

THRESHOLD DYNAMICS OF A STOCHASTIC SEQIRS MODEL WITH SEMI-MARKOV SWITCHING*

Zhijun Zeng[†] and Yilin Liu

Abstract This paper studies the threshold dynamics of a stochastic SEQIRS epidemic model that is subject to two sources of randomness: A white-noise perturbation of the transmission rate and a semi-Markov switching of the environment. The incidence rate has the nonlinear form $\beta S^p f(E)$ with $p \in \mathbb{N}$, and the sojourn time in each environmental regime is allowed to follow an arbitrary distribution rather than an exponential one. A threshold $\mathcal{R}_0^s$ is derived from the stationary distribution of the embedded chain and from the mean sojourn times. The disease dies out almost surely under one of two different extinction mechanisms. The first one combines the threshold requirement $\mathcal{R}_0^s < 1$ with an upper bound on the noise intensity. The second one imposes only a sufficiently large lower bound on the noise intensity and needs no separate threshold assumption, because this lower bound already forces $\mathcal{R}_0^s < 1$. It persists, and the time averages of all five compartments admit positive lower bounds, when $\mathcal{R}_0^s > 1$, the noise intensity lies in a prescribed interval, and (only for $p \geq 2$) the susceptible class is bounded away from zero for all large times. The threshold tends to the basic reproduction number of the corresponding deterministic system as the noise vanishes, and is never larger than it. Simulations carried out for the Holling type II incidence confirm both alternatives. They show in particular that a subsystem with $\mathcal{R}_0^k > 1$ may coexist with extinction of the switched system, so that the sojourn-time distributions cannot be ignored when the fate of an epidemic is assessed.

Keywords Threshold dynamics, stochastic SEQIRS model, semi-Markov switching, disease extinction, disease persistence.

MSC(2010) 34F05, 60H10, 92B05.

1. Introduction

Outbreaks of emerging and re-emerging infections continue to place a heavy burden on public health, and the transmission rate of a pathogen is rarely constant in time: It responds to weather, to the frequency of contacts, and to the control measures in force. A deterministic epidemic model can only follow such variability if the rate is prescribed in advance, whereas the variation actually seen in incidence data is to a large extent random. Stochastic formulations are therefore used whenever the intrinsic randomness of transmission and the uncertainty of the environment cannot be neglected. A convenient way of letting the parameters jump between finitely many regimes is to drive the model by a finite-state Markov chain, and epidemic models of this type have been studied extensively [1, 13, 19, 21]. In a Markov switching model, however, the time at which a switch occurs is decided by a memoryless clock: The probability of leaving a regime

[†]The corresponding author.

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, Jilin, China

*The authors were supported by Science and Technology Development Plan Project of Jilin Province, China under Grant No. 20240101317JC.

Email: zengzj219@nenu.edu.cn(Z. Zeng), 1550296141@qq.com(Y. Liu)

during the next instant is independent of how long that regime has already lasted [2, 15]. For many epidemiological mechanisms this is unrealistic. The chance that an infected individual recovers, for instance, grows with the time elapsed since infection, and so does the chance that an immunized individual becomes susceptible again. The rates appearing in the model studied below are kept constant, but this is a simplification rather than a limitation of the semi-Markov setting: Because the framework keeps track of the duration already spent in the current regime, it is able to accommodate such duration-dependent effects.

Semi-Markov processes relax precisely this restriction. They retain the discrete jump structure of a Markov chain but allow the sojourn time in each regime to follow an arbitrary distribution, so that the switching intensity may depend on the time elapsed since the last transition. This additional flexibility has proved useful in epidemic modeling [7, 16, 18, 20], where quantities such as the duration of the latent or of the infectious period are known to shape the course of an outbreak and are not captured by a constant rate.

Available contributions in the semi-Markov setting treat incidence rates of a specific shape: Bilinear, saturating, or nonlinear but of a form different from the one used here (see, e.g., [20], where the incidence is nonlinear but the model has no quarantined compartment). To the best of our knowledge, no study combines the general incidence $\beta S^p f(E)$ — simultaneously nonlinear in E and of arbitrary polynomial degree p in S — with semi-Markov switching and with a compartment of quarantined individuals. The present paper analyzes a model of exactly this kind. Specifically, we study a stochastic SEQIRS model which incorporates (i) a contact parameter switching according to a semi-Markov process with an arbitrary sojourn-time distribution, (ii) a force of infection of the nonlinear form $\beta S^p f(E)$ with $p \in \mathbb{N}$, and (iii) immunity lost at a constant rate $\ell > 0$. For this model we obtain a threshold $\mathcal{R}_0^s$ expressed through the stationary distribution of the embedded chain and the mean sojourn times, and we show that it governs extinction and persistence. The regimes that enter the threshold may behave in opposite ways, so that the outcome of the switched system is not decided by any single subsystem.

The paper is organized as follows. Section 2 introduces the semi-Markov framework and the model. Section 3 is devoted to the threshold dynamics, Section 4 contains the numerical examples, and Section 5 concludes.

2. Model description and preliminaries

This section collects the semi-Markov tools used below and then states the model.

