

EXISTENCE AND STRUCTURE OF SOLUTION SETS FOR NONLINEAR DISCRETE BOUNDARY VALUE PROBLEMS WITH SPECTRAL PARAMETERS IN BOUNDARY CONDITIONS*

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Abstract By using topological degree methods and bifurcation theory, we prove the existence of the solution and the bifurcation structure at infinity of the second-order difference boundary value problems

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + f(1, u(1)) + h(1), \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + f(N+1, u(N+1)) + h(N+1), \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant, μ_1 is a spectral parameter, $\Delta u(k) := u(k+1) - u(k)$ and $\Delta^2 u(k-1) := \Delta(\Delta u(k-1))$ as the forward difference operator and the second difference operator, respectively.

Keywords Existence, steklov problem, a priori estimate, topological degree methods, bifurcation at infinity.

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1. Introduction

The problem with parameter-dependent boundary condition is widely used in engineering technology, mathematical physics, quantum mechanics and other fields [1, 14, 18, 31]. Therefore, many scholars have made a series of studies on the existence of solutions of problems with spectral parameters in the boundary condition, see [2, 3, 9, 10, 15, 33] and other works. Secondly, the differential equation/difference equation boundary value problem with nonlinear boundary conditions is an important nonlinear problem, which has major applications in the fields of population ecological modeling, fishery management, cryosurgery, etc., [8, 13, 21, 29]. For example, in population dynamics, when species disperse to the boundary, the influence of habitat margins on their dispersal patterns is an important issue in ecology. Boundary data measurements are used in non-invasive testing methods to locate defects in media. In cryosurgery (a minimally invasive treatment used to treat certain types of cancer and certain conditions that can develop into cancer), abnormal tissue is destroyed with a highly exothermic reaction in the thin layer around the border. In view of the important application background of this problem, its research has attracted widespread attention, see Ref. [4, 20, 22, 23, 25, 30], etc., and references.

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In 2002, P. Amster [4] investigated the solvability of the semi-linear second-order ODE with nonlinear boundary conditions, based on the axial deformation problem of a nonlinear elastic beam with friction. After, Jankowski [20] employed the monotone iterative method to investigate the existence of solutions for the following nonlinear boundary value problem

$$\begin{cases} \Delta y(k-1) = f(k, y(k), y(a_1(k)), y(a_2(k)), \dots, y(a_l(k))), k \in [1, N]_{\mathbb{Z}}, \\ g(y(0), y(N)) = 0, \end{cases}$$

where $a_i \in C([1, N]_{\mathbb{Z}}, [0, N]_{\mathbb{Z}})$ and satisfies $a_i(k) \leq k$ ($i = 1, 2, \dots, l$), $f \in C([1, N]_{\mathbb{Z}} \times \mathbb{R}^{l+1}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Meanwhile, N. M. Mavinga [25] analyzed the existence of solutions for elliptic problems with nonlinear boundary conditions

$$\begin{cases} -\Delta u + c(x)u(x) = 0, & x \in \Omega, \\ \frac{\partial v}{\partial n} = g(x, u), & x \in \partial\Omega, \end{cases}$$

when the nonlinear term interacts in a certain sense with the Steklov spectrum, where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, $\partial\Omega \in C^{0,1}(\mathbb{R}^n)$, $\frac{\partial}{\partial n}$ represents the unit outer normal derivative on the boundary, and the function $c : \Omega \rightarrow \mathbb{R}$, $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$. Other significant results can be found in the works of F. Cakoni [6, 7], N. Gavitone [17], et al. In 2014, Lu, Ma [22] utilized Dancer's bifurcation theorem to study the global structure of positive solutions for second-order nonlinear discrete boundary value problems

$$\begin{cases} -\Delta[p(k-1)\Delta y(k-1)] + q(k)y(k) = \lambda a(k)f(y(k)), k \in [1, N]_{\mathbb{Z}}, \\ -\Delta y(0) + \alpha g(y(0)) = 0, \\ \Delta y(N) + \beta g(y(N+1)) = 0, \end{cases}$$

where $\alpha, \beta > 0$ is a constant $p : [0, N]_{\mathbb{Z}} \rightarrow (0, \infty)$, $a, q : [1, N]_{\mathbb{Z}} \rightarrow [0, \infty)$, and $a(k) > 0$, $f, g \in C([0, \infty), [0, \infty))$ holds on the domain.

And, in 2020, Ma and Wang [23] based on bifurcation theory, studied the existence of the positive solution of a kind of semi-positive problem with nonlinear boundary conditions $f(t, \cdot)$ in both superlinear and sublinear cases, respectively. In 2023, K. Umezu [30] obtained the existence and uniqueness of the positive solution for the elliptic problem with indefinite superlinear boundary condition based on the population genetics model.

It is widely recognized in the literature that problems involving nonlinear boundary conditions are intimately connected with the Steklov eigenvalue problem. Ever since its introduction by W. Steklov [28] in 1902, Steklov eigenvalue problem has received considerable scholarly interest.

$$\begin{cases} \Delta v = 0, & x \in \Omega, \\ \frac{\partial v}{\partial n} = \lambda \phi v(x), & x \in \partial\Omega, \end{cases}$$

is considered in the bounded region Ω with smooth boundaries in $\mathbb{R}^2$, where $\frac{\partial}{\partial n} := n \cdot \nabla$ represents the directional derivative of the outer normal vector on the boundary, $\lambda \in \mathbb{R}$ is the spectral parameter, and ϕ is the smooth positive weight function. In particular, the Steklov eigenvalue

problem is related to the elastic membrane model where the masses are uniformly distributed at the boundary [26].

2004, G. Auchmuty [5] consider Steklov eigenvalue problem

$$\begin{cases} Lu := c(x)u(x) - \Delta u(x) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \mu \rho(x)u(x), & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is bounded, and its boundary $\partial\Omega$ is the union of finite closed Lipschitz surfaces that do not intersect with each other, and each surface has a finite surface area, $c(x) \in L^p(\Omega)$, $\rho(x) \in L^\infty(\partial\Omega, d\sigma)$.

In 2012, N. M. Mavinga [24] investigated a generalized Steklov-Robin eigenvalue problem (possibly including a singular weight)

$$\begin{cases} -\Delta u + c(x)u(x) = \mu m(x)u(x), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} + \sigma(x)u(x) = \mu \rho(x)u(x), & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is bounded, $\partial\Omega \in C^{0,1}(\mathbb{R}^n)$, $\frac{\partial}{\partial \nu}$ is the unit outer normal derivative on the boundary, and the function $c : \Omega \rightarrow \mathbb{R}$, $\sigma : \partial\Omega \rightarrow \mathbb{R}$, $f, g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$.

The year 2024, Gao et al. [16] applied the theory of connected components of solution sets to study the existence of solutions for discrete resonance problems

$$\begin{cases} -\Delta[p(t-1)\Delta y(t-1)] + q(t)y(t) = \lambda_k r(t)y(t) \\ + f(t, y(t)) + \gamma \psi_k(t) + \bar{g}(t), & t \in [1, T]_{\mathbb{Z}}, \\ (a_0 \lambda_k + b_0)y(0) = (c_0 \lambda_k + d_0)\Delta y(0), \\ (a_1 \lambda_k + b_1)y(T+1) = (c_1 \lambda_k + d_1)\nabla y(T+1) \end{cases}$$

with spectral parameters in the boundary conditions, where the functions $p : [0, T]_{\mathbb{Z}} \rightarrow (0, \infty)$, $q : [1, T]_{\mathbb{Z}} \rightarrow \mathbb{R}$, $\bar{g} : [1, T]_{\mathbb{Z}} \rightarrow \mathbb{R}$, $r(t) > 0$ and (λ_k, ψ_k) are eigenpairs of the corresponding linear problem.

It is noted that compared to the spectra of continuous problems or classical discrete problems, the study of spectra for discrete problems with parameters in the boundary conditions may be more complex. Coupled with the inherent characteristics of difference equations, the computational complexity of spectral analysis increases significantly, thereby presenting certain challenges in investigating the existence of solutions for nonlinear problems with spectral parameters in the boundary conditions.

Motivated above paper, we first study the spectral structure of the Steklov eigenvalue problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = \mu \sigma(1)u(1), \\ \Delta u(N) = \mu \sigma(N+1)u(N+1). \end{cases} \quad (1.1)$$

Based on the spectral structure of (1.1), we obtain the structure of solution sets of the discrete

boundary value problem with the following nonlinear boundary conditions

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + f(1, u(1)) + h(1), \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + f(N+1, u(N+1)) + h(N+1), \end{cases} \quad (1.2)$$

where $c(k) : [1, N]_{\mathbb{Z}} \rightarrow \mathbb{R}$ and the weight function $\sigma(k) : \{1, N+1\} \rightarrow \mathbb{R}$ satisfy the condition (C_1) .

We make the follow assumptions:

(C_1) $c(k) \geq 0$ and $\sum_{k=1}^N c(k) > 0$, $\sigma(k) > 0$ and $\sigma(1) + \sigma(N+1) \neq 0$.

(C_2) The function $f : \{1, N+1\} \rightarrow \mathbb{R}$ satisfies an asymptotically sublinear condition, in the sense that $\lim_{|u(k)| \rightarrow \infty} \frac{f(k, u(k))}{u(k)} = 0$ uniformly for $k \in \{1, N+1\}$.

(C_3) There exist functions $A, B : \{1, N+1\} \rightarrow \mathbb{R}$, along with constants r and R that satisfy $r < 0 < R$, such that for all $k \in \{1, N+1\}$ and $u \in \mathbb{R}$,

$$f(k, u(k)) \geq A(k), \quad u(k) \geq R, \quad (1.3)$$

and

$$f(k, u(k)) \leq B(k), \quad u(k) \leq r. \quad (1.4)$$

(C_4) $h : \{1, N+1\} \rightarrow \mathbb{R}$ satisfies

$$\begin{aligned} B(1)|\varphi_1(1)| + B(N+1)|\varphi_1(N+1)| &\leq -[h(1)|\varphi_1(1)| + h(N+1)|\varphi_1(N+1)|] \\ &\leq A(1)|\varphi_1(1)| + A(N+1)|\varphi_1(N+1)|, \end{aligned}$$

where $\varphi_1(k)$ is an eigenfunction corresponding to eigenvalue μ_1 of problem (1.1).

