

ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS AND A NONLINEAR NEGATIVE NEUTRAL TERM

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Abstract This study covers the oscillatory characteristics of solutions of delay differential equations. In order to test the oscillation of all solutions, we establish three distinct criteria and take into account the noncanonical situation. The purpose of establishing various criteria is to achieve a number of benefits in the oscillation test, namely applying the criterion to Euler's equation yields the sharp condition, the influence of delay function of the criterion, and applying to some cases that have not been studied previously. In previous works that treated the studied equation in its general form, only classical criteria were obtained for the canonical case. Finally, we demonstrate that our findings are a generalization and extension of earlier findings in the literature through discussion and examples.

Keywords Differential equation, asymptotic and oscillatory behavior, nonlinear negative neutral term, Riccati technique.

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1. Introduction

A branch of mathematics and dynamical system theory known as oscillation theory is concerned with the study of oscillatory behavior in differential equations (DEs). In a wide range of natural modeling of many physical and biological systems [3], oscillations play a fundamental role in the behavior of these systems. Identifying the conditions under which oscillations occur as well as characterizing their properties is an important area of research that has broad applications in science and engineering; see [22, 33, 51].

Oscillation theory encompasses many mathematical tools and techniques aimed at analyzing and describing oscillatory solutions to equations of various types [6, 28, 43]. It is particularly concerned with answering questions like whether or not a system oscillates, how often it oscillates, the amplitude and frequency of the oscillations, and the stability of these oscillatory behaviors, as well as whether they are stable. This theory is an essential part of the qualitative theory, which deals with the qualitative properties of solutions to DEs, for instance, stability, simultaneity,

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periodicity, oscillation, symmetry, and others.

Determining the number of zeros in each interval and the separation between adjacent zeros, investigating zero distribution laws, characterizing the connection between oscillatory and other essential characteristics of solutions to various classes of differential equations, and establishing conditions for the existence of oscillatory are further objectives of oscillation theory. Finding criteria that guarantee there are no affirmative solutions to the studied equations is the main goal of most studies. Oscillation theory has always been and will continue to be a crucial numerical mathematical tool in a variety of cutting-edge technologies (see [23, 30]).

Both linear and nonlinear DEs are studied in the field of oscillation theory, with a specific focus on ordinary DEs (ODEs), partial DEs, functional DEs (FDEs), and other DEs. References [34, 36, 37] have focused their efforts on understanding FDEs. As a specific type of FDE in the oscillation theory, Delay Differential Equations (DDE) have gained a lot of attention in the past few years due to their importance to many applications in science, engineering, and technology; see [24, 41].

Due to the fact that they combine ODEs and delay differential equations (DDEs) into a single framework, these equations are unique and called Neutral Delay Differential Equations (NDDEs), see [25]. They are highly helpful in biology to explain how the human body can self-balance and in robotics to build bipedal robots [32]. Secondly, there are a lot of interesting analytical problems in studying this kind of equation. Several monographs have been published in the last few decades describing oscillation conditions for NDDEs (see, for example, [5, 38, 39, 42]).

1.1. FDEs with negative neutral term

Establishing novel oscillation criteria for the second-order DEs of form

$$(r \cdot (\psi - p \cdot (\psi^\alpha \circ \tau))')'(v) + \sum_{i=1}^m \left(q_i \cdot (\psi^\beta \circ \sigma_i) \right)(v) = 0, \quad v \geq v_0, \quad (1.1)$$

is the aim of this article. During this work, the following requirements are met:

(I1) α, β are ratios of odd positive integer numbers, also m is a positive integer.

(I2) $r \in C^1([v_0, \infty), (0, \infty))$ satisfies

$$\int_{v_0}^{\infty} \frac{1}{r(\varrho)} d\varrho = \infty. \quad (1.2)$$

(I3) $\tau, \sigma_i \in C^1([v_0, \infty), \mathbb{R})$, $\sigma_i(v) \leq v$, $\tau(v) \leq v$, and $\lim_{v \rightarrow \infty} \tau(v) = \lim_{v \rightarrow \infty} \sigma_i(v) = \infty$.