2.1. The semi-Markov process

Environmental conditions such as weather regimes evolve in a way that is rarely memoryless, so a plain Markov chain is often too crude a description for them. The environment is therefore modeled by a semi-Markov process $\{r(t), t \geq 0\}$ taking values in the finite set $\mathcal{M} = \{1, \dots, N\}$. Write $\tau_0 = 0 < \tau_1 < \dots$ for the successive switching epochs and $\sigma_n = \tau_n - \tau_{n-1}$ for the length of the visit that starts at τ_{n-1} . On leaving a state i the process picks its next state j with probability p_{ij} , independently of the past history. Once j has been chosen, the duration of the stay in i is drawn from the distribution function $F_i(\cdot)$. Equivalently, for every $t \geq 0$ and $i, j \in \mathcal{M}$,

$$\mathbb{P}(r(\tau_{n+1}) = j, \sigma_{n+1} \leq t \mid r(\tau_n) = i) = p_{ij} F_i(t).$$

Sampled only at the switching epochs, the process collapses to the chain $\mathfrak{J}_n := r(\tau_n)$, $n = 0, 1, \dots$, which is Markovian and admits $(p_{ij})_{\mathcal{M} \times \mathcal{M}}$ as its one-step transition matrix. Let $f_i(\cdot)$ be a density of $F_i(\cdot)$. The jump intensities carried by the semi-Markov kernel are then given by

$$q_{ij}(t) = p_{ij} \frac{f_i(t)}{1 - F_i(t)} \geq 0 \quad (i \neq j), \quad q_{ii}(t) = - \sum_{j \in \mathcal{M}, j \neq i} q_{ij}(t).$$

The whole paper rests on the hypotheses listed below, which are the ones already used in [7].

- (i) The transition probability matrix $(p_{ij})_{\mathcal{M} \times \mathcal{M}}$ is irreducible, with $p_{ii} = 0$ for all $i \in \mathcal{M}$.
- (ii) For each $i \in \mathcal{M}$, $F_i(\cdot)$ has a continuous and bounded density $f_i(\cdot)$, and for any $t \in (0, \infty)$, $f_i(t) > 0$.
- (iii) For each $i \in \mathcal{M}$, there exists a constant $\epsilon_i > 0$ such that

$$\frac{f_i(t)}{1 - F_i(t)} \geq \epsilon_i, \quad \forall t \in [0, \infty).$$

Since the sojourn times are not exponentially distributed, the value of $r(t)$ at a given instant says little about the residual time still to be spent in the current regime. What matters in addition is how long that regime has already lasted. This motivates recording the elapsed part of the current visit,

$$\zeta(t) = t - \sup\{u < t : r(u) \neq r(t)\},$$

customarily termed the backward recurrence time. Appending it to $r(t)$ restores the Markov property: By [10], the pair $\{(\zeta(t), r(t)), t \geq 0\}$ is a Markov process.

2.2. Model description

Because the parameters of an epidemic respond to environmental noise, Boukanjime et al. [1] proposed the following stochastic model with a general nonlinear incidence rate,

$$\begin{cases} dS(t) = [\Lambda - \beta(1 - \rho_1)(1 - \rho_2)S(t)f(E(t)) + b_1Q(t) - \mu S(t)]dt - \sigma S(t)f(E(t))dB(t), \\ dE(t) = [\beta(1 - \rho_1)(1 - \rho_2)S(t)f(E(t)) - (b_2 + \alpha + \vartheta + \mu)E(t)]dt + \sigma S(t)f(E(t))dB(t), \\ dQ(t) = [b_2E(t) - (b_1 + c + \mu)Q(t)]dt, \\ dI(t) = [\alpha E(t) + cQ(t) - (\eta + \mu + \delta)I(t)]dt, \\ dR(t) = [\eta I(t) + \vartheta E(t) - \mu R(t)]dt. \end{cases} \quad (2.1)$$

All parameters in (2.1) are positive. Their epidemiological meaning is the one used in [12]. Here $S(t)$, $E(t)$, $Q(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, exposed, quarantined, hospitalized infected and immune individuals at time t , respectively, $B(t)$ is a standard Brownian motion and σ measures the intensity of the white noise. The non-negative, twice continuously differentiable function $f(\cdot)$ is required to satisfy the following conditions (H):

(H1) $f(0) = 0$ and $0 < f(x) \leq xf'(0)$, for all $x > 0$.

(H2) $G(x) = \frac{f(x)}{x}$ (with the convention $G(0) = f'(0)$) satisfies the Lipschitz condition on $[0, \frac{\Lambda}{\mu}]$, i.e., for all $u_1, u_2 \in [0, \frac{\Lambda}{\mu}]$, there exists $L > 0$ such that $|G(u_1) - G(u_2)| \leq L|u_1 - u_2|$.

To describe the loss of immunity, we assume that an immune individual returns to the susceptible class at rate $\ell > 0$, so that the mean duration of immunity is $1/\ell$ [7, 11]. Combining this with a semi-Markov environment and with the incidence rate $\beta S^p f(E)$, $p \in \mathbb{N}$, we obtain the stochastic SEQIRS model

$$\begin{cases} dS(t) = [\Lambda - \beta_{r(t)} S^p(t) f(E(t)) + b_1 Q(t) + \ell R(t) - \mu S(t)] dt - \sigma S^p(t) f(E(t)) dB(t), \\ dE(t) = [\beta_{r(t)} S^p(t) f(E(t)) - (b_2 + \alpha + \vartheta + \mu) E(t)] dt + \sigma S^p(t) f(E(t)) dB(t), \\ dQ(t) = [b_2 E(t) - (b_1 + c + \mu) Q(t)] dt, \\ dI(t) = [\alpha E(t) + c Q(t) - (\eta + \mu + \delta) I(t)] dt, \\ dR(t) = [\eta I(t) + \vartheta E(t) - (\mu + \ell) R(t)] dt. \end{cases} \quad (2.2)$$

In (2.2) the transmission coefficient is driven by the environment: $\beta_{r(t)}$ takes the value β_k on the set $\{t \geq 0 : r(t) = k\}$, $k \in \mathcal{M}$, and we assume throughout that the semi-Markov process $\{r(t), t \geq 0\}$ and the Brownian motion $\{B(t), t \geq 0\}$ are independent. Every parameter occurring in (2.2) is strictly positive, with the understanding that the noise intensity σ is allowed to vanish, the case $\sigma = 0$ reducing the discussion to the corresponding deterministic system. The parameters Λ , μ , b_1 , b_2 , α , c , η , ϑ and δ are carried over from (2.1) unchanged: Λ is the constant recruitment rate, μ the natural death rate, b_1 the rate at which quarantined individuals return to the susceptible class, b_2 the quarantine rate of exposed individuals, α and c the hospitalization rates of exposed and of quarantined individuals, η the recovery rate of hospitalized individuals, ϑ the rate at which exposed individuals acquire immunity and δ the disease-induced death rate. The exponent p controls how sharply the force of infection responds to a change in the size of the susceptible class: $p = 1$ returns the classical bilinear rate $\beta S f(E)$, and non-linear incidences of this form are known to generate richer dynamics than classical bilinear rate [9].

