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HOMOCLINIC SOLUTIONS OF NONPERIODIC FOURTH-ORDER ϕ_C-LAPLACIAN PARTIAL DIFFERENCE EQUATIONS*

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Abstract In this paper, we investigate the existence of nontrivial homoclinic solutions for a class of nonperiodic fourth-order partial difference equations involving ϕ_c-Laplacian. By the Mountain Pass Lemma and Fountain Theorem, we establish two key results: (1) Under mild nonlinear growth conditions, the equation admits at least one nontrivial homoclinic solution; (2) for odd symmetric nonlinear terms, the equation possesses infinitely many homoclinic solutions. A critical innovation lies in relaxing the classical Ambrosetti-Rabinowitz (AR) condition to a more general form, enabling the applications of our results to nonlinearities (e.g., logarithmic terms) that fail the AR condition. Additionally, the proposed framework is extendable to higher-order ϕ_c-Laplacian partial difference equations. Two concrete examples are provided to verify the theoretical validity and practical applicability of our theorems.

Keywords Nontrivial homoclinic solution, ϕ_c-Laplacian, fourth-order partial difference equation, Mountain Pass Lemma, Fountain Theorem.

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1. Introduction

Homoclinic solutions are closely related to solitons and play a key role in analyzing chaos in dynamical systems. It is significant to investigate homoclinic solutions of dynamical systems. For $\omega > 0$, we explore the existence of homoclinic solutions of a class of nonperiodic fourth-order ϕ_c-Laplacian partial difference equations expressed by

$$\begin{aligned} &\Delta_1^2(\phi_c(\Delta_1^2 u(\kappa-2,\nu))) + \Delta_2^2(\phi_c(\Delta_2^2 u(\kappa,\nu-2))) - \omega[\Delta_1(\phi_c(\Delta_1 u(\kappa-1,\nu))) \\ &+ \Delta_2(\phi_c(\Delta_2 u(\kappa,\nu-1)))] + g(\kappa,\nu)u(\kappa,\nu) = f\left((\kappa,\nu),u(\kappa,\nu)\right), \qquad (\kappa,\nu)\in\mathbb{Z}^2. \end{aligned} \tag{1.1}$$

Here, the mean curvature operator $\phi_c(\chi) = \frac{\chi}{\sqrt{1+\chi^2}}$ for all $\chi \in \mathbb{R}$. The primitive function of ϕ_c is $\Phi_c(\chi) := \int_0^\chi \phi_c(\iota)d\iota = \sqrt{1+\chi^2} - 1$. $\Delta_1 u(\kappa,\nu) = u(\kappa+1,\nu) - u(\kappa,\nu)$, $\Delta_2 u(\kappa,\nu) = u(\kappa,\nu+1) - u(\kappa,\nu)$ are forward difference operators. $\Delta_i^2 u(\kappa,\nu) = \Delta_i(\Delta_i u(\kappa,\nu))$ for $i = 1,2$. $f((\kappa,\nu),\cdot) \in C(\mathbb{Z}^2\times\mathbb{R},\mathbb{R})$ with $F\left((\kappa,\nu),u\right) = \int_0^u f\left((\kappa,\nu),s\right)ds$ for all $(\kappa,\nu)\in\mathbb{Z}^2$. Commonly, a nontrivial homoclinic solution for (1.1) is referred that $u \neq 0$ solves (1.1) and fulfills $\lim\limits_{|\kappa|+|\nu|\to+\infty} u(\kappa,\nu) = 0$.

Difference equations, serving as discrete counterparts to differential equations, not only boast a long history of research and extensive applications across various fields but also enable accurate modeling of discrete phenomena [16, 27]. Research on the theory of difference equations has been

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extensively conducted ever since Guo and Yu [6] pioneered the application of the variational method to difference equations. And further in-depth studies can be found in relevant literature [1, 2, 9, 17, 26].

Partial difference equations, involving two or more variables, have attracted significant attention from researchers to modeling diverse discrete multi-variable phenomena (e.g., protein dynamics, neural networks, nonlinear transmission lines) [3, 7, 21]. Remarkable progress has been made in studying partial difference equations. For instance, Long [13] studied infinitely many homoclinic solutions of partial difference equations without (AR) condition. Periodic solutions, homoclinic solutions, and boundary value problems of partial difference equations were explored in [10], [11, 12], and [14, 24, 25], respectively.

It is easy to find that the mentioned results do not consider the mean curvature operator ϕ_c. Meanwhile, ϕ_c is of both practical and theoretical significance [4, 5, 8, 19]. It has captured great interest, primarily emanating from seeking radial (singular) solutions in the analysis of capillary surfaces. For example, via the critical point theory, authors [18, 23, 28] formulated results related the Dirichlet boundary value problem, homoclinic solutions, and heteroclinic solutions, respectively. Very recently, for $(\kappa,\nu)\in\mathbb{Z}^2$ and the parameter $\lambda>0$, the following ϕ_c-Laplacian partial difference equation

$$-\Delta_1\left[\phi_c\left(\Delta_1 u(\kappa-1,\nu)\right)\right]-\Delta_2\left[\phi_c\left(\Delta_2 u(\kappa,\nu-1)\right)\right]+b(\kappa,\nu)u(\kappa,\nu)=\lambda f\left((\kappa,\nu),u(\kappa,\nu)\right), \quad (1.2)$$

was addressed in [15], which exhibited multiplicity existence results on homoclinic solutions.

Comparing (1.1) with (1.2), we find that (1.1) generalizes (1.2) and exhibits the following four major features: First, (1.1) harbors latent applications across diverse domains and occurs in discrete models of nonlinear wave propagation, such as protein dynamics and neural networks and nonlinear transmission lines. Second, homoclinic solutions without periodic assumptions are often very hard to handle since they induce infinite-dimensional dynamical systems. Thus, it is necessary to overcome the scarceness of compactness of corresponding functional, which engages us an essential difficulty. Third, our obtained results relax the classical (AR) condition, thereby expanding the scope of applicable problems. Fourth, our conclusions of Theorems 1.1 and 1.2 easily apply to discuss homoclinic solutions of (1.1) in higher-order, broadening the generality of our approach. It is worthy to pointing out that our work constitutes an extension and improvement of some existing research. For the details we refer to Remarks 1.3-1.4.

Based on the aforementioned reasons, we focus on homoclinic solutions of (1.1). Also, two examples are presented to demonstrate theoretical innovations and practical applicability of our results. Our obtained results not only enrich the theoretical system of partial difference equations but also provide a mathematical tool for modeling complex discrete multi-variable phenomena involving mean curvature effects.

Denote $g_*=\min\{g(\kappa,\nu):(\kappa,\nu)\in\mathbb{Z}^2\}>0$ and let $h(\kappa,\nu):\mathbb{Z}^2\to\mathbb{R}^+$ with $\max\limits_{(\kappa,\nu)\in\mathbb{Z}^2}\{h(\kappa,\nu)\}=h_0>g_*$. Write

$$\mu^*=\inf\{\|u\|_E^2:u\in E,\sum_{(\kappa,\nu)\in\mathbb{Z}^2}h(\kappa,\nu)u^2(\kappa,\nu)=1\}, \quad (1.3)$$

where E and $\|u\|_E$ are defined in the following section. We make the following basic assumption:
(G) $\lim\limits_{|\kappa|+|\nu|\to+\infty}g(\kappa,\nu)=+\infty$ for $(\kappa,\nu)\in\mathbb{Z}^2$.