(I4) $p, q_i \in C([v_0, \infty), (0, \infty))$, and there exists a constant $p(v)$ such that

$$0 < p(v) \leq p < 1 \text{ for all } v \geq v_0. \quad (1.3)$$

Let $\Phi := \psi - p \cdot (\psi^\alpha \circ \tau)$. By a solution of (1.1), we mean a real-valued function $\psi \in C([v_\psi, \infty), \mathbb{R})$, $v_\psi \geq v_0$, which has the property $r\Phi' \in C^1([v_0, \infty), (0, 1))$ and satisfies (1.1) on $[v_\psi, \infty)$. In our analysis, we only consider solutions ψ which satisfy $\sup\{|\psi(v)| : v \geq v_a\} > 0$ for all $v_a \geq v_\psi$. Let (1.1) have such a solution called oscillatory if it has arbitrarily large zeros on $[v_\psi, \infty)$ and otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if its solutions are all oscillatory.

1.2. Literature review

In prior years, numerous studies have explored the oscillatory patterns exhibited by various classes of DEs (see [9, 12, 29, 52]). Considerable attention has been given to investigating the oscillatory characteristics of solutions in the presence of nonnegative neutral coefficients in different equation classes, as evidenced by works such as these related references: [31, 45–47]. However, the literature offers relatively better findings for equations featuring nonpositive neutral coefficients, as observed in these references: [17, 26]. This scarcity persists, particularly in the context of exploring the oscillatory and asymptotic behavior of distinct classes of half-linear equations.

In this part, we briefly discuss related results that persuaded our study. Some Philos-type oscillation criteria were established by Philos [40] for the second-order linear differential equation

$$(r((\psi)'))' + q\psi = 0.$$

In [4, 11, 13], by employing a Riccati substitution technique, the authors provided sufficient conditions for DDE

$$\left(a \left(|\psi'|^{\eta-1} \psi'\right)\right)' + q |\psi(\sigma)|^{\eta-1} \psi(\sigma) = 0.$$

Zhang et al. [53] investigated the oscillation of NDDEs

$$(\psi + p\psi(\tau))^n + qf(\psi(\sigma)) = 0, \quad (1.4)$$

also Zafer [50] study (1.4), where $w, m \in C^1([v_0, \infty), (0, \infty))$ such that $w(v) > 0, w'(v) > 0$, and

$$\left|f(v, u, \lambda) \geq m(v) w\left(\frac{|\lambda|}{p(g(v))g^{n-1}(v)}\right)\right|.$$

Elmetwally et al. [14], some oscillation criteria are established in the case $-\delta \leq p \leq 0$, where $\delta \in (0, 1)$

$$\left(a\omega \left(|\psi'|^{n-1}(\psi')\right)\right)' + \int_{\alpha}^{\beta} q(t, v) f(x(g(t, \omega))) d\theta(v) = 0.$$

Wong [48] and Yang et al. [49] examined the oscillation for nonlinear NDDE

$$(r(\psi - p\psi(\tau)')^\alpha)' + qf(\psi(\sigma)) = 0, \quad (1.5)$$

under the assumptions that $0 \leq p \leq p_0 < 1$ and $\tau = t - \tau_0 \leq t, \sigma = t - \sigma_0$.

Li et al. [31] obtained oscillation criteria of the DDE with non-positive neutral term

$$(r(\psi - p\psi(\tau)')^\alpha)' + q\psi(\sigma) = 0, \quad (1.6)$$

where $0 \leq p \leq p_0 < 1$ and

$$\int_{v_0}^{\infty} \frac{d\zeta}{r^{1/\alpha}(\zeta)} = \infty.$$

Results were enhanced by Arul and Shobha [29], improving oscillation results for equation solutions (1.6).

In [19], several findings on oscillation were obtained for NDDE

$$(r(\psi - p\psi(\tau)')^\gamma)' + q\psi^\beta(\sigma) = 0, \quad (1.7)$$

where $\gamma \geq \beta$. In addition, Grace et al. [20] determined some results for oscillation of solutions of the second order DEs on the form

$$\left(r (y')^\beta\right)' + q \psi^\gamma(\sigma) = 0,$$

where

$$y = \psi + p_1 \psi^{\alpha_1}(\tau) - p_2 \psi^{\alpha_2}(\tau).$$

Recently, Baculikova et al. [10] studied the oscillatory behavior of DDE

$$\left(r ((\psi - p\psi^\alpha(\tau))')\right)' + q \psi^\beta(\sigma) = 0,$$

where $0 < \alpha \leq 1$.