The following well-posedness statement follows from the arguments used for Lemmas 2.1 and 2.2 of [6] and for Theorem 2.1 of [5].

Lemma 2.1. *For any initial value $X(0) = (S(0), E(0), Q(0), I(0), R(0)) \in \mathbb{R}_+^5$, when $t \geq 0$, system (2.2) has a unique global positive solution $X(t) = (S(t), E(t), Q(t), I(t), R(t)) \in \mathbb{R}_+^5$ with probability 1, and it satisfies*

$$\limsup_{t \rightarrow \infty} (S(t) + E(t) + Q(t) + I(t) + R(t)) \leq \frac{\Lambda}{\mu}.$$

The uniform bound supplied by Lemma 2.1 justifies the choice of the interval $[0, \Lambda/\mu]$ in hypothesis (H2): Since no compartment can ultimately exceed Λ/μ , this is the smallest interval that is guaranteed to contain $E(t)$ for all large t .

3. Threshold dynamics

In this section we determine the threshold that separates extinction from persistence. Let

$$m_i = \int_0^\infty [1 - F_i(u)] du$$

be the mean sojourn time of $\{r(t), t > 0\}$ in state i , and assume that

- (1) The matrix $(p_{ij})_{\mathcal{M} \times \mathcal{M}}$ is irreducible.

(2) For any i , $m_i < \infty$.

Under these assumptions the semi-Markov process $\{r(t), t > 0\}$ is ergodic [3, 10]. Let $\mathfrak{J}_n = r(\tau_n)$, $n \in \mathbb{N}$, be the embedded chain with transition matrix $P = (p_{ij})_{\mathcal{M} \times \mathcal{M}}$, and let $v = \{v_k; k \in \mathcal{M}\}$ be its unique positive stationary distribution, i.e. $vP = v$. By the ergodic theorem for semi-Markov processes (Theorem 2 of [4]), for every bounded measurable function φ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(r(s)) ds = \frac{\sum_{k \in \mathcal{M}} \varphi(k) v_k m_k}{\sum_{k \in \mathcal{M}} v_k m_k} \quad a.s. \quad (3.1)$$

In the light of (3.1), the threshold is defined as

$$\mathcal{R}_0^s = \frac{\sum_{k \in \mathcal{M}} v_k m_k \beta_k \left(\frac{\Lambda}{\mu}\right)^p f'(0)}{\sum_{k \in \mathcal{M}} v_k m_k [b_2 + \alpha + \vartheta + \mu + \frac{1}{2} \sigma^2 \left(\frac{\Lambda}{\mu}\right)^{2p} (f'(0))^2]}, \quad (3.2)$$

where $\{\beta_k, k \in \mathcal{M}\}$ are the transmission coefficients of the various states. Throughout the paper we write

$$\underline{\beta} = \min_{k \in \mathcal{M}} \beta_k, \quad \bar{\beta} = \max_{k \in \mathcal{M}} \beta_k, \quad (3.3)$$

and, for each state k , we introduce the concave quadratic function

$$\Phi_k(x) = -\frac{\sigma^2}{2} x^2 + \beta_k x - (b_2 + \alpha + \vartheta + \mu), \quad x \geq 0, \quad (3.4)$$

which will be evaluated at the three quantities

$$x(t) = \frac{S^p(t) f(E(t))}{E(t)}, \quad x_1 = \left(\frac{\Lambda}{\mu}\right)^p f'(0), \quad x_2(t) = \frac{S^{p-1}(t) f'(0) \Lambda}{\mu}. \quad (3.5)$$

Since $f(E) \leq f'(0)E$ by (H1) and $S(t) \leq \Lambda/\mu$ by Lemma 2.1, these are ordered according to

$$0 < x(t) \leq x_2(t) \leq x_1 \quad (3.6)$$

for all t large enough. It will be shown below that $\mathcal{R}_0^s$ determines the disease extinction and persistence of system (2.2).

3.1. Extinction of the disease

We first consider the case in which the disease disappears. To this end we impose one of the following two sets of conditions:

$$\begin{aligned} \text{(C1)} \quad \mathcal{R}_0^s < 1 \quad \text{and} \quad \sigma &< \sqrt{\frac{\beta}{\left(\frac{\Lambda}{\mu}\right)^p f'(0)}}, \\ \text{(C2)} \quad \sigma &> \sqrt{\frac{\bar{\beta}^2}{2(b_2 + \alpha + \vartheta + \mu)}}. \end{aligned} \quad (3.7)$$

Theorem 3.1. *Under either of the two sets of conditions (3.7), for any initial value $X(0) \in \mathbb{R}_+^5$, the solution $X(t) \in \mathbb{R}_+^5$ of system (2.2) satisfies*

$$\begin{aligned} \lim_{t \rightarrow \infty} S(t) &= \frac{\Lambda}{\mu} \quad a.s., \\ \lim_{t \rightarrow \infty} E(t) &= \lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} I(t) = \lim_{t \rightarrow \infty} R(t) = 0 \quad a.s. \end{aligned}$$