(C'_4) $h : \{1, N+1\} \rightarrow \mathbb{R}$ is a function with

$$\begin{aligned} f^-(1)|\varphi_1(1)| + f^-(N+1)|\varphi_1(N+1)| &< -[h(1)|\varphi_1(1)| + h(N+1)|\varphi_1(N+1)|] \\ &< f_+(1)|\varphi_1(1)| + f_+(N+1)|\varphi_1(N+1)|, \end{aligned}$$

where $f^-(k) := \limsup_{u(k) \rightarrow -\infty} f(k, u(k))$, $f_+(k) := \liminf_{u(k) \rightarrow +\infty} f(k, u(k))$ and $\varphi_1(k)$ is an eigenfunction

corresponding to eigenvalue μ_1 of problem (1.1).

The rest of the paper is organized as follows. In Section 2, we give some preliminary results and introduce results of the Steklov eigenvalue problem (1.1). In Section 3, we give the main results and their proofs. Finally, we introduce some examples to illustrate the main results.

2. Preliminaries

In this section, we introduce the key notations, definitions, and lemmas required for the analysis to the spectral of (1.1). Denote by $\mathbb{Z}$ the set of integers. For any $a, b \in \mathbb{Z}$ with $a < b$. Let $[a, b]_{\mathbb{Z}} := \{a, a+1, \dots, b\}$. Denote $\sum_{s=a}^b u(s) = 0$, $\prod_{s=a}^b u(s) = 1$ as the case $b < a$. Finally, we introduce $\Delta u(k) := u(k+1) - u(k)$ and $\Delta^2 u(k-1) := \Delta(\Delta u(k-1))$ as the forward difference operator and the second difference operator, respectively.

Define

$$E = \{u \mid u : [0, N+1]_{\mathbb{Z}} \rightarrow \mathbb{R}\}.$$

It can be readily verified that forms an inner product space with respect to the inner product

$$(u, v)_E = \sum_{k=1}^N \Delta u(k) \Delta v(k) + \sum_{k=1}^N c(k) u(k) v(k). \text{ Especially, } E \text{ is a Hilbert space.}$$

Define vector space

$$E_1 = \{u \mid u : [0, N+1]_{\mathbb{Z}} \rightarrow \mathbb{R}, \Delta u(0) + \mu_1 \sigma(1) u(1) = 0, \Delta u(N) - \mu_1 \sigma(N+1) u(N+1) = 0\}.$$

It not difficult to verify that E_1 is an inner product space with respect to $(u, v)_{\sigma} = \sigma(1)u(1)v(1) + \sigma(N+1)u(N+1)v(N+1)$.

From the inner product, we can induce two norms, $\|u\|_E = [\sum_{k=1}^N |\Delta u(k)|^2 + \sum_{k=1}^N c(k) u^2(k)]^{\frac{1}{2}}$, $\|u\|_{\sigma} = [\sigma(1)u^2(1) + \sigma(N+1)u^2(N+1)]^{\frac{1}{2}}$. It is readily verifiable that $\|u\|_E$ is equivalent $\|u\|_{\sigma}$.

Definition 2.1. *Steklov eigenvalue problem (1.1) has an eigenvalue μ with a corresponding eigenfunction φ if the pair $(\mu, \varphi) \in \mathbb{R} \times E, \varphi \not\equiv 0$ satisfies*

$$\sum_{k=1}^N \Delta \varphi(k) \Delta v(k) + \sum_{k=1}^N c(k) \varphi(k) v(k) = \mu [\sigma(1) \varphi(1) v(1) + \sigma(N+1) \varphi(N+1) v(N+1)], \quad v \in E. \quad (2.1)$$

Definition 2.2. ([19, Definition 3.3 of page 47]) Let X and Y are Banach space and let U be an open subset of X . A map $f : U \rightarrow Y$ is said to be Fréchet differentiable at a point $x_0 \in U$ if there exists a bounded linear operator $A \in L(X, Y)$ such that for all $h \in X$ with $x_0 + h \in U$, the following holds

$$f(x_0 + h) - f(x_0) = Ah + \omega(x_0, h),$$

where $\omega(x_0, h) = o(\|h\|)$, i.e.,

$$\lim_{\|h\| \rightarrow 0} \frac{\omega(x_0, h)}{\|h\|} = 0.$$

In this case, the operator Ah is called the Fréchet derivative of f at x_0 , denoted by $d[f(x_0)h]$ or $f'(x_0)h$.

Definition 2.3. A point $(\lambda_{\infty}, \infty)$ is called a bifurcation point emerging from infinity for problem (1.2) if there exists a sequence of solutions (λ_n, u_n) such that $\|u\|_E \rightarrow \infty$ as $n \rightarrow \infty$, and $\lambda_n \rightarrow \lambda_{\infty}$.

Lemma 2.1. ([11, Lemma 2.5], [27]) *If $\mu \in pr_{\mathbb{R}}(\mathcal{G}) \cap \Sigma$ satisfies that the Leray–Schauder degree $deg(I - L(\lambda), B_R(0))$ changes as λ passes through μ , then $\mathcal{S}$ possesses a maximal subcontinuum $\mathcal{C}_{\mu} \subset \overline{\mathcal{G}}$ such that $(\mu, 0) \in \mathcal{C}_{\mu}$, and $\mathcal{C}_{\mu}$ satisfies one of the following three properties*

- (i) $\mathcal{C}_{\mu}$ is unbounded in $\overline{\mathcal{G}}$;
- (ii) $\mathcal{C}_{\mu}$ intersects $\partial \mathcal{G} \setminus \{(\mu, 0)\}$;
- (iii) $\mathcal{C}_{\mu}$ intersects some point $(\bar{\mu}, 0)$ with $\bar{\mu} \in pr_{\mathbb{R}}(\overline{\mathcal{G}}) \cap \Sigma$ and $\bar{\mu} \neq \mu$.

If $\mathcal{G}$ is bounded and $L(\cdot)$ is a linear operator, i.e., $L(\lambda) = \lambda L$ then property (i) does not hold.

Let $V_{\sigma} := \{u \in E : \|u\|_{\sigma} = 0\}$. It can be observed that V_{σ} is a closed linear subspace of E . Denote the E-orthogonal complement of V_{σ} in E as $E_{\sigma} = [V_{\sigma}]^{\perp}$. Thus, E can be decomposed into the direct sum $E = V_{\sigma} \oplus_E E_{\sigma}$. In other words, every $u \in E$ can be uniquely represented as $u = u_1 + u_2$, where $u_1 \in V_{\sigma}$, $u_2 \in E_{\sigma}$ and $(u_1, u_2)_E = 0$.

Lemma 2.2. *when $c(k) = 1, \sigma(\cdot) = 1$ holds, the eigenvalue problem*

$$\begin{cases} -\Delta^2 u(k-1) + u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = \mu u(1), \\ \Delta u(N) = \mu u(N+1) \end{cases}$$

has two distinct eigenvalues

$$\mu_1 = \frac{\sqrt{5}\phi^N + \sqrt{5}\psi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}$$

and

$$\mu_2 = \frac{\sqrt{5}\phi^N + \sqrt{5}\psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}$$

satisfying $0 < \mu_1 < \mu_2$, where $\phi = \frac{3+\sqrt{5}}{2}, \psi = \frac{3-\sqrt{5}}{2}$, the corresponding eigenfunctions

$$\begin{aligned} \varphi_1(k) &= \frac{(1-2\sqrt{5})\psi^N - \phi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}\phi^k + \frac{(1+2\sqrt{5})\phi^N - \psi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}\psi^k, \\ \varphi_2(k) &= \frac{(1-2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}\phi^k + \frac{(1+2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}\psi^k \end{aligned}$$

satisfy $\varphi_1(k) > 0, \varphi_2(k)$ changes sign exactly once, $k \in [1, N]_{\mathbb{Z}}$ and $(\varphi_1, \varphi_2)_E = 0, (\varphi_1, \varphi_2)_\sigma = 0$.

Proof. The general solution is found to be $u(k) = c_1\phi^k + c_2\psi^k$, where c_1, c_2 is an arbitrary constant. Substituting it into the boundary conditions yields that

$$\begin{cases} (1 - (1 + \mu)\phi)c_1 + (1 - (1 + \mu)\psi)c_2 = 0, \\ ((1 - \mu)\phi^{N+1} - \phi^N)c_1 + ((1 - \mu)\psi^{N+1} - \psi^N)c_2 = 0. \end{cases}$$

Therefore, the characteristic determinant of the target eigenvalue problem is

$$\begin{vmatrix} 1 - (1 + \mu)\phi & 1 - (1 + \mu)\psi \\ (1 - \mu)\phi^{N+1} - \phi^N & (1 - \mu)\psi^{N+1} - \psi^N \end{vmatrix}.$$

In order to show that this eigenvalue problem has nontrivial solutions, we set the determinant above equal to zero and subsequently solve for the eigenvalues

$$\mu_1 = \frac{\sqrt{5}\phi^N + \sqrt{5}\psi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)},$$

and

$$\mu_2 = \frac{\sqrt{5}\phi^N + \sqrt{5}\psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)},$$

where $\phi = \frac{3+\sqrt{5}}{2}, \psi = \frac{3-\sqrt{5}}{2}$. For μ_1 and μ_2 , the corresponding eigenfunctions are

$$\varphi_1(k) = \frac{(1 - 2\sqrt{5})\psi^N - \phi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)}\phi^k$$

$$+ \frac{(1 + 2\sqrt{5})\phi^N - \psi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)} \psi^k,$$

and

$$\begin{aligned} \varphi_2(k) &= \frac{(1 - 2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)} \phi^k \\ &+ \frac{(1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{2(\phi^N - \psi^N)} \psi^k, \end{aligned}$$

respectively.

