

UNIQUENESS OF GLOBALLY CONSERVATIVE WEAK SOLUTIONS FOR A WEAK DISSIPATION CAMASSA-HOLM TYPE EQUATION*

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Abstract In this paper, we prove the uniqueness of globally conservative solutions for Camassa-Holm type equation by the analysis of characteristics. Giving a conservative solution, we introduce a set of auxiliary variables tailored to this solution, and prove that these variables satisfy a suitable semilinear system of equations. By proving that this semilinear system has a unique solution, we eventually obtain the uniqueness of the conservative solution of the original equation in time weighted H^1 space.

Keywords Dispersive wave equations, globally conservative solutions, uniqueness.

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1. Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}$$

has stood as a central model in nonlinear dispersive wave theory since its discovery [6]. It is completely integrable [6, 8], admits a bi-Hamiltonian structure [12], and exhibits the remarkable phenomenon of wave breaking, in which the solution remains bounded while its spatial derivative blows up in finite time. This feature, which captures the physical reality of wave overturning, has motivated extensive investigations into local well-posedness [9, 19], wave breaking phenomena [9, 10], and the construction of global weak solutions that continue beyond the breaking point [4, 5, 15].

A natural extension that incorporates weak dissipation and higher-order nonlinearities is given by

$$u_t - u_{txx} + \lambda(u - u_{xx}) + 3u^2u_x = uu_{xxx} + 2u_xu_{xx}, \quad (1.1)$$

where the parameter $\lambda > 0$ models a dissipative effect. In 2020, Freire et al. [13] established the local well-posedness of (1.1) in $H^s(\mathbb{R})$ with $s > 3/2$ using Kato's theory. Through Lie symmetry analysis, they derived conserved quantities and provided a wave breaking criterion: If the initial slope is sufficiently negative, the solution blows up in finite time. Moreover, they showed that the H^1 -norm decays as $\|u(t)\|_{H^1} = e^{-\lambda t}\|u_0\|_{H^1}$, a result crucial for global energy estimates.

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Over the past three decades, the CH equation and its various generalizations have been extensively studied due to their many remarkable properties, see [1, 2, 7, 14, 16, 17, 20, 21] and the reference therein. However, these studies have mainly focused on solution behavior before and during wave breaking. Given that finite-time singularities may arise, it is natural to investigate the behavior of solutions beyond the breaking point. In 2007, using the method of characteristics, Bressan and Constantin first proved that solutions of the CH equation can be continued either as global conservative weak solutions [4] or as global dissipative weak solutions [5]. Nevertheless, a fundamental issue has remained open: Is the global conservative solution unique? The existence results mentioned above did not address uniqueness, leaving a gap in the well-posedness theory. Bressan, Chen and Zhang [3] firstly investigated the uniqueness of conservative solutions for the CH equation using the characteristic method. Following the ideas of [3, 4], Zhou et al. [22, 23] studied the existence and uniqueness of global conservative weak solutions for a class of nonlinear dispersive wave equations.

Following the approach of [4], the global existence of conservative weak solutions for (1.1) after wave breaking was addressed by Deng and Yin [11]. The present paper continues this line of inquiry. Our goal is to prove the uniqueness of global conservative weak solutions for (1.1) in a time-weighted H^1 space. To this end, we introduce the transformation $v = e^{\lambda t}u$, which conserves the H^1 -norm: $\|v\|_{H^1} = \|u_0\|_{H^1}$. As a result, the problem of existence and uniqueness for (1.1) in the time-weighted H^1 space is equivalent to that for the following equation:

$$v_t + e^{-\lambda t}vv_x + \partial_x \Lambda^{-2} \left(-\frac{1}{2}e^{-\lambda t}v^2 + \frac{1}{2}e^{-\lambda t}v_x^2 + e^{-2\lambda t}v^3 \right) = 0. \quad (1.2)$$

Equation (1.1) is equivalent to (1.2). Hence, in this paper we focus on proving that (1.2) admits a unique global conservative solution.

The rest of this paper is organized as follows. In Section 2, we reformulate equation (1.2), review basic definitions and state our main uniqueness result. In Section 3, we prove the uniqueness of characteristic. In Section 4, we give the proof of our main theorem.

Notation. Throughout this paper, all spaces of functions are over $\mathbb{R}$, and for simplicity, we drop $\mathbb{R}$ in our notation of function spaces if there is no confusion. All spaces of functions are over $\mathbb{R}$ and for simplicity, we drop $\mathbb{R}$ in our notation of function spaces if there is no ambiguity. Additionally, we denote by $\|\cdot\|_s$ the norm in the Sobolev space $H^s(\mathbb{R})$, and denote by $\|\cdot\|_{L^p}$ the norm in the Lebesgue space $L^p(\mathbb{R})$.

2. Basic definitions and results

In this section, we focus on the following equation

$$v_t + e^{-\lambda t}vv_x = -P_x, \quad (2.1)$$

which is equivalent to (1.2), where the source term P is given by

$$P = \frac{1}{2}e^{-|x|} * \left(-\frac{1}{2}e^{-\lambda t}v^2 + \frac{1}{2}e^{-\lambda t}v_x^2 + e^{-2\lambda t}v^3 \right). \quad (2.2)$$

For smooth solutions, differentiating (2.1) with respect to x and applying the identity $p_{xx} * h = p * h - h$, we derive

$$v_{tx} + e^{-\lambda t}vv_{xx} + \frac{1}{2}e^{-\lambda t}v_x^2 + \frac{1}{2}e^{-\lambda t}v^2 - e^{-2\lambda t}v^3 = -P. \quad (2.3)$$

Multiplying (2.1) by $2v$ and (2.3) by $2v_x$ respectively yields

$$(v^2)_t + \left(\frac{2}{3} e^{-\lambda t} v^3 + 2vP \right)_x = 2v_x P, \quad (2.4)$$

$$(v_x^2)_t + \left(e^{-\lambda t} v v_x^2 - \frac{1}{2} e^{-2\lambda t} v^4 + \frac{1}{3} e^{-\lambda t} v^3 \right)_x = -2v_x P. \quad (2.5)$$