Proof. Applying Itô's formula to the second equation of system (2.2) and expressing the outcome in the notation of (3.5) and (3.4), we obtain

$$\begin{aligned} d \ln E(t) &= \left[\beta_{r(t)} x(t) - (b_2 + \alpha + \vartheta + \mu) - \frac{1}{2} \sigma^2 x^2(t) \right] dt + \sigma x(t) dB(t) \\ &= \Phi_{r(t)}(x(t)) dt + \sigma x(t) dB(t). \end{aligned}$$

Integrating both sides gives

$$\ln E(t) = \ln E(0) + \int_0^t \Phi_{r(s)}(x(s)) ds + \int_0^t \sigma x(s) dB(s). \quad (3.8)$$

Set $M_1(t) = \int_0^t \sigma x(s) dB(s)$. By (H1) and Lemma 2.1,

$$\langle M_1, M_1 \rangle_t = \int_0^t (\sigma x(s))^2 ds \leq \sigma^2 \left(\frac{\Lambda}{\mu} \right)^{2p} (f'(0))^2 t,$$

so M_1 is a square-integrable martingale, and the strong law of large numbers gives

$$\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0 \quad a.s.$$

Combining this with (3.8), we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln E(t)}{t} = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Phi_{r(s)}(x(s)) ds. \quad (3.9)$$

We now consider the two cases.

Case (C1). Assume that the first line of (3.7) is in force, i.e. $\sigma^2 < \underline{\beta}/x_1$ in the notation of (3.5). Because $\beta_k \geq \underline{\beta}$ for every $k \in \mathcal{M}$, this gives the chain

$$x_1 < \frac{\underline{\beta}}{\sigma^2} \leq \frac{\beta_k}{\sigma^2}, \quad k \in \mathcal{M}.$$

The parabola Φ_k opens downwards and attains its maximum at $x = \beta_k/\sigma^2$. Hence it is increasing on $[0, \beta_k/\sigma^2)$, and in particular on $[0, x_1]$. Since $0 < x(t) \leq x_1$ by (3.6), we conclude that

$$\Phi_k(x(t)) \leq \Phi_k(x_1), \quad \forall k \in \mathcal{M}.$$

From (3.9),

$$\limsup_{t \rightarrow \infty} \frac{\ln E(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Phi_{r(s)}(x_1) ds.$$

It remains to evaluate this ergodic mean. Expanding $\Phi_k(x_1)$ and comparing the outcome with (3.2), whose denominator does not depend on k , we get

$$\frac{\sum_{k \in \mathcal{M}} v_k m_k \Phi_k(x_1)}{\sum_{k \in \mathcal{M}} v_k m_k} = (\mathcal{R}_0^s - 1) \left(b_2 + \alpha + \vartheta + \mu + \frac{1}{2} \sigma^2 x_1^2 \right) < 0,$$

the second factor being strictly positive, so the right-hand side is negative exactly when $\mathcal{R}_0^s < 1$. By the ergodic theorem (3.1),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Phi_{r(s)}(x_1) ds = \frac{\sum_{k \in \mathcal{M}} v_k m_k \Phi_k(x_1)}{\sum_{k \in \mathcal{M}} v_k m_k} < 0.$$

Thus

$$\limsup_{t \rightarrow \infty} \frac{\ln E(t)}{t} < 0,$$

which implies $\lim_{t \rightarrow \infty} E(t) = 0$ almost surely.

With the remaining compartments we argue from the equations. The third equation of (2.2) yields

$$Q(t) = e^{-(b_1+c+\mu)t} \left[Q(0) + b_2 \int_0^t e^{(b_1+c+\mu)s} E(s) ds \right],$$

and since $E(t) \rightarrow 0$ a.s., for every $\varepsilon > 0$ there is a finite random time $T(\varepsilon)$ beyond which $E(s) \leq \varepsilon$. Splitting the integral at $T(\varepsilon)$, the contribution of $[0, T(\varepsilon)]$ is killed by the prefactor $e^{-(b_1+c+\mu)t}$ as $t \rightarrow \infty$, while that of $[T(\varepsilon), t]$ is bounded by $b_2\varepsilon/(b_1+c+\mu)$. Hence $\limsup_{t \rightarrow \infty} Q(t) \leq b_2\varepsilon/(b_1+c+\mu)$ and, ε being arbitrary, $\lim_{t \rightarrow \infty} Q(t) = 0$ a.s. The same representation applied to the fourth equation, whose input is now E and Q , gives $\lim_{t \rightarrow \infty} I(t) = 0$ a.s., and then the fifth equation, driven by E and I , gives $\lim_{t \rightarrow \infty} R(t) = 0$ a.s.

It remains to identify the limit of S . Add the five equations of (2.2) and put $N(t) = S(t) + E(t) + Q(t) + I(t) + R(t)$. All the transfer terms cancel two by two, leaving

$$dN(t) = [\Lambda - \mu N(t) - \delta I(t)] dt,$$

so that

$$N(t) = \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu} \right) e^{-\mu t} - \delta \int_0^t e^{-\mu(t-s)} I(s) ds.$$

The convolution term tends to zero by the same splitting used above: The part of the integral over $[0, T(\varepsilon)]$ is multiplied by $e^{-\mu t}$, and the part over $[T(\varepsilon), t]$ is bounded by ε/μ . Thus $N(t) \rightarrow \Lambda/\mu$ a.s., and subtracting $E(t) + Q(t) + I(t) + R(t) \rightarrow 0$ finally gives

$$\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{\mu} \quad a.s.$$

Case (C2). Assume that the second line of (3.7) is in force. Completing the square gives

$$\Phi_{r(t)}(x(t)) = -\frac{\sigma^2}{2} \left(x(t) - \frac{\beta_{r(t)}}{\sigma^2} \right)^2 + \frac{\beta_{r(t)}^2}{2\sigma^2} - (b_2 + \alpha + \vartheta + \mu) \leq \frac{\beta_{r(t)}^2}{2\sigma^2} - (b_2 + \alpha + \vartheta + \mu).$$

