

STRANGE NONCHAOTIC ATTRACTORS AND COMPLICATED QUASI-PERIODIC CASCADES INDUCED BY BIFURCATIONS IN A DRY-FRICTION OSCILLATOR UNDER QUASI-PERIODIC EXCITATION

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Abstract This paper explores the crossing bifurcation, switching-sliding and grazing-sliding bifurcations in a dry-friction oscillator on a moving belt under quasi-periodic excitation with two frequencies. We present a concrete method to approximate the largest Lyapunov exponent. By applying the method of some numerical experiments, we find that as bifurcation parameter varies, the system shows quite rich dynamical behaviors, including: (1) For crossing and switching-sliding bifurcations, there exists an alternating sequence of chaotic attractors, quasi-periodic toruses and strange nonchaotic attractors (SNAs), in which SNA exists between chaotic attractor and quasi-periodic torus via an evolution route of $\text{torus} \rightarrow \text{SNA} \rightarrow \text{chaos}$. (2) Grazing-sliding bifurcation triggers a direct transition from a pre-grazing three-torus quasi-periodic orbit to an SNA. Furthermore, we also observe torus-adding cascades and *weak torus-adding cascades* in quasi-periodic attractors. The unique properties of the observed SNAs are comprehensively validated by the largest Lyapunov exponent, rational approximation, phase sensitivity property, singular continuous spectrum analysis and separation of nearby trajectories.

Keywords Dry-friction oscillator, quasi-periodic excitation, torus-adding cascades, strange nonchaotic attractor, Lyapunov exponent.

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1. Introduction

As a typical and influential class of nonsmooth dynamical models, piecewise smooth (PWS) dynamical systems have been widely applied in diverse fields, including mechanical transmission and braking systems, power electronic switching circuits, and discontinuous feedback control in control theory [3, 5]. These systems are particularly suitable for describing nonsmooth problems involving collisions, friction and switching, such as the impact behavior in impact oscillators under two-frequency quasi-periodic excitation, frictional effects in dry-friction oscillators and railway wheelset systems and switching dynamics related to sliding bifurcations. Owing to their complex dynamical characteristics and significant research value, discontinuity-induced bifurcations (DIBs) generated by PWS systems, including grazing and sliding bifurcations, chattering,

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have received widespread attention from researchers in recent years.

Chaos, known for their extreme sensitivity to initial conditions, are defined by a positive largest Lyapunov exponent in [10]. A distinct class of attractors was introduced by Grebogi et al. in 1984, termed strange nonchaotic attractors (SNAs) in [12]. SNAs exhibit no sensitivity to initial conditions and are characterized by non-positive Lyapunov exponents. They also possess a fractal geometric structure yet lack chaotic dynamics, placing them in the intermediate dynamical regime between quasiperiodicity and chaos. As can be seen from previous works, the generation mechanisms of SNAs are quite complicated. In [14], Heagy et al. identified the fractalization of quasi-periodic torus as the most prevalent pathway to SNAs. Meanwhile, other classical routes include the Heagy-Hammel route [14], the fractal route [19] and the intermittency route [26]. However, when the scope broadens to more general systems and PWS systems, researchers have discovered new and more complex routes to SNAs. SNAs can not only be co-generated through self-collision of invariant curves and symmetry breaking [27], but also arise from the destruction of regular torus structures via “boundary crisis” [23] and “blowout bifurcation” [34], as well as through unique evolutionary routes such as “formation and merging of bubbles” [31] and “multilayered bubbles” [1]. Due to the lack of analytical methods, SNA identification primarily relies on numerical tools, including Lyapunov exponent [15, 18], rational approximation [10, 24], phase sensitivity property [10, 24, 31], singular continuous spectrum analysis [2, 24, 34], and separation of nearby trajectories [31].

In recent years, many researchers have also investigated the generation mechanisms of SNAs by studying quasi-periodic maps, as well as periodically and quasi-periodically excited systems. Herein, we only present a subset of these mechanisms. Zhang et al. investigated the emergence of SNAs via grazing bifurcation in a quasi-periodically forced PWS map in [37]. Focusing on a different bifurcation type, Zhao and Zhang in [38] investigated the border-collision bifurcation route to SNAs in piecewise linear normal form map. Beyond single-bifurcation routes, Chen et al. in [6] explicitly identified three fundamental routes to SNAs in a quasi-periodically forced Hassell map, such as the torus-doubling route, the fractalization route, and the intermittency route providing a systematic summary of SNA generation in discrete dynamical maps. In [29], Shen et al. investigated the origin and generation mechanisms of hidden strange nonchaotic attractors in a non-smooth map. In addition, researchers have also explored the generation mechanisms of SNAs in periodically excited PWS systems. See, for example, in [35, 36] and the references therein. In [20], Liu et al. verified that the double-grazing bifurcation generates SNAs in a quasi-periodically driven piecewise-linear oscillator, with these SNAs existing in narrow intervals between torus quasi-periodic attractors, or between torus quasi-periodic attractors and chaotic attractor. Shen and Zhang in [28] advanced this understanding by demonstrating that the truncation of border-collision torus-doubling bifurcation in quasi-periodically forced PWS logistic system gives rise to distinct types of SNAs, thereby highlighting the critical role of bifurcation truncation in determining the characteristics of SNAs. Recently, Jiang and Du in [16] analyzed a quasi-periodically excited slender rigid rocking block system and uncovered a novel nonstandard torus-doubling pathway to SNAs, where these attractors may arise either between a quasi-periodic attractor and a chaotic attractor, or between two chaotic attractors. Furthermore, in [17], Jiang and Du also reported SNAs induced by homoclinic bifurcation in a quasi-periodically excited Duffing-type inverted pendulum. In [21], our own prior work explored the dynamics driven by grazing bifurcation in a two-frequency quasi-periodically forced impact oscillator, revealing rich dynamical behaviors including SNAs, chaotic attractors, and complex quasi-periodic cascades as the bifurcation parameter d varies. Dongye and collaborators in [8] carried out research concerning the generation mechanism of SNAs via border-collision

bifurcation in a quasi-periodically driven nonsmooth laser system. Phase space investigations demonstrate that these attractors emerge once the torus loses its transverse stability during the occurrence of border-collision bifurcation.

Dry-friction oscillators are a typical PWS system, which are widely used in engineering practices such as unlubricated gear meshing, brake pads in braking systems, and sliding of sliders on rough surfaces [7, 32, 33]. They are usually modeled as Filippov systems. In existing literature, most works focus on the dynamics of dry-friction oscillators under periodic excitation. As shown in [13], Guo et al. analyzed the Neimark-Sacker bifurcation in a dry-friction oscillator under 1:4 strong resonance condition. Ma and Du in [22] investigated grazing-sliding bifurcation in a moving belt dry-friction oscillator under periodic excitation. Furthermore, the works in [4] showed that sliding bifurcation serves as a critical route for the abrupt transition from regular stick-slip motion to chaos. These works provide important theoretical foundations for understanding dry-friction system dynamics under periodic excitation. Nevertheless, when the excitation form is extended from the periodic case to the more general quasi-periodic one, the existing works on the dry-friction oscillators are obviously insufficient. In particular, the route to SNAs via sliding bifurcations in dry-friction oscillators under quasi-periodic excitation remains to be further explored.

Inspired by the aforementioned studies, we discuss crossing bifurcation, switching-sliding and grazing-sliding bifurcations in a dry-friction oscillator on a moving belt subjected to quasi-periodic excitation with two frequencies, as schematically depicted in Figure 1. By applying the largest Lyapunov exponent and the bifurcation diagrams of system, we unfold the complicated dynamics. The key contributions and novel aspects of our research are outlined as follows.

(1) To our knowledge, our work is the first one to investigate crossing bifurcation, switching-sliding bifurcation and grazing-sliding bifurcation of the dry-friction oscillator under quasi-periodic excitation.

(2) We approximate the largest Lyapunov exponent by extending the periodic case to the quasi-periodic one and present the concrete method.

(3) By some numerical simulations, we find the rich bifurcation scenarios, such as chaos, SNAs and torus. In particular, we observe torus-adding cascades in switching-sliding bifurcation and *weak torus-adding cascades* in crossing bifurcation, demonstrating a clear evolutionary route: Torus $\rightarrow$ SNA $\rightarrow$ chaos. In striking contrast, the grazing-sliding bifurcation directly transforms quasi-periodic torus attractor into SNAs, establishing a unique evolutionary route that completely bypasses the chaotic regime.

The organizational structure of this paper is as follows. In Section 2, the dry-friction oscillator model on the moving belt under quasi-periodic excitation is established, and some relevant preliminaries are given. In Section 3, we present the Poincaré map and give the computation method of the largest Lyapunov exponent. Section 4 presents numerical simulation results that demonstrate the emergence of SNAs, torus-adding and *weak torus-adding cascades*. Section 5 provides a detailed examination of the key properties that characterize these SNAs. Finally, Section 6 concludes the work.

2. The model and preliminaries

We consider a quasi-periodically excited dry-friction oscillator on a moving belt, as illustrated in Figure 1. The oscillator, with mass $M > 0$ and downward pressure N , is connected to a fixed rigid wall via a nonlinear spring with coefficients k_1 and k_2 and a nonlinear damper

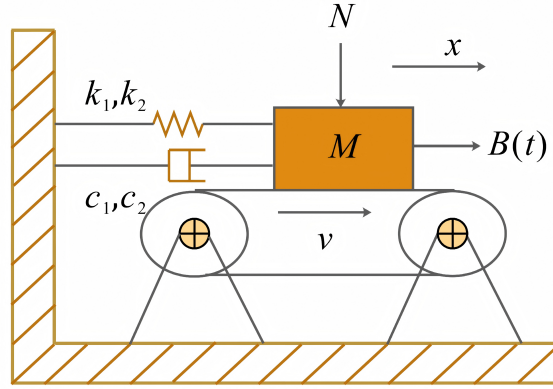


Figure 1. A quasi-periodically excited nonlinear dry-friction oscillator.

with coefficients c_1 and c_2 . The system is driven by an external quasi-periodic force $B(t) = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t$, where F_i and ω_i with $i = 1, 2$ denote the forcing amplitude and frequency, respectively. The frequency ratio ω_1/ω_2 is irrational. Assuming that the belt moves at a constant speed $v > 0$ and the displacement of the oscillator is x , the Coulomb friction force $f_v(\dot{x})$ can be written to the following form

$$f_v(\dot{x}) = \begin{cases} -\mu_k F_N, & \text{if } \dot{x} < v, \\ \in [-\mu_k F_N, \mu_k F_N], & \text{if } \dot{x} = v, \\ \mu_k F_N, & \text{if } \dot{x} > v, \end{cases}$$

where μ_k is the kinetic friction coefficient and $F_N = N + Mg$ is the normal force between the belt and the oscillator. Then the dynamical equation of the oscillator under quasi-periodic excitation is given by

$$M\ddot{x} - c_1\dot{x} + c_2\dot{x}^3 + k_1x + k_2x^3 = -f_v + B(t). \quad (2.1)$$

Let $d_1 = c_1/M$, $d_2 = c_2/M$, $a_1 = k_1/M$, $a_2 = k_2/M$, $\tilde{f}_n = f_v/M$, $e_1 = F_1/M$, $e_2 = F_2/M$, then the system (2.1) can be rewritten to the following form

$$\ddot{x} - d_1\dot{x} + d_2\dot{x}^3 + a_1x + a_2x^3 = -\tilde{f}_n + \beta(t), \quad (2.2)$$

where $\beta(t) = e_1 \cos \omega_1 t + e_2 \cos \omega_2 t$. From the above, it is clear that the non-friction force is

$$F_f = d_1\dot{x} - d_2\dot{x}^3 - a_1x - a_2x^3 + \beta(t).$$

Let $f_n = \mu_k F_N/M$ be the maximum static frictional force. If $|F_f| < f_n$, the oscillator has no relative motion on the belt, which corresponds to a stick motion. If $|F_f| \geq f_n$, the oscillator undergoes non-sticking motion. In this case, the direction of the frictional force f_v acting on the oscillator is determined by the sign of the velocity difference $\dot{x} - v$ between the oscillator and the moving belt. Therefore, the sign function can be employed to represent the frictional force in equation (2.2), and the model can be written as

$$\ddot{x} - d_1\dot{x} + d_2\dot{x}^3 + a_1x + a_2x^3 = f_n \text{sign}(v - \dot{x}) + \beta(t),$$

where

$$\text{sign}(v - \dot{x}) \begin{cases} = 1, & \text{if } v - \dot{x} > 0, \\ \in [-1, 1], & \text{if } v - \dot{x} = 0, \\ = -1, & \text{if } v - \dot{x} < 0. \end{cases}$$

Let $\theta = \omega_1 t$, $\phi = \omega_2 t$ and $\dot{x} = y$. Then system (2.2) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -a_1 x - a_2 x^3 + d_1 y - d_2 y^3 + f_n \text{sign}(v - y) + e_1 \cos \theta + e_2 \cos \phi, \\ \dot{\theta} = \omega_1, \\ \dot{\phi} = \omega_2. \end{cases} \quad (2.3)$$

Let $\mu = (a_1, a_2, d_1, d_2, f_n, v, e_1, e_2, \omega_1, \omega_2)^T \in \mathbb{R}^{10}$ be the parameter variables and $\mathbf{x} = (x, y, \theta, \phi)^T \in \mathbb{R}^4$ be the state variables. Define the switching surface of system (2.3) as

$$\Sigma = \{\mathbf{x} = (x, y, \theta, \phi)^T \in \mathbb{R}^4 : H(\mathbf{x}, v) = v - y = 0\},$$

where $H(\mathbf{x}, v) = v - y$ is a smooth scalar function defined by the system states $\mathbf{x} \in \mathbb{R}^4$ and parameter $\mu \in \mathbb{R}^{10}$. Then it is easy to obtain $\nabla H(\mathbf{x}, v) = (0, -1, 0, 0)^T$, which is a gradient vector normal to Σ . The region $\mathbb{R}^4 \setminus \Sigma$ can be split into two subspaces Σ_1 and Σ_2 :

$$\begin{aligned} \Sigma_1 &= \{\mathbf{x} \in \mathbb{R}^4 : H(\mathbf{x}, v) = v - y > 0\}, \\ \Sigma_2 &= \{\mathbf{x} \in \mathbb{R}^4 : H(\mathbf{x}, v) = v - y < 0\}. \end{aligned}$$

Define

$$F_1(\mathbf{x}, \mu) = \begin{pmatrix} y \\ \varphi_+(\mathbf{x}, \mu) \\ \omega_1 \\ \omega_2 \end{pmatrix}, \quad F_2(\mathbf{x}, \mu) = \begin{pmatrix} y \\ \varphi_-(\mathbf{x}, \mu) \\ \omega_1 \\ \omega_2 \end{pmatrix},$$

where $\varphi_{\pm}(\mathbf{x}, \mu) = -a_1 x - a_2 x^3 + d_1 y - d_2 y^3 \pm f_n + e_1 \cos \theta + e_2 \cos \phi$. Then system (2.3) can be written as

$$\dot{\mathbf{x}} = \begin{cases} F_1(\mathbf{x}, \mu), & \text{if } H(\mathbf{x}, v) > 0, \\ F_2(\mathbf{x}, \mu), & \text{if } H(\mathbf{x}, v) < 0, \end{cases} \quad (2.4)$$

where F_i is a sufficiently smooth vector function on Σ_i for $i = 1, 2$.

If there exists a subset of the switching manifold $\bar{\Sigma} \subset \Sigma$ such that an orbit of system (2.4) hitting $\bar{\Sigma}$ evolves along $\bar{\Sigma}$ until reaching a certain point at which F_1 or F_2 begins to change the vector fields direction, then the $\bar{\Sigma}$ is called as the sliding region. Obviously, the orbit on $\bar{\Sigma}$ satisfies the condition:

$$\langle \nabla H, F_2 \rangle - \langle \nabla H, F_1 \rangle = 2f_n > 0.$$

Based on the above analysis, the orbital motion of system (2.4) that evolves within $\bar{\Sigma}$ until it reaches the boundary of $\bar{\Sigma}$ is called sliding motion. Under Utkin's equivalent control method, the vector field within the sliding region can be defined as a convex combination of F_1 and F_2 as follows

$$F_s(\mathbf{x}, \mu) = \frac{F_2(\mathbf{x}, \mu) - F_1(\mathbf{x}, \mu)}{2} \gamma(\mathbf{x}, \mu) + \frac{F_1(\mathbf{x}, \mu) + F_2(\mathbf{x}, \mu)}{2},$$

where $\gamma(\mathbf{x}, \mu)$ is a control function with a range of $[-1, 1]$ and is obtained by the equation $\langle \nabla H, F_s \rangle = 0$. Applying this condition, we can derive

$$\gamma(\mathbf{x}, \mu) = -\frac{\langle \nabla H, F_1 \rangle + \langle \nabla H, F_2 \rangle}{\langle \nabla H, F_2 \rangle - \langle \nabla H, F_1 \rangle}.$$

We can define the sliding region $\bar{\Sigma}$ as

$$\bar{\Sigma} = \{\mathbf{x} \in \Sigma : |\gamma(\mathbf{x}, \mu)| \leq 1\},$$

and the boundary as

$$\partial \bar{\Sigma} = \partial \bar{\Sigma}^+ \cup \partial \bar{\Sigma}^-,$$

where $\partial \bar{\Sigma}^\pm = \{\mathbf{x} \in \Sigma : \gamma(\mathbf{x}, \mu) = \pm 1\}$. In addition, by direct computation, it is easy to obtain that $F_s(\mathbf{x}, \mu) = (y, 0, \omega_1, \omega_2)^T$.