Examining the literature reveals that there are few findings about the oscillatory behavior of second-order DEs with a sublinear neutral term; see [1, 27, 44, 54]. In addition to the many applications it can be used for, the study of second-order FDEs is quite important from a theoretical standpoint. This reflects the advancement of the study of second-order equations, which exhibit oscillatory behavior see [8, 18, 21]. Inspired by the aforementioned observation, this paper establishes novel oscillation results for Equation (1.1), thereby enhancing and complementing prior findings reported in [10]. We offer fresh parameters for the examined equation's oscillation. The findings acquired here supplement and expand upon certain previously published findings. With the help of the Riccati transformation, we present a new result for the oscillation of (1.1), which improves on the related results reported in [10]. The purpose of this paper is to demonstrate the importance of our results by presenting two examples.

2. Main results

Here, we assume

$$R(v) = \int_{v_1}^v \frac{1}{r(\varrho)} d\varrho \text{ for } v \geq v_1 \geq v_0.$$

The class of all eventually positive solutions is also denoted (1.1) by the symbol $\mathbb{S}$. Note that from equation (1.1) and from the assumptions, the proof for the case of positive solutions is enough to state considering that the proof of the other case is the same.

Lemma 2.1. *Suppose that condition (1.2) holds. If $\psi \in \mathbb{S}$, then the correlated function $\Phi(v)$ eventually satisfies one of the following two cases:*

(I) $\Phi > 0$, $\Phi' > 0$, and $(r(\Phi'))' < 0$,

(II) $\Phi < 0$, $\Phi' > 0$, and $(r(\Phi'))' < 0$,

for all $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

Proof. Suppose that $\psi \in \mathbb{S}$, we may suppose that $\psi(\tau) > 0$ and $\psi(\sigma_i) > 0$ for all $v \geq v_1$ for some $v_1 \geq v_0$, since $\Phi = \psi - p\psi^\alpha(\tau)$, and from (1.1) then,

$$(r(\Phi'))' = - \sum_{i=1}^m q_i \psi^\beta(\sigma_i) < 0. \quad (2.1)$$

It follows that $(r(\Phi'))$ is decreasing and either $\Phi' > 0$ or $\Phi' < 0$ eventually. If $\Phi' < 0$, then $r\Phi' \leq -M < 0$, for $v \geq v_2$, where $M = -r(v_2)\Phi'(v_2) > 0$. Hence, we have

$$\Phi(v) \leq \Phi(v_2) - M \int_{v_2}^v \frac{1}{r(\varrho)} d\varrho.$$

In view of the condition (1.2), we have $\lim_{v \rightarrow \infty} \Phi(v) = -\infty$. The following two cases would be discussed separately now.

Case 1. If ψ is unbounded, as a result there is a sequence $\{v_i\}$ such that $\lim_{i \rightarrow \infty} v_i = \infty$ and $\lim_{i \rightarrow \infty} \psi(v_i) = \infty$ where $\psi(v_i) = \max\{\psi(\varrho) : v_0 \leq \varrho \leq v_i\}$ exists, if $\psi(v)$ is unbounded. Since $\lim_{v \rightarrow \infty} \tau(v) = \infty$, $\tau(v_i) > v_0$ for all sufficiently large i and $\tau(v) \leq v$, we have

$$\psi(\tau(v_i)) = \max\{\psi(\varrho) : v_0 \leq \varrho \leq \tau(v_i)\} \leq \max\{\psi(\varrho) : v_0 \leq \varrho \leq v_i\} = \psi(v_i),$$

where $\tau(v) \leq v$ and $\Phi'(v) > 0$, hence

$$\Phi(v_i) = \psi(v_i) - p(v_i) \psi^\alpha(\tau(v_i)) \geq (1 - p(v_i) \psi^{\alpha-1}(\tau(v_i))) \rightarrow \infty,$$

as $i \rightarrow \infty$ since $0 < \alpha \leq 1$ and p is bounded, which gives result in a contraction with the fact that $\lim_{v \rightarrow \infty} \Phi(v) = -\infty$.

Case 2. If ψ is bounded then Φ is also bounded, since p is bounded, which contradicts $\lim_{v \rightarrow \infty} \Phi(v) = -\infty$. Hence $\Phi(v)$ satisfies one of the cases (I) and (II). Therefore, the proof is complete. $\square$

Lemma 2.2. *Let condition (1.2) hold. Assume that $\psi \in \mathbb{S}$ such that case (I) of Lemma 2.1 holds. Then.*

$$\psi(v) > \Phi(v) > R(v) a(v) \Phi'(v), \quad (2.2)$$

for all $v \geq v_1$ and $\Phi(v)/R(v)$ is eventually decreasing.

