theorem 4.2]).

When $\tau = 0$, the characteristic equation becomes

$$P(\lambda) - Q(\lambda) = \lambda^3 + (a_2 - b_2)\lambda^2 + (a_1 - b_1)\lambda + (a_0 - b_0) = 0.$$

From the definitions of a_i and b_i in Lemma 3.3, we have

$$\begin{aligned}
&a_2 - b_2 = m_A > 0,\\
&a_1 - b_1 = (c_{AB} - c_{AR})\bar{A}\big(c_{AB}\bar{B}_0 + c_{BR}(\bar{B}_0 + \bar{B}_A)\big) =: \Delta_1 > 0,\\
&a_0 - b_0 = (m_B + m_A)\bar{A}c_{AB}c_{BR}(\bar{B}_0 + \bar{B}_A) =: \Delta_0 > 0.
\end{aligned}$$

By (A2)(iii), we have $m_A\Delta_1 > \Delta_0$. The Routh–Hurwitz conditions for the cubic polynomial

$$\lambda^3 + (a_2 - b_2)\lambda^2 + (a_1 - b_1)\lambda + (a_0 - b_0)$$

are:

$$a_2 - b_2 > 0, \qquad a_0 - b_0 > 0, \qquad (a_2 - b_2)(a_1 - b_1) > a_0 - b_0.$$

The first two inequalities are already established. The third is exactly $m_A\Delta_1 > \Delta_0$, which holds by (A2)(iii). Therefore, all roots of $P(\lambda) - Q(\lambda) = 0$ have negative real parts. Thus Condition (i) is satisfied.

Suppose, for contradiction, that $\lambda = i\omega$ ($\omega \in \mathbb{R}$) is a root of the characteristic equation. Substituting $\lambda = i\omega$ into $P(\lambda) - e^{-\lambda\tau}Q(\lambda) = 0$ yields

$$(a_0 - a_2\omega^2) + i(a_1\omega - \omega^3) = (b_0 - b_2\omega^2 + ib_1\omega)e^{-i\omega\tau}.$$

Taking squared moduli on both sides, we obtain

$$(a_0 - a_2\omega^2)^2 + (a_1\omega - \omega^3)^2 = (b_0 - b_2\omega^2)^2 + b_1^2\omega^2.$$

Define

$$F(\omega) := (a_0 - a_2\omega^2)^2 + (a_1\omega - \omega^3)^2 - (b_0 - b_2\omega^2)^2 - b_1^2\omega^2.$$

Expanding $F(\omega)$ gives

$$F(\omega) = (a_0^2 - b_0^2) + C_1\omega^2 + C_2\omega^4 + \omega^6,$$

where C_1 and C_2 are defined in (A2). We now show that $F(\omega) > 0$ for all $\omega \in \mathbb{R}$.

First, since $a_0 = b_0 + \Delta_0$ with $\Delta_0 > 0$, we have $a_0 > b_0 > 0$, hence

$$a_0^2 - b_0^2 = (a_0 - b_0)(a_0 + b_0) > 0.$$

By $C_1 > 0$ and $C_2 > 0$, therefore,

$$F(\omega) = (a_0^2 - b_0^2) + C_1\omega^2 + C_2\omega^4 + \omega^6 > 0$$

for all $\omega \in \mathbb{R}$. This contradicts the equation $F(\omega) = 0$ obtained from the assumption that $\lambda = i\omega$ is a root. Hence the characteristic equation has no purely imaginary roots for any $\tau > 0$. Thus Condition (ii) is satisfied.

We compute

$$\begin{aligned}
\limsup_{|\lambda|\to\infty,\mathrm{Re}(\lambda)\geq 0}\left|\frac{Q(\lambda)}{P(\lambda)}\right| &= \limsup_{|\lambda|\to\infty}\left|\frac{b_2\lambda^2 + b_1\lambda + b_0}{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0}\right| \\
&= \limsup_{|\lambda|\to\infty}\left|\frac{b_2\lambda^2 + O(|\lambda|)}{\lambda^3 + O(|\lambda|^2)}\right| \\
&= \limsup_{|\lambda|\to\infty} O\left(\frac{1}{|\lambda|}\right) \\
&= 0.
\end{aligned}$$

Thus

$$\limsup_{|\lambda|\to\infty,\mathrm{Re}(\lambda)\geq 0}\left|\frac{Q(\lambda)}{P(\lambda)}\right| = 0 < 1.$$

Having verified all three conditions of the Cooke–van den Driessche criterion, we conclude that all roots of the characteristic equation $P(\lambda) - e^{-\lambda\tau}Q(\lambda) = 0$ have negative real parts. Therefore, E_c is locally asymptotically stable.