For dry-friction oscillators, the interaction between trajectories and the boundary of the sliding region gives rise to different motion phenomena. When the relative velocity $\dot{x} - v$ of the mass is zero, the trajectory will cross the switching surface and enter another vector field. On the other hand, when the relative velocity $\dot{x} - v$ of the mass is non-zero, the mass will slide on the moving belt and be in the sliding state. As illustrated in Figure 2, sliding bifurcation can be categorized into four types: Crossing-sliding, switching-sliding, grazing-sliding, and adding-sliding. As the parameter varies, the trajectory exhibits an evolutionary sequence denoted as $a \rightarrow b \rightarrow c$; it should be noted that trajectories a, b and c are independent and coexist simultaneously, and do not represent successive states of the same trajectory progressing from a to b to c . In [3], di Bernardo et al. described four distinct sliding modes in periodically forced Filippov systems. In this paper, we extend periodically forced Filippov systems to the quasi-periodically forced scenario and derive the corresponding conditions for crossing, grazing-sliding and switching-sliding.

Crossing. In a Filippov system, a non-sliding crossing motion occurs when a trajectory intersects the switching surface Σ without being confined to slide along it; instead, it transitions directly from the region governed by vector field F_1 to that governed by F_2 . The critical condition for such a crossing is

$$H_{\mathbf{x}} F_1 H_{\mathbf{x}} F_2 > 0.$$

This inequality ensures that the normal components of F_1 and F_2 relative to the switching surface share the same sign-preventing the trajectory from being trapped in sliding motion and enabling a direct transversal crossing. Similarly, the crossing condition for the transition from vector field F_2 to vector field F_1 in a Filippov system is given by

$$H_{\mathbf{x}} F_2 H_{\mathbf{x}} F_1 > 0.$$

Grazing-sliding. Quasi-periodic grazing refers to the instantaneous tangential contact between a quasi-periodic trajectory and the switching surface Σ at a specific grazing point,

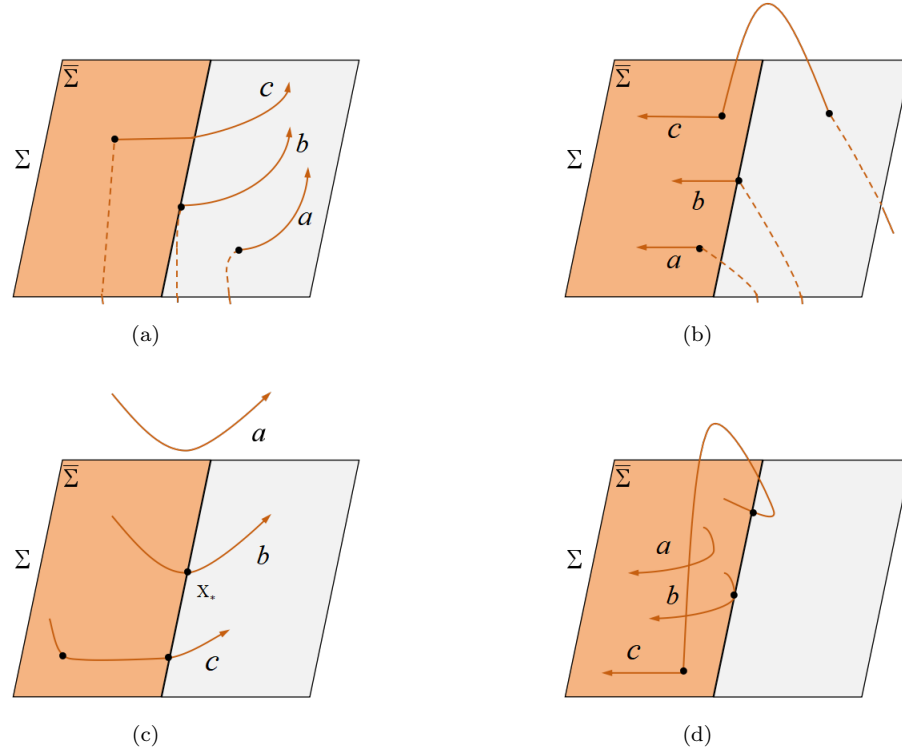


Figure 2. Four cases of sliding bifurcation: (a): Crossing-sliding. (b): Switching-sliding. (c): Grazing-sliding. (d): Adding-sliding. In all cases, Σ is the switching manifold, with F_1 assumed to lie below it and F_2 above it, while $\bar{\Sigma}$ represents the sliding region. The generality of the analysis is preserved even if F_1 and F_2 are interchanged.

see the orbit b shown in Figure 2(c). In Figure 2, as system bifurcation parameter varies, the system undergoes a grazing-sliding bifurcation, where the trajectory may exhibit distinct qualitative behaviors: (1) The orbit in Σ_1 does not intersect Σ as represented by a ; (2) the orbit in Σ_1 arrives at the Σ and has a Short sliding segment as represented by c ; (3) the orbit exhibits a composite dynamical behavior that combines the characteristics of orbits (a) and (c), where segments of quasi-periodic motion away from the switching surface Σ alternate with sliding segments on the surface. The conditions for quasi-periodic grazing-sliding are as follows:

- (1) $H(x_*, v_*) = 0$,
- (2) $\nabla H(x_*, v_*) \neq 0$,
- (3) $\langle \nabla H(x_*, v_*), F_1(x_*, \mu_*) \rangle = 0$,
- (4) $\langle \nabla H(x_*, v_*), (F_1)_x F_1(x_*, \mu_*) \rangle > 0$,

where (x_*, v_*) is grazing point.

Switching-sliding. Now, we describe the switching-sliding bifurcation phenomenon illustrated in Figure 2(b). Trajectory b is a quasi-periodic attractor, and the intersection point between its corresponding torus and the switching surface Σ lies exactly on the sliding boundary $\partial\bar{\Sigma}^-$. When the parameter of system varies, this quasi-periodic trajectory may exhibit two distinct qualitative dynamical behaviors. One is that the quasi-periodic orbit a may enter the sliding region and evolve under the sliding vector field F_s , thereby ensuring the preservation

of quasi-periodicity; the other scenario is that, under the repulsive effect of the sliding region boundary, the quasi-periodic trajectory b undergoes a qualitative change: It first transversely intersects Σ and then enters the sliding region, corresponding to trajectory c . The switching-sliding conditions are

- (1) $H(\mathbf{x}_*, v_*) = 0$,
- (2) $\nabla H(\mathbf{x}_*, v_*) \neq 0$,
- (3) $\langle \nabla H(\mathbf{x}_*, v_*), F_1(\mathbf{x}_*, \mu_*) \rangle = 0$,
- (4) $\langle \nabla H(\mathbf{x}_*, v_*), (F_1)_x F_1(\mathbf{x}_*, \mu_*) \rangle < 0$,

where $(\mathbf{x}_*, v_*)$ is the intersection point between the torus determined by the quasi-periodic orbit and $\partial \bar{\Sigma}^-$.

3. The Poincaré map and Lyapunov exponents

At present, there are two types of Poincaré map methods for studying the dynamics of system (2.4): One is the normal form Poincaré method, and the other is the stroboscopic Poincaré method. In quasi-periodic systems, when the motion trajectories of the system are relatively complex, the normal form Poincaré map is insufficient to simulate the bifurcation phenomena of the system (2.4). Therefore, in this paper we adopt the stroboscopic Poincaré map method to study the dynamical behaviours of system (2.4).

Here, we take ϕ as the angle variable and choose the corresponding Poincaré section defined by $\Sigma_s = \{(x, y, \theta, \phi)^T \in \mathbb{R} \times \mathbb{R} \times S^1 \times S^1 : \phi = 0 \pmod{2\pi}\}$. Element in Σ_s are defined by $(x, y, \theta)^T \in \mathbb{R}^2 \times S^1$. Thus, the stroboscopic Poincaré map $P_s : \Sigma_s \mapsto \Sigma_s$ can be defined as $(x_{n+1}, y_{n+1}, \theta_{n+1})^T = P_s(x_n, y_n, \theta_n)$ for any $(x_n, y_n, \theta_n)^T \in \Sigma_s$, where

$$\begin{cases} x_{n+1} = f_1(x_n, y_n, \theta_n), \\ y_{n+1} = f_2(x_n, y_n, \theta_n), \\ \theta_{n+1} = \theta_n + \frac{2\pi\omega_1}{\omega_2} \pmod{2\pi}, \end{cases} \quad (3.1)$$

where f_1 and f_2 are determined by the flows of system (2.4).

Quasi-periodically forced systems are often accompanied by complex dynamical behaviors. To differentiate chaotic attractors from quasi-periodic attractors and SNAs, we can calculate the largest Lyapunov exponent by using the previously defined Poincaré map (3.1) and check if the corresponding largest Lyapunov exponent is positive, or use an effective 0–1 test method based on time-series data proposed by Gottwald and Melbourne in [11]. In this paper, we use the extended largest Lyapunov exponent method described in [18] to distinguish chaotic attractors and others. Below we present the specific method to calculate the largest Lyapunov exponent of quasi-periodically forced system (2.4).

Let $\mathbf{x}^0 = (1, 0, 0)^T \in \Sigma_s$ be the initial value of system (2.4). We select the orthonormal set $(u_1^0, u_2^0, u_3^0) = (1, 0, 0; 0, 1, 0; 0, 0, 1)$ as the initial vector for the Jacobian matrix DP_s of the Poincaré map P_s , then we can obtain the perturbation vector $(\delta \mathbf{x}_1^0, \delta \mathbf{x}_2^0, \delta \mathbf{x}_3^0)$ as follows

$$\delta \mathbf{x}_i^0 = DP_s(\mathbf{x}^0)u_i^0$$

for $i = 1, 2, 3$. It is important to note that, due to the inherent complexity of the system, within each running time of $2\pi/\omega_2$, the discontinuity arising when the system's trajectory reaches the switching surface Σ renders DP_s non-differentiable. Therefore, when calculating DP_s , we need to compute the system's saltation matrix $Q_{\mathbf{x}}$ at the discontinuity points as the system reaches the switching surface Σ . Let $Q_{\mathbf{x}}^1, Q_{\mathbf{x}}^2, Q_{\mathbf{x}}^3, Q_{\mathbf{x}}^4, Q_{\mathbf{x}}^5$ be the saltation matrices of $F_1 \mapsto F_s, F_s \mapsto F_1, F_1 \mapsto F_2, F_2 \mapsto F_1, F_2 \mapsto F_s$. From [3], we can obtain the expressions of $Q_{\mathbf{x}}^1, Q_{\mathbf{x}}^2, Q_{\mathbf{x}}^3, Q_{\mathbf{x}}^4, Q_{\mathbf{x}}^5$ and have the results as follows:

$$\begin{aligned} Q_{\mathbf{x}}^1 &= I + \frac{(F_s - F_1)\nabla H(\mathbf{x}, v)}{\nabla H(\mathbf{x}, v)F_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ Q_{\mathbf{x}}^2 &= I + \frac{(F_1 - F_s)\nabla H(\mathbf{x}, v)}{\nabla H(\mathbf{x}, v)F_s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1 - \varphi_+(\mathbf{x}, \mu)}{\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ Q_{\mathbf{x}}^3 &= I + \frac{(F_2 - F_1)\nabla H(\mathbf{x}, v)}{\nabla H(\mathbf{x}, v)F_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varphi_-(\mathbf{x}, \mu)}{\varphi_+(\mathbf{x}, \mu)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ Q_{\mathbf{x}}^4 &= I + \frac{(F_1 - F_2)\nabla H(\mathbf{x}, v)}{\nabla H(\mathbf{x}, v)F_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varphi_+(\mathbf{x}, \mu)}{\varphi_-(\mathbf{x}, \mu)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ Q_{\mathbf{x}}^5 &= I + \frac{(F_s - F_2)\nabla H(\mathbf{x}, v)}{\nabla H(\mathbf{x}, v)F_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where I is a three-dimensional identity matrix. It is worth noting that $\nabla H(\mathbf{x}, v)F_1 = 0$, so a sufficiently small constant ϵ is adopted to substitute for $\nabla H(\mathbf{x}, v)F_1$ with $\epsilon = \nabla H(\mathbf{x}, v)F_1$ in the numerical simulations. By performing the Gram-Schmidt orthogonalization method on vector $(\delta u_1^1, \delta u_2^1, \delta u_3^1)$, (u_1^1, u_2^1, u_3^1) is obtained and further serves as the initial value for the next iteration of the Jacobian matrix DP_s . By extending this process, we can further obtain (u_1^n, u_2^n, u_3^n) , where

$$\begin{aligned} \nu_1^n &= \delta \mathbf{x}_1^n, u_1^{(n)} = \nu_1^n / \|\nu_1^n\|, \\ \nu_2^n &= \delta \mathbf{x}_2^n - \langle \delta \mathbf{x}_2^n, u_1^{(n)} \rangle u_1^{(n)}, u_2^{(n)} = \nu_2^n / \|\nu_2^n\|, \\ \nu_3^n &= \delta \mathbf{x}_3^n - \langle \delta \mathbf{x}_3^n, u_1^{(n)} \rangle u_1^{(n)} - \langle \delta \mathbf{x}_3^n, u_2^{(n)} \rangle u_2^{(n)}, \\ u_3^{(n)} &= \nu_3^n / \|\nu_3^n\|, \end{aligned}$$

where $\|\nu_i^n\|$ is the norm of ν_i^n and $\langle \delta \mathbf{x}_i^n, u_j^{(n)} \rangle$ is the standard scalar product for $i, j = 1, 2, 3$.

Then the Lyapunov exponent spectrum of system (2.4) is defined approximately as

$$\lambda_i \approx \frac{1}{n} \sum_{j=1}^n \ln \left\| \nu_i^{(j)} \right\|$$

for $i = 1, 2, 3$, where n is a sufficiently large integer. $\lambda_{\max} := \max\{\lambda_1, \lambda_2, \lambda_3\}$ is the largest Lyapunov exponent. It is worth noting that $\lambda_{\max} > 0 (< 0)$ of system (2.4) represents a chaotic attractor (a quasi-periodic attractor or an SNA).

4. Complicated dynamics

In this section, we investigate the complex dynamical behaviors generated by the crossing bifurcation, grazing-sliding bifurcation and switching-sliding bifurcation of system (2.4) via applying the Poincaré map P_s . Since system (2.4) is nonlinear and lacks available analytical expressions for its solutions, thus we can not obtain the explicit expression of map P_s . In the following, we perform a numerical approximation of P_s . In this paper, the Matlab adaptive ode45 solver, which is based on the fourth-fifth order Runge-Kutta-Fehlberg method, is employed to approximate the stroboscopic map $P_s : \Sigma_s \mapsto \Sigma_s$, in which the relative and absolute errors for solving system (2.4) are both set to 10^{-10} . Additionally, an event-driven algorithm is implemented to detect non-smooth events occurring at the switching line $y = v$ and the sliding boundary $\partial\bar{\Sigma}^\pm$.

4.1. The transition to SNAs

To capture the system's dynamical characteristics as accurately as possible, the largest Lyapunov exponent $\lambda_{\max}$ is computed for each bifurcation parameter by iterating the Poincaré map P_s 3,000 times. For the bifurcation diagrams, the Poincaré map P_s is iterated 1,000 times, with the first 200 iterations discarded to eliminate transient effects. Notably, in the (θ, x) or (ϕ, x) space, a one-torus ($1T$) quasi-periodic attractor corresponds to a single curve, and a two-torus ($2T$) quasi-periodic attractor corresponds to two distinct curves when the attractor exhibits quasi-periodic behavior. By extension, an m -torus (mT) quasi-periodic attractor is also referred to as a torus of multiplicity m .

Example 4.1. Take parameters $\mathcal{P}_1 := (a_1, a_2, d_1, d_2, v, e_1, e_2, \omega_1, \omega_2) = (3.6, 0.1, 2.2, 0.1, 1, 0.1, 2, 6, \sqrt{10})$ and $f_n \in [3, 4.2]$ as the bifurcation parameter. For this set of parameters, through some numerical calculations, we find that as f_n varies from 3 to 4.2 that, there exists a trajectory of system (2.4) on Σ_2 first crosses the switching surface Σ and enters Σ_1 , then enters the sliding region $\bar{\Sigma}$. Here, we give an example $f_n = 3$ for $\mathcal{P}_1$ to show the switching-sliding phenomenon. The results are shown in Figure 4. Thus, this belongs to a part of the switching-sliding bifurcation. By applying the numerical simulation method, we obtain the one-dimensional bifurcation diagram of map $P_s : \Sigma_s \mapsto \Sigma_s$ and the largest Lyapunov exponent $\lambda_{\max}$ of system (2.4). The results are shown in Figure 3(a) and 3(b) respectively. From Figure 3(b), it is easy to see that there exist some alternating windows of $\lambda_{\max} > 0$ and $\lambda_{\max} < 0$, respectively. The windows for $\lambda_{\max} > 0$ indicate that the system exhibits chaotic motion, while the windows for $\lambda_{\max} < 0$ indicate the existence of SNA or quasi-periodic motion.