It follows that

$$E(t) = \int_{\mathbb{R}} (v^2(t, x) + v_x^2(t, x)) dx = E(0). \quad (2.6)$$

For $\bar{v} \in H^1$, Young's inequality implies

$$\begin{aligned} \|P\|_{L^\infty} &\leq \frac{1}{2} \|e^{-|x|}\|_{L^\infty} \left\| -\frac{1}{2} e^{-\lambda t} v^2 + \frac{1}{2} e^{-\lambda t} v_x^2 \right\|_{L^1} + \frac{1}{2} \|e^{-|x|}\|_{L^1} \|e^{-2\lambda t} v^3\|_{L^\infty} \\ &\leq C(E(0) + \sqrt{E(0)}). \end{aligned} \quad (2.7)$$

Similarly, one can show that

$$\|P_x\|_{L^\infty}, \|P\|_{L^2}, \|P_x\|_{L^2} \leq C(E(0) + \sqrt{E(0)}).$$

Definition 2.1. A function $v(t, x)$ is called a conservative solution to (1.2) with initial data $\bar{v}(x)$ if the following hold:

- (i) The mapping $t \rightarrow v(t)$ is Lipschitz continuous from $\mathbb{R}$ to $L^2(\mathbb{R})$ with $v(t, \cdot) \in H^1(\mathbb{R})$ for all $t \geq 0$.
- (ii) The solution $v = v(t, x)$ satisfies the initial datum $\bar{v}(x) \in H^1(\mathbb{R})$ and

$$\int \int_{\Delta} \left(v_x \psi_t + \frac{e^{-\lambda t}}{2} v^2 \psi_x - P_x \psi \right) dx dt + \int_{\mathbb{R}} v(0, x) \psi(0, x) dx = 0, \quad (2.8)$$

for every test function $\psi \in C_c^\infty(\Delta)$, where $\Delta = \{(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$.

- (iii) The balance law (2.5) is satisfied in the following sense.

There exists a family of Radon measures $\{\mu_{(t)} : t \in \mathbb{R}_+\}$, depending continuously on time with respect to the weak convergence topology of measures. For each $t \in \mathbb{R}_+$, $\mu_{(t)}$ decomposes into an absolutely continuous part and a singular part, with the absolutely continuous part having density $v_x(t, \cdot)$ with respect to Lebesgue measure. Moreover, the measure $\mu_{(t)}$ satisfies the balance law (2.5) in the distributional sense, namely

$$\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} (\psi_t + e^{-\lambda t} v \psi_x) d\mu_{(t)} + 2 \int_{\mathbb{R}} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x \psi dx \right) dt + \int_{\mathbb{R}} v_{0x}^2 \psi_0(x) dx = 0 \quad (2.9)$$

for any test function $\psi \in C_c^1(\Delta)$.

Note: The measure $\mu_{(t)}$ corresponds only to the gradient part v_x^2 , rather than the full total energy measure. The full conserved energy satisfies $\int_{\mathbb{R}} (v^2 + v_x^2) dx = \text{constant}$, see Eq. 2.6.

In the work [11], Deng and Yin established the existence of global conservative solutions to equation (1.2) as stated below.

Theorem 2.1. *Assume the initial datum $\bar{v}(x) \in H^1(\mathbb{R})$. Then problem (2.1) with initial datum $\bar{v}(x)$ admits a global conservative solution $v(t, x)$ in the sense of Definition 2.1. Furthermore, the solution $v(t, x)$ enjoys the following properties:*

- (i) *$v(t, x)$ is uniformly Hölder continuous with exponent $\frac{1}{2}$ on both t and x .*
- (ii) *The energy density $v^2 + v_x^2$ is essentially conserved, i.e.,*

$$\|v\|_1^2 = \|\bar{v}\|_1^2, \quad \text{for a.e. } t \in \mathbb{R}_+. \quad (2.10)$$

- (iii) *The solution $v(t, x)$ depends continuously on the initial datum $\bar{v}(x)$. More precisely, let $\bar{v}_n$ be a sequence of initial datum such that*

$$\|\bar{v}_n - \bar{v}\|_1 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then the corresponding solutions $v_n(t, x)$ converge to $v(t, x)$ uniformly on all bounded subsets of (t, x) .

There exists a null set $\mathcal{N} \subset \mathbb{R}$ with $\text{meas}(\mathcal{N}) = 0$ such that for every $t \notin \mathcal{N}$ the measure $\mu_{(t)}$ is absolutely continuous and has density $v_x^2(t, \cdot)$ with respect to Lebesgue measure. At a time $t \in \mathcal{N}$, the measure $\mu_{(t)}$ carries a nontrivial singular component. Since conservative solutions v are generally nonsmooth, we only know that the energy $E(t)$ in (2.6) coincides with a constant, that is,

$$E(t) = E(0) \quad \text{for } t \notin \mathcal{N}, \quad E(t) < E(0) \quad \text{for } t \in \mathcal{N}.$$

We now present the uniqueness result for such conservative solutions.

Theorem 2.2. *Given initial datum $v_0 \in H^1(\mathbb{R})$, there exists a unique global conservative solution $v(t, x)$ to (1.2) satisfies properties (i)–(iii) in Theorem 2.1.*

3. Preliminary lemmas

In this section, we turn to proving the uniqueness of global conservative solutions.

At any time t and for any ξ , define $x(t, \xi)$ to be the unique x such that

$$x(t, \xi) + \mu_{(t)}\{(-\infty, x)\} \leq \xi \leq x(t, \xi) + \mu_{(t)}\{(-\infty, x]\}. \quad (3.1)$$

Accordingly,

$$\xi = x(t, \xi) + \mu_{(t)}\{(-\infty, x)\} + \theta \mu_{(t)}\{x\} \quad (3.2)$$

for some $\theta \in [0, 1]$.

When $\mu_{(t)}$ is absolutely continuous with density v_x^2 , the above definition reduces to

$$\xi = x(t, \xi) + \int_{-\infty}^{x(t, \xi)} v_x^2(t, \tau) d\tau. \quad (3.3)$$

The Lipschitz continuity of x and v as functions of t and ξ is established through a sequence of lemmas.