Proof. Depending on the definition of $\Phi(v)$ and condition (I2), we have $\psi(v) > \Phi(v)$ for all $v \geq v_1 \geq v_0$. Relating to case (I), we have

$$\Phi(v) = \Phi(v_1) + \int_{v_1}^v \frac{r(\varrho) \Phi'(\varrho)}{a(\varrho)} d\varrho > r(v) \Phi'(v) R(v), \quad v \geq v_1. \quad (2.3)$$

Furthermore,

$$\left(\frac{\Phi}{R}\right)' = \frac{r z' R - r R' \Phi}{r(R)^2} = \frac{r \Phi' R - \Phi}{r(R)^2}, \quad v \geq v_1. \quad (2.4)$$

By (2.3), one obtains $\left(\frac{\Phi}{R}\right)' < 0$ for all $v \geq v_1$. Thus, $\frac{\Phi}{R}$ is decreasing stringently for all $\frac{\Phi}{R}$. This completes the proof. $\square$

Lemma 2.3. *Suppose that ψ is a positive solution of (1.1) and Φ satisfies case (II) of Lemma 2.1. Then*

$$\lim_{v \rightarrow \infty} \psi(v) = 0.$$

Proof. The proof follows the similar approach as the proof of [31, Lemma 2], hence it is omitted. $\square$

Lemma 2.4. *Assume that $\beta = \alpha$, $\sigma_i(v) < \tau(v)$ for all i and $\psi \in \mathbb{S}$ and*

$$\limsup_{v \rightarrow \infty} \int_{\tau^{-1}(\sigma(v))}^v \frac{1}{r(\varrho)} \left(\int_{v_2}^{\varrho} \sum_{i=1}^m q_i(\zeta) d\zeta \right) d\varrho > \frac{1}{p}. \quad (2.5)$$

Then $\Phi(v)$ can not satisfy case (II) of Lemma 2.1.

Proof. Suppose that $\Phi(v)$ satisfies case (II) of Lemma 2.1, then it follows from the definition of $\Phi(v)$ that

$$-\Phi(v) = -\psi(v) + p(v)\psi^\alpha(\tau(v)) \leq p\psi^\alpha(\tau(v)),$$

that is,

$$\psi(v) \geq \left(-\frac{1}{p}\right)^{1/\alpha} \Phi^{1/\alpha}(\tau^{-1}(\sigma_i(v))). \quad (2.6)$$

This with (1.1) and $\beta = \alpha$ leads to

$$(r(v)\Phi'(v))' + \left(-\frac{1}{p}\right) \sum_{i=1}^m q_i(v)\Phi(\tau^{-1}(\sigma_i(v))) \leq 0, \quad (2.7)$$

integrating (2.7) from v_2 to v , we get

$$r(v)\Phi'(v) - r(v_2)\Phi'(v_2) - \frac{1}{p} \int_{v_2}^v \sum_{i=1}^m q_i(\zeta)\Phi(\tau^{-1}(\sigma_i(\zeta)))d\zeta \leq 0,$$

employing the non-decreasing property of $(r(v)\Phi'(v))$, considering the increasing nature of $\Phi(v)$ and $\sigma_i(v) < \tau(v)$, we obtain

$$-\Phi'(v) \leq \frac{1}{p} \frac{1}{r(v)} \int_{v_2}^v \Phi(\zeta) \sum_{i=1}^m q_i(\zeta)d\zeta \leq 0, \quad (2.8)$$

integrating (2.8) from $\tau^{-1}(\sigma(v))$ to v , we get

$$\Phi(\tau^{-1}(\sigma(v))) - \Phi(v) \leq \frac{1}{p} \Phi(\tau^{-1}(\sigma(v))) \int_{\tau^{-1}(\sigma(v))}^v \frac{1}{r(\varrho)} \left(\int_{v_2}^{\varrho} \sum_{i=1}^m q_i(\zeta)d\zeta \right) d\varrho \leq 0. \quad (2.9)$$