In order to investigate the torus-adding cascades in section 4.2, in Figure 5, Figure 6, Figure 7, Figure 9, Figure 11 and Figure 27, we set a red zero line to distinguish the tours for $y > 0$ and $y < 0$.

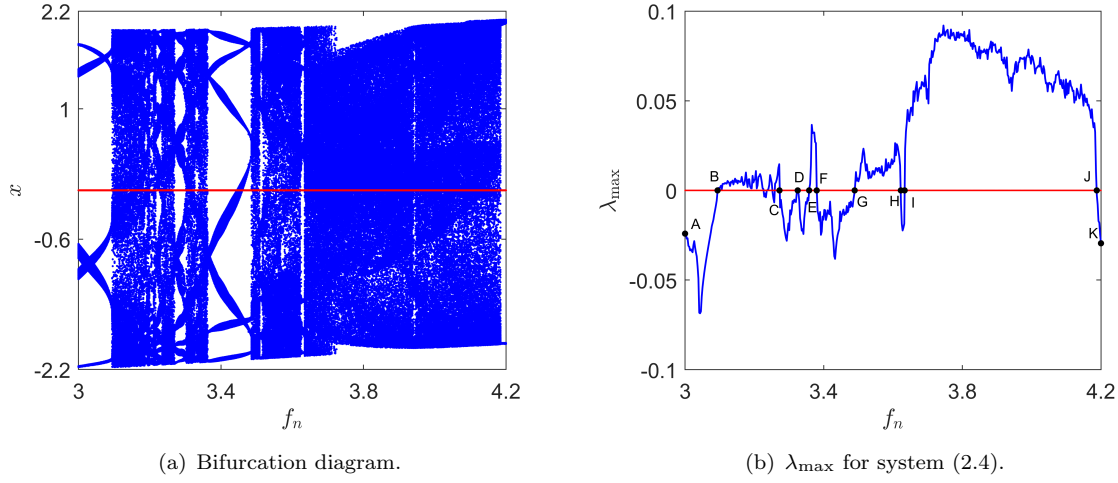


Figure 3. Bifurcation diagram in the (f_n, x) space and the corresponding largest Lyapunov exponent $\lambda_{\max}$ of system (2.4) with $a_1 = 3.6, a_2 = 0.1, d_1 = 2.2, d_2 = 0.1, v = 1, e_1 = 0.1, e_2 = 2, \omega_1 = 6, \omega_2 = \sqrt{10}$ and $f_n \in [3, 4.2]$.

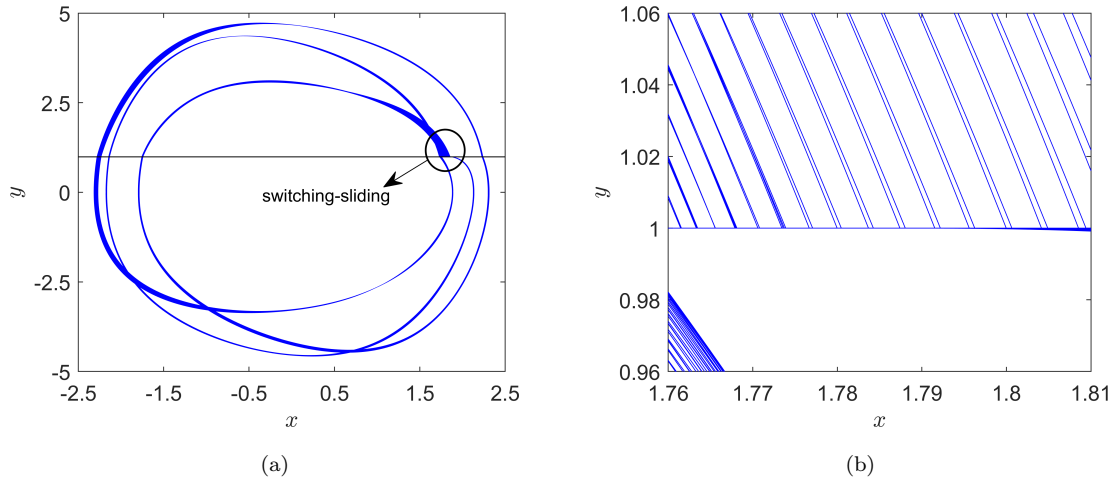


Figure 4. The phase diagram of system (2.4) with $a_1 = 3.6, a_2 = 0.1, d_1 = 2.2, d_2 = 0.1, v = 1, e_1 = 0.1, e_2 = 2, \omega_1 = 6, \omega_2 = \sqrt{10}$ and $f_n = 3$. **(a)** The phase diagram for $(x, y) \in (-2.5, 2.5) \times (-5, 5)$. **(b)** Enlargement of Figure 4(a).

For $f_n \in [3, 4.2]$, we label the different bifurcation parameter values f_n as $A, B, C, \dots, K$, where $A = 3.0000, B = 3.0956, C = 3.2724, D = 3.3216, E = 3.3559, F = 3.3798, G = 3.4911, H = 3.6235, I = 3.6351, J = 4.1878, K = 4.2000$. Here, we present the phase diagrams on the Poincaré section Σ_s of system (2.4) in the (θ, x) -plane and (x, y) -plane of system (2.4) for different f_n with $f_n \in [3, 4.2]$.

In Figure 3(b), we take $f_n = 3.081 \in (A, B)$ and find that the largest Lyapunov exponent satisfies $\lambda_{\max} < 0$, confirming the nonchaotic nature of the attractor. As seen in Figure 5(a), five distinct frequency curves appear in the (θ, x) -plane, which indicates that system (2.4) is a five-torus ($5T$) quasi-periodic attractor, as further illustrated by the corresponding (x, y) -plane in Figure 5(b).

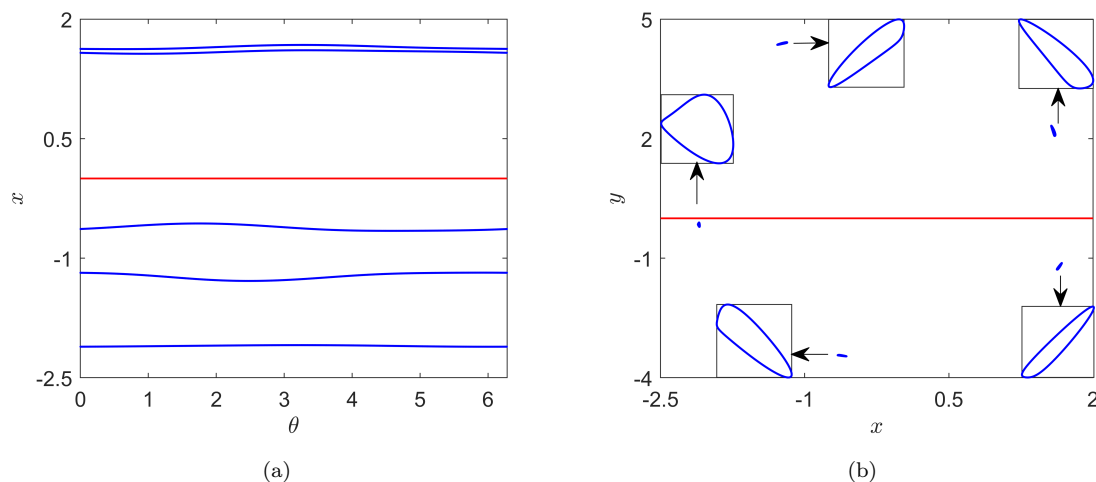


Figure 5. The five-torus ($5T$) quasi-periodic attractor at $f_n = 3.081$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

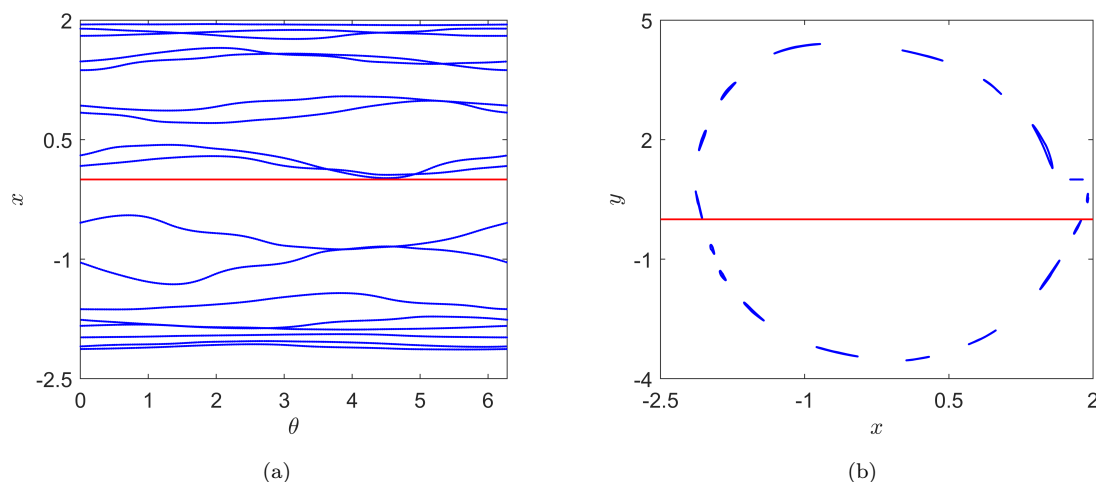


Figure 6. The seventeen-torus ($17T$) quasi-periodic attractor at $f_n = 3.2264$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

As f_n increases to the interval (B, C) , Figure 3(b) reveals both $\lambda_{\max} > 0$ and $\lambda_{\max} < 0$. Numerical simulations confirm that $\lambda_{\max} < 0$ corresponds to a seventeen-torus ($17T$) quasi-periodic attractor. For $f_n = 3.2264 \in (B, C)$, the phase diagrams in the (θ, x) and (x, y) -plane are presented in Figure 6, which verifies the existence of a seventeen-torus ($17T$) quasi-periodic attractor.

As f_n enters the interval (C, D) , the corresponding $\lambda_{\max} < 0$ remains negative, as shown in Figure 3(b). For $f_n = 3.27956 \in (C, D)$, Figure 7(a) displays twelve distinct frequency curves in the (θ, x) -plane, confirming that the system exhibits a twelve-torus ($12T$) quasi-periodic attractor in the (x, y) -plane, see Figure 7(b).

As f_n increases to $3.333 \in (D, E)$, it can be seen that from Figure 3(b) that $\lambda_{\max}$ converges

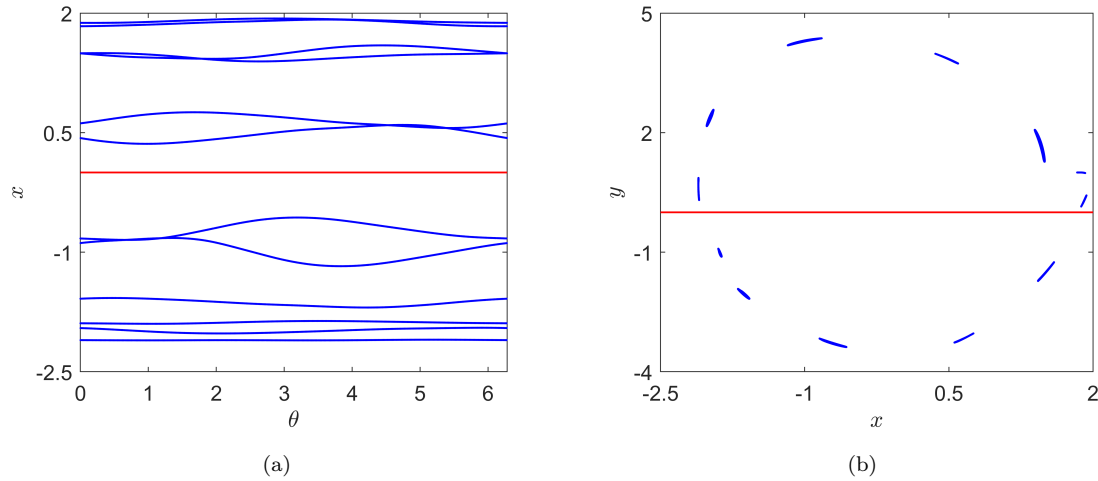


Figure 7. The twelve-torus ($12T$) quasi-periodic attractor at $f_n = 3.27956$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

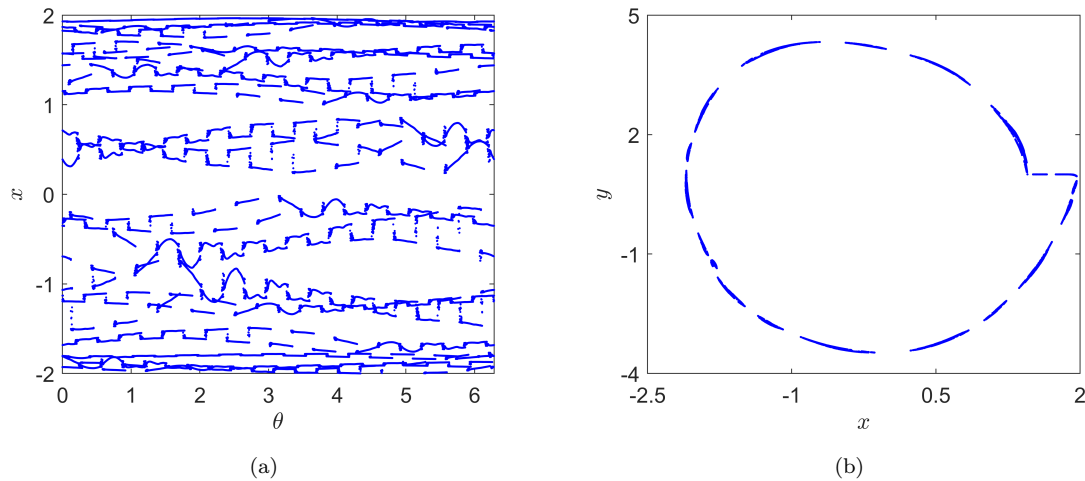


Figure 8. SNA attractor at $f_n = 3.333$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

to -0.09336456 . It implies the system (2.4) is nonchaotic. Figure 8(a) reveals the frequency curves of system (2.4) in the (θ, x) -plane become quite wrinkled and discontinuous. As shown in Figure 8(b), the attractor loses smooth completely and obtains a distinct fractal structure. In the next section, the attractor is confirmed to be an SNA.

As f_n increases into the interval (F, G) , Figure 3(b) shows that $\lambda_{\max} < 0$. At $f_n = 3.421 \in (F, G)$, Figure 9(a) displays seven frequency curves in the (θ, x) -plane, which indicates that the system (2.4) is a seven-torus ($7T$) quasi-periodic attractor, as further illustrated by the corresponding (x, y) -plane, see Figure 9(b).