Since $\sigma^2 > \bar{\beta}^2/(2(b_2 + \alpha + \vartheta + \mu))$ by the second line of (3.7) and $\beta_k \leq \bar{\beta}$ by (3.3), we obtain the explicit chain

$$\frac{\beta_k^2}{2\sigma^2} - (b_2 + \alpha + \vartheta + \mu) \leq \frac{\bar{\beta}^2}{2\sigma^2} - (b_2 + \alpha + \vartheta + \mu) < 0, \quad k \in \mathcal{M},$$

the last step being just a rewriting of the condition on σ . Hence

$$\Phi_{r(t)}(x(t)) < 0 \quad \text{for all } t \geq 0.$$

Therefore,

$$\limsup_{t \rightarrow \infty} \frac{\ln E(t)}{t} \leq \frac{\sum_{k \in \mathcal{M}} v_k m_k \left[\frac{\beta_k^2}{2\sigma^2} - (b_2 + \alpha + \vartheta + \mu) \right]}{\sum_{k \in \mathcal{M}} v_k m_k} < 0,$$

which again implies $\lim_{t \rightarrow \infty} E(t) = 0$ almost surely. The remaining conclusions follow as in Case (C1). $\square$

Remark 3.1. The two lines of (3.7) are of a different nature: (C1) combines the threshold condition with an upper bound on the noise intensity, while (C2) places no restriction on $\mathcal{R}_0^s$ at all. No separate threshold assumption is needed in (C2), because the lower bound on σ there already forces $\mathcal{R}_0^s$ below one. To see this, put $\kappa = b_2 + \alpha + \vartheta + \mu$ and let $\langle \beta \rangle = \sum_{k \in \mathcal{M}} v_k m_k \beta_k / \sum_{k \in \mathcal{M}} v_k m_k$ be the mean transmission coefficient, so that $0 < \langle \beta \rangle \leq \bar{\beta}$ and the denominator of (3.2) does not depend on k . The threshold can therefore be written as

$$\mathcal{R}_0^s = \frac{\langle \beta \rangle x_1}{\kappa + \frac{1}{2} \sigma^2 x_1^2}.$$

Assume the lower bound of (C2). Then $\frac{1}{2} \sigma^2 x_1^2 > \bar{\beta}^2 x_1^2 / (4\kappa)$, while

$$\kappa + \frac{\bar{\beta}^2 x_1^2}{4\kappa} \geq 2\sqrt{\kappa \cdot \frac{\bar{\beta}^2 x_1^2}{4\kappa}} = \bar{\beta} x_1 \geq \langle \beta \rangle x_1$$

by the arithmetic–geometric mean inequality. Combining the two displays gives $\mathcal{R}_0^s < 1$. Condition (C2) is consequently not a sub-case of (C1). The two lines of (3.7) describe different mechanisms: (C1) is extinction driven by a subcritical threshold under weak noise, whereas (C2) is extinction driven by the size of the noise itself, which can occur although the transmission coefficients are large. This distinction is also visible in the proof of Theorem 3.1, where Case (C2) uses only the pointwise bound $\Phi_k(x) \leq \beta_k^2 / (2\sigma^2) - \kappa < 0$ and never the value of $\mathcal{R}_0^s$. The converse implication does not hold: $\mathcal{R}_0^s < 1$ is compatible with arbitrarily small noise intensities, and it is then (C1) rather than (C2) that applies.

Remark 3.2. The threshold $\mathcal{R}_0^s$ increases with the transmission coefficients β_k , so that a higher infection coefficient promotes the spread of the disease. It decreases with the quarantine rate b_2 , the hospitalization rate α and the immunization rate ϑ . Intensifying these three measures therefore helps to suppress an outbreak.

Remark 3.3. The basic reproduction number of the deterministic system corresponding to (2.2) is

$$\mathcal{R}_0 = \frac{\sum_{k \in \mathcal{M}} v_k m_k \beta_k \left(\frac{\Lambda}{\mu} \right)^p f'(0)}{\sum_{k \in \mathcal{M}} v_k m_k (b_2 + \alpha + \vartheta + \mu)}.$$

Clearly $\mathcal{R}_0^s < \mathcal{R}_0$ and $\lim_{\sigma \rightarrow 0} \mathcal{R}_0^s = \mathcal{R}_0$, so the extinction condition for the stochastic model is stronger than the one for the corresponding deterministic model.

3.2. Persistence of the disease

We now show that, under the complementary assumptions, the disease persists and the time averages of all five compartments admit positive lower bounds.

Theorem 3.2. *Assume that $\mathcal{R}_0^s > 1$, that there exist a constant $\varrho \in (0, \Lambda/\mu)$ and an almost surely finite random time T_ϱ such that*

$$S(t) \geq \varrho \quad \text{for all } t \geq T_\varrho, \text{ a.s.}, \quad (3.10)$$

and that the noise intensity satisfies

$$\sqrt{\frac{2\bar{\beta}}{\frac{\varrho^{p-1}\Lambda}{\mu}f'(0) + x_1}} \leq \sigma < \sqrt{\frac{2\bar{\beta}\mu}{\varrho^{p-1}f'(0)\Lambda}}, \quad (3.11)$$

where $\underline{\beta}, \bar{\beta}$ are the bounds of (3.3) and x_1 is the constant of (3.5). Set

$$K_\varrho = \bar{\beta} - \frac{\sigma^2 \varrho^{p-1} f'(0) \Lambda}{2\mu}, \quad \Lambda_\Phi = \frac{\sum_{k \in \mathcal{M}} v_k m_k \Phi_k(x_1)}{\sum_{k \in \mathcal{M}} v_k m_k}. \quad (3.12)$$