Remark 1.1. The assumption (G) ensures the coercivity of the potential term, which help us overcome the core difficulty of noncompactness in nonperiodic cases to obtain the compact embedding of E into l^2 (Lemma 2.3).

Now, we are in the position to display our main results.

Theorem 1.1. *Let (G) hold and $\eta \geq 1$. Assume that*
(A_1) *$f((\kappa,\nu),u) \equiv 0$ for all $u<0$ and $(\kappa,\nu) \in \mathbb{Z}^2$, and there exists $a(\kappa,\nu) : \mathbb{Z}^2 \to \mathbb{R}^+$ with $\max_{(\kappa,\nu)\in\mathbb{Z}^2}\{a(\kappa,\nu)\} = a_0 < \frac{g_*}{2}$ such that*

$$\lim_{u\to 0^+} \frac{f((\kappa,\nu),u)}{u^\eta} = a(\kappa,\nu) \quad \text{for all} \quad (\kappa,\nu) \in \mathbb{Z}^2,$$

and

$$\frac{f((\kappa,\nu),u)}{u^\eta} \geq a(\kappa,\nu) \quad \text{for all} \quad u>0 \quad \text{and} \quad (\kappa,\nu) \in \mathbb{Z}^2;$$

(A_2) *$\lim_{u\to+\infty} \frac{f((\kappa,\nu),u)}{u^\eta} = \frac{2(16+4\omega)+g_*}{g_*} h(\kappa,\nu)$ for all $(\kappa,\nu) \in \mathbb{Z}^2$;*
(A_3) *there exist constants $\theta > 2$ and $0 < L < \frac{g_*(\theta-2)}{2\theta}$ such that*

$$F((\kappa,\nu),u) - \frac{1}{\theta} f((\kappa,\nu),u)u \leq Lu^2 \quad \text{for all} \quad u>0 \quad \text{and} \quad (\kappa,\nu) \in \mathbb{Z}^2.$$

Then either (Λ_1) $\eta = 1$ and $\mu^* < 1$ or (Λ_2) $\eta > 1$, (1.1) admits at least one nontrivial homoclinic solution.

Theorem 1.2. *Let (G) hold. Moreover,*
(A_4) *there exist constants $d>0$ and $\alpha > 2$ such that*

$$|f((\kappa,\nu),u)| \leq d(|u| + |u|^{\alpha-1}) \qquad \text{for all} \quad (\kappa,\nu) \in \mathbb{Z}^2 \quad \text{and} \quad u \in \mathbb{R};$$

(A_5) *$\lim_{|u|\to+\infty} \frac{f((\kappa,\nu),u)}{u} = +\infty$ for all $(\kappa,\nu) \in \mathbb{Z}^2$;*
(A_6) *there exist three constants b_0, $b_1 > 0$ and $\theta_0 < 2$ such that*

$$0 < \left(2 + \frac{1}{b_0 + b_1|u|^{\theta_0}}\right) F((\kappa,\nu),u) \leq uf((\kappa,\nu),u) \qquad \text{for all} \quad u \in \mathbb{R}\backslash\{0\} \quad \text{and} \quad (\kappa,\nu) \in \mathbb{Z}^2;$$

(A_7) *$f((\kappa,\nu),-u) = -f((\kappa,\nu),u)$ for all $((\kappa,\nu),u) \in \mathbb{Z}^2 \times \mathbb{R}$.*
Then (1.1) possesses infinitely many homoclinic solutions.

Remark 1.2. μ^* defined by (1.3) is reasonable. Similar to the fashion of Lemma 8 in [22], we state the proof in Lemma 3.2 for the sake of clarity.

Remark 1.3. If the classical Ambrosetti-Rabinwitz condition (AR) there is $\mu > 2$ such that

$$0 < \mu F((\kappa,\nu),u) \leq uf((\kappa,\nu),u) \quad \text{for all} \quad (\kappa,\nu) \in \mathbb{Z}^2 \quad \text{and} \quad u \in \mathbb{R}\backslash\{0\}.$$

Then (A_6) is also true by choosing $b_0 > \frac{1}{\mu-2}$, $b_1 > 0$, and $\theta_0 < 2$. We assert that (A_6) is more applicable than (AR). For example, consider (1.1) with

$$F((\kappa,\nu),u) = u^2 \ln(1+|u|).$$

By straightforward computation, for all $(\kappa,\nu) \in \mathbb{Z}^2$ and $u \in \mathbb{R}\backslash\{0\}$, we have

$$uf((\kappa,\nu),u) = 2u^2\ln(1+|u|) + \frac{u^3}{1+|u|} \cdot \frac{u}{|u|} \geq \left(2 + \frac{1}{1+|u|}\right) F((\kappa,\nu),u) > 0,$$

which manifests (A_6) with $b_0 = b_1 = \theta_0 = 1$. Meanwhile, For any $\mu > 2$,

$$uf((\kappa,\nu),u) - \mu F((\kappa,\nu),u) \to -\infty \qquad \text{as} \quad |u| \to +\infty.$$

Thus, F fails to fulfill (AR), and (A_6) relaxes (AR) somehow. (A_4)-(A_6) are used as more general conditions for improving (AR), which make the results applicable to more types of nonlinear terms (e.g., logarithmic nonlinearity). It has improved some existing literature.

Remark 1.4. Theorems 1.1 and 1.2 are easily applied to partial difference equations of the form (1.1) with higher-order ϕ_c-Laplacian. The key distinction lies in the need to adjust the order at this point. Nevertheless, this variation does not hinder our discussion in the higher-order case.

An outline of our paper reads as follows. In Section 2, we construct the corresponding variational structure to (1.1) and provide some primal lemmas. Section 3 is devoted to the detailed proofs of Theorems 1.1 and 1.2. Also, two concrete examples are presented to bridge the gap between theory and application in Section 4. Finally, we draw an conclusion as an end of the present paper in Section 5.

2. Variational functional and preliminaries

In this section, we state some preliminary notations and construct the associated variational functional of (1.1).

Let $\mathcal{H}$ be a bidirectional real-valued vector space of $u = \{u(\kappa,\nu)\}_{(\kappa,\nu)\in\mathbb{Z}^2}$. Namely,

$$\mathcal{H} = \left\{u = \{u(\kappa,\nu)\} | u(\kappa,\nu) \in \mathbb{R}, (\kappa,\nu) \in \mathbb{Z}^2\right\}.$$

Define the subspace E of $\mathcal{H}$ as

$$E = \big\{u \in \mathcal{H} | \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)|u(\kappa,\nu)|^2 < +\infty\big\},$$

with the norm

$$\|u\|_E = \left(\sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)|u(\kappa,\nu)|^2\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in E.$$

Then $(E, \|\cdot\|_E)$ is a Hilbert space.