Since $\Phi(v) < 0$, we have

$$\int_{\tau^{-1}(\sigma(v))}^v \frac{1}{r(\varrho)} \left(\int_{v_2}^{\varrho} \sum_{i=1}^m q_i(\zeta)d\zeta \right) d\varrho \leq p,$$

which contradicts (2.5). The proof has been completed. $\square$

Theorem 2.1. Let $\sigma_i(v) < \tau(v)$ for all $i = 1, 2, \dots, m$, and condition (1.2) holds. If

$$\int_{v_1}^{\infty} \sum_{i=1}^m q_i(\zeta) R^\beta(\sigma_i(\zeta)) d\zeta = \infty, \quad (2.10)$$

and

$$\limsup_{v \rightarrow \infty} \sum_{i=1}^m \int_{\tau^{-1}(\sigma_i(v))}^v q_i(\zeta) R^{\beta/\alpha}(\tau^{-1}(\sigma_i(\zeta))) d\zeta \begin{cases} > 1, & \beta = \alpha, \\ = \infty, & \beta < \alpha, \end{cases} \quad (2.11)$$

in this case, equation (1.1) oscillates.

Proof. Assume that $\psi \in \mathbb{S}$, without loss of generality; therefore, it can be assumed that $\psi(\tau) > 0$ and $\psi(\sigma_i) > 0$ for $v \geq v_1 \geq v_0$. Lemma 2.1 states that the corresponding function Φ must satisfy either (I) or (II).

Let Φ satisfy case (I). According to the definition of Φ , we have

$$\psi^\beta > \Phi^\beta.$$

This with (1.1), leads to

$$r\Phi'' + \sum_{i=1}^m q_i \Phi^\beta(\sigma_i) \leq 0, \quad (2.12)$$

it follows from (2.2) that

$$r\Phi'' + \sum_{i=1}^m q_i (R(\sigma_i a(\sigma_i) \Phi'(\sigma_i)))^\beta \leq 0. \quad (2.13)$$

Since $r\Phi$ is non-increasing and $\sigma_i \leq v$, then

$$(r(\sigma_i) \Phi'(\sigma_i))^\beta \geq r(\Phi')^\beta \quad \text{for all } i = 1, 2, \dots, m.$$

It follows that

$$r\Phi'' + \sum_{i=1}^m q_i R^\beta(\sigma_i) (r(\Phi'))^\beta \leq 0. \quad (2.14)$$

Define $w := r\Phi'$ and combining with (2.14), we get

$$w + \sum_{i=1}^m q_i R^\beta(\sigma_i) w^\beta(\sigma) \leq 0. \quad (2.15)$$

On the contrary, by [16, Theorem 3.9.3] condition (2.10) achieves that there is no eventually positive solution in (2.15), a contradiction.

Let Φ satisfy case (II) of Lemma 2.1. Putting $\chi = -\Phi > 0$, then $\chi' < 0$, and (1.1) can be formulated as follows

$$[r\chi']' \geq \sum_{i=1}^m q_i \psi^\beta(\sigma_i) \geq 0. \quad (2.16)$$

Since

$$\begin{aligned} \chi &= -\Phi = p\psi^\alpha - \psi, \\ \chi &\leq p\psi^\alpha(\tau), \end{aligned}$$

due to the validity of (1.3), we can conclude that

$$\chi^\alpha(\tau^{-1}(\sigma_i)) \leq \psi(\sigma_i). \quad (2.17)$$

Subsequently, the inequalities (2.16) and (2.17) result in

$$[r\chi']' \geq \sum_{i=1}^m q_i \chi^{\beta/\alpha}(\tau^{-1}(\sigma_i)). \quad (2.18)$$

For $v_2 \leq v$, we have

$$\begin{aligned}\chi(v) - \chi(v_2) &= \int_{v_2}^v \frac{1}{r(\zeta)} (-r(\zeta)\chi'(\zeta)) d\zeta, \\ \chi(v) - \chi(v_2) &\geq R(v)(-r(v)\chi'(v)),\end{aligned}\tag{2.19}$$

this with (2.18) leads to

$$[a\chi']' \geq \sum_{i=1}^m q_i R^{\beta/\alpha}(\tau^{-1}(\sigma_i)) (-r(\tau^{-1}(\sigma_i))\chi'(\sigma_i))^{\beta/\alpha},\tag{2.20}$$

integrating (2.20) from σ_i to v , and using the non-increasing fact of $-a(\sigma_i)\chi'(\sigma_i)$ and $\sigma_i < \tau$ for all i , we get