Lemma 3.1. *Let $v = v(t, x)$ be a conservative solution of (2.1). Then, for any fixed $t \geq 0$, the maps*

$$\xi \mapsto x(t, \xi) \quad \text{and} \quad \xi \mapsto v(t, \xi) := v(t, x(t, \xi))$$

implicitly defined by (3.1) are Lipschitz continuous with Lipschitz constant 1. The map $t \mapsto x(t, \xi)$ is also Lipschitz continuous with a constant depending only on $\|v_0\|_1$.

Proof. The proof proceeds in three steps.

Step 1. Fix any $t \geq 0$. The map $x \mapsto \xi(t, x)$ is right-continuous and strictly increasing, so it admits a well-defined, continuous, nondecreasing inverse $\xi \mapsto x(t, \xi)$. If $\xi_1 < \xi_2$, then

$$x(t, \xi_2) - x(t, \xi_1) + \mu_{(t)}\{(x(t, \xi_1), x(t, \xi_2))\} \leq \xi_2 - \xi_1, \quad (3.4)$$

which implies

$$x(t, \xi_2) - x(t, \xi_1) \leq \xi_2 - \xi_1. \quad (3.5)$$

This shows that $\xi \mapsto x(t, \xi)$ is a contraction.

Step 2. We claim that $\xi \mapsto v(t, x(t, \xi))$ is Lipschitz continuous. Indeed, for $\xi_1 < \xi_2$, one have

$$\begin{aligned} |v(t, x(t, \xi_2)) - v(t, x(t, \xi_1))| &\leq \int_{x(t, \xi_1)}^{x(t, \xi_2)} |v_x| dy \\ &\leq \frac{1}{2} \int_{x(t, \xi_1)}^{x(t, \xi_2)} (1 + v_x^2) dy \\ &\leq \frac{1}{2} (x(t, \xi_2) - x(t, \xi_1) + \mu_{(t)}\{(x(t, \xi_1), x(t, \xi_2))\}) \\ &\leq \frac{1}{2} (\xi_2 - \xi_1). \end{aligned} \quad (3.6)$$

Step 3. We prove that $t \mapsto x(t, \xi)$ is Lipschitz continuous. Let $x(\tau, \xi) = y$. From (2.9), one get

$$\left\| 2 \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x \right\|_{L^1} \leq 2 \|v_x\|_{L^2} \left\| e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right\|_{L^2} \leq C_k, \quad (3.7)$$

where C_k is a constant depending only on the H^1 norm of v . By the Sobolev embedding $\|v\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|v\|_1$, for $t \geq \tau$ we have

$$\mu_{(t)}\{(-\infty, y - C_\infty(t - \tau))\} \leq \mu_{(\tau)}\{(-\infty, y)\} + C_k(t - \tau),$$

with $C_\infty = \frac{1}{\sqrt{2}} \|v\|_1$.

Define $y^-(t) := y - (C_\infty + C_k)(t - \tau)$. Then

$$\begin{aligned} y^-(t) + \mu_{(t)}\{(-\infty, y^-(t))\} &\leq y - (C_\infty + C_k)(t - \tau) + \mu_{(\tau)}\{(-\infty, y)\} + C_k(t - \tau) \\ &\leq y + \mu_{(\tau)}\{(-\infty, y)\} \\ &\leq \xi, \end{aligned} \quad (3.8)$$

which implies $x(t, \xi) \geq y^-$ for all $t > \tau$.

Similarly, one can show $x(t, \xi) \leq y^+ := y + (C_\infty + C_k)(t - \tau)$. Thus the uniform Lipschitz continuity of $t \mapsto x(t, \xi)$ is verified. This completes the proof of Lemma 3.1. $\square$

In what follows, we show that characteristic curves are uniquely determined by an integral equation that combines the characteristic ODE with the balance law for $\mu_{(t)}$.

Lemma 3.2. *Let $v = v(t, x)$ be a conservative solution of (2.1). Then, for any $x_0 \in \mathbb{R}$, there exists a unique Lipschitz continuous map $t \mapsto x(t)$ satisfying*

$$\frac{dx(t)}{dt} = e^{-\lambda t} v(t, x(t)), \quad x(0) = x_0 \quad (3.9)$$

and

$$\frac{d}{dt} \int_{-\infty}^{x(t,\xi)} v_x^2 dx = 2 \int_{-\infty}^{x(t,\xi)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx. \quad (3.10)$$

In addition, for any $0 \leq \tau \leq t$,

$$v(t, x(t)) - v(\tau, x(\tau)) = - \int_{\tau}^t P_x(s, x(s)) ds. \quad (3.11)$$

Proof. The argument is divided into six steps.

Step 1. In the adapted coordinates (t, ξ) , we represent the characteristic emanating from x_0 as $t \mapsto x(t, \xi(t))$, where $\xi(\cdot)$ is a map to be determined. Combining (3.9)–(3.10) and integrating in time yields

$$\begin{aligned} \xi(t) &= x(t) + \mu_{(t)}\{(-\infty, x(t))\} + \theta(t)\mu_{(t)}\{x(t)\} \\ &= \bar{\xi} + \int_0^t \int_{-\infty}^{x(s)} \left(e^{-\lambda s} v_x + 2 \left(e^{-2\lambda s} v^3 - \frac{1}{2} e^{-\lambda s} v^2 - P \right) v_x \right) dx ds, \end{aligned} \quad (3.12)$$

where $\bar{\xi} = x_0 + \int_{-\infty}^{x_0} v_{0x}^2 dx$.