For $f_n \in \mathcal{EJ} := (E, F) \cup (G, H) \cup (I, J)$, Figure 3(b) reveals that $\lambda_{\max} > 0$, implying the presence of chaotic attractors in these parameter ranges. For instance, at $f_n = 3.8507 \in \mathcal{EJ}$,

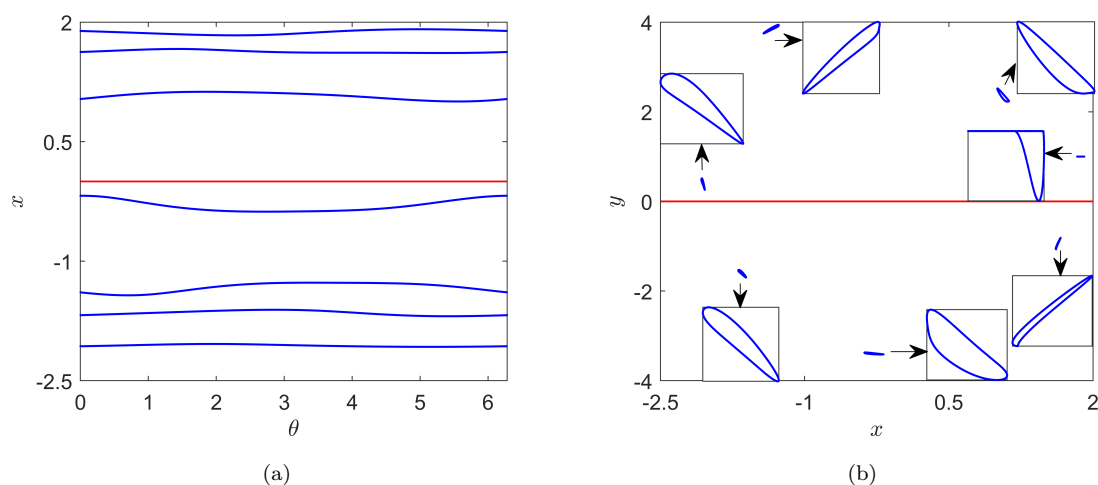


Figure 9. The seven-torus ($7T$) quasi-periodic attractor at $f_n = 3.421$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

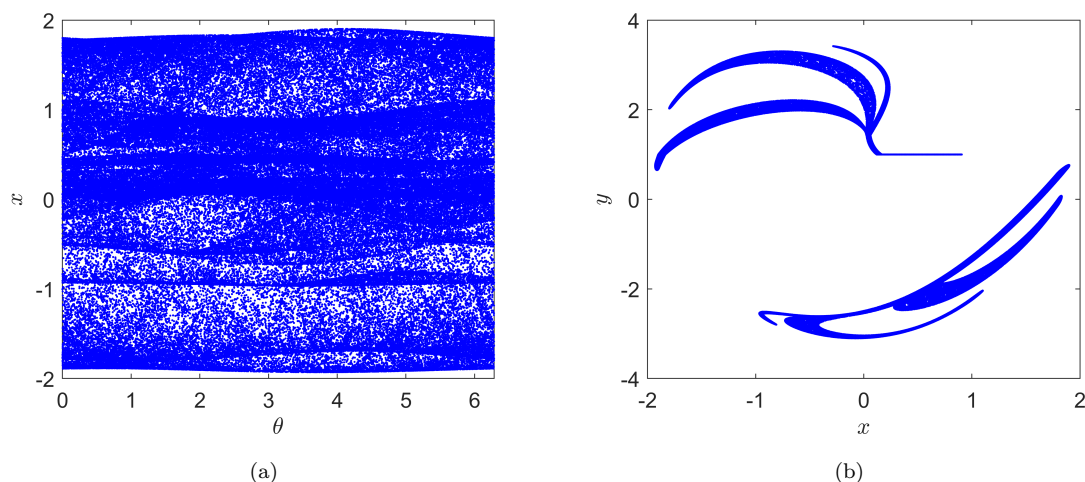


Figure 10. Chaotic attractor at $f_n = 3.8507$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

Figure 10 shows the attractor loses its wrinkled structure and becomes completely non-smooth. Thus the attractor is chaotic.

In addition, at $f_n = 3.634 \in (H, I)$, Figure 11 reveals that system (2.4) exhibits a nine-torus ($9T$) quasi-periodic attractor. At $f_n = 4.2 \in (J, K)$, system (2.4) exhibits a two-torus ($2T$) quasi-periodic attractor, see Figure 12.

To summarize, through the discussion above, the evolution process of switching-sliding bifurcation from 3 to 4.2 can be described as follows: $5T$ quasi-periodic attractor for $(A, B) \rightarrow$ chaotic attractor or $17T$ quasi-periodic attractor for $(B, C) \rightarrow 12T$ quasi-periodic attractor for $(C, D) \rightarrow$ SNA for $(D, E) \rightarrow$ chaotic attractor for $(E, F) \rightarrow 7T$ quasi-periodic attractor for $(F, G) \rightarrow$ chaotic attractor for $(G, H) \rightarrow 9T$ quasi-periodic attractor for $(H, I) \rightarrow$ chaotic attractor for

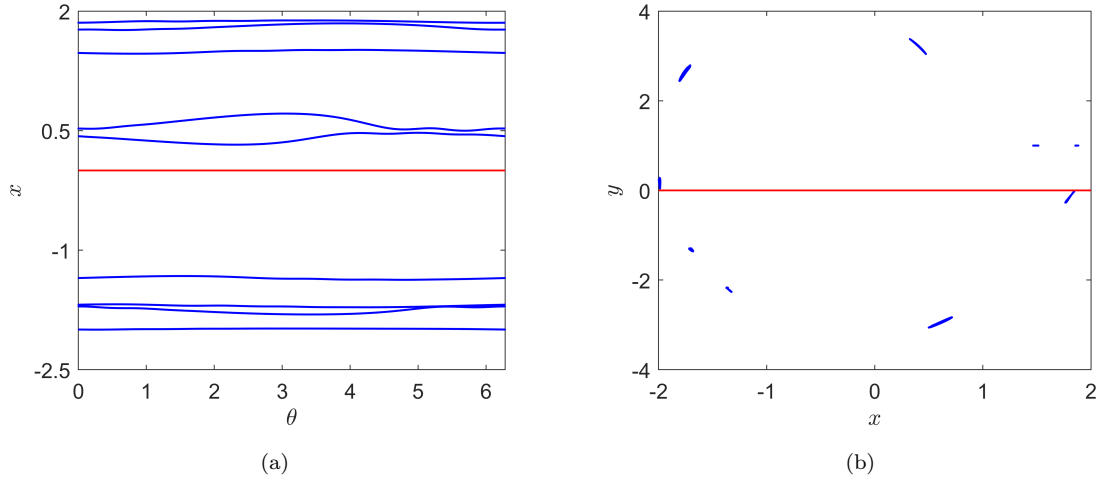


Figure 11. The nine-torus ($9T$) quasi-periodic attractor at $f_n = 3.634$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

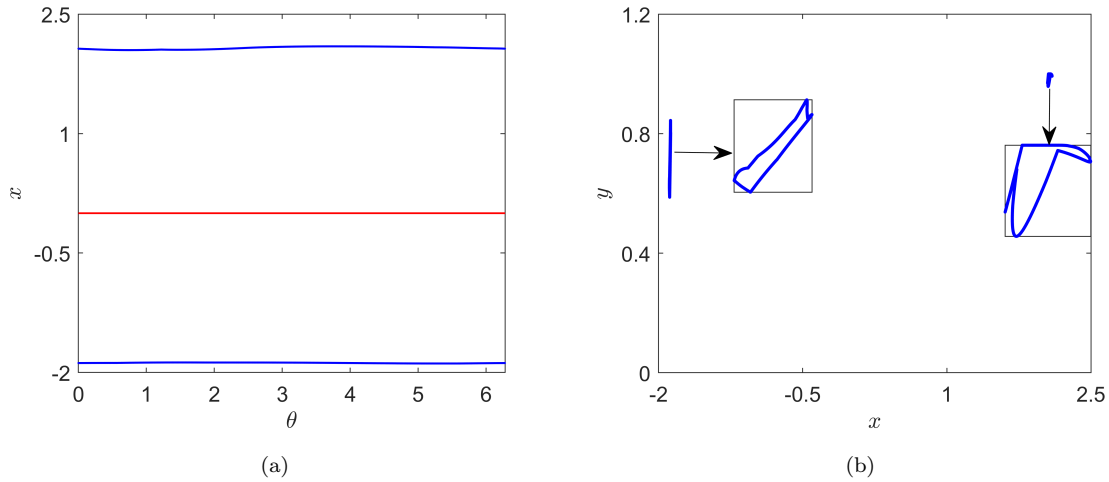


Figure 12. The two-torus ($2T$) quasi-periodic attractor at $f_n = 4.2$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

$(I, J) \rightarrow 2T$ quasi-periodic attractor for (J, K) .

Example 4.2. Take parameters $\mathcal{P}_2 = (a_2, d_1, d_2, f_n, v, e_1, e_2, \omega_1, \omega_2) = (1.23, 2.5, 0.4, 1.2, 0.5, 0.1, 1.9, 2, \sqrt{10})$ and $a_1 \in [3.5, 3.62]$ as the bifurcation parameter. For this set of parameters, by using some numerical calculations, it is easy to see that these transversal intersection points between the trajectories of system (2.4) and the switching surface Σ satisfy the crossing conditions $H_X F_1 H_X F_2 > 0$ ($H_X F_2 H_X F_1 < 0$). Here, we give an example $a_1 = 3.5$ for $\mathcal{P}_2$ to show the cross phenomenon with results visualized in Figure 14. We further plot the one-dimensional bifurcation diagram of map $P_s : \Sigma_s \mapsto \Sigma_s$ and the largest Lyapunov exponent $\lambda_{\max}$ of system (2.4), as shown in Figures 13(a) and 13(b) respectively. Figure 13(b) clearly shows some alternating windows of $\lambda_{\max} > 0$ and $\lambda_{\max} < 0$. Next, we present the phase diagram of system (2.4)

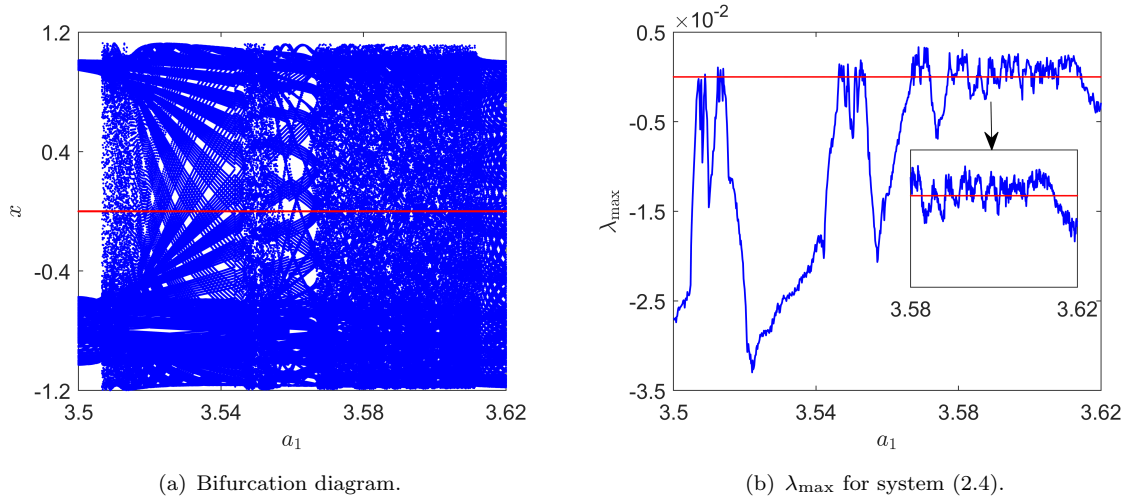


Figure 13. Bifurcation diagram in the (a_1, x) space and the corresponding largest Lyapunov exponent $\lambda_{\max}$ of system (2.4) with $a_2 = 1.23, d_1 = 2.5, d_2 = 0.4, f_n = 1.2, v = 0.5, e_1 = 0.1, e_2 = 1.9, \omega_1 = 2, \omega_2 = \sqrt{10}$ and $a_1 \in [3.5, 3.62]$.

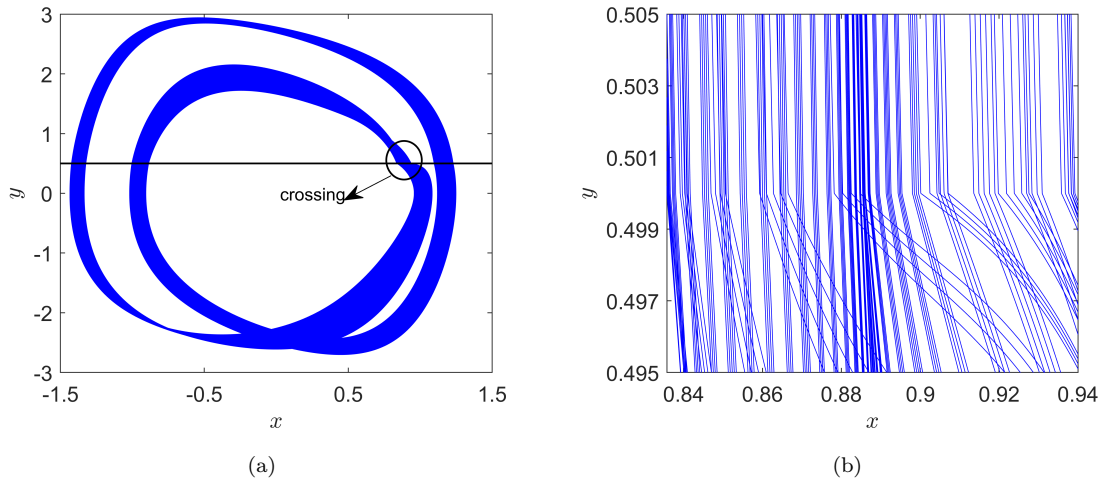


Figure 14. The phase diagram of system (2.4) with $a_2 = 1.23, d_1 = 2.5, d_2 = 0.4, f_n = 1.2, v = 0.5, e_1 = 0.1, e_2 = 1.9, \omega_1 = 2, \omega_2 = \sqrt{10}$ and $a_1 = 3.5$. **(a)** The phase diagram for $(x, y) \in (-1.5, 1.5) \times (-3, 3)$. **(b)** Enlargement of Figure 14(a).

on the Poincaré section Σ_s for $\mathcal{P} = \mathcal{P}_2$ in both the (θ, x) -plane and the (x, y) -plane.

When $a_1 = 3.5$, three frequency curves can be clearly observed in the (θ, x) -plane from Figure 15(a). Accordingly, the system (2.4) generates a three-torus ($3T$) quasi-periodic attractor in the (x, y) -plane, see Figure 15(b). As a_1 increases to 3.51, eight frequency curves appear in the (θ, x) -plane in Figure 16, which corresponds to an eight-torus ($8T$) quasi-periodic attractor of system (2.4) in the (x, y) -plane, see Figure 16(b). Furthermore, when $a_1 = 3.54$, Figure 17(a) presents five frequency curves in the (θ, x) -plane. This indicates that the system (2.4) possesses a five-torus ($5T$) quasi-periodic attractor in the (x, y) -plane, see Figure 17(b).

When $a_1 = 3.548$, it is easy to see from Figure 13(b) that $\lambda_{\max}$ converges to -0.00827981 .

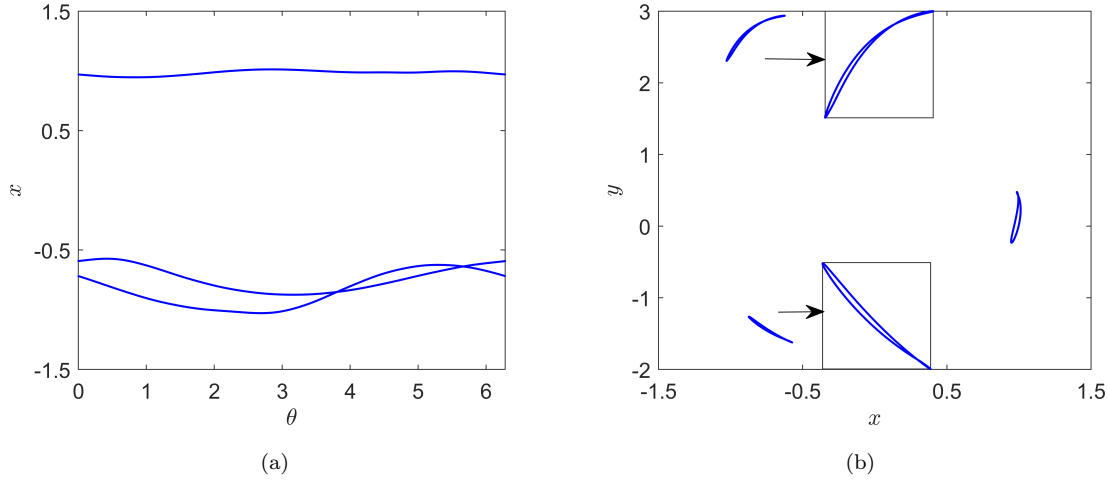


Figure 15. The three-torus ($3T$) quasi-periodic attractor at $a_1 = 3.5$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

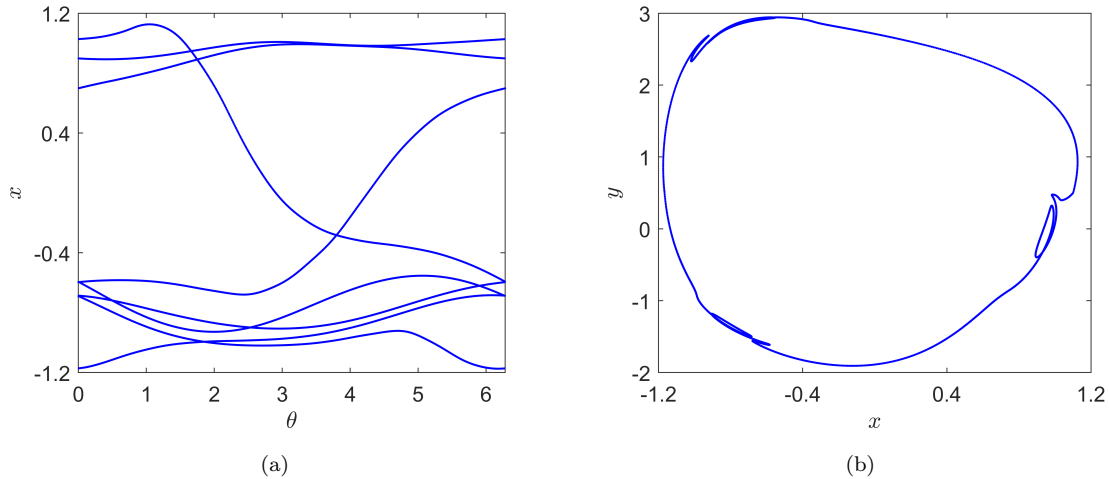


Figure 16. The eight-torus ($8T$) quasi-periodic attractor at $a_1 = 3.51$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

It implies the system (2.4) is nonchaotic. From Figure 18, we can observe that the frequency curves of system (2.4) in the (θ, x) -plane become highly wrinkled, while the attractor loses smooth completely in the (x, y) -plane and generates the strange fractal property. This property will be verified in the next section. Thus the attractor is an SNA.