For $1 \le p < +\infty$, let

$$l^p(\mathbb{Z}^2,\mathbb{R}) = \left\{u \in \mathcal{H} : \sum_{(\kappa,\nu)\in\mathbb{Z}^2} |u(\kappa,\nu)|^p < +\infty\right\}, \qquad \|u\|_{l^p} = \left(\sum_{(\kappa,\nu)\in\mathbb{Z}^2} |u(\kappa,\nu)|^p\right)^{\frac{1}{p}}.$$

Denote

$$l^\infty(\mathbb{Z}^2,\mathbb{R}) = \left\{u \in \mathcal{H} : \sup_{(\kappa,\nu)\in\mathbb{Z}^2} |u(\kappa,\nu)| < +\infty\right\},$$
$$\|u\|_\infty = \sup_{(\kappa,\nu)\in\mathbb{Z}^2} |u(\kappa,\nu)| \quad \text{for all} \quad u \in l^\infty(\mathbb{Z}^2,\mathbb{R}).$$

Then $(l^p, \|\cdot\|_{l^p})$ is a reflexive Banach space. Moreover, there holds

$$l^\tau \subseteq l^\iota \quad \text{and} \quad \|u\|_{l^\iota} \leq \|u\|_{l^\tau}, \qquad 1 \leq \tau \leq \iota \leq +\infty. \tag{2.1}$$

Specially,

$$\|u\|_\infty \leq \|u\|_{l^2} \leq g_*^{-\frac{1}{2}} \|u\|_E, \qquad \forall u \in E. \tag{2.2}$$

Define the energy functional $\ell : E \to \mathbb{R}$ related to (1.1) as

$$\begin{aligned} \ell(u) = &\sum_{(\kappa,\nu)\in\mathbb{Z}^2} \left[\Phi_c\big(\Delta_1^2 u(\kappa-2,\nu)\big) + \Phi_c\big(\Delta_2^2 u(\kappa,\nu-2)\big)\right] + \frac{1}{2}\sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)|u(\kappa,\nu)|^2 \\ &+ \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \omega[\Phi_c(\Delta_1 u(\kappa-1,\nu)) + \Phi_c(\Delta_2 u(\kappa,\nu-1))] - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} F((\kappa,\nu),u(\kappa,\nu)). \end{aligned} \tag{2.3}$$

Standard arguments show that the functional ℓ is well-defined and of class C^1 on E. Further, the derivative of ℓ is given by

$$\begin{aligned} \langle \ell'(u), v\rangle = &\sum_{(\kappa,\nu)\in\mathbb{Z}^2} [\phi_c(\Delta_1^2 u(\kappa-2,\nu))\Delta_1^2 v(\kappa-2,\nu) + \phi_c(\Delta_2^2 u(\kappa,\nu-2))\Delta_2^2 v(\kappa,\nu-2)] \\ &+ \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \omega[\phi_c(\Delta_1 u(\kappa-1,\nu))\Delta_1 v(\kappa-1,\nu) + \phi_c(\Delta_2 u(\kappa,\nu-1))\Delta_2 v(\kappa,\nu-1)] \\ &+ \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)u(\kappa,\nu)v(\kappa,\nu) - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} f((\kappa,\nu),u(\kappa,\nu))v(\kappa,\nu) \\ = &\sum_{(\kappa,\nu)\in\mathbb{Z}^2} [\Delta_1^2(\phi_c(\Delta_1^2 u(\kappa-2,\nu))) + \Delta_2^2(\phi_c(\Delta_2^2 u(\kappa,\nu-2)))]v(\kappa,\nu) \\ &- \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \omega[\Delta_1(\phi_c(\Delta_1 u(\kappa-1,\nu))) + \Delta_2(\phi_c(\Delta_2 u(\kappa,\nu-1)))]v(\kappa,\nu) \\ &+ \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)u(\kappa,\nu)v(\kappa,\nu) - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} f((\kappa,\nu),u(\kappa,\nu))v(\kappa,\nu). \end{aligned} \tag{2.4}$$

Then, for any $v \in E$, (2.4) ensures that $\langle \ell'(u), v\rangle = 0$ if and only if

$$\begin{aligned} &\Delta_1^2(\phi_c(\Delta_1^2 u(\kappa-2,\nu))) + \Delta_2^2(\phi_c(\Delta_2 u(\kappa,\nu-2))) - \omega[\Delta_1(\phi_c(\Delta_1 u(\kappa-1,\nu))) \\ &+ \Delta_2(\phi_c(\Delta_2 u(\kappa,\nu-1)))] + g(\kappa,\nu)u(\kappa,\nu) - f((\kappa,\nu),u(\kappa,\nu)) = 0, \qquad \forall u \in E, \end{aligned}$$

which is (1.1). Thereby, the problem on seeking non-trivial homoclinic solutions of (1.1) has been transformed into looking for non-zero critical points of the functional ℓ on E.

Now we introduce the definition of the $(C)_c$ condition and two theorems that make crucial contributions to the subsequent proofs.

Definition 2.1. Let C^1-functional ℓ be defined on real Banach space E. ℓ satisfies the $(C)_c$ condition refers that any $(C)_c$ sequence $\{y_m\} \subset E$, i.e., $\ell(y_m) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|y_m\|)\|\ell'(y_m)\| \to 0$ as $m \to \infty$, possesses a convergent subsequence.

Lemma 2.1. (Mountain Pass Lemma [20]) *Let E be a reflexive Banach space. Assume that $\ell : E \to \mathbb{R}$ is a C^1 functional, and there exist constants γ, $\varsigma > 0$ and $e \in E \backslash B_\gamma$ such that*

$$\ell|_{\partial B_\gamma} \geq \varsigma, \quad \ell(e) \leq 0.$$

Then there exists a $(C)_c$ sequence $\{y_m\} \subset E$ at level c, where

$$c = \inf_{q\in\Gamma} \max_{s\in[0,1]} \ell(q(s)) \geq \varsigma, \qquad \Gamma = \{q \in C([0,1], E) | q(0) = 0, q(1) = e\}.$$

Furthermore, ℓ gains a critical value c if ℓ meets the $(C)_c$ condition.

We provide several notations antecedent to the presentation of Fountain Theorem. Let reflexive Banach space $E = \overline{\bigoplus_{m\in\mathbb{N}} E_m}$ with $\dim E_m < +\infty$ for every $m \in \mathbb{N}$. Set

$$Y_j = \bigoplus\nolimits_{m=0}^{j} E_m \quad \text{and} \quad Z_j = \overline{\bigoplus\nolimits_{m=j}^{\infty} E_m}.$$

Lemma 2.2. (Fountain Theorem [29]) *Suppose that $\ell : E \to \mathbb{R}$ is a C^1 functional and fulfills $\ell(-u) = \ell(u)$ for $u \in E$. For every $j \in \mathbb{N}$, if there exist $\sigma_j > \tau_j > 0$ such that*
(W_1) ℓ meets the $(C)_c$ condition for some $c \in \mathbb{R}$;
(W_2) $s_j := \inf_{u\in Z_j, \|u\|_E=\tau_j} \ell(u) \to \infty$ as $j \to \infty$;
(W_3) $t_j := \max_{u\in Y_j, \|u\|_E=\sigma_j} \ell(u) \leq 0$.
Then ℓ achieves an unbounded sequence of critical values.