Introduce the function

$$G(t, \xi) = \int_{-\infty}^{x(t,\xi)} \left(e^{-\lambda s} v_x + 2 \left(e^{-2\lambda s} v^3 - \frac{1}{2} e^{-\lambda s} v^2 - P \right) v_x \right) dx. \quad (3.13)$$

Then equation (3.12) is reduced to

$$\xi(t) = \bar{\xi} + \int_0^t G(s, \xi(s)) ds. \quad (3.14)$$

Step 2. We claim that $G(t, \xi)$ is uniformly Lipschitz continuous in ξ for each fixed $t \geq 0$. From (3.2) and (3.13),

$$\begin{aligned} G_{\xi}(t, \xi) &= \left(e^{-\lambda t} v_x + 2 \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x \right) x_{\xi} \\ &= \frac{e^{-\lambda t} v_x + 2 \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x}{1 + v_x^2} \\ &\in [-C, C], \quad \mu_{(t)}\{x(t, \xi)\} = 0, \end{aligned} \quad (3.15)$$

for some positive constant C depending only on the H^1 norm of v . If $\mu_{(t)}\{x(t, \xi)\} \neq 0$, then $G_{\xi}(t, \xi) = 0$.

Moreover, the definition of $G(t, \xi)$ implies that $\xi \mapsto G(t, \xi)$ is uniformly bounded and absolutely continuous.

Step 3. Thanks to the Lipschitz continuity of $G(t, \xi)$, a standard fixed point argument ensures that the integral equation (3.14) has a unique solution. The details are analogous to those in [3] and are omitted.

Step 4. We verify that $t \mapsto x(t) := x(t, \xi(t))$ solves (3.9). By construction, $t \mapsto x(t)$ provides the unique solution to (3.12). Since both $\xi(t)$ and $x(t, \xi(t))$ are Lipschitz continuous, they are differentiable almost everywhere. We only consider times at which $x(t)$ is differentiable.

We assert that (3.9) holds for almost every time. Suppose otherwise, so that

$$x'(\tau) \neq e^{-\lambda\tau}v(\tau, x(\tau)).$$

Without loss of generality, assume

$$x'(\tau) = e^{-\lambda\tau}v(\tau, x(\tau)) + 2\epsilon_0$$

for some $\epsilon_0 > 0$. The case $\epsilon_0 < 0$ is entirely similar. To derive a contradiction, choose $\delta > 0$ small enough so that

$$x^+(t) = x(\tau) + (t - \tau) \left(e^{-\lambda\tau}v(\tau, x(\tau)) + \epsilon_0 \right) < x(t), \quad t \in (\tau, \tau + \delta]. \quad (3.16)$$

Note that (2.9) remains valid for Lipschitz continuous test functions ψ with compact support.

For small $\epsilon > 0$, we use the test functions from [3]:

$$\rho^\epsilon(s, y) = \begin{cases} 0, & y \leq -\epsilon^{-1}, \\ y + \epsilon^{-1}, & -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\ 1, & 1 - \epsilon^{-1} \leq y \leq x^+(s), \\ 1 - \epsilon^{-1}(y - x(s)), & x^+(s) \leq y \leq x^+(s) + \epsilon, \\ 0, & y \geq x^+(s) + \epsilon, \end{cases} \quad (3.17)$$

$$\chi^\epsilon(s) = \begin{cases} 0, & s \leq \tau - \epsilon, \\ \epsilon^{-1}(s - \tau + \epsilon), & \tau - \epsilon \leq s \leq \tau, \\ 1, & \tau \leq s \leq t, \\ 1 - \epsilon^{-1}(s - t), & t \leq s \leq t + \epsilon, \\ 0, & s \geq t + \epsilon. \end{cases} \quad (3.18)$$

Define

$$\varphi^\epsilon(s, y) := \min\{\rho^\epsilon(s, y), \chi^\epsilon(s)\}.$$

Substituting φ^ϵ into (2.9) gives

$$\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} \left(\varphi_t^\epsilon + e^{-\lambda t} v \varphi_x^\epsilon \right) d\mu_{(t)} + 2 \int_{\mathbb{R}} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x \varphi^\epsilon dx \right) dt = 0. \quad (3.19)$$

For $|t - \tau|$ sufficiently small and $s \in [\tau + \epsilon, t - \epsilon]$,

$$0 = \varphi_t^\epsilon + (e^{-\lambda t} v(\tau, x(\tau)) + \epsilon_0) \varphi_x^\epsilon \leq \varphi_t^\epsilon + e^{-\lambda t} v(s, x) \varphi_x^\epsilon, \quad (3.20)$$

where we used the Hölder continuity of v and $\varphi_x^\epsilon \leq 0$. It follows that

$$\lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} \left(\varphi_t^\epsilon + e^{-\lambda t} v \varphi_x^\epsilon \right) d\mu_{(s)} ds \geq 0. \quad (3.21)$$

Since the family $\mu_{(t)}$ depends continuously on t in the weak topology, taking $\epsilon \rightarrow 0$ in (3.19) yields

$$0 = -\mu_{(t)}\{(-\infty, x^+(t)]\} + \mu_{(\tau)}\{(-\infty, x^+(\tau)]\} + \lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} \left(\varphi_t^\epsilon + e^{-\lambda t} v \varphi_x^\epsilon \right) d\mu_{(s)} ds$$

$$\begin{aligned}
& + 2 \int_{\tau}^t \int_{-\infty}^{x^+(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds \\
& \geq 2 \int_{\tau}^t \int_{-\infty}^{x^+(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds - \mu_{(t)}\{(-\infty, x^+(t))\} + \mu_{(\tau)}\{(-\infty, x^+(\tau))\} \\
& = 2 \int_{\tau}^t \int_{-\infty}^{x(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds \\
& \quad - 2 \int_{\tau}^t \int_{x^+(s)}^{x(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds \\
& \quad - \mu_{(t)}\{(-\infty, x^+(t))\} + \mu_{(\tau)}\{(-\infty, x^+(\tau))\}. \tag{3.22}
\end{aligned}$$

Using (3.16) and the Lipschitz continuity of $t \mapsto x(t)$, we deduce

$$\begin{aligned}
|o_1(t - \tau)| & = 2 \left| \int_{\tau}^t \int_{x^+(s)}^{x(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds \right| \\
& \leq 2 \left\| \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) \right\|_{L^\infty} \int_{\tau}^t \int_{x^+(s)}^{x(s)} |v_x| dx ds \\
& \leq 2 \left\| \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) \right\|_{L^\infty} \|v_x\|_{L^2} \int_{\tau}^t (x(s) - x^+(s))^{\frac{1}{2}} ds \\
& \leq C(t - \tau)^{\frac{3}{2}}, \tag{3.23}
\end{aligned}$$

where $\frac{o_1(t-\tau)}{t-\tau} \rightarrow 0$ as $t \rightarrow \tau$.