When $a_1 = 3.56$, seven frequency curves can be observed in the (θ, x) -plane from Figure 19(a), indicating that the system (2.4) exhibits a seven-torus ($7T$) quasi-periodic attractor in the (x, y) -plane, see Figure 19(b). As a_1 increases to 3.575, Figure 20(a) shows that there exist nine frequency curves in the (θ, x) -plane. Thus the system (2.4) exhibits a nine-torus ($9T$) quasi-periodic attractor in the (x, y) -plane, see Figure 20(b). When $a_1 = 3.585$, Figure 21(a) presents eleven frequency curves in the (θ, x) -plane, implying the existence of an eleven-torus

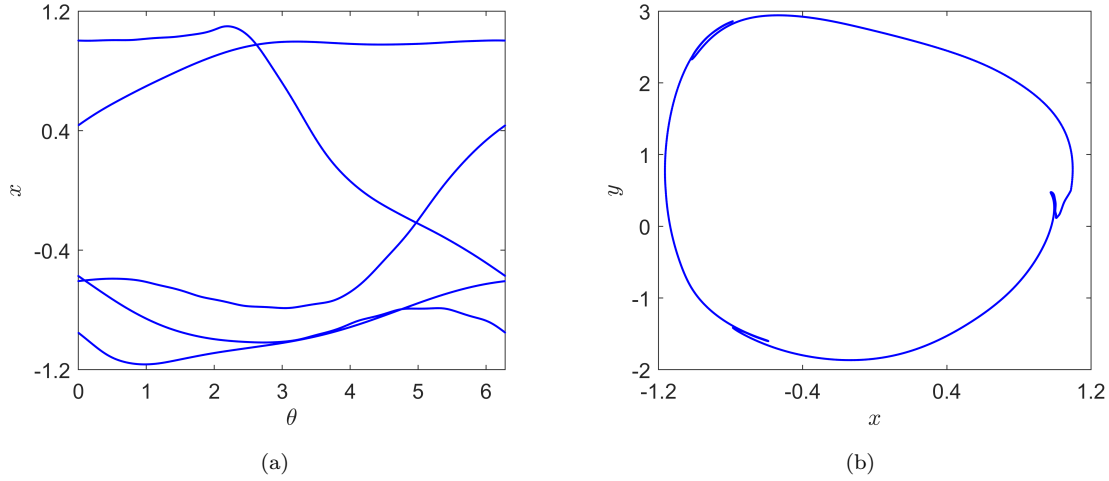


Figure 17. The five-torus ($5T$) quasi-periodic attractor at $a_1 = 3.54$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

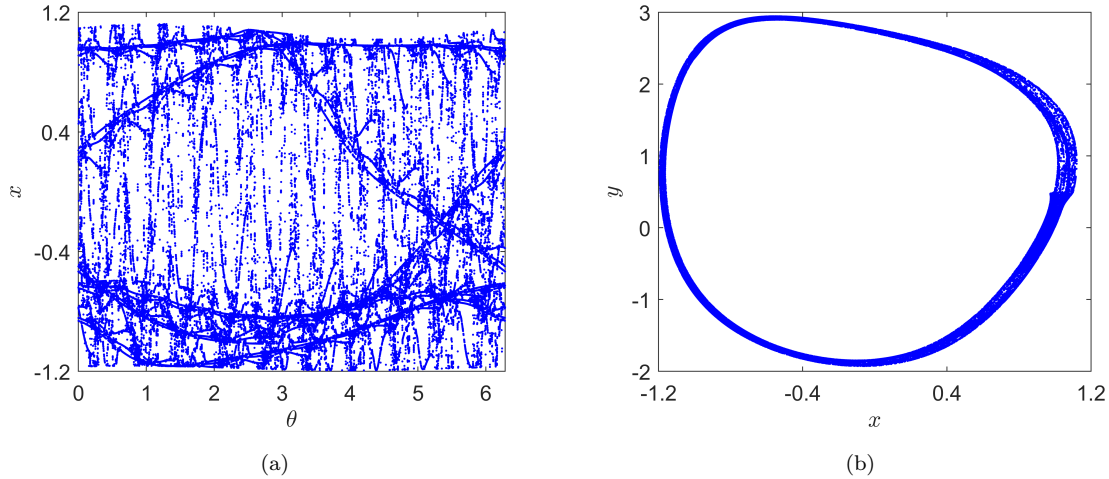


Figure 18. SNA attractor at $a_1 = 3.548$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

($11T$) quasi-periodic attractor in the (x, y) -plane, see Figure 21(b).

When $a_1 = 3.61$, Figure 13(b) shows that $\lambda_{\max} > 0$, which indicates the attractor is chaotic. The phase diagrams in the (θ, x) -plane and the (x, y) -plane are provided in Figure 22, which clearly shows the attractor is completely nonsmooth. In addition, as a_1 increases to 3.62, from Figure 23, the system (2.4) exhibits a two-torus ($2T$) quasi-periodic attractor.

Example 4.3. Take $a_1 = 2.1$, $a_2 = 3.5$, $d_1 = 2.5$, $d_2 = 0.8$, $f_n = 1.5$, $e_1 = 0.285323$, $e_2 = 2.5$, $\omega_1 = 3$, $\omega_2 = \sqrt{10}$. By applying the shooting method (see Chap. 7 of [30]), it can be easy to find a three-torus ($3T$) quasi-periodic grazing orbit with grazing point $\mathbf{x}_* = (-0.40204326, 2.42228001, 6.08242938, 0.19090991)$ at $v \approx 2.42228001$. By using simple numeri-

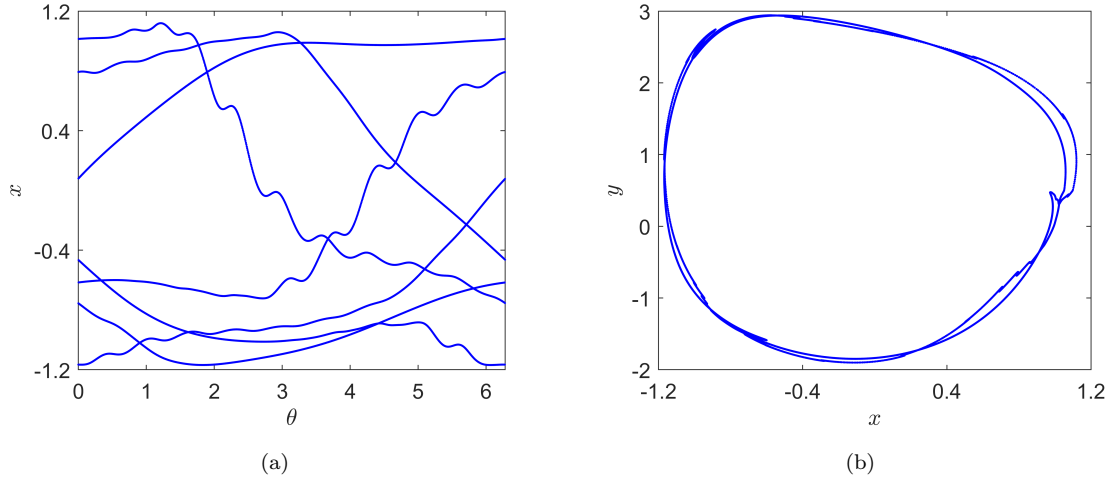


Figure 19. The seven-torus ($7T$) quasi-periodic attractor at $a_1 = 3.56$. **(a)** The phase diagram in the (θ, x) -plane. **(b)** The phase diagram in the (x, y) -plane.

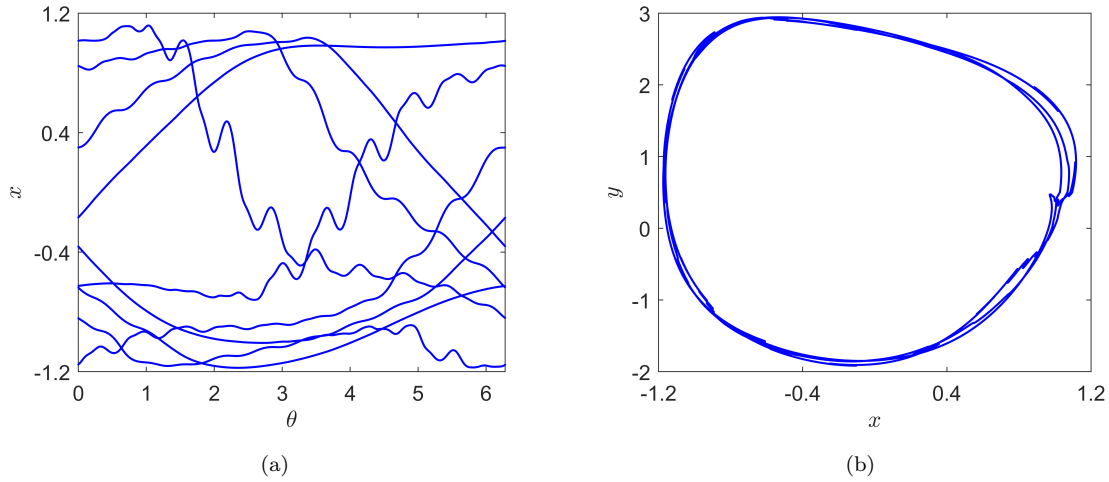


Figure 20. The nine-torus ($9T$) quasi-periodic attractor at $a_1 = 3.575$. **(a)** The phase diagram in the (θ, x) -plane. **(b)** The phase diagram in the (x, y) -plane.

cal computation, we have $H(\mathbf{x}_*, v_*) = 0$, $\langle \nabla H(\mathbf{x}_*, v_*), F_1(\mathbf{x}_*, \mu_*) \rangle = 0$, $\nabla H(\mathbf{x}_*, v_*) \neq 0$, and

$$\langle \nabla H(\mathbf{x}_*, v_*), (F_1)_x F_1(\mathbf{x}_*, \mu_*) \rangle \approx 10.0150 > 0.$$

It implies that $\gamma(\mathbf{x}_*, \mu_*) = -1$, and the system (2.4) undergoes a grazing-sliding bifurcation at $v \approx 2.42228001$. In Figure 24, we plot the phase diagrams on the Poincaré section Σ_s of the grazing three-torus quasi-periodic orbit of system (2.4) in the (x, y) -plane, as well as the phase diagrams of map P_s in the (θ, x) -plane and in the (x, y) -plane.

Take parameters $\mathcal{P}_3 := (a_1, a_2, d_1, d_2, f_n, v, e_2, \omega_1, \omega_2) = (2.1, 3.5, 2.5, 0.8, 1.5, 2.42228001, 2.5, 3, \sqrt{10})$ and e_1 as the bifurcation parameter. In Figure 25, we plot the one-dimensional bifurcation diagram of map $P_s : \Sigma_s \mapsto \Sigma_s$ and the largest Lyapunov exponent $\lambda_{\max}$ of system

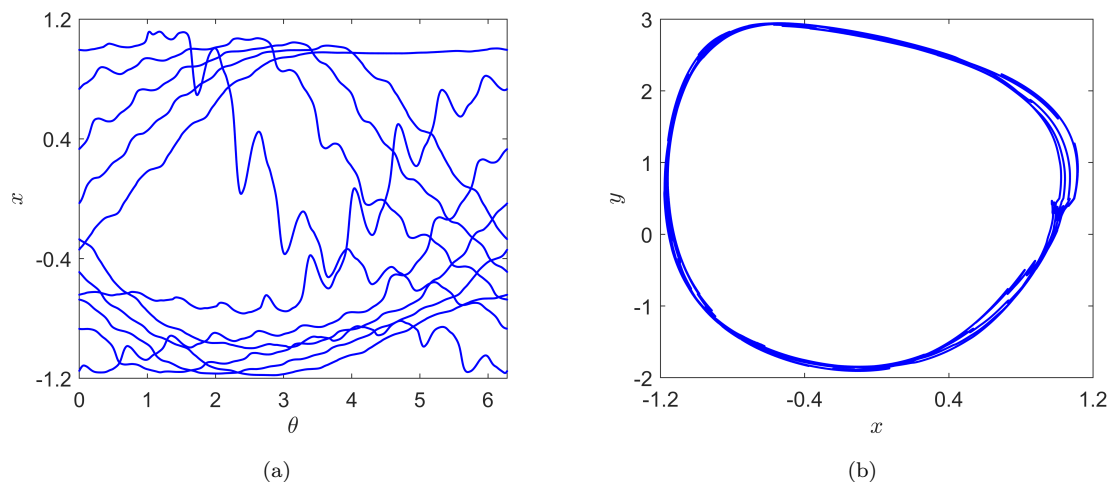


Figure 21. The eleven-torus (11T) quasi-periodic attractor at $a_1 = 3.585$. **(a)** The phase diagram in the (θ, x) -plane. **(b)** The phase diagram in the (x, y) -plane.

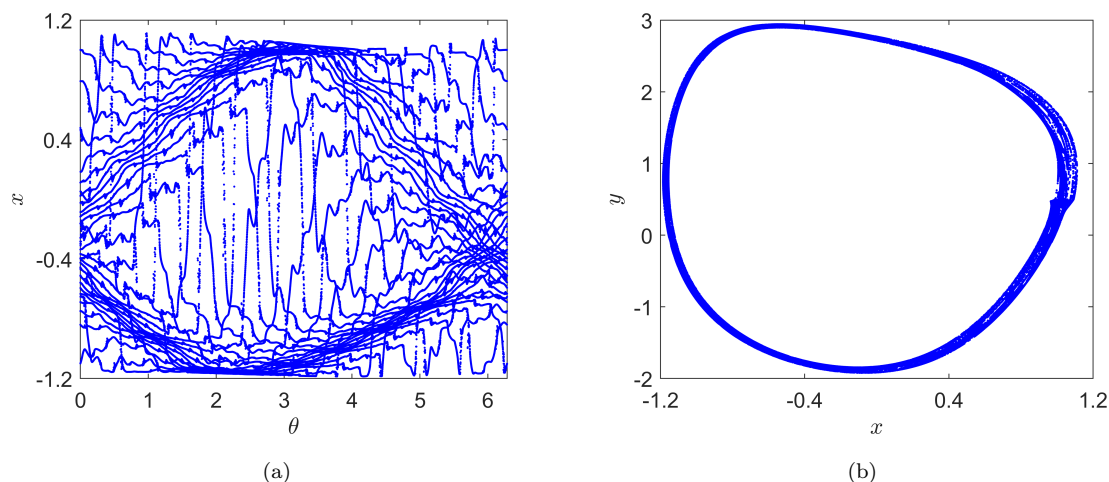


Figure 22. Chaotic attractor at $a_1 = 3.61$. **(a)** The phase diagram in the (θ, x) -plane. **(b)** The phase diagram in the (x, y) -plane.

(2.4), respectively. From Figure 25(b), we have $\lambda_{\max} < 0$, implying the attractor is nonchaotic. Take $e_1 = 0.28535$, It can be observed from Figure 26(b) that the frequency curves of system (2.4) become very wrinkled. Meanwhile, the attractor loses smooth entirely and generates the strange fractal property. In the next section, we will confirm the strange fractal property of attractor is SNA. Thus the attractor is an SNA.