Then for any $X(0) \in \mathbb{R}_+^5$ the solution $X(t) \in \mathbb{R}_+^5$ of system (2.2) satisfies

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(s) ds &\geq \frac{\Lambda}{\bar{\beta}x_1 + \mu} > 0 \quad \text{a.s.}, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t E(s) ds &\geq \frac{\mu \Lambda_\Phi}{K_\varrho f'(0) \left(\frac{\Lambda}{\mu}\right)^{p-1} \left(\frac{\Lambda\gamma}{f'(0)} + \bar{\beta}x_1\right)} \triangleq E^* > 0 \quad \text{a.s.}, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(s) ds &\geq \frac{b_2 E^*}{b_1 + c + \mu} \triangleq Q^* > 0 \quad \text{a.s.}, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds &\geq \frac{\alpha E^* + c Q^*}{\eta + \mu + \delta} \triangleq I^* > 0 \quad \text{a.s.}, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(s) ds &\geq \frac{\eta I^* + \vartheta E^*}{\mu + \ell} > 0 \quad \text{a.s.}, \end{aligned}$$

where $\gamma > 0$ is a constant such that $f'(0) - f(E(t))/E(t) \leq \gamma E(t)$ for all large t .

Proof. Throughout the proof we fix $\varepsilon > 0$ and work on the event of full probability on which Lemma 2.1 holds. All the estimates below are valid for t large enough that $S(t) \leq \frac{\Lambda}{\mu} + \varepsilon$ and $E(t) \leq \frac{\Lambda}{\mu} + \varepsilon$. The constants Λ and Λ/μ enter the bounds only through such inequalities, so replacing them by $\Lambda + \mu\varepsilon$ and $\frac{\Lambda}{\mu} + \varepsilon$ changes the final constants continuously. Since ε is arbitrary, we let $\varepsilon \downarrow 0$ at the end and write the clean values throughout.

By (H1), $f(E(t)) \leq f'(0)E(t)$, and since $E(t) \leq \Lambda/\mu + \varepsilon$ and $S(t) \leq \Lambda/\mu + \varepsilon$ while $\beta_{r(t)} \leq \bar{\beta}$,

$$\begin{aligned} dS(t) &\geq [\Lambda - \beta_{r(t)} S^p(t) f(E(t)) - \mu S(t)] dt - \sigma S^p(t) f(E(t)) dB(t) \\ &\geq \left[\Lambda - \bar{\beta} f'(0) \left(\frac{\Lambda}{\mu} + \varepsilon \right) \left(\frac{\Lambda}{\mu} + \varepsilon \right)^{p-1} S(t) - \mu S(t) \right] dt - \sigma S^p(t) f(E(t)) dB(t). \end{aligned}$$

Integrating, dividing by t and using the strong law of large numbers for the martingale $M_2(t) = \int_0^t \sigma S^p(s) f(E(s)) dB(s)$, whose quadratic variation grows at most linearly because S and E are bounded, together with the boundedness of S , we get

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(s) ds \geq \frac{\Lambda}{\bar{\beta} f'(0) \left(\frac{\Lambda}{\mu} + \varepsilon \right)^p + \mu},$$

and letting $\varepsilon \downarrow 0$ and recalling $x_1 = (\Lambda/\mu)^p f'(0)$ yields the first assertion,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(s) ds \geq \frac{\Lambda}{\bar{\beta} x_1 + \mu} > 0 \quad \text{a.s.}$$

We next compare $\Phi_{r(t)}(x(t))$ with $\Phi_{r(t)}(x_1)$. By (3.6), $0 < x(t) \leq x_2(t) \leq x_1$. For the quadratic

$$\Phi_k(u) = -\frac{\sigma^2}{2} u^2 + \beta_k u - (b_2 + \alpha + \vartheta + \mu),$$

one has the identity

$$\Phi_k(u) - \Phi_k(w) = \left[\beta_k - \frac{\sigma^2}{2} (u + w) \right] (u - w).$$

Taking $(u, w) = (x_2(t), x_1)$ gives

$$\Phi_{r(t)}(x_2(t)) - \Phi_{r(t)}(x_1) = \left[\beta_{r(t)} - \frac{\sigma^2}{2} (x_1 + x_2(t)) \right] (x_2(t) - x_1).$$

Here $x_2(t) - x_1 \leq 0$ by (3.6), while $S(t) \geq \varrho$ in (3.10) yields $x_2(t) \geq \varrho^{p-1} f'(0) \Lambda / \mu$, so that the left inequality of (3.11) gives

$$\frac{\sigma^2}{2} (x_1 + x_2(t)) \geq \frac{\sigma^2}{2} \left(x_1 + \frac{\varrho^{p-1} \Lambda}{\mu} f'(0) \right) \geq \bar{\beta} \geq \beta_{r(t)}.$$

The bracket in the previous display is therefore non-positive and, its second factor being non-positive as well, the product is non-negative:

$$\Phi_{r(t)}(x_2(t)) - \Phi_{r(t)}(x_1) \geq 0.$$

Hence

$$\Phi_{r(t)}(x_2(t)) \geq \Phi_{r(t)}(x_1).$$

Next, taking $(u, w) = (x(t), x_2(t))$ gives

$$\Phi_{r(t)}(x(t)) - \Phi_{r(t)}(x_2(t)) = \left[\beta_{r(t)} - \frac{\sigma^2}{2} (x(t) + x_2(t)) \right] (x(t) - x_2(t)).$$

Since $x(t) - x_2(t) \leq 0$ and

$$\beta_{r(t)} - \frac{\sigma^2}{2} (x(t) + x_2(t)) \leq \bar{\beta} - \frac{\sigma^2}{2} x_2(t) \leq \bar{\beta} - \frac{\sigma^2 \varrho^{p-1} f'(0) \Lambda}{2\mu},$$

we obtain

$$\Phi_{r(t)}(x(t)) - \Phi_{r(t)}(x_2(t)) \geq - \left(\bar{\beta} - \frac{\sigma^2 \varrho^{p-1} f'(0) \Lambda}{2\mu} \right) (x_2(t) - x(t)) = -K_\varrho (x_2(t) - x(t))$$

by (3.12). Summing the two inequalities yields

$$\Phi_{r(t)}(x(t)) \geq \Phi_{r(t)}(x_1) - K_\varrho (x_2(t) - x(t)) \quad (3.13)$$

for all $t \geq T_\varrho$.