Combining (3.22) and (3.23),

$$\begin{aligned}
& \mu_{(t)}\{(-\infty, x^+(t))\} \\
& \geq 2 \int_{\tau}^t \int_{-\infty}^{x(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds + \mu_{(\tau)}\{(-\infty, x^+(\tau))\} + o_1(t - \tau). \tag{3.24}
\end{aligned}$$

From (3.16) and (3.24), for t close enough to τ ,

$$\begin{aligned}
\xi(t) & = x(t) + \mu_{(t)}\{(-\infty, x(t))\} + \theta(t)\mu_{(t)}\{x(t)\} \\
& \geq x(\tau) + (t - \tau)(e^{-\lambda t} v(\tau, x(\tau)) + \epsilon_0) + \mu_{(\tau)}\{(-\infty, x(\tau))\} \\
& \quad + 2 \int_{\tau}^t \int_{-\infty}^{x(s)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx ds + o_1(t - \tau). \tag{3.25}
\end{aligned}$$

On the other hand, a linear approximation of (3.13)–(3.14) gives

$$\xi(t) = \xi(\tau) + (t - \tau) \left(e^{-\lambda t} v(\tau, x(\tau)) + 2 \int_{-\infty}^{x(\tau)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx \right) + o_2(t - \tau) \tag{3.26}$$

with $\frac{o_2(t-\tau)}{t-\tau} \rightarrow 0$ as $t \rightarrow \tau$.

From (3.25) and (3.26), we get

$$\xi(\tau) + (t - \tau) \left(e^{-\lambda t} v(\tau, x(\tau)) + 2 \int_{-\infty}^{x(\tau)} \left(e^{-2\lambda t} v^3 - \frac{1}{2} e^{-\lambda t} v^2 - P \right) v_x dx \right) + o_2(t - \tau)$$

$$\begin{aligned} &\geq x(\tau) + (t - \tau)(e^{-\lambda t}v(\tau, x(\tau)) + \epsilon_0) + \mu_{(\tau)}\{(-\infty, x(\tau)]\} \\ &\quad + 2 \int_{\tau}^t \int_{-\infty}^{x(s)} \left(e^{-2\lambda t}v^3 - \frac{1}{2}e^{-\lambda t}v^2 - P \right) v_x dx ds + o_1(t - \tau). \end{aligned} \quad (3.27)$$

Subtracting common terms, dividing both sides by $t - \tau$ and letting $t \rightarrow \tau$, then we obtain a contradiction. Thus it must be case in (3.9).

Step 5. We now prove (3.11). For any test function $\psi \in C_c^\infty$,

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(v\psi_t + \frac{e^{-\lambda t}}{2}v^2\psi_x - P_x\psi \right) dx dt + \int_{\mathbb{R}} v_0(x)\psi(0, x) dx = 0. \quad (3.28)$$

Let $\phi_x = \psi$ with $\phi \in C_c^\infty$. Integrating by parts in x and using the absolute continuity of $x \mapsto v(t, x)$, one get

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(v_x\phi_t + e^{-\lambda t}vv_x\phi_x - P_x\phi_x \right) dx dt + \int_{\mathbb{R}} v_{0x}(x)\phi(0, x) dx = 0. \quad (3.29)$$

By approximation, this identity extends to all Lipschitz continuous test functions with compact support.

For small $\epsilon > 0$, define

$$\omega^\epsilon(s, y) = \begin{cases} 0, & y \leq -\epsilon^{-1}, \\ y + \epsilon^{-1}, & -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\ 1, & 1 - \epsilon^{-1} \leq y \leq x(s), \\ 1 - \epsilon^{-1}(y - x(s)), & x(s) \leq y \leq x(s) + \epsilon, \\ 0, & y \geq x(s) + \epsilon, \end{cases}$$

and

$$\Phi^\epsilon(s, y) := \min\{\omega^\epsilon(s, y), \chi^\epsilon(s)\}, \quad (3.30)$$

where $\chi^\epsilon(s)$ is given in (3.18). Substitute Φ^ϵ into (3.29) and let $\epsilon \rightarrow 0$. Since P_x is continuous,

$$\begin{aligned} \int_{-\infty}^{x(t)} v_x(t, x) dx &= \int_{-\infty}^{x(\tau)} v_x(\tau, x) dx - \int_{\tau}^t P_x(s, x(s)) ds \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t}v\Phi_x^\epsilon \right) v_x dx ds. \end{aligned} \quad (3.31)$$

To obtain (3.11), it suffices to show that

$$\lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t}v\Phi_x^\epsilon \right) v_x dx ds = 0. \quad (3.32)$$

In what follows, we divided (3.32) into two parts.

$$\lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t}v\Phi_x^\epsilon \right) v_x dx ds$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{\tau} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds + \lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_t^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds.
\end{aligned} \tag{3.33}$$

First, we estimate the first and third terms as follows:

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \left| \int_{\tau-\epsilon}^{\tau} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds \right| + \lim_{\epsilon \rightarrow 0} \left| \int_t^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds \right| \\
&\leq \lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{\tau} \left(\int_{x(s)}^{x(s)+\epsilon} v_x^2 dx \right)^{\frac{1}{2}} \left(\int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right)^2 dx \right)^{\frac{1}{2}} ds \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_t^{t+\epsilon} \left(\int_{x(s)}^{x(s)+\epsilon} v_x^2 dx \right)^{\frac{1}{2}} \left(\int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right)^2 dx \right)^{\frac{1}{2}} ds \\
&\leq 2 \lim_{\epsilon \rightarrow 0} \epsilon \|v\|_1 \left(\int_{x(s)}^{x(s)+\epsilon} 4\epsilon^{-2} (1 + \|v\|_{L^\infty})^2 dx \right)^{\frac{1}{2}} \\
&\leq C \epsilon^{\frac{1}{2}} \\
&\rightarrow 0.
\end{aligned} \tag{3.34}$$