$$(-r(v)\chi'(v)) \geq (-r(v)\chi')^{\beta/\alpha} \int_{\tau^{-1}(\sigma_i(v))}^v \sum_{i=1}^m q_i(v) R^{\beta/\alpha}(\tau^{-1}(\sigma_i(v))),\tag{2.21}$$

hence,

$$(F(v))^{1-\frac{\beta}{\alpha}} \geq \sum_{i=1}^m \int_{\tau^{-1}(\sigma_i(v))}^v q_i(\zeta) R^{\beta/\alpha}(\tau^{-1}(\sigma_i(\zeta))) d\zeta,$$

where $F = -r\chi'$. Taking the limsup of both sides of this inequality as $v \rightarrow \infty$ leads to a contradiction with (2.11), the proof is therefore complete. $\square$

Theorem 2.2. *Let $\beta = 1$ and (1.2) hold. If*

$$\liminf_{v \rightarrow \infty} \sum_{i=1}^m \int_{\sigma_i(v)}^v q_i(\zeta) R(\sigma_i(\zeta)) d\zeta > \frac{1}{e},\tag{2.22}$$

then every solution of (1.1) is either oscillatory or $\lim_{v \rightarrow \infty} \psi(v) = 0$.

Proof. The proof follows the similar approach as the proof of [10, Theorem 2.2], hence it is omitted. $\square$

Theorem 2.3. *Suppose that $\beta > 1$ and condition (1.2) hold. If there exists a positive nondecreasing function $\rho(v)$ such that*

$$\limsup_{v \rightarrow \infty} \left(\rho(v)Q(v) + \int_{v_2}^v \left(\rho(\zeta) \sum_{i=1}^m q_i(\zeta) - \frac{(\rho'_+(\zeta))^2 r(\sigma(\zeta))}{4\beta M^{\beta-1} \rho(\zeta) \sigma'(\zeta)} \right) d\zeta \right) = \infty,\tag{2.23}$$

where $\rho'_+(v) = \max\{\rho'(v), 0\}$ and $Q(v) = \int_v^\infty \sum_{i=1}^m q_i(\zeta) d\zeta$, then every solution of equation (1.1) is either oscillatory or tends to zero as $v \rightarrow \infty$.

Proof. Suppose that there exists a nonoscillatory solution ψ of equation (1.1), say, $\psi > 0$, $\psi(\tau) > 0$, and $\psi(\sigma) > 0$ for all $v \geq v_1 \geq v_0$ so that for Φ one of the cases (I) and (II) holds. Suppose that Φ fulfills Lemma 2.1 (I), we can arrive (2.12). Integrating (2.12) from v to ϱ , letting $\varrho \rightarrow \infty$ and leveraging the monotonically increasing nature of Φ , we obtain

$$r\Phi'^\beta(\sigma) \int_v^\infty \sum_{i=1}^m q_i(\zeta) d\zeta = Q\Phi^\beta(\sigma),\tag{2.24}$$

where $\sigma = \min\{\sigma_i, i = 1, 2, \dots, m\}$. Now, Define

$$\omega = \rho \frac{r\Phi'}{\Phi^\beta(\sigma)}, \quad v \geq v_1. \quad (2.25)$$

It is clear that $\omega > 0$ and

$$\begin{aligned} \omega' &= \rho \left(\frac{(r\Phi')'}{\Phi^\beta(\sigma)} - \frac{\beta r\Phi'}{\Phi^{\beta+1}(\sigma)} \Phi'(\sigma) \sigma' \rho \right) + \rho' \frac{r\Phi'}{\Phi^\beta(\sigma)}, \\ \omega' &\leq -\rho \sum_{i=1}^m q_i + \frac{\rho'_+}{\rho} \omega - \frac{\beta r\Phi'}{\Phi^{\beta+1}(\sigma)} \Phi'(\sigma) \sigma' \rho. \end{aligned} \quad (2.26)$$

By utilizing the non-increasing property of $(r\Phi')$, we can establish $(r(\sigma)\Phi'(\sigma)) \geq (r\Phi')$. Consequently, equation (2.26) assumes the following form

$$\omega' \leq -\rho \sum_{i=1}^m q_i + \frac{\rho'_+}{\rho} \omega - \frac{\beta \sigma'}{\rho r(\sigma)} \omega^2 \Phi^{\beta-1}(\sigma). \quad (2.27)$$