4.2. Torus-adding cascades

The numerical experiments in Section 4.1 reveal that the crossing and switching-sliding bifurcations in system (2.4) give rise to diverse dynamical behaviors. Building on these findings, this

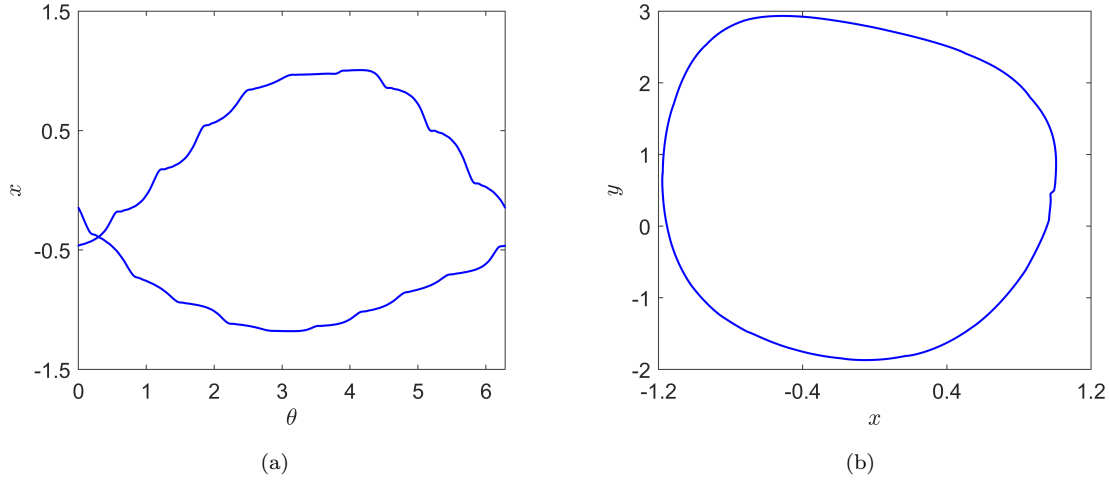


Figure 23. The two-torus ($2T$) quasi-periodic attractor at $a_1 = 3.62$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

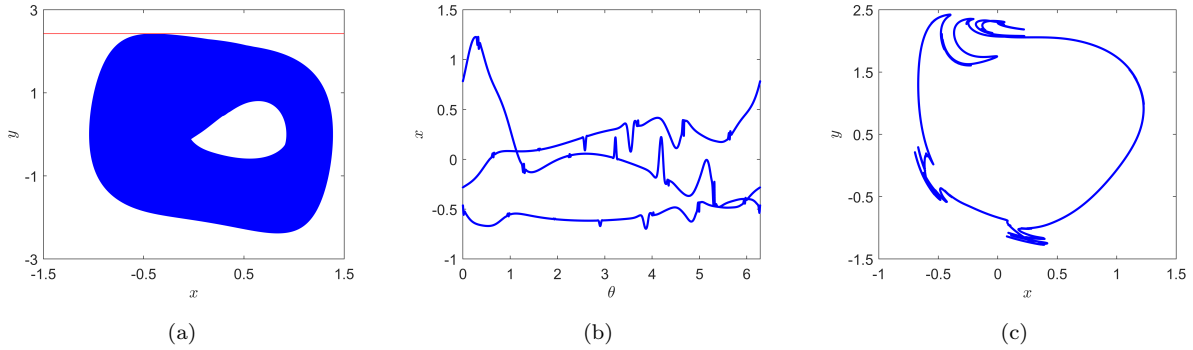


Figure 24. The three-torus ($3T$) quasi-periodic attractor with $a_1 = 2.1, a_2 = 3.5, d_1 = 2.5, d_2 = 0.8, f_n = 1.5, e_1 = 0.285323, e_2 = 2.5, \omega_1 = 3, \omega_2 = \sqrt{10}$ and $v \approx 2.42228001$. (a) The phase diagram of the grazing quasi-periodic orbit of system (2.4) in the (x, y) -plane. (b) The phase diagram of map P_s in the (θ, x) -plane. (c) The phase diagram of map P_s in the (x, y) -plane.

section further conducts numerical simulations to show the existence of torus-adding cascades. This phenomenon is analogous to the period-adding cascade that arises from grazing bifurcations in periodically excited impact systems [5, Chapter 6].

The torus-adding cascades in system (2.4) can be analyzed via the rotation number. For a stable $(m_1 + m_2)T$ quasi-periodic torus in system (2.4), the rotation number can be defined as $\rho = m_1/(m_1 + m_2)$, where the system (2.4) exhibits an m_1 -torus quasi-periodic attractor on the $y > 0$ side and an m_2 -torus quasi-periodic attractor on the $y < 0$ side in the (x, y) -plane, respectively. By the Farey tree structure, for any given two quasi-periodic windows corresponding to nonnegative rotation numbers p_1/q_1 and p_2/q_2 , there exists a quasi-periodic window with the rotation number $(p_1 + p_2)/(q_1 + q_2)$, which satisfies the following the Farey summation:

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2},$$

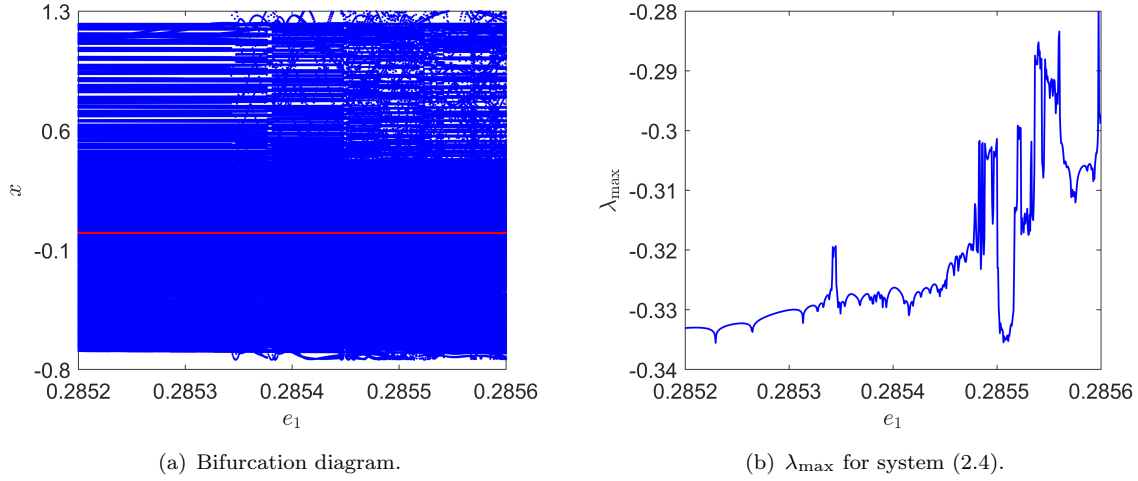


Figure 25. Bifurcation diagram in the (e_1, x) space and the corresponding largest Lyapunov exponent $\lambda_{\max}$ of system (2.4) with $a_1 = 2.1, a_2 = 3.5, d_1 = 2.5, d_2 = 0.8, f_n = 1.5, v \approx 2.42228001, e_2 = 2.5, \omega_1 = 3, \omega_2 = \sqrt{10}$ and $e_1 \in [0.2852, 0.2856]$.

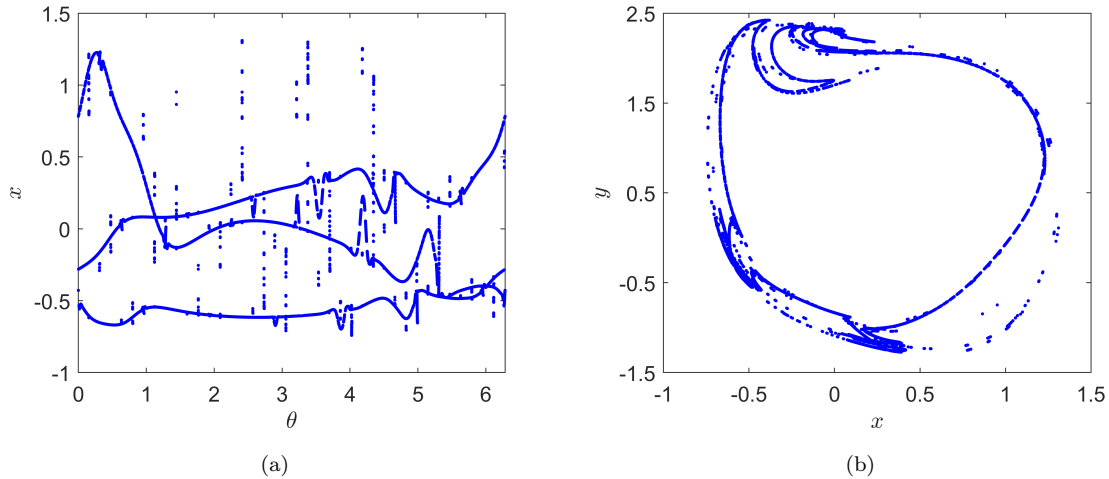


Figure 26. SNA attractor at $e_1 = 0.28535$. (a) The phase diagram in the (θ, x) -plane. (b) The phase diagram in the (x, y) -plane.

where $p_i \geq 0$ and $q_i \geq 0$ for $i = 1, 2$.

For $\mathcal{P} = \mathcal{P}_1$, we take $f_n = 3$ and $f_n = 3.48$ and plot the phase diagrams of map P_s in the (x, y) -plane, respectively. The corresponding results are shown in Figure 27. As seen from the Figure 27, system (2.4) displays a five-torus (5T) quasi-periodic attractor with $f_n = 3$ and a seven-torus (7T) quasi-periodic attractor with $f_n = 3.48$, respectively.

As $\mathcal{P} = \mathcal{P}_1$ and f_n increases from 3 to 4.2, system (2.4) presents a completed bifurcation phenomenon, for example quasi-periodic torus, SNA and chaotic attractors. From Figure 5(b), Figure 6(b), Figure 7(b), Figure 9(b), Figure 11(b), Figure 12(b) and Figure 27, we derive the rotation numbers for $\mathcal{P} = \mathcal{P}_1$ as f_n varies, as presented in Table 1. Moreover, torus-adding cascades are observed from the Table 1, as detailed below:

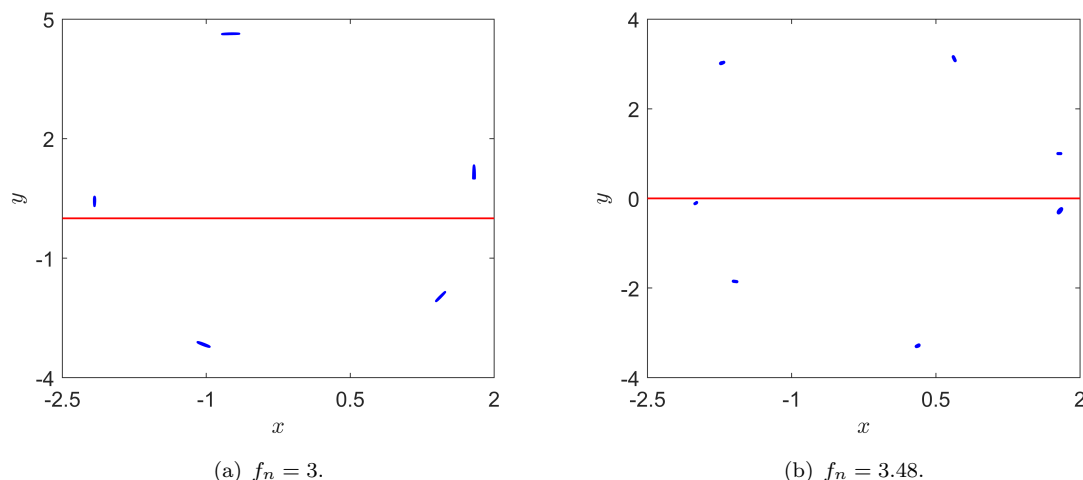


Figure 27. The phase diagram of map P_s in the (x, y) -plane.

Table 1. The rotation numbers of system (2.4) with $\mathcal{P} = \mathcal{P}_1$.

f_n	3	3.081	3.2264	3.27956	3.421	3.48	3.634	4.2
ρ	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{9}{17}$	$\frac{7}{12}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{2}$

- (1) Between the $5T$ -torus orbit with the rotation number $2/5$ and the $2T$ -torus orbit with the rotation number $2/2$, there exists a $7T$ -torus orbit with the rotation number $2/5 \oplus 2/2 = 4/7$.
- (2) Between the $5T$ -torus orbit with the rotation number $2/5$ and $12T$ -torus orbit with the rotation number $7/12$, there exists a $17T$ -torus orbit with the rotation number $2/5 \oplus 7/12 = 9/17$.
- (3) Between the $5T$ -torus orbit with the rotation number $3/5$ and the $7T$ -torus orbit with the rotation number $4/7$, there exists a $12T$ -torus orbit with the rotation number $3/5 \oplus 4/7 = 7/12$.
- (4) Between the $7T$ -torus orbit with the rotation number $3/7$ and the $2T$ -torus orbit with the rotation number $2/2$, there exists a $9T$ -torus orbit with the rotation number $3/7 \oplus 2/2 = 5/9$.

Due to the strong nonlinearity of the system (2.4), the torus generated by crossing bifurcations of system (2.4) do not satisfy torus-adding cascades. Thus, we redefine a type of torus cascades based on the definition of torus-adding cascades. This torus cascades can be termed a *weak torus-adding cascades*. Specifically, between the k_1T -torus quasi-periodic window and the k_2T -torus quasi-periodic window, there exists a kT -torus quasi-periodic window, where $k = k_1 + k_2$.

As $\mathcal{P} = \mathcal{P}_2$ and a_1 increases from 3.5 to 3.62, system (2.4) exhibits an alternating sequence of quasi-periodic torus, SNA and chaotic attractors. In Table 2, we present kT -torus quasi-periodic attractor for different values of a_1 . Thus the *weak torus-adding cascades* are observed from table

Table 2. The kT -torus quasi-periodic window with $\mathcal{P} = \mathcal{P}_2$.

a_1	3.5	3.51	3.54	3.56	3.575	3.585	3.62
k	3	8	5	7	9	11	2

2, as detailed below:

- (1) Between the $3T$ -torus orbit and the $5T$ -torus orbit, there exists a $8T$ -torus orbit.
- (2) Between the $3T$ -torus orbit and the $2T$ -torus orbit, there exists a $5T$ -torus orbit.
- (3) Between the $5T$ -torus orbit and the $2T$ -torus orbit, there exists a $7T$ -torus orbit.
- (4) Between the $7T$ -torus orbit and the $2T$ -torus orbit, there exists a $9T$ -torus orbit.
- (5) Between the $9T$ -torus orbit and the $2T$ -torus orbit, there exists a $11T$ -torus orbit.

5. The strange properties of SNAs

In Section 4, the non-chaotic properties of the SNAs were confirmed by applying the method that their largest Lyapunov exponent $\lambda_{\max}$ is negative. Here, we further characterize the strange properties of the three SNAs corresponding to $\mathcal{P} = \mathcal{P}_1$ with $f_n = 3.333$, $\mathcal{P} = \mathcal{P}_2$ with $a_1 = 3.548$ and $\mathcal{P} = \mathcal{P}_3$ with $e_1 = 0.28535$ by using rational approximation [10], phase sensitivity property [10], singular continuous spectrum [24, 25] and separation of nearby trajectories [9].

(1) Rational approximation.

Rational approximation provides a method to approximately represent irrational numbers using rational numbers, and it is often used to distinguish between quasi-periodic attractor, SNA, and chaotic attractor in quasi-periodic system. Any irrational number ω can be uniquely written as an infinite continued fraction:

$$\omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} := [a_0; a_1, a_2, \cdots], \quad (5.1)$$

where $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Z}^+$ for $i \geq 1$. By truncating the continued fraction, a finite continued fraction can be obtained:

$$\omega_n = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + a_n}},$$

where ω_n denotes the n -th order continued fraction approximation. As n increases, ω_n converges to ω at an extremely fast rate. Thus, this allows system (2.4) with irrational frequency ratio $\omega = \omega_1/\omega_2$ to be approximated by a sequence of periodically forced systems with rational approximation frequencies ω_1 and ω_1/ω_n ($n = 1, 2, \cdots$).

For an SNA, as n is sufficiently small, we can observe some periodic points. In contrast, as n tends to infinity, the number of periodic points increases without bound. At this time, the approximating system exhibits dynamics identical to those of the original system.

For the three SNAs of system (2.4) described in section 4, we have $\omega_2 = \sqrt{10}$. By applying the continued fraction approximation method, it is easy to obtain $\sqrt{10} := [3; 6, 6, 6, \cdots]$. Thus, the system (2.4) with irrational ratio $\omega = \omega_1/\omega_2$ can be approximated by a family of systems with rational frequencies ω_1 and ω_1/ω_n .