Second, we estimate the second term as follows:

$$\begin{aligned}
&\left| \int_{\tau}^t \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right) v_x dx ds \right| \\
&\leq \int_{\tau}^t \left(\int_{x(s)}^{x(s)+\epsilon} v_x^2 dx \right)^{\frac{1}{2}} \left(\int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right)^2 dx \right)^{\frac{1}{2}} ds.
\end{aligned} \tag{3.35}$$

In what follows, we introduce the function

$$\nu^\epsilon(s) = \left(\sup_{x \in \mathbb{R}} \int_{x(s)}^{x(s)+\epsilon} v_x^2(s, y) dy \right)^{\frac{1}{2}}. \tag{3.36}$$

Owing to $v \in H^1$, we obtain that all functions $\nu^\epsilon(s)$ are uniformly bounded. Furthermore, for $\epsilon \rightarrow 0$, we have $\nu^\epsilon(s) \rightarrow 0$ pointwise at a.e. time t . Therefore,

$$\lim_{\epsilon \rightarrow 0} \int_{\tau}^t \left(\int_{x(s)}^{x(s)+\epsilon} v_x^2 dx \right)^{\frac{1}{2}} ds \leq \lim_{\epsilon \rightarrow 0} \int_{\tau}^t \nu^\epsilon(s) ds = 0. \tag{3.37}$$

In addition, for every time $s \in [\tau, t]$, by the construction of Φ^ϵ one has

$$\Phi_x^\epsilon(s, y) = \epsilon^{-1}, \quad \Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon = 0, \quad x(s) < y < x(s) + \epsilon. \tag{3.38}$$

Then

$$\int_{x(s)}^{x(s)+\epsilon} \left| \Phi_t^\epsilon + e^{-\lambda t} v \Phi_x^\epsilon \right|^2 dy = \epsilon^{-2} \int_{x(s)}^{x(s)+\epsilon} |v(s, y) - v(s, x(s))|^2 dy$$

$$\begin{aligned}
&\leq \epsilon^{-1} \left(\max_{x(s) \leq y \leq x(s)+\epsilon} |v(s, y) - v(s, x(s))| \right)^2 \\
&\leq \epsilon^{-1} \left(\int_{x(s)}^{x(s)+\epsilon} |v_x(s, y)| dy \right)^2 \\
&\leq C_1 \epsilon^{-1} \left(\int_{x(s)}^{x(s)+\epsilon} |v_x(s, y)| dy \right)^2 \\
&\leq C_1 \epsilon^{-1} (\epsilon^{\frac{1}{2}} \|v_x\|_{L^2})^2 \\
&\leq C \|v\|_1^2.
\end{aligned} \tag{3.39}$$

Combining (3.37) and (3.39), we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x(s)}^{x(s)+\epsilon} \left(\Phi_t^{\epsilon} + e^{-\lambda t} v \Phi_x^{\epsilon} \right) v_x dx ds = 0. \tag{3.40}$$

The above analysis proves that (3.20) holds.

Step 6. Following arguments identical to those in [3], we obtain the uniqueness of $x(t)$.

This finishes the proof of Lemma 3.2. $\square$

Analogously to [3], one can readily prove the Lipschitz continuity of u in t under the new coordinates, as well as the Lipschitz dependence of solutions $\xi(\cdot)$ to (3.14) on initial data. We state the following lemmas without proof.

Lemma 3.3. *Let $v = v(t, x)$ be a conservative solution of (2.1). Then the map*

$$(t, \xi) \mapsto v(t, \xi) := v(t, x(t, \xi))$$

is Lipschitz continuous with a constant depending only on $\|v_0\|_1$.

Lemma 3.4. *Let $v = v(t, x)$ be a conservative solution of (2.1). Denote by $t \mapsto \xi(t; \tau, \bar{\xi})$ the solution to the integral equation*

$$\xi(t) = \bar{\xi} + \int_{\tau}^t G(\tau, \xi(\tau)) d\tau. \tag{3.41}$$

4. Proof of Theorem 2.2

In this section, we establish the uniqueness of conservative solutions to equation (1.1). The proof is divided into six steps.

Step 1. From Lemmas 3.1 and 3.3, the map $(t, \xi) \mapsto (x, v)(t, \xi)$ is Lipschitz continuous. By the same reasoning, one have

$$\xi \mapsto G(t, \xi) := G(t, x(t, \xi)) \quad \text{and} \quad \xi \mapsto P_x(t, \xi) := P_x(t, x(t, \xi))$$

are also Lipschitz continuous. By Rademacher's theorem, the partial derivatives $x_t, x_{\xi}, v_t, v_{\xi}, P_{x, \xi}$ exist almost everywhere, and almost every point (t, ξ) is a Lebesgue point for these derivatives. Let $t \mapsto \xi(t, \bar{\xi})$ be the unique solution to (3.14). Then for almost every $\bar{\xi}$, the following holds:

(GC) For almost every $t > 0$, the point $(t, \xi(t, \bar{\xi}))$ is a Lebesgue point for $x_t, x_{\xi}, v_t, v_{\xi}, P_{x, \xi}$, and

$x_\xi(t, \xi(t, \bar{\xi})) > 0$ for a.e. $t > 0$.

If (GC) holds, we call $t \mapsto \xi(t, \bar{\xi})$ a good characteristic.

Step 2. We derive an ODE system describing the evolution of v_ξ and x_ξ along good characteristics. As in Lemma 3.4, let $t \mapsto \xi(t; \tau, \bar{\xi})$ solve (3.41). For $\tau, t \notin \mathcal{N}$, assume $\xi(\cdot; \tau, \bar{\xi})$ is a good characteristic. Differentiating (3.41) with respect to $\bar{\xi}$, one obtain

$$\frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) = 1 + \int_\tau^t G_\xi(s, \xi(s; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(s, \xi(s; \tau, \bar{\xi})) ds. \quad (4.1)$$

Integrating (3.9) over (τ, t) ,

$$x(t, \xi(t; \tau, \bar{\xi})) = x(\tau, \bar{\xi}) + \int_\tau^t e^{-\lambda s} v(s, \xi(s; \tau, \bar{\xi})) ds, \quad (4.2)$$

where $v(s, \xi(s; \tau, \bar{\xi})) = v(s, x(\xi(s; \tau, \bar{\xi})))$.