Lemma 2.3. *If $g(\kappa, \nu)$ verifies (G), then the embedding map from E into l^2 is compact.*

Proof. Let $\{u_j\} \subset E$ be a bounded sequence. Namely, there exists a constant $D_1 > 0$ such that $\|u_j\|_E^2 < D_1$ for all $j \in \mathbb{N}$. Due to the reflexivity of E, going if necessary to a subsequence, one has

$$u_j \rightharpoonup u \qquad \text{in} \quad E.$$

For the sake of simplicity, we assume that $u = 0$, in particular $u_j(\kappa, \nu) \to 0$ as $j \to \infty$ for all $(\kappa, \nu) \in \mathbb{Z}^2$. Subsequently, we show that

$$u_j \to 0 \qquad \text{in} \quad l^2.$$

Applying (G), for any $\varepsilon > 0$, there exists $V_1 \in \mathbb{N}$ such that

$$g(\kappa, \nu) \geq \frac{1 + D_1}{\varepsilon} \quad \text{for all} \quad |\kappa| + |\nu| > V_1.$$

Then

$$\sum_{|\kappa|+|\nu|>V_1} |u_j(\kappa, \nu)|^2 \leq \frac{\varepsilon}{1 + D_1} \sum_{|\kappa|+|\nu|>V_1} g(\kappa, \nu)|u_j(\kappa, \nu)|^2 \leq \frac{D_1\varepsilon}{1 + D_1}. \tag{2.5}$$

Further, the continuity of the finite sum ensures that there exists $J_1 \in \mathbb{N}$ such that

$$\sum_{|\kappa|+|\nu|\leq V_1} |u_j(\kappa, \nu)|^2 \leq \frac{\varepsilon}{1 + D_1} \quad \text{for all} \quad j > J_1. \tag{2.6}$$

Put $\widehat{N} = \max\{V_1, J_1\}$. For $j > \widehat{N}$, it follows from (2.5) and (2.6) that

$$\sum_{(\kappa,\nu)\in\mathbb{Z}^2} |u_j(\kappa, \nu)|^2 = \sum_{|\kappa|+|\nu|\leq\widehat{N}} |u_j(\kappa, \nu)|^2 + \sum_{|\kappa|+|\nu|>\widehat{N}} |u_j(\kappa, \nu)|^2 \leq \frac{\varepsilon}{1 + D_1} + \frac{D_1\varepsilon}{1 + D_1} = \varepsilon. \tag{2.7}$$

Consequently, sequence $\{u_j\}$ strongly converges to 0 in l^2 and $E \hookrightarrow l^2$ is compact. $\square$

3. Proofs of main results

In this section, we are to display detailed proofs of Theorems 1.1 and 1.2 with aid of Mountain Pass Lemma and Fountain Theorem, respectively.

Lemma 3.1. *Suppose that* (G) *and* $(A_1)-(A_2)$ *hold. Then there exist* γ, $\varsigma > 0$ *such that*

$$\ell(u)|_{\|u\|_E=\gamma} \geq \varsigma > 0.$$

Proof. For every $\varepsilon > 0$, (A_1) implies that there is a constant $\delta > 0$ such that

$$f((\kappa,\nu),u) \leq (a_0+\varepsilon)u^{\eta} \leq (a_0+\varepsilon)u \qquad \text{for all} \quad (\kappa,\nu)\in\mathbb{Z}^2 \quad \text{and} \quad 0<u<\delta.$$

From (A_2), we have $\lim\limits_{u\to+\infty}\frac{f((\kappa,\nu),u)}{u^{\eta+1}}=0$. Thus, there appears $M>0$ such that

$$f((\kappa,\nu),u) \leq \varepsilon u^{\eta+1} \qquad \text{for all} \quad (\kappa,\nu)\in\mathbb{Z}^2 \quad \text{and} \quad u>M.$$

For $\delta \leq u \leq M$, the continuity of $f((\kappa,\nu),u)$ in u indicates that there exists $D>0$ such that $\frac{f((\kappa,\nu)u)}{u^{\eta+1}} \leq D$. Then

$$f((\kappa,\nu),u) \leq (a_0+\varepsilon)u+\varepsilon u^{\eta+1}+Du^{\eta+1} \qquad \text{for all} \quad (\kappa,\nu)\in\mathbb{Z}^2 \quad \text{and} \quad u\in\mathbb{R}. \tag{3.1}$$

Subsequently, there are $D_\varepsilon>0$ and $i=\eta+2>\max\{2,\eta\}$ such that

$$F((\kappa,\nu),u) \leq \frac{a_0+\varepsilon}{2}u^2+\frac{D_\varepsilon}{i}u^i \qquad \text{for all} \quad (\kappa,\nu)\in\mathbb{Z}^2 \quad \text{and} \quad u\in\mathbb{R}. \tag{3.2}$$

Notice that $\Phi_c(\chi)\geq 0$ for $\chi\in\mathbb{R}$. Then (2.3) and (2.7) lead to

$$\begin{aligned}
\ell(u) \geq& \frac{1}{2}\|u\|_E^2-\sum_{(\kappa,\nu)\in\mathbb{Z}^2}F((\kappa,\nu),u(\kappa,\nu)) \\
\geq& \frac{1}{2}\|u\|_E^2-\frac{a_0+\varepsilon}{2}\sum_{(\kappa,\nu)\in\mathbb{Z}^2}|u(\kappa,\nu)|^2-\frac{D_\varepsilon}{i}\sum_{(\kappa,\nu)\in\mathbb{Z}^2}|u(\kappa,\nu)|^i \\
\geq& \frac{1}{2}\left(1-\frac{a_0+\varepsilon}{g_*}\right)\|u\|_E^2-\frac{D_\varepsilon}{ig_*^{\frac{i}{2}}}\|u\|_E^i, \qquad \forall u\in E.
\end{aligned} \tag{3.3}$$

Select $\varepsilon=\frac{g_*}{2}-a_0>0$ and $\|u\|_E=\gamma=\left(\frac{8D_\varepsilon g_*^{-\frac{i}{2}}}{i}\right)^{\frac{1}{2-i}}$. Then (3.3) gives

$$\ell(u) \geq \frac{1}{8}\|u\|_E^2=\varsigma>0, \qquad \text{for} \quad u\in\partial B_\gamma \triangleq \{u\in E:\|u\|_E=\gamma\}.$$

Therefore, there exists $\varsigma>0$ such that $\ell(u)|_{\|u\|_E=\gamma}\geq\varsigma>0$. The proof is finished. $\square$

Lemma 3.2. *Suppose* (G), (A_1), *and* (A_2) *hold for* $\eta\geq 1$. *Then for either* (Λ_1) $\eta=1$ *and* $\mu^*<1$ *or* (Λ_2) $\eta>1$, *there is* $e\in E\backslash B_\gamma$ *such that* $\ell(e)<0$, *where* γ *given by Lemma 3.1.*

Proof. The proof will be done by two cases.