We now prove that $\varphi_1(k) > 0$ holds. From the expression of $\phi, \psi, \varphi_1(k)$, it suffices to show that the numerator of $\varphi_1(k)$ is greater than zero. When $N > \log_4 3 \approx 0.7925$ holds, it follows that $8 > 24\psi^{2N}$ and thus $\phi^{2N} + \psi^{2N} + 18 > \phi^{2N} + 10 + 25\psi^{2N} = (\phi^N + 5\psi^N)^2$, furthermore

$$\begin{aligned} (1 - 2\sqrt{5})\psi^N - \phi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18} &> (1 - 2\sqrt{5})\psi^N - \phi^N + \sqrt{(\phi^N + 5\psi^N)^2} \\ &= (6 - 2\sqrt{5})\psi^N \\ &> 0. \end{aligned}$$

Similarly, when $N > \log_{\frac{25}{4}} \frac{3}{2} \approx 0.2213$ holds, it yields that $12 < 8\phi^{2N}$ and then $\phi^{2N} + \psi^{2N} + 18 < 9\phi^{2N} + 6 + \psi^{2N} = (3\phi^N + \psi^N)^2$, furthermore

$$\begin{aligned} (1 + 2\sqrt{5})\phi^N - \psi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18} &> (1 + 2\sqrt{5})\phi^N - \psi^N - \sqrt{(3\phi^N + \psi^N)^2} \\ &= (2\sqrt{5} - 2)\phi^N - 2\psi^N \\ &> 0. \end{aligned} \tag{2.2}$$

Therefore, when $N \geq 1$ holds, $\varphi_1(k) > 0$ is true.

Next, we discuss the sign of $\varphi_2(k)$. Note that $\varphi_2(0) = \sqrt{5} > 0$, it's not difficult to compute that when $k \geq 1$, $((1 - 2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18})\phi^k < 0$, $((1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18})\psi^k > 0$. We thus proceed to compare the magnitude of the former with that of the latter

$$\left| (1 - 2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18} \right| = \phi^N + (2\sqrt{5} - 1)\psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18} > 2\phi^N.$$

By the same argument as in equation (2.2), it can be shown that when $N > \log_{\frac{25}{4}} \frac{3}{2} \approx 0.2213$, $(1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18} < (4 + 2\sqrt{5})\phi^N$ is satisfied.

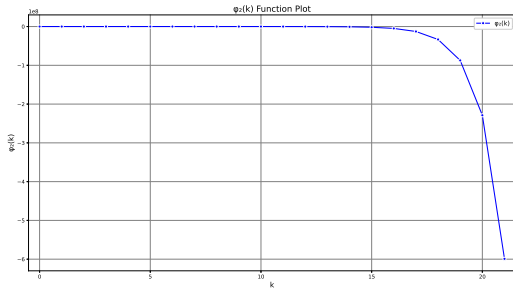
Hence, we have that

$$\begin{aligned} &\frac{\left| (1 - 2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18} \right|}{(1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}} \cdot \left(\frac{\phi}{\psi} \right)^k \\ &= \frac{\phi^N + (2\sqrt{5} - 1)\psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}}{(1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18}} \cdot \left(\frac{\phi}{\psi} \right)^k \\ &> \frac{2\phi^N}{(4 + 2\sqrt{5})\phi^N} \cdot \frac{\phi}{\psi} \approx 1.618 > 1, \end{aligned}$$

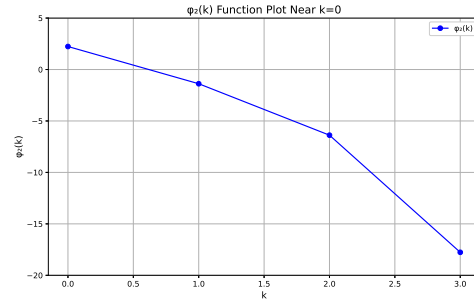
i.e.,

$$\left| (1 - 2\sqrt{5})\psi^N - \phi^N - \sqrt{\phi^{2N} + \psi^{2N} + 18} \right| \phi^k > ((1 + 2\sqrt{5})\phi^N - \psi^N + \sqrt{\phi^{2N} + \psi^{2N} + 18})\psi^k.$$

Therefore, $\varphi_2(k)$ changes sign exactly once, and it can be readily verified from $\varphi_1(k)$, $\varphi_2(k)$, and the definition of the inner product that $(\varphi_1, \varphi_2)_E = 0$, $(\varphi_1, \varphi_2)_\sigma = 0$. $\square$



(a) Plot of $\varphi_2(k)$.



(b) Behavior of $\varphi_2(k)$ near its zero.

Figure 1. Plot of the function $\varphi_2(k)$ for $N = 20$.

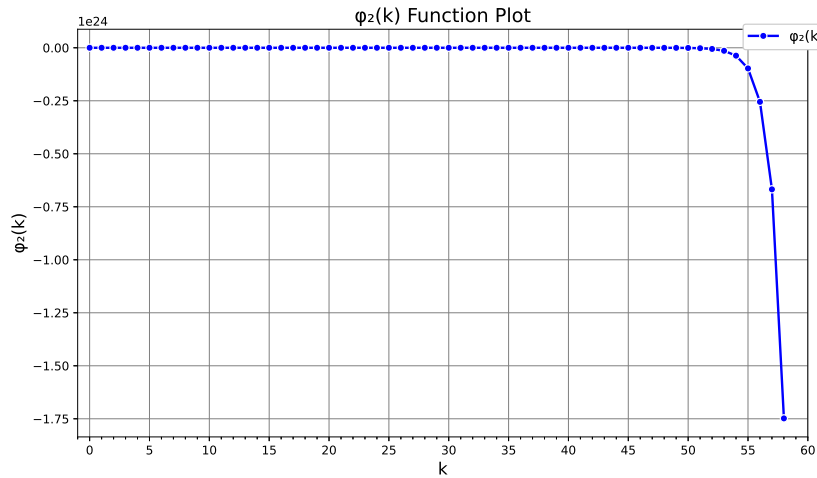


Figure 2. Plot of the function $\varphi_2(k)$ for $N = 57$.

Lemma 2.3. When $c(k) = 1$, $\sigma(\cdot) = 1$, the nonhomogeneous problem

$$\begin{cases} -\Delta^2 u(k-1) + u(k) = 0, \\ -\Delta u(0) = (\mu_k + \lambda)u(1) + h(1), \\ \Delta u(N) = (\mu_k + \lambda)u(N+1) + h(N+1) \end{cases} \quad (2.3)$$

possesses a unique solution at $\lambda \neq 0$.

Proof. The equation in problem (2.3) has two linearly independent solutions $u_1(k) = \phi^k$ and $u_2(k) = \psi^k$, where ϕ and ψ are defined by Lemma 2.2. Substituting these into (2.3) yields that

$$\begin{aligned} U_1(u_1) &:= u_1(0) - (\mu_k + \lambda + 1)u_1(1) = 1 - (\mu_k + \lambda + 1)\phi, & U_1(u_2) &:= 1 - (\mu_k + \lambda + 1)\psi, \\ U_2(u_1) &:= (1 - \mu_k - \lambda)\phi^{N+1} - \phi^N, & U_2(u_2) &:= (1 - \mu_k - \lambda)\psi^{N+1} - \psi^N. \end{aligned}$$

Subsequently, it follows that

$$\begin{aligned} \kappa &= U_1(u_1)U_2(u_2) - U_1(u_2)U_2(u_1) \\ &= (\mu_k + \lambda)^2(\psi^N - \phi^N) + (\mu_k + \lambda)(-\psi^{N+1} + \psi^{N-1} + \phi^{N+1} - \phi^{N-1}) \\ &\quad + (\psi^{N+1} - 2\psi^N + \psi^{N-1} - \phi^{N+1} + 2\phi^N - \phi^{N-1}) \\ &= \mu_k^2(\psi^N - \phi^N) + \mu_k(-\psi^{N+1} + \psi^{N-1} + 2\lambda(\psi^N - \phi^N) + \phi^{N+1} - \phi^{N-1}) \\ &\quad + [(1 - \lambda)\psi^{N+1} - 2(\psi^N - \phi^N) + \lambda^2(\psi^N - \phi^N) + (\lambda + 1)(\psi^{N-1} - \phi^{N-1}) + (\lambda - 1)\phi^{N+1}]. \end{aligned}$$

If $\lambda \neq 0$, then $\kappa \neq 0$. By [12, Theorem 1.1], the solution of the problem (2.3) exists and is unique.

Assume that the general solution of the equation in (2.3) is given by $u(k) = c_1 u_1(k) + c_2 u_2(k)$. Then

$$\begin{aligned} c_1 U_1(u_1) + c_2 U_1(u_2) &= h(1), \\ c_1 U_2(u_1) + c_2 U_2(u_2) &= h(N + 1), \end{aligned}$$

that is,

$$\begin{aligned} [1 - (\mu_k + \lambda + 1)\phi]c_1 + [1 - (\mu_k + \lambda + 1)\psi]c_2 &= h(1), \\ [(1 - \mu_k - \lambda)\phi^{N+1} - \phi^N]c_1 + [(1 - \mu_k - \lambda)\psi^{N+1} - \psi^N]c_2 &= h(N + 1), \end{aligned}$$

which implies that

$$\begin{aligned} c_1 &= \frac{h(1)U_2(u_2) - h(N + 1)U_1(u_2)}{\kappa}, \\ c_2 &= \frac{h(N + 1)U_1(u_1) - h(1)U_2(u_1)}{\kappa}. \end{aligned}$$

Consequently, the solution of the nonhomogeneous boundary value problem is

$$u(k) = \frac{h(1)U_2(u_2) - h(N + 1)U_1(u_2)}{\kappa}\phi^k + \frac{h(N + 1)U_1(u_1) - h(1)U_2(u_1)}{\kappa}\psi^k.$$

□

From Lemma 2.2, we know that the eigenvalue problem (1.1) has only two eigenvalues when the coefficient function and the weight function take the specific values $c(k) = 1, \sigma(\cdot) = 1$, respectively. A natural question arises: Under the general condition (C_1) for $c(k), \sigma(\cdot)$ does problem (1.1) still have only two eigenvalues? From (C_1) , it follows that the boundary condition $\Delta u(N) = \mu\sigma(N + 1)u(N + 1)$, we further consider two cases:

Case 1. When $\mu = \frac{1}{\sigma(N+1)}$, the eigenvalue problem (1.1) reduces to the following boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \frac{\sigma(1)}{\sigma(N+1)}u(1), \\ u(N) = 0. \end{cases}$$

When $c(k) = 1$ holds, this problem has only the trivial solution.