Differentiating (4.2) in $\bar{\xi}$,

$$x_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) = x_\xi(\tau, \bar{\xi}) + \int_\tau^t e^{-\lambda s} v_\xi(s, \xi(s; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(s; \tau, \bar{\xi}) ds. \quad (4.3)$$

Similarly, differentiating (3.11) with respect to $\bar{\xi}$, we get

$$v_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) = v_\xi(\tau, \bar{\xi}) - \int_\tau^t P_{x, \xi}(s, \xi(s; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(s; \tau, \bar{\xi}) ds. \quad (4.4)$$

From (4.1), (4.3), and (4.4), we obtain the ODE system

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) \right) = G_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t, \xi(t; \tau, \bar{\xi})), \\ \frac{d}{dt} \left(x_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) \right) = e^{-\lambda t} v_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}), \\ \frac{d}{dt} \left(v_\xi(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}) \right) = -P_{x, \xi}(t, \xi(t; \tau, \bar{\xi})) \frac{\partial}{\partial \bar{\xi}} \xi(t; \tau, \bar{\xi}). \end{cases} \quad (4.5)$$

In particular, the bracketed terms on the left are absolutely continuous. Using

$$P_{xx} = P - \left(-\frac{1}{2} e^{-\lambda t} v^2 + \frac{1}{2} e^{-\lambda t} v_x^2 + e^{-2\lambda t} v^3 \right),$$

we derive from (4.5) that

$$\begin{cases} \frac{d}{dt} x_\xi + G_\xi x_\xi = e^{-\lambda t} v_\xi, \\ \frac{d}{dt} v_\xi + G_\xi v_\xi = \left(-\frac{1}{2} e^{-\lambda t} v^2 - \frac{1}{2} e^{-\lambda t} + e^{-2\lambda t} v^3 - P \right) x_\xi + \frac{e^{-\lambda t}}{2}. \end{cases} \quad (4.6)$$

Step 3. We return to the original (t, x) coordinates and derive an evolution equation for v_x along good characteristics. Fix a point $(\tau, \bar{x})$ with $\tau \notin \mathcal{N}$ and assume $\bar{x}$ is a Lebesgue point for $x \mapsto v_x(\tau, x)$. Let $\bar{\xi}$ satisfy $\bar{x} = x(\tau, \bar{\xi})$, and suppose $t \mapsto \xi(t; \tau, \bar{\xi})$ is a good characteristic so that (GC) holds.

From (3.3),

$$v_x^2(\tau, \bar{x}) = \frac{1}{x_\xi(\tau, \bar{\xi})} - 1 \geq 0, \quad x_\xi(\tau, \bar{\xi}) > 0. \quad (4.7)$$

As long as $x_\xi(\tau, \bar{\xi}) > 0$, the derivative v_x along the characteristic is

$$v_x(t, x(t, \xi(t; \tau, \bar{\xi}))) = \frac{v_\xi(t, x(t, \xi(t; \tau, \bar{\xi})))}{x_\xi(t, x(t, \xi(t; \tau, \bar{\xi})))}. \quad (4.8)$$

Using (4.8),

$$\frac{dv_x}{dt} = \frac{d}{dt} \left(\frac{v_\xi}{x_\xi} \right) = -\frac{1}{2}e^{-\lambda t}v^2 - \frac{1}{2}e^{-\lambda t} + e^{-2\lambda t}v^3 - P + \frac{e^{-\lambda t}}{2} \frac{1}{x_\xi} - e^{-\lambda t} \frac{v_\xi^2}{x_\xi^2}. \quad (4.9)$$

Thus $t \mapsto v_x$ is absolutely continuous as long as $x_\xi \neq 0$.

Furthermore, for $x_\xi > 0$, we have

$$\begin{aligned} \frac{d}{dt} \arctan v_x(t, x(t, \xi(t; \tau, \bar{\xi}))) &= \frac{1}{1 + v_x^2} \frac{d}{dt} v_x \\ &= \left(-\frac{1}{2}e^{-\lambda t}v^2 - \frac{1}{2}e^{-\lambda t} + e^{-2\lambda t}v^3 - P + \frac{e^{-\lambda t}}{2} \frac{1}{x_\xi} - e^{-\lambda t} \frac{v_\xi^2}{x_\xi^2} \right) x_\xi \\ &= \left(-\frac{1}{2}e^{-\lambda t}v^2 - \frac{1}{2}e^{-\lambda t} + e^{-2\lambda t}v^3 - P \right) x_\xi + \frac{e^{-\lambda t}}{2} - \frac{e^{-\lambda t}v_\xi^2}{x_\xi} \\ &= \left(-\frac{1}{2}e^{-\lambda t}v^2 - \frac{1}{2}e^{-\lambda t} + e^{-2\lambda t}v^3 - P - e^{-\lambda t}v_x^2 \right) x_\xi + \frac{e^{-\lambda t}}{2} \\ &= \left(-\frac{1}{2}e^{-\lambda t}v^2 + \frac{1}{2}e^{-\lambda t} + e^{-2\lambda t}v^3 - P \right) x_\xi - \frac{e^{-\lambda t}}{2}. \end{aligned} \quad (4.10)$$

Step 4. Introduce the function

$$f := \begin{cases} 2 \arctan v_x, & 0 < x_\xi < 1, \\ \pi, & x_\xi = 0, \end{cases} \quad (4.11)$$

which implies

$$x_\xi = \frac{1}{1 + v_x^2} = \cos^2 \frac{f}{2}, \quad \frac{v_x}{1 + v_x^2} = \frac{1}{2} \sin f, \quad \frac{v_x^2}{1 + v_x^2} = \sin^2 \frac{f}{2}. \quad (4.12)$$

We regard v as a function taking values in the interval $\mathcal{I} := [-\pi, \pi]$ with endpoints identified. Along each good characteristic, $t \mapsto v(t) = v(t, x(t, \xi(t; \tau, \bar{\xi})))$ is absolutely continuous and satisfies

$$\frac{df(t)}{dt} = \left(-e^{-\lambda t}v^2 + e^{-\lambda t} + 2e^{-2\lambda t}v^3 - 2P \right) \cos^2 \frac{f}{2} - e^{-\lambda t}. \quad (4.13)$$

In fact, we denote the values of x_ξ, v_ξ and v_x along this particular characteristic by $x_\xi(t), v_\xi(t)$ and $v_x(t) = \frac{v_\xi(t)}{x_\xi(t)}$. By (GC), we have $x_\xi > 0$ for a.e. $t > 0$.