Case 1. If $\eta = 1$ and $\mu^* < 1$, we first certify that μ^* defined as (1.3) is rational. Let $u \in E$ fulfill $\sum\limits_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)u^2(\kappa,\nu) = 1$. Then

$$1 = \sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)u^2(\kappa,\nu) \le h_0 \sum_{(\kappa,\nu)\in\mathbb{Z}^2} u^2(\kappa,\nu) = h_0\|u\|_{l^2}^2, \tag{3.4}$$

which manifests $\|u\|_{l^2}^2 \ge \frac{1}{h_0}$. Together with (2.2), we get that

$$\|u\|_E^2 \ge g_*\|u\|_{l^2}^2 \ge \frac{g_* h_0}{>} 0 \quad \text{and} \quad \mu^* \ge \frac{g_*}{h_0} > 0. \tag{3.5}$$

To show μ^* is attainable, we let $\{u_j\} \subset E$ be a minimizing sequence of (1.3). Then $\{u_j\}$ is bounded and $\sum\limits_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)u_j^2(\kappa,\nu) = 1$. Take a subsequence, still denoted by $\{u_j\}$, such that $u_j \rightharpoonup \zeta_0$ in E for some $\zeta_0 \in E$ and $u_j \to \zeta_0$ in l^2. By Lemma 2.3, we obtain

$$u_j \to \zeta_0 \qquad \text{in} \quad l^2.$$

Let $j \to \infty$, there hold

$$\sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)u_j^2(\kappa,\nu) \to \sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\zeta_0^2(\kappa,\nu) \quad \text{and} \quad \sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\zeta_0^2(\kappa,\nu) = 1. \tag{3.6}$$

On the other hand,

$$\mu^* \le \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)\zeta_0^2(\kappa,\nu) \le \liminf_{j\to\infty} \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)u_j^2(\kappa,\nu) = \mu^*.$$

Then it suffices to show $\mu^* = \sum\limits_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)\zeta_0^2(\kappa,\nu) = \|\zeta_0\|_E^2$.

Owing to $\mu^* < 1$, we choose $0 < \mathfrak{K} \in E$ with $\sum\limits_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\mathfrak{K}^2(\kappa,\nu) = 1$ such that $0 < \|\mathfrak{K}\|_E^2 < 1$. Remember $\Phi_c(u) \le \frac{1}{2}u^2$ for $u \in \mathbb{R}$. Jointly with (2.2) and (2.3), we get that

$$\begin{aligned} I(u_j) \le& \frac{1}{2}\Big(\sum_{(\kappa,\nu)\in\mathbb{Z}^2} |\Delta_1^2 u_j(\kappa-2,\nu)|^2 + \sum_{(\kappa,\nu)\in\mathbb{Z}^2} |\Delta_2^2 u_j(\kappa,\nu-2)|^2 \Big) \\ &+ \frac{\omega}{2}\Big(\sum_{(\kappa,\nu)\in\mathbb{Z}^2} |\Delta_1 u_j(\kappa-1,\nu)|^2 + \sum_{(\kappa,\nu)\in\mathbb{Z}^2} |\Delta_2 u_j(\kappa,\nu-1)|^2 \Big) \\ &+ \frac{1}{2} \sum_{(\kappa,\nu)\in\mathbb{Z}^2} g(\kappa,\nu)|u_j(\kappa,\nu)|^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} F((\kappa,\nu),u_j(\kappa,\nu)) \\ \le& \Big(\frac{16+4\omega}{g_*} + \frac{1}{2}\Big)\|u_j\|_E^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} F((\kappa,\nu),u_j(\kappa,\nu)). \end{aligned} \tag{3.7}$$

Let $u = t\mathfrak{K}$, where t is a real number. Then $t\mathfrak{K} \to +\infty$ as $t \to +\infty$. Taking advantage of (A_2), one arrives at

$$\lim_{t\to+\infty} \frac{\ell(t\mathfrak{K})}{t^2} \le \left(\frac{16+4\omega}{g_*} + \frac{1}{2}\right)\|\mathfrak{K}\|_E^2 - \lim_{t\to+\infty} \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \frac{F((\kappa,\nu),t\mathfrak{K}(\kappa,\nu))}{t^2}$$

$$
\begin{aligned}
&= \left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) \|\mathfrak{K}\|_E^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \lim_{t\to+\infty} \frac{f((\kappa,\nu), t\mathfrak{K}(\kappa,\nu)) \cdot \mathfrak{K}(\kappa,\nu)}{2t} \\
&= \left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) \|\mathfrak{K}\|_E^2 - \left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) \sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\mathfrak{K}^2(\kappa,\nu) \\
&= \left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) (\|\mathfrak{K}\|_E^2 - 1) \\
&<0,
\end{aligned}
$$

which means that $\ell(t\mathfrak{K}) \to -\infty$ as $t \to +\infty$. Consequently, there is an $e \in E$ with $\|e\|_E > \gamma$ such that $\ell(e) < 0$.

Case 2. For $\eta > 1$. According to $h(\kappa,\nu) : \mathbb{Z}^2 \to \mathbb{R}^+$, there is $0 < \mathfrak{J} \in E$ such that

$$
\sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\mathfrak{J}^{\eta+1}(\kappa,\nu) > 0.
$$

By means of a fashion resembling case, there holds

$$
\begin{aligned}
\lim_{t\to+\infty} \frac{\ell(t\mathfrak{J})}{t^{\eta+1}} &\le \lim_{t\to+\infty} \frac{\left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) \|t\mathfrak{J}\|_E^2 - \sum\limits_{(\kappa,\nu)\in\mathbb{Z}^2} F((\kappa,\nu), t\mathfrak{J}(\kappa,\nu))}{x^{\eta+1}} \\
&= \lim_{u\to+\infty} \frac{\left(\frac{16+4\omega}{g_*} + \frac{1}{2} \right) \|\mathfrak{J}\|_E^2}{t^{\eta-1}} - \lim_{t\to+\infty} \frac{F((\kappa,\nu), u\mathfrak{J}(\kappa,\nu))}{t^{\eta+1}} \\
&= -\frac{2(16+4\omega) + g_*}{g_*(\eta+1)} \sum_{(\kappa,\nu)\in\mathbb{Z}^2} h(\kappa,\nu)\mathfrak{J}^{\eta+1}(\kappa,\nu) \\
&<0.
\end{aligned}
$$

Thus, there exists $e \in E\backslash B_\gamma$ such that $\ell(e) < 0$. The proof has achieved. $\square$

In light of Lemmas 3.1 and 3.2, all conditions of Lemma 2.1 are met. Subsequently, ℓ possesses a $(C)_c$ sequence $\{u_j\} \subset E$ for the mountain pass level c such that

$$
\ell(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|_E)\|\ell'(u_j)\|_E \to 0 \qquad \text{as} \quad j \to \infty, \tag{3.8}
$$

where

$$
c = \inf_{q\in\Gamma} \max_{s\in[0,1]} \ell(q(s)) \ge \varsigma > 0 \quad \text{and} \quad \Gamma = \{q \in C([0,1], E) | q(0) = 0, q(1) = e\}. \tag{3.9}
$$

In what follows, we test the compactness property of the functional ℓ.

Lemma 3.3. *Let (G) and (A_3) hold. Then ℓ satisfies the $(C)_c$ condition for c given by (3.8).*

Proof. Firstly, we prove the $(C)_c$ sequence $\{u_j\}_{j\in\mathbb{N}} \subset E$ of ℓ, defined by (3.8), is bounded. Since $\Phi_c(\chi) - \frac{1}{2}\phi_c(\chi)\chi = \sqrt{1+\chi^2} - 1 - \frac{\chi^2}{2\sqrt{1+\chi^2}}$ is even and increases in χ for all $\chi \in [0,\infty)$, we have $\Phi_c(\chi) \geq \frac{1}{2}\phi_c(\chi)\chi$ for $\chi \in \mathbb{R}$. By (2.3), (2.4), and (A_3), for sufficiently large j, we have

$$\begin{aligned}
c+1 \geq& \ell(u_j) - \frac{1}{\theta}\langle \ell'(u_j), u_j\rangle \\
\geq& (\frac{1}{2}-\frac{1}{\theta})\|u_j\|_E^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2}\left[F((\kappa,\nu),u_j(\kappa,\nu)) - \frac{1}{\theta}f((\kappa,\nu),u_j)u_j(\kappa,\nu)\right] \\
\geq& \frac{\theta-2}{2\theta}\|u_j\|_E^2 - L\sum_{(\kappa,\nu)\in\mathbb{Z}^2} u_j^2(\kappa,\nu) \geq \frac{\theta-2}{2\theta}\|u_j\|_E^2 - \frac{L}{g_*}\|u_j\|_E^2 \\
=& \left(\frac{\theta-2}{2\theta} - \frac{L}{g_*}\right)\|u_j\|_E^2.
\end{aligned} \tag{3.10}$$

For $\theta > 2$ and $0 < L < \frac{g_*(\theta-2)}{2\theta}$, (3.10) implies that $\{u_j\}$ is bounded in E.