Since Φ is increasing and $\beta > 1$, there exists a constant $M > 0$ such that $\Phi^{\beta-1} \geq M^{\beta-1}$ for all $v \geq v_2 \geq v_1$. Consequently, (2.27) can be expressed as follows

$$\begin{aligned} \omega' &\leq -\rho \sum_{i=1}^m q_i + \frac{\rho'_+}{\rho} \omega - \frac{\beta M^{\beta-1} \sigma'}{\rho r(\sigma)} \omega^2, \\ &\leq -\rho \sum_{i=1}^m q_i + \frac{(\rho'_+)^2 r(\sigma)}{4\beta M^{\beta-1} \rho \sigma'}, \end{aligned} \quad (2.28)$$

integrating (2.28) from v_2 to v , we get

$$\omega(v) \leq \omega(v_2) - \int_{v_2}^v \left(\rho(\zeta) \sum_{i=1}^m q_i(\zeta) + \frac{(\rho'_+(\zeta))^2 r(\sigma(\zeta))}{4\beta M^{\beta-1} \rho(\zeta) \sigma'(\zeta)} \right) d\zeta, \quad (2.29)$$

in view of (2.24) and (2.25) inequality (2.29) can be formulated as

$$\rho(v)Q(v) + \int_{v_2}^v \left(\rho(\zeta) \sum_{i=1}^m q_i(\zeta) - \frac{(\rho'_+(\zeta))^2 r(\sigma(\zeta))}{4\beta M^{\beta-1} \rho(\zeta) \sigma'(\zeta)} \right) d\zeta \leq \omega(v_2),$$

which contradicts (2.23). If Φ satisfies Lemma 2.1 (II), then $\lim_{v \rightarrow \infty} \psi(v) = 0$ follows from Lemma 2.3. This concludes the proof. $\square$

Corollary 2.1. Suppose that $\alpha = \beta$, $\sigma_i(v) < \tau(v)$ for all $i = 1, 2, \dots, m$, and condition (2.5) hold. If condition (2.10) holds, then (1.1) is oscillatory.

Corollary 2.2. Suppose that $\beta = 1$, and condition (2.5) hold. If condition (2.22) holds, then (1.1) is oscillatory.

Corollary 2.3. Assume that $\beta > 1$, and condition (2.5) hold. If condition (2.23) holds, then (1.1) is oscillatory.

Example 2.1. Consider the second order neutral DE

$$(v(\psi(v) - \frac{1}{2}\psi(v/2))')' + v \psi(v/3) = 0, \quad v \geq v_1. \quad (2.30)$$

Here $r(v) = v$, $p(v) = \frac{1}{2}$, $q(v) = v$, $\tau(v) = \frac{v}{2}$, $\sigma(v) = \frac{v}{3}$ for $v \geq v_1$, $\alpha = \beta = 1$ and $R(v) = \ln(v)$. Then it is clear to see that

$$\limsup_{v \rightarrow \infty} \int_{\tau^{-1}(\sigma(v))}^v \frac{1}{r(\varrho)} \left(\int_{v_2}^{\varrho} \sum_{i=1}^m q_i(\zeta) d\zeta \right) d\varrho = \limsup_{v \rightarrow \infty} \int_{2v/3}^v \frac{1}{\varrho} \left(\int_{v_2}^{\varrho} \zeta d\zeta \right) d\varrho = \infty.$$

Applying condition (2.22), based on Corollary 2.2, (2.30) is oscillatory. The graphical result is showed in Figure 1. The solutions, as demonstrated, show distinct oscillating shapes around the zero axis, suggesting that they neither divergence nor approach a steady state. As the independent variable rises, the oscillations stay limited and their amplitude eventually decreases or stabilizes, indicating asymptotic stability. These numerical results confirm that oscillations result from the presence of numerous delays and a negative nonlinear neutral variable, which is fully consistent with the analytical conditions defined in Theorems and corollaries. Overall, the graphs show that the developed oscillation criteria are reliable and useful for describing the asymptotic and oscillatory nature of the system's solutions.

Example 2.2. Consider the second order neutral DE

$$((\psi(v) - \frac{1}{4}\psi^{1/3}(v/2))')^{1/5}(v/3) + \mu^{1/5}v\psi(v/4) = 0, \quad v \geq v_1, \quad (2.31)$$

where λ and μ are positive constants.