If τ is any time where $x_\xi > 0$, then there exists a neighborhood $\Omega = [\tau - \delta, \tau + \delta]$ such that $x_\xi > 0$ on Ω . By (4.10) and (4.12), we obtain that (4.13) holds and $v = 2 \arctan \frac{v_\xi}{x_\xi}$ is absolutely continuous constrained to Ω .

To prove our claim, it suffices to verify that $t \mapsto v(t)$ is continuous on the null set $\mathcal{N}$. Let t_0 be such that $x_\xi(t_0) = 0$. Then there is a sequence $\{t_n\}$ satisfying that $\lim_{n \rightarrow \infty} t_n = t_0$ and $x_\xi(t_n) > 0$, which implies that

$$v_x^2(t) = \frac{1 - x_\xi(t)}{x_\xi(t)}. \quad (4.14)$$

It is clear that $v_x^2(t) \rightarrow \infty$ as $t \rightarrow t_0$ and $x_\xi(t) \rightarrow 0$. This means that

$$f(t) = 2 \arctan v_x(t) \rightarrow \pm\pi. \quad (4.15)$$

Since we identify the points $\pm\pi$ in $\mathcal{I}$, this establishes the continuity of $f(t)$ for all $t \geq 0$. This proves our claim.

Step 5. For any conservative solution $v = v(t, x)$, the variables x, v, f in terms of t and ξ solve the semilinear system

$$\begin{cases} \frac{d}{dt} \xi(t, \bar{\xi}) = G(t, \xi(t, \bar{\xi})), \\ \frac{d}{dt} x(t, \xi(t, \bar{\xi})) = e^{-\lambda t} v(t, \xi(t, \bar{\xi})), \\ \frac{d}{dt} v(t, \xi(t, \bar{\xi})) = -P_x(t, \xi(t, \bar{\xi})), \\ \frac{d}{dt} f(t, \xi(t, \bar{\xi})) = \left(-e^{-\lambda t} v^2 + e^{-\lambda t} + 2e^{-2\lambda t} v^3 - 2P \right) \cos^2 \frac{f}{2} - e^{-\lambda t}, \end{cases} \quad (4.16)$$

with initial conditions

$$\begin{cases} \xi(0, \bar{\xi}) = \bar{\xi}, \\ x(0, \bar{\xi}) = x(0, \bar{\xi}), \\ v(0, \bar{\xi}) = v_0(x(0, \bar{\xi})), \\ f(0, \bar{\xi}) = 2 \arctan v_{0x}(x(0, \bar{\xi})). \end{cases} \quad (4.17)$$

Recall that P and G are defined in (2.2) and (3.13). By (4.12), we derive the functions P and P_x under the new variable ξ as follows

$$\begin{aligned} P(x(\xi)) &= \frac{1}{2} \int_{\mathbb{R}} e^{-\left| \int_{\xi'}^{\xi} \cos^2 \frac{f(s)}{2} ds \right|} \left(-\frac{1}{2} e^{-\lambda t} v^2 + \frac{1}{2} e^{-\lambda t} v_x^2 + e^{-2\lambda t} v^3 \right) \frac{1}{1 + v_x^2} d\xi' \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{-\left| \int_{\xi'}^{\xi} \cos^2 \frac{f(s)}{2} ds \right|} \left(\left(-\frac{1}{2} e^{-\lambda t} v^2 + e^{-2\lambda t} v^3 \right) \cos^2 \frac{f(\xi')}{2} + \frac{1}{2} e^{-\lambda t} \sin^2 \frac{f(\xi')}{2} \right) d\xi', \end{aligned} \quad (4.18)$$

$$\begin{aligned} P_x(x(\xi)) &= \frac{1}{2} \left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi} \right) e^{-\left| \int_{\xi'}^{\xi} \cos^2 \frac{f(s)}{2} ds \right|} \left(\left(-\frac{1}{2} e^{-\lambda t} v^2 + e^{-2\lambda t} v^3 \right) \cos^2 \frac{f(\xi')}{2} \right. \\ &\quad \left. + \frac{1}{2} e^{-\lambda t} \sin^2 \frac{f(\xi')}{2} \right) d\xi'. \end{aligned} \quad (4.19)$$

By the previous lemmas, it is easy to prove the Lipschitz continuity of the right hand sides for (4.16). Consequently, the Cauchy problem (4.16) with initial data (4.17) has a unique global solution defined for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Step 6. To conclude uniqueness, suppose there exist two conservative solutions v and $\tilde{v}$ to (1.2) with the same initial data $v_0 \in H^1(\mathbb{R})$. For almost every $t > 0$, the Lipschitz maps $\xi \mapsto x(t, \xi)$ and $\xi \mapsto \tilde{x}(t, \xi)$ are strictly increasing and thus admit continuous inverses

$$x \mapsto \xi^*, \quad x \mapsto \tilde{\xi}^*.$$

From the preceding analysis, $(t, \xi) \mapsto (x, v, f)(t, \xi)$ is uniquely determined by v_0 . Therefore

$$x(t, \xi) = \tilde{x}(t, \xi), \quad v(t, \xi) = \tilde{v}(t, \xi).$$

For almost every $t > 0$, this implies

$$v(t, x) = v(t, \xi^*(t, x)) = \tilde{v}(t, \tilde{\xi}^*(t, x)) = \tilde{v}(t, x).$$

The uniqueness of the solution is thus established. This completes the proof of Theorem 2.2.

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