Table 3. The rational approximation sequence with $\omega = \omega_1/\omega_2 = 6/\sqrt{10}$.

n	$\sqrt{10}$	ω_n	$ \omega - \omega_n $
1	3	2	$1.02633403 \times 10^{-1}$
2	$\frac{19}{6}$	$\frac{36}{19}$	$2.62975400 \times 10^{-3}$
3	$\frac{117}{37}$	$\frac{74}{39}$	$6.93013349 \times 10^{-5}$
4	$\frac{721}{228}$	$\frac{1368}{721}$	$1.82494985 \times 10^{-6}$
5	$\frac{4443}{1405}$	$\frac{8430}{4443}$	4.8058324×10^{-8}
6	$\frac{27379}{8658}$	$\frac{51948}{27379}$	$1.26556977 \times 10^{-9}$

Table 4. The rational approximation sequence with $\omega = \omega_1/\omega_2 = 2/\sqrt{10}$.

n	$\sqrt{10}$	ω_n	$ \omega - \omega_n $
1	3	$\frac{2}{3}$	$3.42111346 \times 10^{-2}$
2	$\frac{19}{6}$	$\frac{12}{19}$	$8.76584667 \times 10^{-4}$
3	$\frac{117}{37}$	$\frac{74}{117}$	$2.31004450 \times 10^{-5}$
4	$\frac{721}{228}$	$\frac{456}{721}$	$6.08316616 \times 10^{-7}$
5	$\frac{4443}{1405}$	$\frac{2810}{4443}$	$1.60194414 \times 10^{-8}$
6	$\frac{27379}{8658}$	$\frac{17316}{27379}$	$4.21856589 \times 10^{-10}$

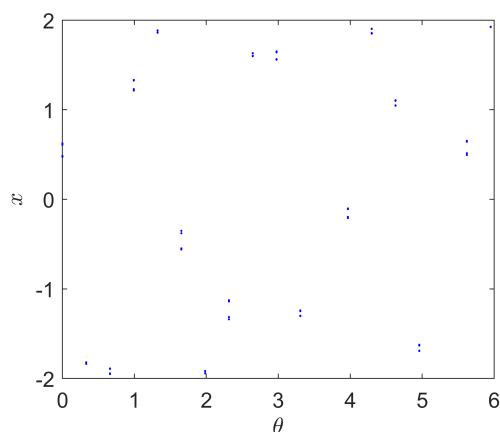
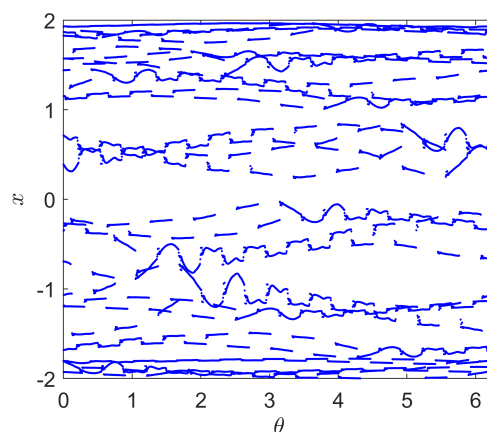
For the SNA of system (2.4) corresponding to $\mathcal{P} = \mathcal{P}_1$ with $f_n = 3.333$, we have $\omega_1 = 6$ and $\omega_2 = \sqrt{10}$. Then the ratio $\omega = \omega_1/\omega_2 = 6/\sqrt{10}$ can be approximated by a sequence of rational numbers $\{\omega_n \in \mathbb{Q} : n \in \mathbb{Z}_+\}$, see Table 3. The phase diagrams of P_s in the (θ, x) -plane are shown in Figure 28 for $n = 2, 6$. At $n = 2$, we observe that there exist only discrete points for the attractor in Figure 28(a). As n increases to 6, from Figure 28(b), the number of discrete points rises rapidly, and the geometric structure of attractor becomes consistent with the one reported in Figure 8(a). This confirms that the attractor is an SNA.

For the SNA of system (2.4) corresponding to $\mathcal{P} = \mathcal{P}_2$ with $a_1 = 3.548$, we have $\omega_1 = 2$ and $\omega_2 = \sqrt{10}$. Then the ratio $\omega = \omega_1/\omega_2 = 2/\sqrt{10}$ is approximated by a family of rational numbers $\{\omega_n \in \mathbb{Q} : n \in \mathbb{Z}_+\}$, see Table 4. From Figure 29, it is easy to see that the result for this parameter is analogous to the case of system (2.4) with $\mathcal{P} = \mathcal{P}_1$ and $f_n = 3.333$.

For the SNA of system (2.4) corresponding to $\mathcal{P} = \mathcal{P}_3$ with $e_1 = 0.28535$, we have $\omega_1 = 3$ and $\omega_2 = \sqrt{10}$. Then the ratio $\omega = \omega_1/\omega_2 = 3/\sqrt{10}$ is approximated by a family of rational

Table 5. The rational approximation sequence with $\omega = \omega_1/\omega_2 = 3/\sqrt{10}$.

n	$\sqrt{10}$	ω_n	$ \omega - \omega_n $
1	3	1	$5.13167020 \times 10^{-2}$
2	$\frac{19}{6}$	$\frac{18}{19}$	$1.31487700 \times 10^{-3}$
3	$\frac{117}{37}$	$\frac{111}{117}$	$3.46506674 \times 10^{-5}$
4	$\frac{721}{228}$	$\frac{684}{721}$	$9.12474924 \times 10^{-7}$
5	$\frac{4443}{1405}$	$\frac{4215}{4443}$	$2.40291621 \times 10^{-8}$
6	$\frac{27379}{8658}$	$\frac{25974}{27379}$	$6.32784883 \times 10^{-10}$

(a) $\omega_2 = 36/19$.(b) $\omega_6 = 51948/27379$.**Figure 28.** Rational approximation of the SNA for $\mathcal{P} = \mathcal{P}_1$ with $f_n = 3.333$ of system (2.4).

numbers $\{\omega_n \in \mathbb{Q} : n \in \mathbb{Z}_+\}$, see Table 5. From Figure 30, it can be seen that the results for this case are analogous to those observed in the previous two cases of system (2.4).

(2) *Phase sensitivity property.*

Phase sensitivity to the initial conditions of external excitation serves as a critical criterion for distinguishing between SNAs and quasi-periodic attractors. Generally, the dependence of SNA on the system's initial phase is stronger than that of quasi-periodic attractor. In the following, we present the basic method for the dependence of the map P_s of system (2.4) on the initial phase.

Take (x_k, y_k, θ_k) as initial value of map P_s starting from (x_0, y_0, θ_0) . For a given error bound σ , there exists an n_0 such that $\sigma_0 := |\theta_{n_0} - \theta_0| < \sigma$, further has $|\theta_{k+n_0} - \theta_k| = \sigma_0$ for any $k \geq 0$.

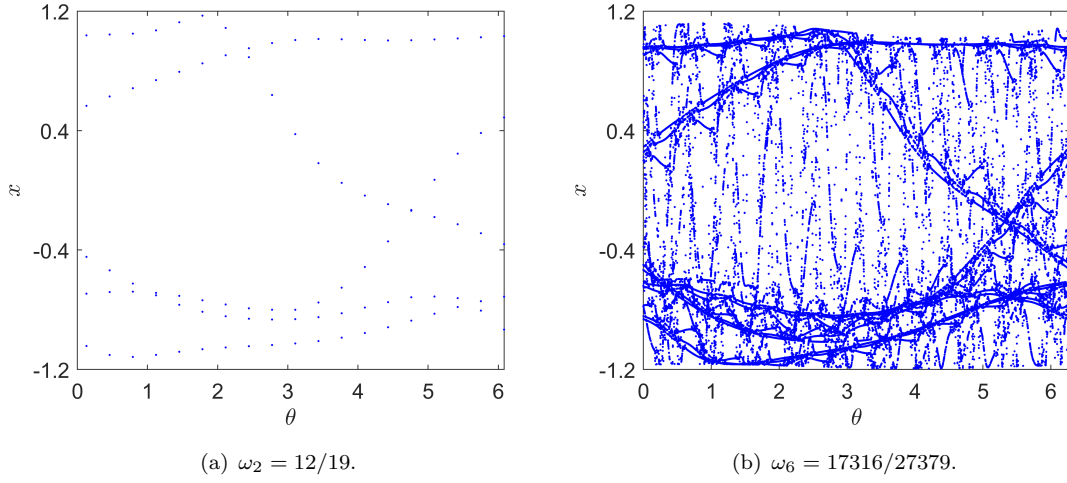


Figure 29. Rational approximation of the SNA for $\mathcal{P} = \mathcal{P}_2$ with $a_1 = 3.548$ of system (2.4).

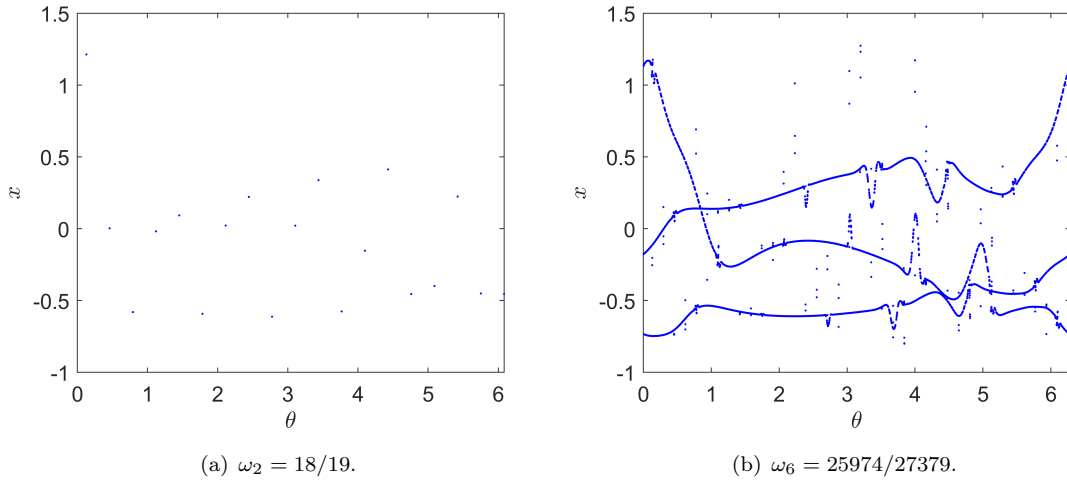


Figure 30. Rational approximation of the SNA for $\mathcal{P} = \mathcal{P}_3$ with $e_1 = 0.28535$ of system (2.4).

On this basis, the derivative of phase sensitivity can be formulated as

$$S_i^N = \sum_{k=1}^{N-n_0} \left| \frac{f_i(k+n_0) - f_i(k)}{\theta(k+n_0) - \theta(k)} \right|,$$

for $i = 1, 2$, where $f_i(n)$ and $\theta(n)$ are the n -th iteration of map P_s , N is the total number of iterations. For SNA and quasi-periodic attractor, if for $i = 1, 2$, $\tau_i^N := \max\{S_i^j : 1 \leq j \leq N\}$ increases rapidly as N increases, which shows that the attractor has a strong dependence on the initial phase and is strange, then it is an SNA; otherwise, it is a quasi-periodic attractor.

For system (2.4) with $\mathcal{P} = \mathcal{P}_1$, we take $(x_0, y_0, \theta_0) = (1, 0, 0)$ as the initial value and take $\sigma = 5 \times 10^{-3}$ as the error bound. By direct numerical computation, for both $f_n = 3.333$ and $f_n = 3.081$, we found $n_0 = 381$ satisfying $|\theta_{n_0} - \theta_0| \approx 4.35727346 \times 10^{-3}$. In Figure 31, we plot τ_1^N for $f_n = 3.333, 3.081$ with blue line and red line, respectively. Figure 31 shows that as

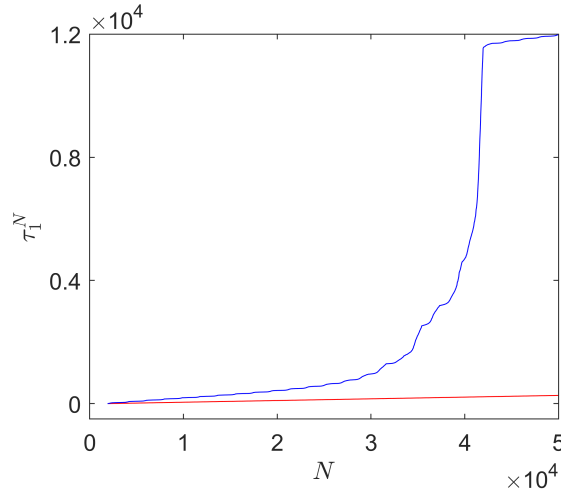


Figure 31. Phase sensitivity of attractors of system (2.4) at $\mathcal{P} = \mathcal{P}_1$. The red line corresponds to the five-torus quasi-periodic attractor at $f_n = 3.081$, while blue line represents the SNA at $f_n = 3.333$.

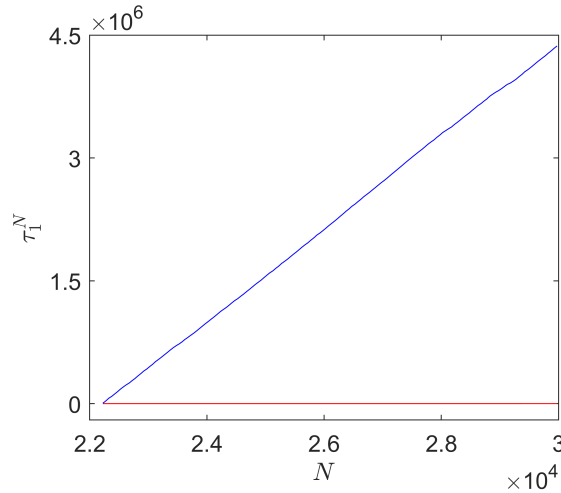


Figure 32. Phase sensitivity of attractors of system (2.4) at $\mathcal{P} = \mathcal{P}_2$. The red line corresponds to the three-torus quasi-periodic attractor at $a_1 = 3.5$, while the blue line represents the SNA at $a_1 = 3.548$.

N increases, τ_1^N for the SNA rises much faster than that for the five-torus ($5T$) quasi-periodic attractor at $f_n = 3.081$, implying the SNA is non-smooth and strange. Moreover, at $f_n = 3.081$, the five-torus ($5T$) quasi-periodic attractor exhibits convergence of τ_1^N to 266.6012 as the number of iterations N approaches infinity.

When $\mathcal{P} = \mathcal{P}_2$, we take $(x_0, y_0, \theta_0) = (1, 0, 0)$ as the initial value and take $\sigma = 8 \times 10^{-3}$ as the error bound. For both $a_1 = 3.5$ and $a_1 = 3.548$, we found $n_0 = 420$ satisfying $|\theta_{n_0} - \theta_0| \approx 7.11305517 \times 10^{-3}$. Figure 32 shows τ_1^N for $a_1 = 3.5, 3.548$ with red line and blue line, respectively. From Figure 32, it is clear that the SNA at $a_1 = 3.548$ leads to a rapidly growing τ_1^N with N , confirming its non-smooth and strange properties. Moreover, at $a_1 = 3.5$, the three-torus ($3T$) quasi-periodic attractor exhibits convergence of τ_1^N to 128.7828 as the number of iterations N approaches infinity.

When $\mathcal{P} = \mathcal{P}_3$, we take $(x_0, y_0, \theta_0) = (-0.40204326, 2.42228001, 6.08242938)$ as the initial

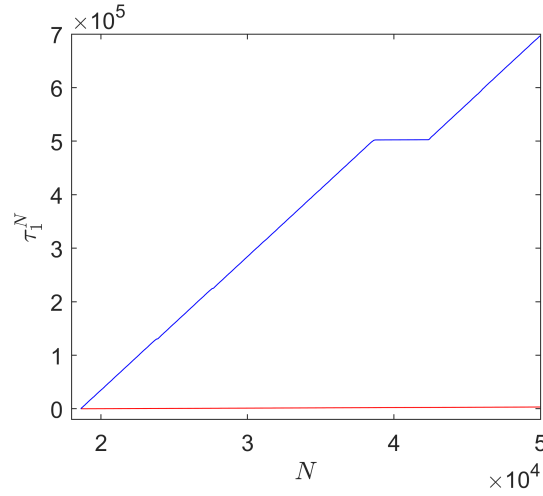


Figure 33. Phase sensitivity of attractors of system (2.4) at $\mathcal{P} = \mathcal{P}_3$. The red line corresponds to the three-torus quasi-periodic attractor at $e_1 = 0.285323$, while the blue line represents the SNA at $e_1 = 0.28535$.

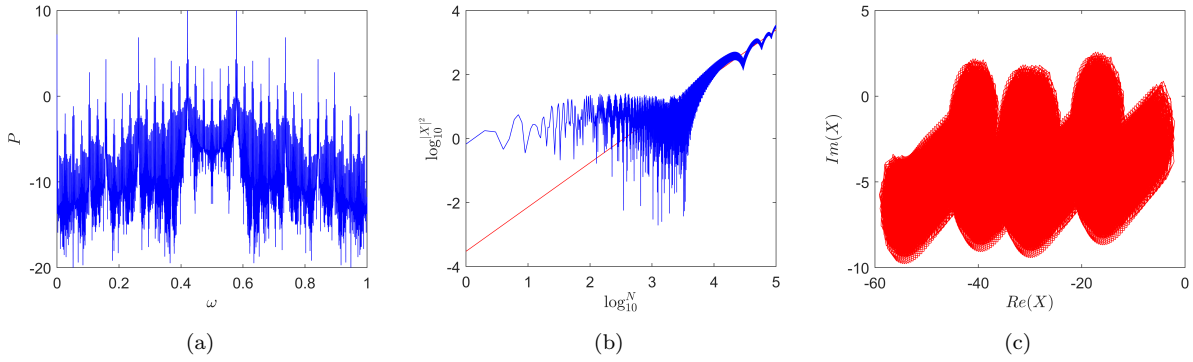


Figure 34. SNA for $\mathcal{P} = \mathcal{P}_1$ and $f_n = 3.333$. **(a)** the power spectrum $P := P(\omega)$. **(b)** plot of $\log_{10} |X|^2 := \log_{10} |X(\omega, N)|^2$ versus $\log_{10} N$, with the blue and red lines denoting the numerical values and the power law fit. **(c)** the fractal paths in the $(\text{Re}(X), \text{Im}(X))$ plane.

value and take $\sigma = 5 \times 10^{-3}$ as the error bound. For both $e_1 = 0.285323$ and $e_1 = 0.28535$, we found $n_0 = 722$ satisfying $|\theta_{n_0} - \theta_0| \approx 4.13367256 \times 10^{-3}$. Figure 33 shows τ_1^N for $e_1 = 0.285323, 0.28535$ with red line and blue line, respectively. From Figure 33, we can observe that τ_1^N of the SNA rises rapidly with N at $e_1 = 0.28535$, showing its non-smooth and strange properties. Moreover, at $e_1 = 0.285323$, the three-torus (3T) quasi-periodic attractor exhibits convergence of τ_1^N to 3723 as the number of iterations N approaches infinity.