Subsequently, we verify $\{u_j\}$ possesses a convergent subsequence. We set the subsequence to be the sequence, such that $u_j \rightharpoonup u$ in E. By Lemma 2.3, we get

$$u_j \to u \qquad \text{in} \quad l^2.$$

Using (2.3) and (2.4), straightforward calculations yield that

$$\begin{aligned}
&\langle \ell'(u_j) - \ell'(u), u_j - u\rangle \\
=&\langle \ell'(u_j), u_j - u\rangle - \langle \ell'(u), u_j - u\rangle \\
=&\sum_{(\kappa,\nu)\in\mathbb{Z}^2}[\phi_c(\Delta_1^2 u_j(\kappa-2,\nu)) - \phi_c(\Delta_1^2 u(\kappa-2,\nu))][\Delta_1^2 u_j(\kappa-2,\nu) - \Delta_1^2 u(\kappa-2,\nu)] \\
&+\sum_{(\kappa,\nu)\in\mathbb{Z}^2}[\phi_c(\Delta_2^2 u_j(\kappa,\nu-2)) - \phi_c(\Delta_2^2 u(\kappa,\nu-2))][\Delta_2^2 u_j \kappa,\nu-2) - \Delta_2^2 u(\kappa,\nu-2)] \\
&+\sum_{(\kappa,\nu)\in\mathbb{Z}^2}\omega[\phi_c(\Delta_1 u_j(\kappa-1,\nu)) - \phi_c(\Delta_1 u(\kappa-1,\nu))][\Delta_1 u_j(\kappa-1,\nu) - \Delta_1 u(\kappa-1,\nu)] \\
&+\sum_{(\kappa,\nu)\in\mathbb{Z}^2}\omega[\phi_c(\Delta_2 u_j(\kappa,\nu-1)) - \phi_c(\Delta_2 u(\kappa,\nu-1))][\Delta_2 u_j(\kappa,\nu-1) - \Delta_2 u(\kappa,\nu-1)] \\
&+\|u_j - u\|_E^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2}[f((\kappa,\nu),u_j(\kappa,\nu)) - f((\kappa,\nu),u(\kappa,\nu))][u_j(\kappa,\nu) - u(\kappa,\nu)] \\
\geq&\|u_j - u\|_E^2 - \sum_{(\kappa,\nu)\in\mathbb{Z}^2}[f((\kappa,\nu),u_j(\kappa,\nu)) - f((\kappa,\nu),u(\kappa,\nu))][u_j(\kappa,\nu) - u(\kappa,\nu)].
\end{aligned} \tag{3.11}$$

Since $f((\kappa,\nu),\cdot) \in C(\mathbb{Z}^2 \times \mathbb{R}, \mathbb{R})$ and $u_j \to u$ in l^2, we have

$$\sum_{(\kappa,\nu)\in\mathbb{Z}^2}[f((\kappa,\nu),u_j(\kappa,\nu)) - f((\kappa,\nu),u(\kappa,\nu))][u_j(\kappa,\nu) - u(\kappa,\nu)] \to 0, \quad \text{as} \quad j\to\infty. \tag{3.12}$$

Moreover, $(1 + \|u_j\|_E)\|\ell'(u_j)\|_E \to 0$ and $u_j \rightharpoonup u$ in E as $j \to \infty$. Thus,

$$\langle \ell'(u_j) - \ell'(u), u_j - u\rangle = \langle \ell'(u_j), u_j\rangle - \langle \ell'(u_j), u\rangle - \langle \ell'(u), u_j - u\rangle \to 0 \qquad \text{as} \quad j\to\infty. \tag{3.13}$$

By (3.11), we have $\|u_j - u\|_E \to 0$ as $j \to \infty$ and $\ell : E \to \mathbb{R}$ meets the $(C)_c$ condition. □

Proof of Theorem 1.1. With the help of Lemmas 3.1, 3.2, and 3.3, Lemma 2.1 ensures that a nontrivial critical point u of ℓ as the corresponding critical value $c \geq \varsigma > 0$. Thus, (1.1) possesses a nontrivial homoclinic solution. This finishes the proof of Theorem 1.1. □

In sequence, we present the detailed proof of Theorem 1.2 by Lemma 2.2. To certify (W_1) in Lemma 2.2, we need

Lemma 3.4. *Let (G) and (A_6) hold. Then ℓ satisfies the $(C)_{\widetilde{c}}$ condition for any given $\widetilde{c}$.*

Proof. Let sequence $\{u_j\}_{j\in\mathbb{N}} \subset E$ fulfill

$$\ell(u_j) \to \widetilde{c} \quad \text{and} \quad (1+\|u_j\|_E)\|\ell'(u_j)\|_E \to 0 \qquad \text{as} \quad j\to\infty. \tag{3.14}$$

Write $C_0 = 2\widetilde{c}+1$. By (2.3), (2.4), and (3.14), we obtain that

$$C_0 \geq 2\ell(u_j) - \langle \ell'(u_j), u_j\rangle \geq \sum_{(\kappa,\nu)\in\mathbb{Z}^2} \left[u_j(\kappa,\nu)f((\kappa,\nu),u_j(\kappa,\nu)) - 2F((\kappa,\nu),u_j(\kappa,\nu))\right]. \tag{3.15}$$

Jointly with (A_6), we obtain that

$$u_j f((\kappa,\nu),u_j) \geq 2F((\kappa,\nu),u_j) \geq 0, \quad \text{for} \quad ((\kappa,\nu),u)\in\mathbb{Z}^2\times\mathbb{R}, \tag{3.16}$$

$$F((\kappa,\nu),u_j) \leq (b_0+b_1|u|^{\theta_0})\left[u_j f((\kappa,\nu),u_j) - 2F((\kappa,\nu),u_j)\right], \quad \text{for} \quad ((\kappa,\nu),u)\in\mathbb{Z}^2\times\mathbb{R}\backslash\{0\}. \tag{3.17}$$