We next isolate $x_2(t) - x(t)$ in the drift of the S -equation. By the definitions in (3.5),

$$\Lambda = \frac{\mu}{S^{p-1}(t)f'(0)} x_2(t), \quad \frac{\mu}{S^{p-1}(t)f'(0)} x(t) = \frac{\mu S(t)f(E(t))}{E(t)f'(0)},$$

so that, discarding the non-negative terms $b_1 Q(t) + \ell R(t)$, the drift of the first equation of (2.2) becomes

$$\frac{\mu}{S^{p-1}(t)f'(0)} (x_2(t) - x(t)) - \frac{\mu S(t)}{f'(0)} \left(f'(0) - \frac{f(E(t))}{E(t)} \right) - \beta_{r(t)} S^p(t) f(E(t)).$$

Bounding the last two terms by means of $f'(0) - f(E(t))/E(t) \leq \gamma E(t)$, $\mu S(t) \leq \Lambda$ and $\beta_{r(t)} S^p(t) f(E(t)) \leq \bar{\beta} x_1 E(t)$, we get

$$dS(t) \geq \left[\frac{\mu}{S^{p-1}(t)f'(0)} (x_2(t) - x(t)) - \left(\frac{\Lambda \gamma}{f'(0)} + \bar{\beta} x_1 \right) E(t) \right] dt - \sigma S^p(t) f(E(t)) dB(t). \quad (3.14)$$

Integrating (3.14) and using $S^{p-1}(t) \leq (\Lambda/\mu)^{p-1}$, i.e. $\mu/(S^{p-1}(t)f'(0)) \geq 1/c_p$ with $c_p = (\Lambda/\mu)^{p-1} f'(0)/\mu$, gives

$$\begin{aligned} \int_0^t (x_2(s) - x(s)) ds &\leq c_p \left[\left(\frac{\Lambda \gamma}{f'(0)} + \bar{\beta} x_1 \right) \int_0^t E(s) ds + (S(t) - S(0)) \right. \\ &\quad \left. + \int_0^t \sigma S^p(s) f(E(s)) dB(s) \right]. \end{aligned}$$

Substituting (3.13) and the last bound into (3.8) gives

$$\begin{aligned} \ln E(t) &\geq \ln E(0) + \int_0^t \Phi_{r(s)}(x_1) ds - K_\varrho c_p \left(\frac{\Lambda \gamma}{f'(0)} + \bar{\beta} x_1 \right) \int_0^t E(s) ds \\ &\quad + \int_0^t \sigma x(s) dB(s) - K_\varrho c_p (S(t) - S(0)) - K_\varrho c_p \int_0^t \sigma S^p(s) f(E(s)) dB(s). \end{aligned}$$

Dividing by t and solving for $\frac{1}{t} \int_0^t E(s) ds$, the term $-\frac{1}{t} \ln E(t)$ appears on the right-hand side. No claim is made that this term converges: $E(t)$ may come arbitrarily close to 0, in which case it is large and positive. What we do have, on the event on which we are working and for all large t , is $E(t) \leq \frac{\Lambda}{\mu} + \varepsilon$, hence

$$-\frac{1}{t} \ln E(t) \geq -\frac{1}{t} \ln \left(\frac{\Lambda}{\mu} + \varepsilon \right) \rightarrow 0,$$

so that its liminf is non-negative and the term may be dropped without weakening the lower bound. By the strong law of large numbers for martingales, $M_1(t)/t \rightarrow 0$ and $M_2(t)/t \rightarrow 0$ a.s., while the boundedness of S gives $(S(t) - S(0))/t \rightarrow 0$. Letting finally $\varepsilon \downarrow 0$, we obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t E(s) ds \geq \frac{\Lambda_\Phi}{K_\varrho c_p \left(\frac{\Lambda_\gamma}{f'(0)} + \bar{\beta} x_1 \right)} \triangleq E^* > 0,$$

where, by the ergodic theorem (3.1),

$$\Lambda_\Phi = \frac{\sum_{k \in \mathcal{M}} v_k m_k \Phi_k(x_1)}{\sum_{k \in \mathcal{M}} v_k m_k} = (\mathcal{R}_0^s - 1) \left(b_2 + \alpha + \vartheta + \mu + \frac{1}{2} \sigma^2 x_1^2 \right) > 0$$

because $\mathcal{R}_0^s > 1$.

Finally, integrating the Q -, I -, and R -equations of (2.2) and using boundedness gives

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(s) ds &\geq \frac{b_2 E^*}{b_1 + c + \mu} \triangleq Q^* > 0, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds &\geq \frac{\alpha E^* + c Q^*}{\eta + \mu + \delta} \triangleq I^* > 0, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(s) ds &\geq \frac{\eta I^* + \vartheta E^*}{\mu + \ell} > 0 \quad \text{a.s.} \end{aligned}$$

This completes the proof. $\square$

Remark 3.4. The two constants K_ϱ and Λ_Φ introduced in (3.12) are positive: $K_\varrho > 0$ follows from the right-hand inequality of (3.11), and $\Lambda_\Phi > 0$ is equivalent to $\mathcal{R}_0^s > 1$.