(3) Singular continuous spectrum analysis.

The singular continuous spectrum constitutes another key tool for analyzing SNAs. In quasi-periodic systems, the power spectrum is discrete for quasi-periodic attractors, while chaotic systems exhibit a continuous power spectrum. For an SNA, however, the spectrum is a hybrid of discrete and continuous components, occupying an intermediate state between the two cases.

The map P_s is iterated from the initial value (x_0, y_0, θ_0) to generate a time series $\{x_n\}$ of

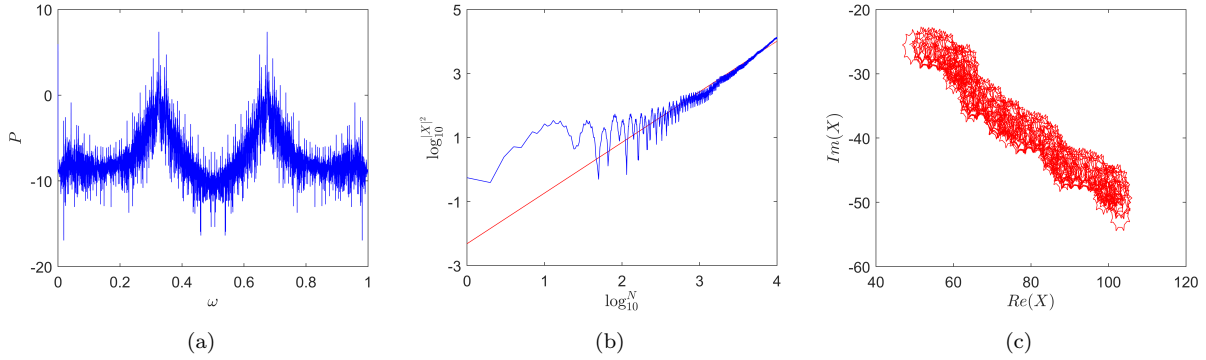


Figure 35. SNA for $\mathcal{P} = \mathcal{P}_2$ and $a_1 = 3.548$. **(a)** the power spectrum $P := P(\omega)$. **(b)** plot of $\log_{10} |X|^2 := \log_{10} |X(\omega, N)|^2$ versus $\log_{10} N$, with the blue and red lines denoting the numerical values and the power law fit. **(c)** the fractal paths in the $(\text{Re}(X), \text{Im}(X))$ plane.

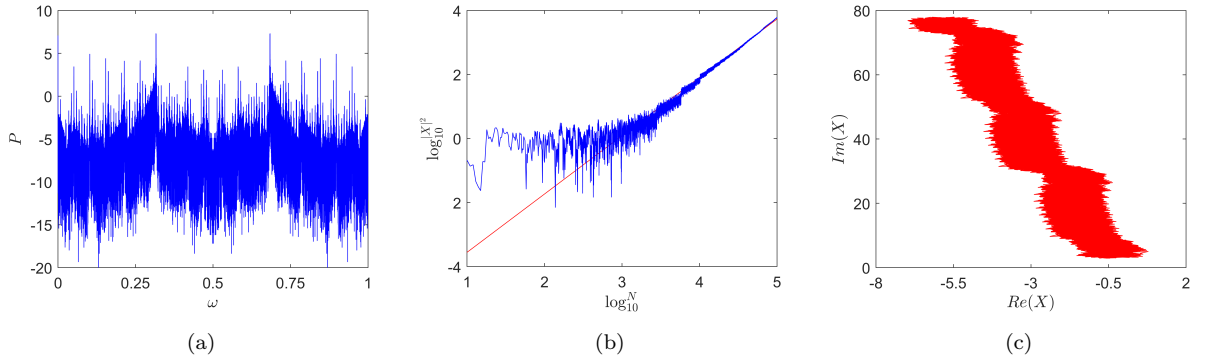


Figure 36. SNA for $\mathcal{P} = \mathcal{P}_3$ and $e_1 = 0.28535$. **(a)** the power spectrum $P := P(\omega)$. **(b)** plot of $\log_{10} |X|^2 := \log_{10} |X(\omega, N)|^2$ versus $\log_{10} N$, with the blue and red lines denoting the numerical values and the power law fit. **(c)** the fractal paths in the $(\text{Re}(X), \text{Im}(X))$ plane.

length N . The definition of the power spectrum can be referred to (see [25])

$$P(\omega) = \lim_{N \rightarrow \infty} |X(\omega, N)/N|^2,$$

where ω denotes the frequency and $X(\omega, N)$ is the Fourier transformation of $\{x_n\}$, given by

$$X(\omega, N) = \sum_{n=1}^N x_n \exp(i2\pi n\omega).$$

According to [10, p.69-73], the scaling relation $|X(\omega, N)|^2 \sim N^\eta$ holds, where η denotes the scaling exponent. For periodic or quasi-periodic sequences $\{x_n\}$, $\eta = 2$. For chaotic sequences $\{x_n\}$, $\eta = 1$. An intermediate value $1 < \eta < 2$ is characteristic of an SNA.

For three SNAs of system (2.4) corresponding to $\mathcal{P} = \mathcal{P}_1$ with $f_n = 3.333$, $\mathcal{P} = \mathcal{P}_2$ with $a_1 = 3.548$ and $\mathcal{P} = \mathcal{P}_3$ with $e_1 = 0.28535$, we plot their power spectrum diagrams, and the results are shown in Figure 34(a), Figure 35(a) and Figure 36(a), respectively. From Figure 34(a), Figure 35(a) and Figure 36(a), we can clearly observe that the corresponding power spectrum for three SNAs exhibit a hybrid of discrete and continuous components. From Figure

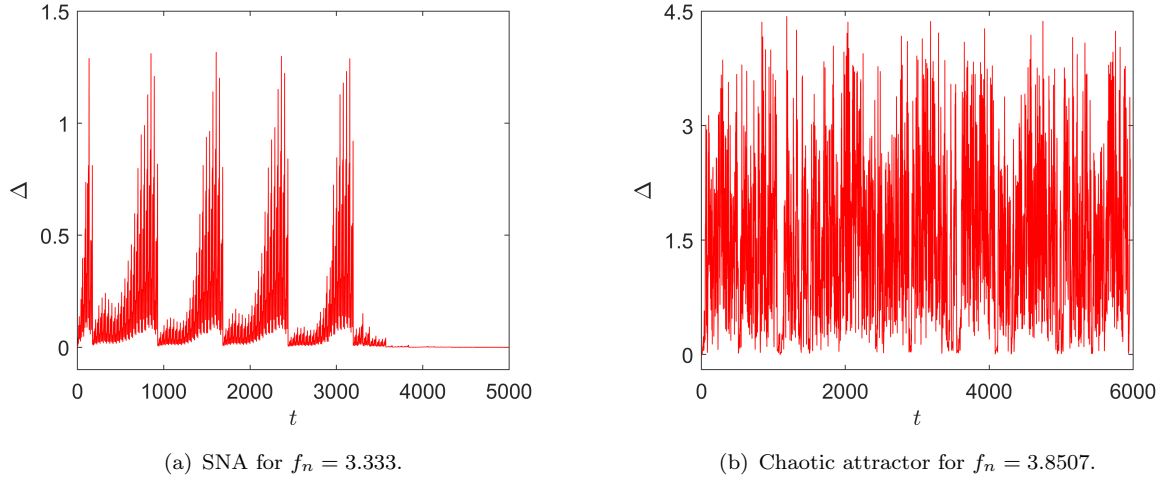


Figure 37. Separation of two nearby trajectories $(x_i(t), y_i(t))$, $(x_j(t), y_j(t))$ defined as $\Delta := \Delta(t) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ with $\mathcal{P} = \mathcal{P}_1$.

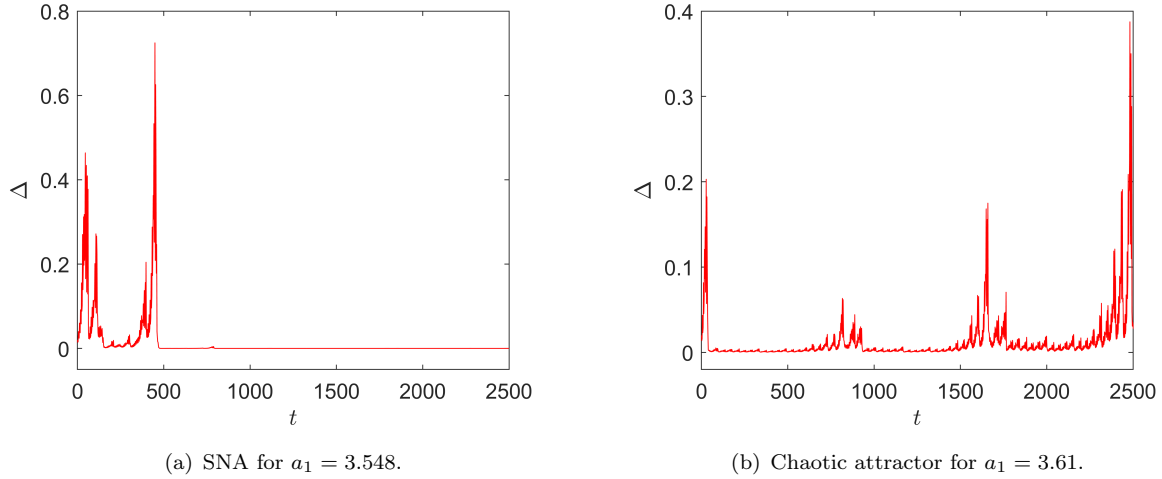


Figure 38. Separation of two nearby trajectories $(x_i(t), y_i(t))$, $(x_j(t), y_j(t))$ defined as $\Delta := \Delta(t) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ with $\mathcal{P} = \mathcal{P}_2$.

34(b), Figure 35(b) and Figure 36(b), the logarithm of $|X(\omega, N)|^2$ increases linearly with the logarithm of N for all three SNAs. By performing least square fitting, the scaling exponent η is calculated as 1.38 for $\mathcal{P} = \mathcal{P}_1$ at $f_n = 3.333$, 1.58 for $\mathcal{P} = \mathcal{P}_2$ at $a_1 = 3.548$, 1.82 for $\mathcal{P} = \mathcal{P}_3$ at $e_1 = 0.28535$. All these values fall within the interval $(1, 2)$. Meanwhile, the fractal paths in the plane of $(\text{Re}(X(\omega, N)), \text{Im}(X(\omega, N)))$ corresponding to the three cases are displayed in Figure 34(c), Figure 35(c) and Figure 36(c), respectively.

(4) Separation of nearby trajectories.

The method of separating nearby trajectories can effectively distinguish between chaotic attractors and SNAs. We select two initially close (x_i, y_i) and (x_j, y_j) , then calculate their separation as $\Delta := \Delta(t) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ by iterating the map P_s . If Δ exhibits

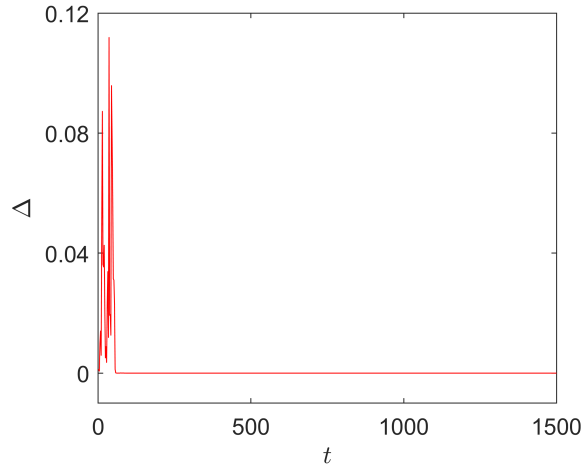


Figure 39. Separation of two nearby trajectories $(x_i(t), y_i(t))$, $(x_j(t), y_j(t))$ defined as $\Delta := \Delta(t) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ for SNA with $\mathcal{P} = \mathcal{P}_3$ and $e_1 = 0.28535$.

sustained and non-decaying fluctuations as $t \rightarrow \infty$, it indicates a chaotic attractor. If Δ decays to zero as $t \rightarrow \infty$, it corresponds to a periodic or quasi-periodic attractor. If Δ rapidly decreases and tends to zero, it shows an SNA.

When $\mathcal{P} = \mathcal{P}_1$, we take initial value $\Delta = 0.001$ and plot time-dependent evolution of Δ for SNA at $f_n = 3.333$ and for chaotic attractor at $f_n = 3.8507$ as illustrated in Figure 37(a) and Figure 37(b), respectively. When $\mathcal{P} = \mathcal{P}_2$, we take initial value $\Delta = 0.02$ and plot time-dependent evolution of Δ for SNA at $a_1 = 3.548$ and for chaotic attractor at $a_1 = 3.61$ as illustrated in Figure 38(a) and Figure 38(b), respectively. When $\mathcal{P} = \mathcal{P}_3$, we take initial value $\Delta = 0.001$ and plot time-dependent evolution of Δ for SNA at $e_1 = 0.28535$ as illustrated in Figure 39.

6. Conclusions

In this study, we investigated crossing bifurcation, switching-sliding bifurcation, and grazing-sliding bifurcation in the dry-friction system (2.4) under quasi-periodic excitation. Since the system is a piecewise-smooth nonlinear dry-friction system, its exact analytical solution is unavailable. Therefore, the present work is mainly carried out by means of numerical simulations. We first gave the criteria for the occurrence of crossing, switching-sliding and grazing-sliding at the switching manifold of system (2.4). Then we constructed the stroboscopic Poincaré map $P_s : \Sigma_s \mapsto \Sigma_s$, and further approximated the largest Lyapunov exponent. As confirmed by numerical simulations, adjusting the bifurcation induces a wealth of complex dynamical behaviours in system (2.4):

(1) For crossing and switching-sliding bifurcations, we observed that a succession of alternating states including chaotic attractors, SNAs along with complicated quasi-periodic cascades, such as torus-adding cascades and *weak torus-adding cascades*. SNA exists between a chaotic attractor and a quasi-periodic attractor via an evolution route of $\text{torus} \rightarrow \text{SNA} \rightarrow \text{chaos}$.

(2) For grazing-sliding bifurcation, we also observed that the pre-grazing three-torus quasi-periodic orbit gives way to an SNA.

The switching manifold acts as the inherent non-smooth boundary of the dry-friction system,

but it is not chosen as the bifurcation parameter in the present work. The primary objective of the present paper is to investigate the complex dynamical evolution induced by quasi-periodic excitation in the dry-friction oscillator. To separately and clearly reveal the evolutionary mechanisms corresponding to switching-sliding, crossing and grazing-sliding bifurcations, we select f_n , a_1 and e_1 as the bifurcation parameters for the three types of bifurcations, respectively. This parametric strategy allows us to concentrate on the dynamical evolution involving strange nonchaotic attractors, torus-adding cascades, and weak torus-adding cascades, and to fully demonstrate the rich dynamic phenomena generated by the coupling of quasi-periodic excitation and dry friction. In addition, three groups of parameters $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are selected by combining the physical background of the belt-driven dry-friction oscillator and extensive numerical experiments. The stiffness, damping, friction coefficient, belt speed and other mechanical parameters are set to physically reasonable values matching real friction transmission and braking devices, and two incommensurate excitation frequencies are adopted to satisfy the quasi-periodic condition in engineering applications.

To the best of our knowledge, the dynamical behaviors of quasi-periodically excited dry-friction systems, especially the SNAs induced by crossing, switching-sliding and grazing-sliding bifurcations, have not been explored in existing studies. In fact, compared with periodically excited systems, dry-friction systems under quasi-periodic excitation exhibit much richer dynamical behaviors. In future work, we will explore further dynamical behaviors of such systems, such as hysteresis effects, and theoretically investigate crossing-sliding and adding-sliding bifurcations in quasi-periodically excited dry-friction systems. We expect that the results in this paper will provide meaningful insights for related research.

Authors contributions.

Jiaojiao Zhou: Investigation, Methodology, Validation, Data curation, Software, Writing—original draft. Fei Luo: Conceptualization, Investigation, Methodology, Validation, Data curation, Writing—original draft, Writing – reviewing and editing. Jinkai Jiang: Investigation, Methodology, Validation, Data curation, Software, Writing—original draft. Yundong Li: Investigation, Methodology, Validation, Data curation, Software.

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