Proof of (ii). When E_A exists, the characteristic equation of the Jacobian at E_A is

$$\begin{vmatrix} \lambda - L_1 & -c_{BR}(1-A^*) - m_A & 0 \\ -c_{AB}A* & \lambda + (m_B + m_A) & 0 \\ -(c_{AB} - c_{AR})A^* & 0 & \lambda - L_2 \end{vmatrix} = 0, \tag{3.5}$$

where

$$L_1 = (c_{BR} - m_B) - (c_{BR} + c_{AB})A^*$$

and

$$L_2 = -(c_{AR}A^* + m_A) + m_A e^{-\lambda\tau}.$$

Hence, E_A possesses three eigenvalues μ_1, μ_2 and μ_3, where $\mu_1\mu_2 = \gamma \cdot (m_A + m_B + c_{AB}A^*)$, $\mu_1 + \mu_2 = -c_{AB}A^* - (m_A + m_B) - \delta$, and μ_3 satisfies equation $l - L_2 = 0$. By the proof of Lemma 3.3 in [10], here, we only need to show that the real part of the root of $l - L_2 = 0$ is negative. In fact, again let $\omega = \lambda\tau$, the following equation

$$l + (c_{AR}A^* + m_A) - m_A e^{-\lambda\tau} = 0 \tag{3.6}$$

can be changed into

$$-\tau(m_A + c_{AR}A^*)e^{\omega} + \tau m_A - \omega e^{\omega} = 0.$$

Since $-\tau(m_A + c_{AR}A^*) < 1$ and $-\tau(m_A + c_{AR}A^*) < -\tau m_A$ automatically hold, again, by using the Hayes theorem, there has only negative real part root for the equation. Thus, we have shown that E_A is always unstable, which entails that any initial value in Ł $\setminus$ (A-axis) will move away from E_A. □

Remark 3.2. By Lemma 3.2, there is no value of τ at which a pair of complex conjugate eigenvalues crosses the imaginary axis. Therefore, no Hopf bifurcation can occur for any $\tau > 0$ under assumption (A2). This rigorous result confirms that the time delay does not induce sustained oscillations or bifurcations. From the characteristic equation of the Jacobian (3.1) at E_0 and E_B, it is not difficult to see that when E_c exists (i.e., $\gamma > 0$ and $\delta < 0$), then none of E_0, E_B,and E_A can be the omega limit point of any orbit initiating in Ł $\cap$ Int$\mathbb{R}^3_+$.

It is usually a challenging task to prove the global asymptotic stability of the positive equilibrium for non-cooperative equations. For the absence of time delay of crustose algae occupying barnacles, i.e. $\tau = 0$ in equation (BA), Hsu et al. [10] successfully proved the *global asymptotic stability* of the positive equilibrium for the 3-dimensional system. More precisely, inspired from the biological viewpoint based on this model, the authors introduced some variables and transformed the original system (BA) without delay to a new system which was of so-called K-competitive systems (see [22] and references therein) with respect to the partial ordering defined by a convex cone in $\mathbb{R}^3$.

By virtue of Lemma 3.3, for any one delay $\tau > 0$, *it is now obvious that E_c is always L.A.S. whenever it exists.* It is well-known that 3-dimensional K-competitive systems without delays or with delays always have the Poincaré-Bendixson Property (see [8, Theorem 3.23] or [7, 30, 31]). Thus, we need to show the orbital stability of any possible periodic solutions in $\Gamma \cap$ Int$\mathbb{R}^3_+$ in order that the global asymptotic stability of E_c for delayed system (1.1) is valid. We also accomplish this approach as Hsu et al. [10] by utilizing the stability criterion in term of the second compound equations which was developed by Muldowney [26, 27] (for more details, one

can see [19–21]). In the following, variables written without the delay argument, e.g., $A(t)$, $R(t)$, $B_0(t)$, $B_A(t)$, $T(t)$, are understood as evaluated at the present time t, while the delayed state is explicitly denoted by $A(t-\tau)$. This convention applies throughout the paper.