Together with the definition of ℓ and (3.14)-(3.17), we have

$$\begin{aligned}
\frac{1}{2}\|u_j\|_E^2 \leq& \ell(u_j) + \sum_{(\kappa,\nu)\in\mathbb{Z}^2} F((\kappa,\nu),u_j(\kappa,\nu)) \\
\leq& \widetilde{c} + \sum_{(\kappa,\nu)\in\mathbb{Z}^2}\Big(b_0+b_1|u_j(\kappa,\nu)|^{\theta_0}\Big)\left[u_j(\kappa,\nu)f((\kappa,\nu),u_j(\kappa,\nu)) - 2F((\kappa,\nu),u_j(\kappa,\nu))\right] \\
\leq& \widetilde{c} + \Big(b_0+b_1\|u_j\|_\infty^{\theta_0}\Big)\sum_{(\kappa,\nu)\in\mathbb{Z}^2}\left[u_j(\kappa,\nu)f((\kappa,\nu),u_j(\kappa,\nu)) - 2F((\kappa,\nu),u_j(\kappa,\nu))\right] \\
\leq& \widetilde{c} + C_0\big(b_0+b_1\|u_j\|_\infty^{\theta_0}\big) \\
\leq& \widetilde{c} + C_0\big(b_0+b_1 g_*^{-\frac{\theta_0}{2}}\|u_j\|^{\theta_0}\big).
\end{aligned} \tag{3.18}$$

For $\theta_0 < 2$ is arbitrary, (3.18) ensures that $\{u_j\}$ is bounded in E. In the same manner as the proof of Lemma 3.3, we attain that $\{u_j\}$ admits a convergent subsequence. Therefore, ℓ satisfies the $(C)_{\widetilde{c}}$ condition. The proof is completed. $\square$

Proof of Theorem 1.2. Thanks to (A_7) and Lemma 3.4, $\ell\in C^1(E,\mathbb{R})$ is an even function and satisfies the $(C)_{\widetilde{c}}$ condition. Next, we show that (W_2) and (W_3) validate.

For $(\chi,\varrho)\in\mathbb{Z}^2$, endow $\beta_{(\chi,\varrho)} = \{\beta_{(\chi,\varrho)}(\kappa,\nu)\}$ with

$$\beta_{(\chi,\varrho)}(\kappa,\nu) = \begin{cases} 1, & \text{if} \quad (\chi,\varrho) = (\kappa,\nu), \\ 0, & \text{if} \quad (\chi,\varrho)\neq(\kappa,\nu). \end{cases}$$

Then $E = \overline{\text{span}\{\beta_{(\chi,\varrho)} : (\chi,\varrho)\in\mathbb{Z}^2\}}$. Additionally, write

$$\mathcal{B}_{(\chi,\varrho)} = \text{span}\{\beta_{(\chi,\varrho)} : (\chi,\varrho)\in\mathbb{Z}^2\}, \quad Z_j = \overline{\bigoplus_{|\chi|+|\varrho|=j}^{\infty}\mathcal{B}_{(\chi,\varrho)}}, \quad Y_j = \bigoplus_{|\chi|+|\varrho|=0}^{j}\mathcal{B}_{(\chi,\varrho)}.$$

Define

$$\Im_j = \sup_{u\in T_j, \|u\|_E=1}\|u\|_{l^2}.$$

For α given by (A_4), $\lim\limits_{j\to\infty}\Im_j=0$ and $\|u\|_{l^\alpha}\leq\|u\|_{l^2}\leq\Im_j\|u\|_E$. Let $\tau_j=\left(\frac{4d\Im_j^\alpha}{\alpha}\right)^{\frac{1}{2-\alpha}}>0$. Then $\lim\limits_{j\to\infty}\tau_j=+\infty$ for $\alpha>2$. Remember $\Phi_c(u)\geq 0$. In view of (A_4), we have

$$|F((\kappa,\nu),u)|\leq\frac{d}{2}|u|^2+\frac{d}{\alpha}|u|^\alpha\quad\text{for all}\quad((\kappa,\nu),u)\in\mathbb{Z}^2\times\mathbb{R}.\tag{3.19}$$

Combining (2.3) with (3.19), we get

$$\begin{aligned}\ell(u)=&\sum_{(\kappa,\nu)\in\mathbb{Z}^2}[\Phi_c(\Delta_1^2u(\kappa-2,\nu))+\Phi_c(\Delta_2^2u(\kappa,\nu-2))]+\frac{1}{2}\sum_{(\kappa,\nu)\in\mathbb{Z}^2}g(\kappa,\nu)|u(\kappa,\nu)|^2\\&+\sum_{(\kappa,\nu)\in\mathbb{Z}^2}\omega[\Phi_c(\Delta_1u(\kappa-1,\nu))+\Phi_c(\Delta_2u(\kappa,\nu-1))]-\sum_{(\kappa,\nu)\in\mathbb{Z}^2}F((\kappa,\nu),u(\kappa,\nu))\\&\geq\frac{1}{2}\|u\|_E^2-\frac{d}{2}\|u\|_{l^2}^2-\frac{d}{\alpha}\|u\|_{l^\alpha}^\alpha.\end{aligned}\tag{3.20}$$

For $u\in T_j$, put $\|u\|_E=\tau_j$. Then

$$s_j=\inf_{u\in Z_j,\|u\|_E=\tau_j}\ell(u)\geq\left(\frac{1}{2}-\frac{d\Im_j^2}{2}-\frac{d\Im_j^\alpha}{\alpha}\tau_j^{\alpha-2}\right)\tau_j^2=\left(\frac{1}{4}-\frac{d\Im_j^2}{2}\right)\tau_j^2\to+\infty\qquad\text{as}\quad j\to\infty,$$

which indicates that (W_2) in Lemma 2.2 is satisfied.

Due to the finite-dimension of Y_j, there exists $Q_j>0$ such that $\|u\|_E\leq Q_j\|u\|_\infty$. By (A_5), there is $K_0>0$ satisfying

$$F((\kappa,\nu),u)>Q_j^2\left(\frac{16+4\omega}{g_*}+\frac{1}{2}\right)u^2\quad\text{for all}\quad((\kappa,\nu),u)\in\mathbb{Z}^2\times\mathbb{R}\quad\text{with}\quad|u|>K_0.\tag{3.21}$$

Choose $(\kappa_0,\nu_0)\in\mathbb{Z}^2$ such that $\|u\|_\infty=|u(\kappa_0,\nu_0)|>K_0$. Then, for u large enough with $|u|>K_0$, by (3.7) and (3.21), we have

$$\begin{aligned}\ell(u)\leq&\left(\frac{16+4\omega}{g_*}+\frac{1}{2}\right)\|u\|_E^2-\sum_{(\kappa,\nu)\in\mathbb{Z}^2}F((\kappa,\nu),u(\kappa,\nu))\\\leq&Q_j^2\left(\frac{16+4\omega}{g_*}+\frac{1}{2}\right)\|u\|_\infty^2-F((\kappa_0,\nu_0),u(\kappa_0,\nu_0))\\<&Q_j^2\left(\frac{16+4\omega}{g_*}+\frac{1}{2}\right)|u(\kappa_0,\nu_0)|^2-Q_j^2\left(\frac{16+4\omega}{g_*}+\frac{1}{2}\right)|u(\kappa_0,\nu_0)|^2\\=&0.\end{aligned}\tag{3.22}$$

Take sufficiently large σ_j with $\sigma_j>\tau_j>0$ such that

$$t_j=\max_{u\in Y_j,\|u\|_E=\sigma_j}\ell(u)\leq 0.$$

Then (3.22) guarantees that (W_3) of Lemma 2.2 is fulfilled. Thus, by Lemma 2.2, ℓ has a sequence of critical points $\{u_j\}\subset E$ such that $\ell(u_j)\to\infty$ as $j\to\infty$. The proof of Theorem 1.2 is done. □

4. Examples

Here, two examples, which are not just theoretical demonstrations, but also are actionable tools for engineers and physicists working with discrete nonlinear systems, are presented.