Case 2. When $\mu \neq \frac{1}{\sigma(N+1)}$, problem (1.1) can be rewritten as $D_N(\mu) = 0$, where

$$D_N = \begin{pmatrix} 1+c(1)-\mu\sigma(1) & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2+c(2) & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+c(3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+c(N-2) & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2+c(N-1) & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+c(N)-\frac{1}{1-\mu\sigma(N+1)} \end{pmatrix}.$$

Through calculation, we obtain

$$\begin{aligned} D_N(\mu) \\ &:= \det D_N \\ &= \frac{(\sigma(1)\sigma(N+1)|B|)\mu^2 + (\sigma(N+1)(|B|-|A_N|) + \sigma(1)(|C|-|B|))\mu + (|A_N|-|A_{N-1}|-|B|+|C|)}{1-\mu\sigma(N+1)}. \end{aligned}$$

This transforms the problem of finding eigenvalues of (1.1) into the problem of finding the roots of the polynomial $D_N(\mu)$, where

$$\begin{aligned} A_N &= \begin{pmatrix} 2+c(1) & -1 & \cdots & 0 & 0 \\ -1 & 2+c(2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2+c(N-1) & -1 \\ 0 & 0 & \cdots & -1 & 2+c(N) \end{pmatrix}, \\ B &= \begin{pmatrix} 2+c(2) & -1 & \cdots & 0 & 0 \\ -1 & 2+c(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2+c(N-1) & -1 \\ 0 & 0 & \cdots & -1 & 2+c(N) \end{pmatrix}, \\ C &= \begin{pmatrix} 2+c(2) & -1 & \cdots & 0 & 0 \\ -1 & 2+c(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2+c(N-2) & -1 \\ 0 & 0 & \cdots & -1 & 2+c(N-1) \end{pmatrix}. \end{aligned}$$

Let $D_N(\mu) = 0$. Since the denominator of $D_N(\mu)$ is nonzero, the numerator must be zero, i.e., $(\sigma(1)\sigma(N+1)|B|)\mu^2 + (\sigma(N+1)(|B| - |A_N|) + \sigma(1)(|C| - |B|))\mu + (|A_N| - |A_{N-1}| - |B| + |C|) = 0$, which is clearly a quadratic polynomial in μ . By the Fundamental Theorem of Algebra, this polynomial has two roots.

Theorem 2.1. *If problem (1.1) satisfies condition (C_1) , then the eigenvalue problem yields the following conclusions:*

(1) *The eigenvalue problem (1.1) possesses two real eigenvalues $0 < \mu_1 \leq \mu_2$, and each eigenvalue corresponds to a finite-dimensional eigenspace. The eigenfunctions $\{\varphi_j\}_{j=1}^2$ associated with the eigenvalue μ_j form both a E -orthogonal set and a σ -orthogonal set.*

(2) *The normalized eigenfunctions constitute a complete E -orthogonal basis. Moreover, every function $u \in E$ can be uniquely expressed as $u(k) = c_1\varphi_1 + c_2\varphi_2$, where $c_j := \frac{1}{\mu_j}(u, \varphi_j)_E = (u, \varphi_j)_\sigma$ and satisfies the norm conditions*

$$\|u\|_E^2 = \sum_{j=1}^2 \mu_j |c_j|^2 \quad \text{and} \quad \|u\|_\sigma^2 = \sum_{j=1}^2 |c_j|^2.$$

Proof. (1) Define the functional

$$\begin{aligned} P(u) &= \sum_{k=1}^N |\Delta u(k)|^2 + \sum_{k=1}^N c(k)u^2(k), \\ Q(u) &= \sigma(1)u^2(1) + \sigma(N+1)u^2(N+1) - 1 \end{aligned}$$

for $u \in E$, where P and Q are C^1 functional satisfying

$$\begin{aligned} P'(u)(v) &= \lim_{h \rightarrow 0} \frac{P(u+hv) - P(u)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sum_{k=1}^N |\Delta(u(k) + hv(k))|^2 + \sum_{k=1}^N c(k)(u(k) + hv(k))^2 - \sum_{k=1}^N |\Delta u(k)|^2 - \sum_{k=1}^N c(k)u^2(k)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sum_{k=1}^N 2h\Delta u(k) \cdot \Delta v(k) + h^2|\Delta v(k)|^2 + \sum_{k=1}^N c(k)(2hu(k)v(k) + h^2v^2(k))}{h} \\ &= 2\left[\sum_{k=1}^N \Delta u(k) \cdot \Delta v(k) + \sum_{k=1}^N c(k)u(k)v(k)\right], \end{aligned} \tag{2.4}$$

$$Q'(u)(v) = \lim_{h \rightarrow 0} \frac{Q(u+hv) - Q(u)}{h} = 2[\sigma(1)u(1)v(1) + \sigma(N+1)u(N+1)v(N+1)], \tag{2.5}$$

for $u, v \in E$ and $P'(u), Q'(u)$ denotes the *Fréchet* derivative.

We now show that $P(u)$ attains its minimum on the constraint set $W = \{u \in E_\sigma : Q(u) = 0\}$. Set $\eta = \inf_{u \in W} P(u)$. Since the zero set of a continuous linear functional is a closed linear subspace, it is known that W is a closed linear subspace of E_σ . Clearly, W is bounded, and a bounded closed set is compact. By $P(u) = \|u\|_E^2$ and the lower semi-continuity of $\|u\|_E$, it follows that $P(u)$ is weakly lower semicontinuous on W . According to [32, Theorem 1.4.1], there exists $\varphi_1 \in W$, such that $P(\varphi_1) = \eta = \inf_{u \in W} P(u)$, i.e., $P(u)$ attains its minimum at φ_1 .

Construct the Lagrange function $\mathfrak{L}(u, \mu) = P(u) - \mu Q(u) = \|u\|_E^2 - \mu(\|u\|_\sigma^2 - 1)$, where μ is the Lagrange multiplier. The minimizer φ_1 satisfies $\mathfrak{L}_u(u, \mu)(v) = P'(u)(v) - \mu Q'(u)(v) = 0$, i.e., $P'(\varphi_1)(v) - \mu_1 Q'(\varphi_1)(v) = 0$. Substituting equation (2.4), (2.5) into the above expression, we obtain that φ_1 satisfies the Euler-Lagrange equation

$$\sum_{k=1}^N \Delta \varphi_1(k) \cdot \Delta v(k) + \sum_{k=1}^N c(k) \varphi_1(k) v(k) = \mu_1 (\sigma(1) \varphi_1(1) v(1) + \sigma(N+1) \varphi_1(N+1) v(N+1)), \quad (2.6)$$

where $v \in E$, it follows that (μ_1, φ_1) satisfies equation (2.1) and $\varphi_1 \in E_\sigma$. Set $v = \varphi_1$ in equation (2.6) gives

$$\sum_{k=1}^N |\Delta \varphi_1(k)|^2 + \sum_{k=1}^N c(k) \varphi_1^2(k) = \mu_1 [\sigma(1) \varphi_1^2(1) + \sigma(N+1) \varphi_1^2(N+1)].$$

Since $\varphi_1 \in W$, it follows that $Q(\varphi_1) = 0$, i.e., $\sigma(1) \varphi_1^2(1) + \sigma(N+1) \varphi_1^2(N+1) = 1$, and thus $\eta = P(\varphi_1) = \mu_1$. Therefore, μ_1 can be defined via the Rayleigh quotient

$$\mu_1 = \inf_{u \in E_\sigma, u \neq 0} \frac{\|u\|_E^2}{\sigma(1)u^2(1) + \sigma(N+1)u^2(N+1)}. \quad (2.7)$$

Clearly, $\mu_1 = P(\varphi_1) \geq 0$. In fact, then $P(\varphi_1) = 0$, $|\Delta \varphi_1| = 0$ on $[1, N]_{\mathbb{Z}}$. Hence, φ_1 must be a constant, but this contradicts condition (C_1) . Therefore, $\mu_1 > 0$.

We now examine the existence of other eigenvalues. Define $W_1 = \{u \in E_\sigma : (u, \varphi_1)_\sigma = 0\}$. Since W_1 is the kernel of a continuous functional $(\cdot, \varphi_1)_\sigma$ on E_σ , and W_1 is a closed subspace of E_σ , it is a Hilbert space under the same inner product $(\cdot, \cdot)_E$.

$$\mu_2 = \inf\{P(u) : u \in W_1 \text{ and } \|u\|_\sigma = 1\} = \inf_{u \in W_1, u \neq 0} \frac{\|u\|_E^2}{\sigma(1)u^2(1) + \sigma(N+1)u^2(N+1)}. \quad (2.8)$$

Due to $W_1 \subset W$, we have $\mu_1 \leq \mu_2$. Furthermore, analogous to the case where $P(u)$ attains its minimum μ_1 at φ_1 , it can be shown that $P(u)$ attains its minimum μ_2 at φ_2 , where the minimizer (μ_2, φ_2) satisfies equation (2.1). Therefore, from equations (2.7) and (2.8), it follows that the eigenvalue problem (1.1) admits two real eigenvalues $0 < \mu_1 \leq \mu_2$.