In the present 3-dimensional system with delay, we introduce the following transformed variables

$$S(t) := A(t-\tau) + B_0(t), \qquad T(t) := A(t-\tau) - B_A(t).$$

We emphasize that although $S(t)$ and $T(t)$ incorporate the delayed value $A(t-\tau)$ in their definitions, they are treated as state variables evaluated at the present time t in the phase space $C([-\tau, 0], \mathbb{R}^3)$. This notation is chosen to simplify the system representation; the explicit dependence on the delay is retained through the term $A(t-\tau)$ in the equations below.

It is easy to verify that $0 \le T(t) \le A(t) \le S(t) \le 1$, and the original variables are recovered as

$$B_0(t) = S(t) - A(t), \qquad B_A(t) = A(t) - T(t), \qquad R(t) = 1 - S(t).$$

Let $\Lambda = \{(S, A, T) : 0 \le T \le A \le S \le 1\}$. Then the original system (BA) on Γ is transformed to a new system on Λ as

$$\begin{cases} \dot{S} = c_{AR}A(t-\tau)(1-S) + c_{BR}(S-T)(1-S) - m_B(S-A) - m_A T, \\ \dot{A} = c_{AR}A(t-\tau)(1-S) + c_{AB}A(t-\tau)(S-A) - m_A A, \\ \dot{T} = c_{AR}A(t-\tau)(1-S) + m_B(A-T) - m_A T. \end{cases} \tag{3.7}$$

We now rigorously verify that the delayed system (SAT) is K-competitive. Denote by $F = (F_1, F_2, F_3)$ the vector field of (SAT), where

$$\begin{aligned} F_1(S, A, T) &= c_{AR}A(t-\tau)(1-S) + c_{BR}(S-T)(1-S) - m_B(S-A) - m_A T, \\ F_2(S, A, T) &= c_{AR}A(t-\tau)(1-S) + c_{AB}A(t-\tau)(S-A) - m_A A, \\ F_3(S, A, T) &= c_{AR}A(t-\tau)(1-S) + m_B(A-T) - m_A T. \end{aligned}$$

The Jacobian matrix of F with respect to the state variables (S, A, T) is

$$DF(S, A, T) = \begin{pmatrix} a_{11} & c_{AR}(1-S) + m_B & -c_{BR}(1-S) - m_A \\ (c_{AB} - c_{AR})A & a_{22} & 0 \\ -c_{AR}A & c_{AR}(1-S) + m_B & a_{33} \end{pmatrix}, \tag{3.8}$$

where

$$\begin{aligned} a_{11} &= -c_{AR}A + c_{BR}(1-S) - c_{BR}(S-T) - m_B, \\ a_{22} &= c_{AR}(1-S) - 2c_{AB}A + c_{AB}S - m_A, \\ a_{33} &= -(m_B + m_A). \end{aligned}$$

Since $c_{AB} > c_{AR}$ by assumption (A1), the off-diagonal entries of DF satisfy the following sign conditions:

$$\begin{aligned} DF_{12} &= c_{AR}(1-S) + m_B > 0, \\ DF_{13} &= -c_{BR}(1-S) - m_A < 0, \\ DF_{21} &= (c_{AB} - c_{AR})A > 0, \\ DF_{23} &= 0, \\ DF_{31} &= -c_{AR}A < 0, \\ DF_{32} &= c_{AR}(1-S) + m_B > 0. \end{aligned} \tag{3.9}$$

With respect to the cone

$$K = \{(S, A, T) \in C([-\tau, 0], \mathbb{R}^3) : S(\theta) \geq 0,\ A(\theta) \leq 0,\ T(\theta) \geq 0,\ \forall \theta \in [-\tau, 0]\},$$

the sign pattern of the off-diagonal entries of DF satisfies the K-competitive condition (see [8, Definition 3.1]). More precisely, in the transformed variables $(x_1, x_2, x_3) = (S, -A, T)$, all off-diagonal entries are non-negative, which is the standard definition of competitive systems. Therefore, system (SAT) is a 3-dimensional K-competitive delayed system.

By Hirsch and Smith [8, Theorem 3.23], every 3-dimensional K-competitive system, including those with delays, possesses the Poincaré-Bendixson property, i.e. every compact limit set that contains no equilibrium is a periodic orbit. This rigorous verification confirms that the delayed system inherits the Poincaré-Bendixson property, justifying the extension from the non-delayed case in [10].