Example 4.1. Let $\omega = 6$, $a(\kappa,\nu) = \frac{117}{5\pi}\text{arccot}(\kappa+\nu)$ and $h(\kappa,\nu) = \frac{54}{\pi}\text{arccot}(\kappa+\nu)$. For any $(\kappa,\nu) \in \mathbb{Z}^2$ and $\eta \geq 1$, consider (1.1) with $g(\kappa,\nu)$ and $f((\kappa,\nu),u)$ given by

$$g(\kappa,\nu) = \kappa^2+\nu^2+50, \quad f((\kappa,\nu),u) = \begin{cases} \dfrac{1404\text{arccot}(\kappa+\nu)}{5\pi^2} u^\eta \arctan(u+2-\sqrt{3}), & u \geq 0, \\ 0, & u < 0. \end{cases} \tag{4.1}$$

Given $g(\kappa,\nu)$, $a(\kappa,\nu)$ and $h(\kappa,\nu)$ as above, it follows that $\lim\limits_{|\kappa|+|\nu|\to+\infty} g(\kappa,\nu) = +\infty$ with $g_* = 50$, $a(\kappa,\nu) \in (\mathbb{Z}^2\times\mathbb{R}^+)$ with $a_0 = \frac{117}{5} < \frac{g_*}{2}$, and $h(\kappa,\nu) \in (\mathbb{Z}^2\times\mathbb{R}^+)$ with $h_0 = 54 > 50 = g_*$.

Moreover, (4.1) implies that

$$F((\kappa,\nu),u) = \begin{cases} \dfrac{1404\text{arccot}(\kappa+\nu)}{5\pi^2} \displaystyle\int_0^u t^\eta \arctan(t+2-\sqrt{3})dt, & u \geq 0, \\ 0, & u < 0. \end{cases} \tag{4.2}$$

Then for all $(\kappa,\nu) \in \mathbb{Z}^2$, we have

$$\lim_{u\to0^+} \frac{f((\kappa,\nu),u)}{u^\eta} = \lim_{u\to0^+} \frac{\frac{1404\text{arccot}(\kappa+\nu)}{5\pi^2} u^\eta \arctan(u+2-\sqrt{3})}{u^\eta} \frac{117}{5\pi}\text{arccot}(\kappa+\nu) = a(\kappa,\nu),$$

$$\frac{f((\kappa,\nu),u)}{u^\eta} \geq \frac{117}{5\pi}\text{arccot}(\kappa+\nu) = a(\kappa,\nu) \quad \text{for all} \quad u > 0,$$

$$\lim_{u\to+\infty} \frac{f((\kappa,\nu),u)}{u^\eta} = \lim_{u\to+\infty} \frac{\frac{1404\text{arccot}(\kappa+\nu)}{5\pi^2} u^\eta \arctan(u+2-\sqrt{3})}{u^\eta} = \frac{702\text{arccot}(\kappa+\nu)}{5\pi} = \frac{13}{5}h(\kappa,\nu),$$

which validate (A_1) and (A_2).

In addition, for $u \geq 0$ in (4.2), we obtain

$$\begin{aligned} F((\kappa,\nu),u) =& \frac{1404\text{arccot}(\kappa+\nu)}{5\pi^2} \int_0^u t^\eta \arctan(t+2-\sqrt{3})dt \\ =& \frac{1404\text{arccot}(\kappa+\nu)}{5\pi^2(\eta+1)} \left[u^{\eta+1}\arctan(u+2-\sqrt{3}) - \int_0^u \frac{t^{\eta+1}}{1+(t+2-\sqrt{3})^2}dt \right]. \end{aligned}$$

Notice that $\frac{1404\text{arccot}(\kappa+\nu)}{5\pi^2(\eta+1)} > 0$ for all $(\kappa,\nu) \in \mathbb{Z}^2$. Then $\frac{t^{\eta+1}}{1+(t+2-\sqrt{3})^2} > 0$ for $t > 0$ ensures $\int_0^u \frac{t^{\eta+1}}{1+(t+2-\sqrt{3})^2}dt \geq 0$. For $\eta = 1$, we just require that $\theta > 2$ and $\frac{117(\theta-2)}{10\theta} \leq L < \frac{50(\theta-2)}{2\theta}$. Specially, take $\theta = 3$ and $L = \frac{20}{3}$, (A_3) is satisfied. Thus, Theorem 1.1 implies that (1.1) admits at least one nontrivial homoclinic solution.

Remark 4.1. (1.1) with nonlinear term f given by (4.1), possesses the following core features: Unidirectional nonlinearity (non-negative response), saturation via arctangent, and single homoclinic solution. Therefore, these make it ideal for modeling systems where signals/energy propagate in one direction and stabilize to a resting state.

Example 4.2. Let (κ, ν) represent time slots κ and fiber segments ν. Denote the optical intensity and dispersion of pulse shape by $u(\kappa, \nu)$ and the fourth-order difference term, respectively. Choose $g(\kappa, \nu) = \kappa^2 + \nu^2 + 9$ and

$$f((\kappa, \nu), u) = u^3 \arctan u^2 \tag{4.3}$$

to model nonlinear saturation of the Kerr effect. Then (1.1) with (4.3) models discrete optical pulse propagation in non-uniform fibers.

By (4.3), we find (A_7) validates. Moreover,

$$F((\kappa, \nu), u) = \frac{1}{4} u^4 \arctan u^2 + \frac{1}{4} \arctan u^2 - \frac{1}{4} u^2. \tag{4.4}$$

Take $d = \frac{\pi}{2}$ and $\alpha = 4$. Then

$$|f((\kappa, \nu), u)| \leq \frac{\pi}{2}(|u|^3 + |u|) \qquad \text{for all} \quad (\kappa, \nu) \in \mathbb{Z}^2 \quad \text{and} \quad u \in \mathbb{R},$$

which ensures that (A_4) is satisfied. Further,

$$\lim_{|u| \to +\infty} \frac{f((\kappa, \nu), u)}{u} = \lim_{|u| \to \infty} u^2 \arctan u^2 \to +\infty.$$

Then (A_5) is valid. Moreover, for any $(\kappa, \nu) \in \mathbb{Z}^2$, there holds

$$0 < \left(2 + \frac{1}{2 + |u|}\right) F((\kappa, \nu), u) \leq f((\kappa, \nu), u)u \qquad \text{for all} \quad u \in \mathbb{R} \backslash \{0\}, (\kappa, \nu) \in \mathbb{Z}^2.$$

Thus, (A_6) holds with $b_0 = 2$ and $b_1 = \theta_0 = 1$. Consequently, Theorem 1.2 guarantees that (1.1) admits infinitely many homoclinic solutions.

5. Conclusion

This paper studies nontrivial homoclinic solutions of nonperiodic fourth-order ϕ_c-Laplacian partial difference equations. We obtain one nontrivial homoclinic solution under weak nonlinear conditions by Mountain Pass Lemma and infinitely many such solutions for odd nonlinearities by Fountain Theorem. Our generalized growth condition relaxes the classic (AR) restriction to non-AR nonlinearities like logarithmic terms. Also, the theorems are ready to extend to higher-order cases. Two practical examples are provided to verify our theorems, which enrich relevant partial difference equation theories. Applying our theoretical findings to solve practical engineering problems will be our future research.

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