Let $S(\mu_i) \subset E$, denote the eigenspace spanned by the eigenfunctions by φ_i corresponding μ_i . E is isomorphic to $\mathbb{R}^{N+2}$, and $S(\mu_i)$ is a linear subspace of E , so $S(\mu_i)$ is also finite-dimensional. Since φ_2 is the minimum value of $P(u)$ on the constraint set W_1 , φ_j is σ -orthogonal and satisfies equation (2.1), hence φ_j is also E -orthogonal.

(2) Let $\psi_j = \frac{1}{\sqrt{\mu_j}} \varphi_j$ satisfy $\|\psi_j\|_E = 1$. We shall show that the sequence $\{\psi_j\}_{j=1}^2$ is a complete E -orthogonal set for E . If $\mu_1 \neq \mu_2$ are two different eigenvalues, then their corresponding eigenfunctions φ_1, φ_2 are linearly independent, and it is impossible to find a third eigenfunction that is linearly independent from them. If $\mu_1 = \mu_2$ are double roots, i.e., the algebraic multiplicity of the eigenvalue is two, then the geometric multiplicity is less than or equal to two, and there are at most two linearly independent eigenfunctions corresponding to them. Thus, the sequence $\{\psi_j\}_{j=1}^2$ provides a complete E -orthonormal basis for E ; that is, for any $u(k) \in E$, it can be expressed as a linear combination of $\{\psi_j\}_{j=1}^2$ in the form

$$u(k) = \sum_{j=1}^2 \alpha_j \psi_j(k), \quad (2.9)$$

where $(u, \psi_j)_E = (\sum_{i=1}^2 \alpha_i \psi_i, \psi_j)_E = \sum_{i=1}^2 \alpha_i (\psi_i, \psi_j)_E = \alpha_j$, i.e., $\alpha_j = (u, \psi_j)_E$. From (2.9), we obtain that

$$\|u\|_E^2 = (u, u)_E = (\sum_{j=1}^2 \alpha_j \psi_j, \sum_{j=1}^2 \alpha_j \psi_j)_E = \sum_{j=1}^2 |\alpha_j|^2 (\psi_j, \psi_j)_E = \sum_{j=1}^2 |\alpha_j|^2.$$

Substituting $\psi_j = \frac{1}{\sqrt{\mu_j}} \varphi_j$ into (2.9) yields that

$$u(k) = \sum_{j=1}^2 \alpha_j \frac{1}{\sqrt{\mu_j}} \varphi_j(k) = \sum_{j=1}^2 c_j \varphi_j(k), \quad (2.10)$$

where $c_j := \frac{1}{\sqrt{\mu_j}} \alpha_j$. This concludes that

$$\|u\|_E^2 = \sum_{j=1}^2 |\alpha_j|^2 = \sum_{j=1}^2 \mu_j |c_j|^2,$$

and moreover

$$c_j = \frac{1}{\sqrt{\mu_j}} \alpha_j = \frac{1}{\sqrt{\mu_j}} (u, \psi_j)_E = \frac{1}{\sqrt{\mu_j}} (u, \frac{1}{\sqrt{\mu_j}} \varphi_j)_E = \frac{1}{\mu_j} (u, \varphi_j)_E = (u, \varphi_j)_\sigma.$$

According to equation (2.10), $\varphi_1 \in W$ together with (2.9) gives that

$$\|u\|_\sigma^2 = (\sum_{j=1}^2 c_j \varphi_j, \sum_{j=1}^2 c_j \varphi_j)_\sigma = \sum_{j=1}^2 |c_j|^2 (\varphi_j, \varphi_j)_\sigma = \sum_{j=1}^2 |c_j|^2.$$

□

Corollary 2.1. *Suppose that $c(k)$ and $\sigma(k)$ satisfy condition (C_1) . Then the following conclusions hold*

(a) *For all $u \in E$, it follows that*

$$\mu_1 (\sigma(1)u^2(1) + \sigma(N+1)u^2(N+1)) \leq \sum_{k=1}^N |\Delta u(k)|^2 + \sum_{k=1}^N c(k)u^2(k) \quad (2.11)$$

holds, where $\mu_1 > 0$ is the minimal Steklov eigenvalue of problem (1.1). If equality is attained in (2.11), then $u(k)$ is a constant multiple of an eigenfunction $\varphi_1(k)$ of problem (1.1).

(b) *For each $v(k) \in \bigoplus_{i \leq 2} S(\mu_i)$, the inequality*

$$\|v\|_E^2 \leq \mu_2 \|v\|_\sigma^2, \quad (2.12)$$

holds, where $S(\mu_i)$ is the eigenspace corresponding to μ_i , and $\bigoplus_{i \leq 2} S(\mu_i)$ is the space spanned by the eigenfunctions corresponding to eigenvalues not exceeding μ_2 .

Proof. (a) If $u(k) = 0$, then inequality (2.11) trivially holds. Otherwise, if $0 \neq u \in E$, holds, then $u(k) = u_1(k) + u_2(k)$, where $u_1 \in E_\sigma$ and $u_2 \in V_\sigma$. Therefore, from the E-orthogonality of $u_1(k), u_2(k)$ and the definition of μ_1 in (2.7), we obtain that

$$\sum_{k=1}^N |\Delta u(k)|^2 + \sum_{k=1}^N c(k)u^2(k)$$

$$\begin{aligned}
&= \|u\|_E^2 = \|u_1\|_E^2 + \|u_2\|_E^2 \\
&= \sum_{k=1}^N [|\Delta u_1(k)|^2 + c(k)u_1^2(k)] + \sum_{k=1}^N [|\Delta u_2(k)|^2 + c(k)u_2^2(k)] \\
&\geq \mu_1 (\sigma(1)u_1^2(1) + \sigma(N+1)u_1^2(N+1) + \sigma(1)u_2^2(1) + \sigma(N+1)u_2^2(N+1)) \\
&= \mu_1 [\sigma(1)(u_1^2(1) + u_2^2(1)) + \sigma(N+1)(u_1^2(N+1) + u_2^2(N+1))] \\
&= \mu_1 [\sigma(1)u^2(1) + \sigma(N+1)u^2(N+1)].
\end{aligned}$$

Hence, inequality (2.11) is satisfied.

(b) For $v(k) \in \bigoplus_{i \leq 2} S(\mu_i)$, we have $\|v\|_E^2 = \sum_{i=1}^2 \mu_i |c_i|^2 \leq \mu_2 \sum_{i=1}^2 |c_i|^2 = \mu_2 \|v\|_\sigma^2$. Consequently, inequality (2.12) holds. $\square$

Lemma 2.4. *The principal eigenvalue μ_1 is simple, and its corresponding eigenfunction $\varphi_1(k)$ does not change sign in $[1, N]_{\mathbb{Z}}$.*

Proof. First, we prove that $\varphi_1(k)$ does not change sign in $[1, N]_{\mathbb{Z}}$. Assume by contradiction that φ_1 changes sign. Define

$$\varphi_1^+(k) = \begin{cases} > 0, & \varphi_1(k) > 0, \\ = 0, & \varphi_1(k) \leq 0, \end{cases} \quad \varphi_1^-(k) = \begin{cases} < 0, & \varphi_1(k) < 0, \\ = 0, & \varphi_1(k) \geq 0, \end{cases}$$

where $\varphi_1^+, \varphi_1^- \in E$. Since (μ_1, φ_1) are the eigenvalue and eigenfunction of the eigenvalue problem (1.1), they satisfy (2.1), i.e., $(\varphi_1, \varphi_1)_E = \mu_1(\varphi_1, \varphi_1)_\sigma$. According to the definition of μ_1 in (2.7), $\varphi_1^+(k)$ and $\varphi_1^-(k)$ satisfy the eigenvalue problem (1.1), and thus they are also eigenfunctions corresponding to μ_1 . Due to $\varphi_1^+(k) \geq 0$ and $\varphi_1^-(k) \leq 0$, this leads to a contradiction with the assumption. Therefore, $\varphi_1(k)$ does not change sign in $[1, N]_{\mathbb{Z}}$.

Next, we prove that μ_1 is simple if and only if $\varphi_1(k)$ does not change sign. We establish this by proving its contrapositive.

If $\varphi_1(k)$ changes sign, then $\varphi_1^+(k)$ and $\varphi_1^-(k)$ are also eigenfunctions corresponding to μ_1 and they are linearly independent. Hence, μ_1 is not simple..

On the other hand, if μ_1 is not simple, let $\varphi(k)$ and $\gamma(k)$ be two eigenfunctions corresponding to μ_1 . Then they are linearly independent. If both $\varphi(k)$ and $\gamma(k)$ change sign, the conclusion follows. Otherwise, without loss of generality, assume $\varphi(k)$ and $\gamma(k)$ are positive. Suppose, for contradiction, that for all $\alpha \in \mathbb{R}$, $\varphi(k) + \alpha\gamma(k)$ does not change sign. Define a function $g : \mathbb{R} \rightarrow \mathbb{R}$ as $g(\alpha) = \sum_{k=1}^N \varphi(k) + \alpha \sum_{k=1}^N \gamma(k)$. Since g is continuous, there exists $\alpha_0 \in \mathbb{R}$ such that $g(\alpha_0) = \sum_{k=1}^N \varphi(k) + \alpha_0 \sum_{k=1}^N \gamma(k) = 0$, i.e., $\varphi(k) = -\alpha_0\gamma(k)$, which contradicts the fact that $\varphi(k)$ and $\gamma(k)$ are linearly independent. Therefore, $\varphi(k) + \alpha_0\gamma(k)$ changes sign. $\square$

3. The main result

Next, we analyze the structure of the solutions set for the nonlinear perturbation of problem (1.1). First, consider structure of the solutions set for the linear nonhomogeneous problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, \\ -\Delta u(0) = (\mu_k + \lambda)\sigma(1)u(1) + h(1), \\ \Delta u(N) = (\mu_k + \lambda)\sigma(N+1)u(N+1) + h(N+1) \end{cases} \quad (3.1)$$

here, $\lambda \in \mathbb{R}$, μ_k are *Steklov* eigenvalues of problem (1.1), $h(\cdot) \in \mathbb{R}$. By the *Fredholm* alternative,

- (i) If $\mu_k + \lambda$ is not an eigenvalue of (1.1), then problem (3.1) admits a unique solution for every $h(\cdot)$.
- (ii) If $\mu_k + \lambda$ is an eigenvalue of (1.1), then problem (3.1) has a solution if and only if $h(\cdot)$ is orthogonal to the eigenspace corresponding to $\mu_k + \lambda$.