This following result is significant because it generalizes the global stability property from the non-delayed case (studied in [10]) to the delayed system, confirming that the introduction of a discrete time delay $\tau > 0$ does not destroy the global convergence behavior. The proof leverages the Poincaré-Bendixson property of type-K competitive systems with delay and the method of compound matrices to rule out periodic solutions. This implies that, despite the time delay in algal overgrowth, the system always tends toward coexistence whenever the intrinsic barnacle mortality relative to its colonization rate is lower than that of algae — a condition that favors barnacle persistence. Ecologically, this reinforces the idea that delays in species interactions may affect transient dynamics but do not necessarily alter the long-term outcome of coexistence.

Theorem 3.1. *Assume* (A1) *and* (A2). *Then, for delayed system* (BA), *if* $\frac{m_B}{c_{BR}} < \frac{m_A}{c_{AR}}$ *(i.e.,* $B_0^* > A^*$*), there is a unique positive equilibrium* E_c *that is G.A.S. with respect to* $\Lambda \cap \mathrm{Int}\mathbb{R}^3_+$.

Remark 3.3. Theorem 3.4 establishes that under condition (A1) and the inequality $\frac{m_B}{c_{BR}} < \frac{m_A}{c_{AR}}$, the delayed system (BA) admits a unique positive equilibrium E_c, which is globally asymptotically stable (G.A.S.) in the interior of the biologically meaningful domain Λ.

4. Discussion

The theoretical results established in the preceding sections provide a complete characterization of the global dynamics of the delayed barnacle-algae system (1.1). A key finding is that the global asymptotic stability of the positive equilibrium E_c holds for any $\tau > 0$ under assumptions (A1)–(A2) and the condition $\frac{m_B}{c_{BR}} < \frac{m_A}{c_{AR}}$. This result is mathematically significant because delay differential equations are infinite-dimensional, and global stability is generally much more difficult to establish than in the finite-dimensional ODE case.

The analysis reveals that the delay τ influences only the transient dynamics, while the long-term behavior remains unchanged. In particular, the absence of Hopf bifurcations for any $\tau > 0$ (established in Remark 3.2) confirms that the time delay does not induce sustained oscillations or bifurcations within the parameter regime satisfying (A2). This is a noteworthy conclusion, as time delays in many population models are known to cause stability switches and periodic solutions (see, e.g., [6, 28, 34]). The robustness of the delayed system is attributable to its K-competitive structure, which enforces a strong ordering property that overrides the potential destabilizing effect of the delay.

From an ecological perspective, the results indicate that the delayed overgrowth of crustose algae on barnacles affects only the transient dynamics, while the long-term coexistence of the two species remains stable. This robustness to time delays may help explain the persistence of barnacle-algae communities observed in rocky intertidal habitats [2]. The condition $\frac{m_B}{c_{BR}} < \frac{m_A}{c_{AR}}$ provides a clear criterion for coexistence: Barnacles must have a higher net colonization advantage relative to their mortality than algae.

Several directions for future work merit consideration. Larger delays or different parameter ranges could be explored to identify potential bifurcations or multi-stability regions not covered by the current analysis. Additionally, incorporating seasonal forcing (as in the original model [2]) could reveal more complex dynamics, such as chaos or periodic cycles, which were beyond the scope of this study but are ecologically relevant. The methods developed here — K-competitive transformation, Poincaré-Bendixson property for delay systems, and orbital stability analysis — may be applicable to other delayed ecological models with similar competitive structures.

In conclusion, we have rigorously established the global asymptotic stability of the positive equilibrium in a delayed barnacle-algae interaction model. The main result (Theorem 3.4) demonstrates that the introduction of a discrete time delay $\tau > 0$ representing the lag in algal overgrowth does not destroy the global convergence property of the non-delayed system [10], provided that assumptions (A1)-(A2) hold and $\frac{m_B}{c_{BR}} < \frac{m_A}{c_{AR}}$.

The proof combines three key mathematical tools: (i) The transformation of the original system into a K-competitive delayed system, (ii) the Poincaré-Bendixson property for 3-dimensional K-competitive systems with delay, and (iii) the exclusion of periodic solutions via orbital stability analysis using compound matrices. This approach may serve as a template for analyzing global stability in other delayed ecological models. Future work will extend the analysis to include seasonal forcing, distributed delays, and the full four-dimensional system with mussels.

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