Here $r(v) = 1$, $p(v) = \frac{1}{4}$, $q_1(v) = \lambda v$, $q_2(v) = \mu v$, $\tau(v) = \frac{v}{2}$, $\sigma_1(v) = \frac{v}{3}$, $\sigma_2(v) = \frac{v}{4}$ for $v \geq v_1$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$ and $R(v) = v - 1$. Clearly, all conditions of Theorem 2.1 are achieved, hence

$$\int_{v_1}^{\infty} \sum_{i=1}^m q_i(\zeta) R^{\beta}(\sigma_i(\zeta)) d\zeta = \int_{v_1}^{\infty} [\lambda\zeta(\zeta/3 - 1)^{1/5} + \mu\zeta(\zeta/4 - 1)^{1/5}] d\zeta = \infty,$$

and

$$\limsup_{v \rightarrow \infty} \left(\int_{2v/3}^v \lambda\zeta(2\zeta/3 - 1)^{3/5} d\zeta + \int_{3v/5}^v \lambda\zeta(\zeta/2 - 1)^{3/5} d\zeta \right) = \infty,$$

for $\lambda, \mu > 0$. It is clear that (2.31) is oscillatory.

3. Conclusions

In this work, in the canonical case, we explored the oscillatory behavior of a second-order DE with a negative sublinear neutral term. We obtained some new oscillation properties for (1.1) in the canonical case. It is evident from these results that all solutions to the equation studied oscillate. In addition, some of our results extend and complement those published previously, see [17, 26, 31, 45–47]. A growing number of researchers have been interested in the oscillatory behavior of higher order DEs in recent years (see, for example, [7, 15, 35]). Hence, we are interested in extending our results to higher-order equations of the form (1.1).

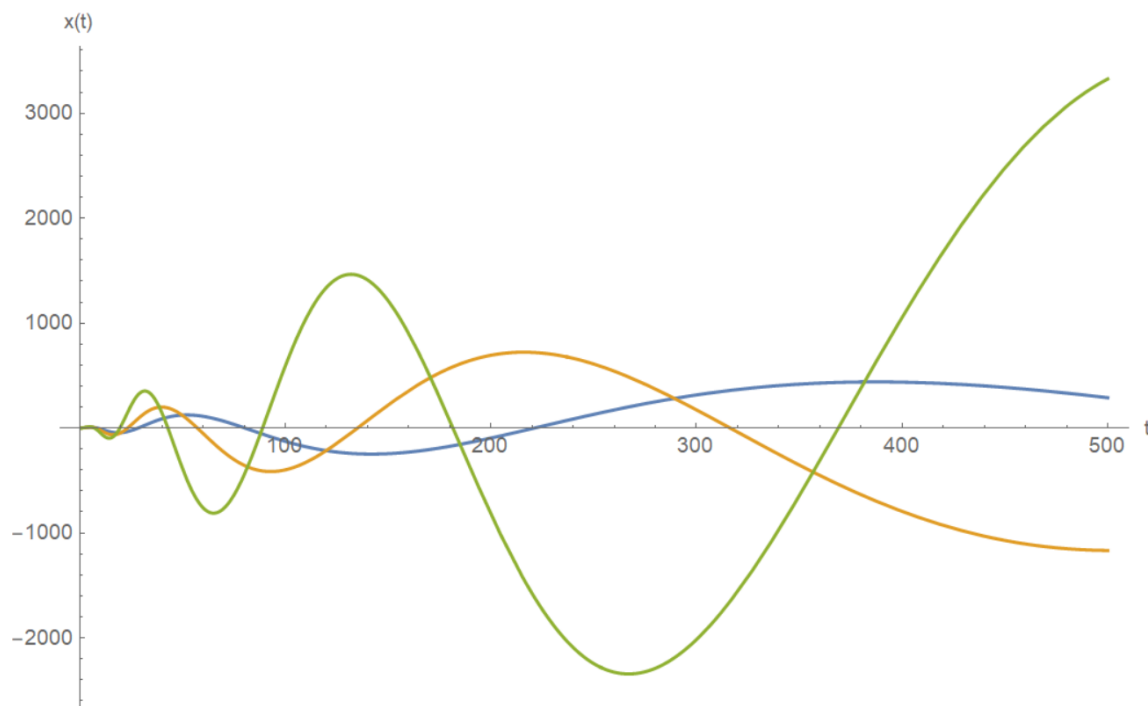


Figure 1. Some numerical solutions to equation (1.1).

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