Now, consider the nonlinear perturbation of equation (1.1), denoted as

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + f(1, u(1)) + h(1), \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + f(N+1, u(N+1)) + h(N+1), \end{cases} \quad (3.2)$$

where the real parameter λ is chosen in a neighborhood of 0, μ_1 is the first eigenvalue of problem (1.1), and $c(k), \sigma(\cdot)$ satisfies condition (C_1) . Assume that the nonlinear term $f(\cdot, u(\cdot))$ and the nonhomogeneous term $h(\cdot)$ satisfy conditions $(C_2), (C_3)$ and (C_4) .

When the nonlinear term $f \not\equiv 0$, the structure of the solution set for problem (3.2) may differ significantly from that of the nonhomogeneous problem (3.1). Therefore, we present below some results on the structure of the solution set for the nonlinear problem (3.2) with λ in a neighborhood of 0.

First, define the linear boundary operator $\mathfrak{B} : E_1 \rightarrow \{u(0), u(N+1)\}$,

$$\mathfrak{B}u := \left(-\Delta u(0) - \mu_1 \sigma(1)u(1), \Delta u(N) - \mu_1 \sigma(N+1)u(N+1) \right),$$

where $\text{Dom}(\mathfrak{B}) = \{u \in E_1 : -\Delta^2 u(k-1) + c(k)u(k) = 0, \quad k \in [1, N]_{\mathbb{Z}}\}$.

Define the nonlinear *Nemytskii* operator

$$\mathcal{N} : \text{Dom}(\mathfrak{B}) \subset E_1 \rightarrow E_1, \quad \mathcal{N}u = f(\cdot, u(\cdot)).$$

Then problem (3.2) is equivalent to the operator equation

$$\mathfrak{B}u = \lambda \sigma(\cdot)u + \mathcal{N}u + h, \quad u \in \text{Dom}(\mathfrak{B}). \quad (3.3)$$

To establish the existence of solutions using degree theory, we first derive a priori bounds for all possible homotopy solutions of problem (3.2).

Proposition 3.1. (Case $\lambda > 0$) Suppose $(C_2), (C_3)$ and (C_4) hold. Fix constant $\lambda_0 \in \mathbb{R}$ satisfying $0 < \lambda_0 < \mu_2 - \mu_1$. Then there exists constant $R := R(\lambda_0) > 0$, such that when $0 < \lambda \leq \lambda_0$, all possible solutions $u(k)$ of problem (3.2) satisfy

$$\|u\|_E \leq R,$$

that is, when $0 < \lambda \leq \lambda_0$ all possible solutions of problem (3.2) are (uniformly) bounded, independently of λ .

Proof. Assume by contradiction that when $0 < \lambda \leq \lambda_0$ the solutions of problem (3.2) are not uniformly bounded. Then there exist a real sequence $\{\lambda_n\} \subset (0, \lambda_0]$ and a solution sequence $\{u_n\} \subset E_1$, such that for $n \in \mathbb{N}$, there exists $\|u_n\|_E \geq n$ satisfying

$$\mathfrak{B}u_n = \lambda_n \sigma(\cdot)u_n + \mathcal{N}u_n + h. \quad (3.4)$$

It is readily verified that the operator $\mathfrak{B}$ is compact. Define $v_n = \frac{u_n}{\|u_n\|_E}$, then $\|v_n\|_E = 1$, and $v_n \in E_1$. Substituting $u_n = \|u_n\|_E v_n$ into (3.4) and dividing both sides by $\|u_n\|_E$ yields that

$$\mathfrak{B}v_n = \lambda_n \sigma(\cdot)v_n + \frac{\mathcal{N}\|u_n\|_E v_n}{\|u_n\|_E} + \frac{h}{\|u_n\|_E}, \quad (3.5)$$

when $n \rightarrow \infty$, it follows from $(C_2), (C_4)$ that $\frac{\mathcal{N}\|u_n\|_E v_n}{\|u_n\|_E} + \frac{h}{\|u_n\|_E} \rightarrow 0$.

By condition (C_2) , for each $\varepsilon > 0$, there exists a constant $r_\varepsilon > 0$, such that for $k \in \{1, N+1\}$ and all $u(k) \in \mathbb{R}$, when $|u(k)| \geq r_\varepsilon$, $|f(k, u(k))| \leq \varepsilon |u(k)|$. Taking $\varepsilon = 1$, for $k \in [1, N]_{\mathbb{Z}}$ and all $u \in \mathbb{R}$, we have $|f(k, u(k))| \leq |u(k)|$. Therefore, $\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|_E} \right\}$ is bounded, and there exists a constant $\eta > 0$ (independent of n) such that

$$\frac{|f(\cdot, u_n(\cdot))|}{\|u_n\|_E} \leq |v_n| \leq \eta. \quad (3.6)$$

Moreover, $\lambda_n \in (0, \lambda_0]$, by the compactness of the boundary operator, there exists $\lambda^* \in [0, \lambda_0]$ such that $\lambda_n \rightarrow \lambda^*$. The bounded sequence $\{v_n\}$ has a convergent subsequence, still denoted as $\{v_n\}$, i.e., $v_n \rightarrow v$ ($n \rightarrow \infty$), where $v \in E_1$. From equation (3.6), we deduce $\frac{f(\cdot, u_n(\cdot))}{\|u_n\|_E} \rightarrow 0$ ($n \rightarrow \infty$), where $v(k) = 0$. Note that when $v(k) > 0$, $u_n(k) \rightarrow +\infty$; when $v(k) < 0$, $u_n(k) \rightarrow -\infty$. Therefore, for $k \in [1, N]_{\mathbb{Z}}$ such that $v(k) \neq 0$, by condition (C_2) , we have that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|_E} = \left(\frac{f(\cdot, u_n(\cdot))}{u_n} \right) \cdot v_n(k) \rightarrow 0 \cdot v(k) = 0 \quad (n \rightarrow \infty). \quad (3.7)$$

From (3.5), $v_n \rightarrow v$ and the compactness of operator $\mathfrak{B}$, when $n \rightarrow \infty$,

$$\mathfrak{B}v = \lambda^* \sigma(\cdot)v, \quad (3.8)$$

the equivalent problem is

$$\begin{cases} -\Delta^2 v(k-1) + c(k)v(k) = 0, \\ -\Delta v(0) = (\mu_1 + \lambda^*)\sigma(1)v(1), \\ \Delta v(N) = (\mu_1 + \lambda^*)\sigma(N+1)v(N+1). \end{cases} \quad (3.9)$$

Since $\|v\|_E = 1$, it follows that $v(k) > 0$ or $v(k) < 0$. Multiplying both sides of the equation in (3.9) by $\varphi_1(k)$ and summing yields that

$$\sum_{k=1}^N \Delta \varphi_1(k) \Delta v(k) + \varphi_1(1) \Delta v(0) - \varphi_1(N+1) \Delta v(N) + \sum_{k=1}^N c(k) \varphi_1(k) v(k) = 0. \quad (3.10)$$

When $(\mu, u) = (\mu_1, \varphi_1)$ in problem (1.1), a similar argument gives that

$$\sum_{k=1}^N \Delta v(k) \Delta \varphi_1(k) + v(1) \Delta \varphi_1(0) - v(N+1) \Delta \varphi_1(N) + \sum_{k=1}^N c(k) v(k) \varphi_1(k) = 0. \quad (3.11)$$

Subtracting (3.10) from (3.11) yields that

$$-\varphi_1(1)\Delta v(0) + \varphi_1(N+1)\Delta v(N) = -v(1)\Delta\varphi_1(0) + v(N+1)\Delta\varphi_1(N). \quad (3.12)$$

Multiplying the second and third equations of (3.9) by $\varphi_1(1), \varphi_1(N+1)$ and adding them gives that

$$-\varphi_1(1)\Delta v(0) + \varphi_1(N+1)\Delta v(N) = (\mu_1 + \lambda^*)\sigma(1)v(1)\varphi_1(1) + (\mu_1 + \lambda^*)\sigma(N+1)v(N+1)\varphi_1(N+1). \quad (3.13)$$

Since $\varphi_1(k)$ is an eigenfunction corresponding to eigenvalue μ_1 and satisfies problem (1.1), similarly we obtain that

$$-v(1)\Delta\varphi_1(0) + v(N+1)\Delta\varphi_1(N) = \mu_1\sigma(1)\varphi_1(1)v(1) + \mu_1\sigma(N+1)\varphi_1(N+1)v(N+1). \quad (3.14)$$

From equations (3.8), (3.12)-(3.14), it follows that $\lambda^* \equiv 0$, i.e., $\mathfrak{B}v = 0$, so $v(k)$ is an eigenfunction corresponding to μ_1 . Moreover, by Lemma 2.4, μ_1 is simple; hence, $v(k) = t\varphi_1(k), t \in \mathbb{R}, t \neq 0$. Without loss of generality, assume $v(k) > 0$, the case $v(k) < 0$ can be proved similarly. By Lemma 2.4, $\varphi_1(k) > 0$ or $\varphi_1(k) < 0$, so there exists a constant $\varepsilon_0 > 0$ such that $v(k) = t\varphi_1(k) \geq \varepsilon_0 > 0$. When $\varphi_1(k) > 0, t > 0$; when $\varphi_1(k) < 0, t < 0$.

Since $v_n \rightarrow v$ holds uniformly on $k \in [1, N]_{\mathbb{Z}}$, it follows that $u_n(k) = v_n(k)||u_n||_E \rightarrow \infty$ holds uniformly on $k \in [1, N]_{\mathbb{Z}}$. Therefore, there exists $N_0 \in \mathbb{N}$, such that for any $n \geq N_0$, we have that

$$u_n(k) \geq R, \quad k \in [1, N]_{\mathbb{Z}}, \quad (3.15)$$

where R is the constant given in condition (C_3) . Multiplying the second and third equations of (3.2) by $\varphi_1(1), \varphi_1(N+1)$ and adding them yields that

$$\begin{aligned} & -\varphi_1(1)\Delta u_n(0) + \varphi_1(N+1)\Delta u_n(N) \\ &= (\mu_1 + \lambda_n)\sigma(1)u_n(1)\varphi_1(1) + (\mu_1 + \lambda_n)\sigma(N+1)u_n(N+1)\varphi_1(N+1) \\ & \quad + f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) + h(1)\varphi_1(1) \\ & \quad + h(N+1)\varphi_1(N+1). \end{aligned} \quad (3.16)$$

Combining (3.12), (3.14), and (3.16) gives that

$$\begin{aligned} & -\lambda_n[\sigma(1)u_n(1)\varphi_1(1) + \sigma(N+1)u_n(N+1)\varphi_1(N+1)] \\ &= f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) \\ & \quad + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1). \end{aligned} \quad (3.17)$$

Since $0 < \lambda_n \leq \lambda_0$, from (3.15), (3.17) and (C_3) , for each $n \geq N_0$, when $\varphi_1(k) > 0$, we have that

$$\begin{aligned} 0 &> -\lambda_n[\sigma(1)u_n(1)\varphi_1(1) + \sigma(N+1)u_n(N+1)\varphi_1(N+1)] \\ &= f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &\geq A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1), \end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] > A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1)$. This contradicts condition (C_4) .

When $\varphi_1(k) < 0$, we have that

$$\begin{aligned} 0 &< -\lambda_n[\sigma(1)u_n(1)\varphi_1(1) + \sigma(N+1)u_n(N+1)\varphi_1(N+1)] \\ &= f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &\leq A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1), \end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] < A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1)$. This contradicts condition (C_4) . In summary, when $0 < \lambda \leq \lambda_0$, all possible solutions of problem (3.2) are uniformly bounded, independent of λ . $\square$

Proposition 3.2. (Case $\lambda < 0$) *Suppose conditions (C_2) hold. Fix constant $\lambda_0, \lambda_1 \in \mathbb{R}$ satisfying $-\infty < \lambda_0 < \lambda_1 < 0$. Then there exists constant $R := R(\lambda_0, \lambda_1) > 0$ such that when $\lambda_0 \leq \lambda \leq \lambda_1$ all possible solutions $u(k)$ of problem (3.2) satisfy*

$$\|u\|_E \leq R,$$

that is, when $\lambda_0 \leq \lambda \leq \lambda_1 < 0$, all possible solutions of problem (3.2) are (uniformly) bounded, independently of λ .

Proof. The initial part of the proof follows the same reasoning as the steps preceding equation (3.8) in Proposition 3.1. At this stage $\lambda^* \in [\lambda_0, \lambda_1]$. However, $\lambda_1 < 0$, so λ^* is not an eigenvalue. Hence, $v(k) \equiv 0$. This contradicts $\|v\|_E = 1$. Therefore, when $\lambda_0 \leq \lambda \leq \lambda_1 < 0$, all possible solutions of problem (3.2) are uniformly bounded, independent of λ . $\square$

Theorem 3.1. (Existence of solutions) *Suppose conditions that (C_2) , (C_3) and (C_4) hold. Then for all $\lambda < \mu_2 - \mu_1$, problem (3.2) admits at least one solution. Moreover, for $0 < \lambda \leq \lambda_0$ and $\lambda_0 < \mu_2 - \mu_1$, all solutions of problem (3.2) are uniformly bounded, independent of λ .*

Proof. Consider the case where $\lambda > 0$ is fixed. Choose $\delta \in \mathbb{R}$ satisfying $0 < \delta < \mu_2 - \mu_1$, and define the homotopy equation

$$\mathfrak{B}u - \delta\sigma(\cdot)u = \theta[(\lambda - \delta)\sigma(\cdot)u + \mathcal{N}u + h], \quad u \in \text{Dom}(\mathfrak{B}), \quad (3.18)$$

where $\theta \in [0, 1]$. Clearly, when $\theta = 0$, the homotopy equation (3.18) reduces to the homogeneous linear problem $\mathfrak{B}u - \delta\sigma(\cdot)u = 0$. If $\theta = 1$, then the homotopy equation (3.18) coincides with the nonlinear problem (3.2).

Evidently, the linear operator $\mathfrak{B} - \delta\sigma(\cdot)I : \text{Dom}(\mathfrak{B}) \rightarrow E_1$ is bounded. By the *Fredholm* alternative, the linear operator $\mathfrak{B} - \delta\sigma(\cdot)I$ has a unique solution, i.e., the linear operator $\mathfrak{B} - \delta\sigma(\cdot)I$ is invertible. Therefore, the homotopy equation (3.18) is equivalent to the fixed-point homotopy equation

$$u = \theta(\mathfrak{B} - \delta\sigma(\cdot)I)^{-1}((\lambda - \delta)\sigma(\cdot)Iu + \mathcal{N}u + h) := \mathcal{T}_\theta(u), \quad u \in \text{Dom}(\mathfrak{B}). \quad (3.19)$$

This equation is equivalent to $u = \mathcal{T}_\theta(u)$.

When $0 \leq \theta < 1$, $0 < (1 - \theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} := \lambda_0 < \mu_2 - \mu_1$ holds. By Proposition 3.1, all solutions of equation (3.19) are uniformly bounded, independent of $\theta \in [0, 1]$.

When $\theta = 0$, $\mathcal{T}_0(u) = 0$ has only the trivial solution, so $\deg(I - \mathcal{T}_0, B_R, 0) = 1$. When $\theta = 1$ by the homotopy invariance of the topological degree, we obtain that

$$\deg(I - \mathcal{T}_1, B_R, 0) = 1 \neq 0,$$

i.e., $\mathcal{T}_1(u)$ has a fixed point, and problem (3.2) admits at least one solution. The first part of Theorem 3.1 is thus proved. The second part follows directly from Proposition 3.1. $\square$

Theorem 3.2. (Existence of solutions) *Suppose condition (C_2) holds. Then for all $\lambda < 0$, problem (3.2) admits at least one solution. Moreover, for $\lambda_0 \leq \lambda \leq \lambda_1$ and $-\infty < \lambda_0 < \lambda_1 < 0$, all solutions of problem (3.2) are uniformly bounded, independent of λ .*

Proof. Assume in the homotopy transformation (3.18) that $\delta < 0$ and $\theta \in [0, 1]$. Note that when $0 \leq \theta \leq 1$, $\lambda_0 := \min\{\lambda, \delta\} \leq (1 - \theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} := \lambda_1 < 0$. By Proposition 3.2, all solutions of equation (3.19) are uniformly bounded, independent of $\theta \in [0, 1]$. Following a reasoning similar to the first part of Theorem 3.1, it can be shown that problem (3.2) has at least one solution. $\square$

Proposition 3.3. *Suppose (C_2) , the strict inequality in (1.3) and (1.4) of (C_3) and (C_4) hold. When $\lambda = 0$, all possible solutions of problem (3.2) are uniformly bounded.*

Proof. It suffices to show that all possible solutions of the $\lambda = 0, \theta = 1$ homotopy family are uniformly bounded. The initial part of the proof follows the same reasoning as the steps preceding equation (3.15) in Proposition 3.1. Substituting $\lambda_n = 0$ into (3.17), and combining with condition (C_3) we find that, when $\varphi_1(k) > 0$, it follows that

$$\begin{aligned} 0 &= f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &> A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1), \end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] > A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1)$. This contradicts condition (C_4) .

When $\varphi_1(k) < 0$, we have that

$$\begin{aligned} 0 &= f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &< A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1), \end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] < A(1)\varphi_1(1) + A(N+1)\varphi_1(N+1)$. This again contradicts condition (C_4) . In summary, when $\lambda = 0$, all possible solutions of problem (3.2) are uniformly bounded, independent of λ . That is, Proposition 3.1 holds for all $\lambda \in [0, \lambda_0]$. $\square$

Theorem 3.3. (Strong resonance case: Bifurcation from infinity) *Suppose conditions (C_2) , (C_4) and the strict inequality in (1.3) and (1.4) of (C_3) is satisfied. Then there exists constant $\lambda_- < 0$, such that for all $\varepsilon \in (0, |\lambda_-|)$, problem (3.2) admits at least two distinct solutions, denoted by $(\lambda_\varepsilon^+, u_\varepsilon)$ and $(\lambda_\varepsilon^-, v_\varepsilon)$, where $-\varepsilon < \lambda_\varepsilon^\pm < 0$, satisfying*

$$\lim_{\varepsilon \rightarrow 0^+} \min\{\|u_\varepsilon\|_E, \|v_\varepsilon\|_E\} = \infty;$$

when $\varepsilon \rightarrow 0^+$, $\lambda_\varepsilon^\pm \rightarrow 0$, and thus $(0, \infty)$ is a bifurcation point from infinity.

Moreover, when $0 < \lambda \leq \lambda_0$, all possible solutions of problem (3.2) are uniformly bounded, independent of λ . Therefore, bifurcation from infinity occurs only on the left side of the eigenvalue μ_1 .

Proof. Consider the fixed-point equation

$$u = (\mathfrak{B} - \delta\sigma(\cdot)I)^{-1}((\lambda - \delta)\sigma(\cdot)Iu + \mathcal{N}u + h).$$

Define

$$\mu := \lambda - \delta, \quad Lu := [(\mathfrak{B} - \delta\sigma(\cdot)I)^{-1}\sigma(\cdot)I]u, \quad \text{and} \quad Hu := (\mathfrak{B} - \delta\sigma(\cdot)I)^{-1}(\mathcal{N}u + h).$$

Then the above fixed-point equation is equivalent to the nonlinear equation

$$u = \mu Lu + H(u).$$

It is easy to see that when $\lambda = 0$, $\mu^{-1} = -\delta^{-1}$ is the principal eigenvalue of L , and by the compactness of the boundary operator, L is a compact linear operator. One may readily verify that H is a completely continuous operator. From condition (C_2) , it follows that $\|f(\cdot, u)\|_E = o(\|u\|_E)$. Let $\beta = H(u)$, $u \in E_1$, i.e., u satisfies the operator equation $(\mathfrak{B} - \delta\sigma(\cdot)I)\beta = (\mathcal{N}u + h)$. There exists $\rho > 0$ (independent of u) such that

$$\begin{aligned}\|\beta\|_E &= \|(\mathfrak{B} - \delta\sigma(\cdot)I)^{-1}(\mathcal{N}u + h)\|_E \\ &\leq \rho\|\mathcal{N}u + h\|_E \\ &\leq \rho\varepsilon\|u\|_E + \rho\|h\|_E,\end{aligned}$$

i.e., $\frac{\|\beta\|_E}{\|u\|_E} \leq \rho\varepsilon + \rho\frac{\|h\|_E}{\|u\|_E} = \varepsilon$. Hence, when $\|u\|_E \rightarrow \infty$, $H(u) = o(\|u\|_E)$. By the *Rabinowitz* bifurcation theorem [27] and Lemma 2.1, we conclude that $\lambda = 0$ is a bifurcation point from infinity. Properties (ii) and (iii) in Lemma 2.1 do not occur, by the similar argument of [11, 27]. From Propositions 3.1, 3.2, and 3.3, when $0 \leq \lambda \leq \lambda_0 < \mu_2 - \mu_1$ and $\lambda_0 \leq \lambda \leq \lambda_1 < 0$, the solutions of problem (3.2) are uniformly bounded. Therefore, there exists $\lambda_- < 0$ such that for each $0 < \varepsilon < |\lambda_-|$, there exists two distinct solutions $(\lambda_\varepsilon^+, u_\varepsilon) \in \mathfrak{C}^+$ and $(\lambda_\varepsilon^-, v_\varepsilon) \in \mathfrak{C}^-$ satisfying $\mathfrak{C}^+ \cap U_\varepsilon \neq \emptyset$, $\mathfrak{C}^- \cap U_\varepsilon \neq \emptyset$, where $U_\varepsilon = \{(\lambda_\varepsilon^\pm, u_\varepsilon), (\lambda_\varepsilon^\pm, v_\varepsilon) \in \mathbb{R} \times E_1 : -\varepsilon < \lambda_\varepsilon^\pm < 0, \min\{\|u_\varepsilon\|_E, \|v_\varepsilon\|_E\} > 1/\varepsilon\}$. This implies that when $\varepsilon \rightarrow 0^+$, $\lambda_\varepsilon^\pm \rightarrow 0$, and $\min\{\|u_\varepsilon\|_E, \|v_\varepsilon\|_E\} \rightarrow \infty$. $\square$

Theorem 3.4. (Resonance case: Bifurcation from infinity) *Suppose (C_2) , (C_3) and (C'_4) hold, then $(0, \infty)$ is a bifurcation point from infinity.*

Proof. First, we verify that when $\lambda = 0$, all possible solutions of problem (3.2) are uniformly bounded. The proof follows a procedure similar to that of Proposition 3.3. By condition (C_3) and *Fatou* Lemma, when $\varphi_1(k) > 0$, we have that

$$\begin{aligned}0 &= \liminf_{n \rightarrow +\infty} [f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1)] \\ &\quad + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &\geq \liminf_{n \rightarrow +\infty} f(1, u_n(1))\varphi_1(1) + \liminf_{n \rightarrow +\infty} f(N+1, u_n(N+1))\varphi_1(N+1) \\ &\quad + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &= f_+(1)\varphi_1(1) + f_+(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1),\end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] \geq f_+(1)\varphi_1(1) + f_+(N+1)\varphi_1(N+1)$, which contradicts condition (C'_4) .

When $\varphi_1(k) < 0$, it follows that

$$\begin{aligned}0 &= \liminf_{n \rightarrow +\infty} [f(1, u_n(1))\varphi_1(1) + f(N+1, u_n(N+1))\varphi_1(N+1)] \\ &\quad + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &\leq \liminf_{n \rightarrow +\infty} f(1, u_n(1))\varphi_1(1) + \liminf_{n \rightarrow +\infty} f(N+1, u_n(N+1))\varphi_1(N+1) \\ &\quad + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1) \\ &= f_+(1)\varphi_1(1) + f_+(N+1)\varphi_1(N+1) + h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1),\end{aligned}$$

and thus $-[h(1)\varphi_1(1) + h(N+1)\varphi_1(N+1)] \leq f_+(1)\varphi_1(1) + f_+(N+1)\varphi_1(N+1)$, which again contradicts condition (C'_4) .

Second, by the same argument of Theorem 3.3, $(0, \infty)$ is a bifurcation point from infinity. $\square$

4. Applications

In this section, we introduce some examples to illustrate our main results.

Example 4.1. Consider the existence of solutions for problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + f(1, u(1)) + \frac{1}{8}, \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + f(N+1, u(N+1)) + N + \frac{1}{8}, \end{cases} \quad (4.1)$$

where

$$f(k, u(k)) = \begin{cases} k\sqrt{u(k)}, & u(k) \geq 1, \\ k\sqrt[3]{u(k)}, & u(k) \leq -1, \\ ku(k), & |u(k)| < 1. \end{cases}$$

Clearly, $\lim_{u(k) \rightarrow +\infty} \frac{k\sqrt{u(k)}}{u(k)} = \lim_{u(k) \rightarrow +\infty} \frac{k}{\sqrt{u(k)}} = 0$, $\lim_{u(k) \rightarrow -\infty} \frac{k\sqrt[3]{u(k)}}{u(k)} = \lim_{u(k) \rightarrow -\infty} \frac{k}{\sqrt[3]{u^2(k)}} = 0$, (C_2) is satisfied. There exists $A(k) = k - \frac{1}{4}$, $B(k) = -k + \frac{1}{3}$ and a constant $r = -1$, $R = 1$, such that for all $k \in \{1, N+1\}$ and $u(k) \in \mathbb{R}$, when $u(k) \geq 1$ holds, $f(k, u(k)) \geq k - \frac{1}{4}$ follows; when $u(k) \leq -1$ holds, $f(k, u(k)) \leq -k + \frac{1}{3}$ are valid. Condition (C_3) is true. From $A(k)$, $B(k)$, $h(\cdot)$, it follows that $-\frac{2}{3}|\varphi_1(1)| - (n + \frac{2}{3})|\varphi_1(N+1)| \leq -\frac{1}{8}|\varphi_1(1)| - (\frac{1}{8} + N)|\varphi_1(N+1)| \leq \frac{3}{4}|\varphi_1(1)| + (N + \frac{3}{4})|\varphi_1(N+1)|$ and (C_4) holds. It follows from the Theorem 3.1 that problem (4.1) has at least a solution. More precisely, when $u(k) \geq 1$ holds, $f(k, u(k)) > k - \frac{1}{4}$ follows; when $u(k) \leq -1$, $f(k, u(k)) < -k + \frac{1}{3}$ is satisfied, that is, strict inequalities (1.3) and (1.4) in condition (C_3) are verified. By Theorem 3.3, we conclude that problem (4.1) has at least two solutions.

Example 4.2. Consider the existence of solutions for problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + \ln(1 + |u(1)|) + h(1), \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + (N+1)^2 \ln(1 + |u(N+1)|) + h(N+1). \end{cases} \quad (4.2)$$

$\lim_{|u(k)| \rightarrow \infty} \frac{k^2 \ln(1 + |u(k)|)}{u(k)} = k^2 \lim_{|u(k)| \rightarrow \infty} \frac{1}{(1 + |u(k)|)u(k)} = 0$, (C_2) is valid. By Theorem 3.2, problem (4.2) has at least one solution.

Example 4.3. Consider the bifurcation structure of the solution set at infinity for problem

$$\begin{cases} -\Delta^2 u(k-1) + c(k)u(k) = 0, & k \in [1, N]_{\mathbb{Z}}, \\ -\Delta u(0) = (\mu_1 + \lambda)\sigma(1)u(1) + \arctan u(1) - 1, \\ \Delta u(N) = (\mu_1 + \lambda)\sigma(N+1)u(N+1) + (N+1) \arctan u(N+1) - (N+1). \end{cases} \quad (4.3)$$

Clearly, $\lim_{|u(k)| \rightarrow \infty} \frac{k \arctan u(k)}{u(k)} = \lim_{|u(k)| \rightarrow \infty} \frac{k}{(1 + u^2(k))u(k)} = 0$, (C_2) holds. There exists functions $A(k) = \frac{\pi}{3}k$, $B(k) = -\frac{\pi}{4}k$ and a constant $r = -1$, $R = \sqrt{3}$, such that for all $k \in$

$\{1, N+1\}$ and $u(k) \in \mathbb{R}$, when $u(k) \geq \sqrt{3}$ holds, $f(k, u(k)) \geq \frac{\pi}{3}k$ follows; when $u(k) \leq -1$ holds, $f(k, u(k)) \leq -\frac{\pi}{4}k$, (C_3) is satisfied. $f^-(k) = \limsup_{u(k) \rightarrow -\infty} (k \arctan u(k)) = -\frac{\pi}{2}k$, $f_+(k) =$

$\liminf_{u(k) \rightarrow +\infty} (k \arctan u(k)) = \frac{\pi}{2}k$. Hence, the nonhomogeneous term $h(1) = -1$, $h(N+1) = -(N+1)$ satisfies

$$-\frac{\pi}{2}|\varphi_1(1)| - \frac{\pi}{2}(N+1)|\varphi_1(N+1)| < |\varphi_1(1)| + (N+1)|\varphi_1(N+1)| < \frac{\pi}{2}|\varphi_1(1)| + \frac{\pi}{2}(N+1)|\varphi_1(N+1)|,$$

(C'_4) is valid. By Theorem 3.4, problem (4.3) has at least two solutions.

Conflict of interest

The authors of this work declare that they have no conflicts of interest.

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