The constant γ of Theorem 3.2 exists because f is twice continuously differentiable with $f(0) = 0$ and $f'(0) > 0$, so that $x \mapsto (f'(0) - f(x)/x)/x$ extends continuously by $-f''(0)/2$ at $x = 0$ and is therefore bounded on $[0, \Lambda/\mu]$.

Remark 3.5. The role of ϱ in Theorem 3.2 deserves a comment. For $p \geq 2$, (3.10) is a genuine extra requirement: Positivity of the solution guarantees $S(t) > 0$ for every finite t , but no uniform positive lower bound follows from Lemma 2.1, which only controls $S(t)$ from above. The need for (3.10) comes from the algebra of the proof rather than from the degeneracy of the noise: The estimate (3.13) is expressed through the constant K_ϱ of (3.12), and that constant is obtained from the lower bound $x_2(t) = S^{p-1}(t) f'(0) \Lambda / \mu \geq \varrho^{p-1} f'(0) \Lambda / \mu$. For $p \geq 2$ this bound requires $S(t) \geq \varrho$, whereas Lemma 2.1 supplies only the opposite, upper bound. Writing

$$L(\varrho) = \sqrt{\frac{2\bar{\beta}}{\varrho^{p-1} \frac{\Lambda}{\mu} f'(0) + x_1}}, \quad R(\varrho) = \sqrt{\frac{2\bar{\beta}\mu}{\varrho^{p-1} f'(0) \Lambda}},$$

the requirement (3.11) is simply $L(\varrho) \leq \sigma < R(\varrho)$. For $p \geq 2$ both endpoints are decreasing in ϱ , so lowering ϱ relaxes the upper bound but raises the lower one. Since $\varrho \in (0, \Lambda/\mu)$ is required, the interval $[L(\varrho), R(\varrho)]$ is admissible only for $\varrho < \Lambda/\mu$, and $L(\Lambda/\mu)$ is an infimum that is not attained. No value of ϱ is uniformly optimal, and a given σ is admissible as soon as $[L(\varrho), R(\varrho)]$ contains it for *one* choice of $\varrho \in (0, \Lambda/\mu)$. For $p = 1$, on the other hand, hypothesis (3.10) can be dropped altogether. The reason is that $p = 1$ gives $S^{p-1}(t) \equiv 1$ and $\varrho^{p-1} \equiv 1$, so that $x_2(t) \equiv x_1$ by (3.5), $c_p = f'(0)/\mu$ and $K_\varrho = \bar{\beta} - \sigma^2 f'(0) \Lambda / (2\mu)$ no longer involve ϱ , and every step of the proof goes through with ϱ replaced by any positive number.

4. Numerical simulations

We now illustrate the results above with the Holling type II incidence rate $f(E) = E/(1 + aE)$, for which $f'(0) = 1$. With $p = 2$, system (2.2) becomes

$$\begin{cases} dS(t) = \left[\Lambda - \beta_{r(t)} S^2(t) \frac{E(t)}{1 + aE(t)} + b_1 Q(t) + \ell R(t) - \mu S(t) \right] dt - \sigma S^2(t) \frac{E(t)}{1 + aE(t)} dB(t), \\ dE(t) = \left[\beta_{r(t)} S^2(t) \frac{E(t)}{1 + aE(t)} - (b_2 + \alpha + \vartheta + \mu) E(t) \right] dt + \sigma S^2(t) \frac{E(t)}{1 + aE(t)} dB(t), \\ dQ(t) = [b_2 E(t) - (b_1 + c + \mu) Q(t)] dt, \\ dI(t) = [\alpha E(t) + c Q(t) - (\eta + \mu + \delta) I(t)] dt, \\ dR(t) = [\eta I(t) + \vartheta E(t) - (\mu + \ell) R(t)] dt, \end{cases} \quad (4.1)$$

with $a > 0$, the driving process $\{r(t), t \geq 0\}$ being semi-Markov on $\mathcal{M} = \{1, 2\}$. Freezing the environment in a fixed state $k \in \mathcal{M}$ (that is, replacing $\beta_{r(t)}$ by β_k in (4.1)) gives the k -th subsystem, whose trajectories we denote by $(S_k(t), E_k(t), Q_k(t), I_k(t), R_k(t))$. These are the curves plotted in panels (b) and (c) of the figures below. Substituting β_k for $\beta_{r(t)}$ in the computation leading to (3.2) and taking for v the degenerate stationary distribution concentrated at k , the basic reproduction number of the k -th subsystem reads

$$\mathcal{R}_0^k = \frac{\beta_k \left(\frac{\Lambda}{\mu}\right)^2}{b_2 + \alpha + \vartheta + \mu + \frac{1}{2}\sigma^2 \left(\frac{\Lambda}{\mu}\right)^4}.$$

Let $v = (\frac{1}{2}, \frac{1}{2})$ be the unique stationary probability distribution of the embedded Markov chain $\{\mathfrak{J}_n, n \in \mathbb{N}\}$ associated with $\{r(t), t \geq 0\}$ [7]. Then (3.2) gives

$$\mathcal{R}_0^s = \frac{\sum_{k \in \mathcal{M}} m_k \beta_k \left(\frac{\Lambda}{\mu}\right)^2}{\sum_{k \in \mathcal{M}} m_k \left\{ b_2 + \alpha + \vartheta + \mu + \frac{1}{2}\sigma^2 \left(\frac{\Lambda}{\mu}\right)^4 \right\}},$$

where $m_k = \int_0^\infty [1 - F_k(u)] du$.

Take the state space $\mathcal{M} = \{1, 2\}$ and suppose that the sojourn time in state i is Gamma distributed with shape ν_i and rate λ_i , so that its density is $f_i(t) = \frac{\lambda_i^{\nu_i}}{\Gamma(\nu_i)} t^{\nu_i-1} e^{-\lambda_i t}$ and its mean is $m_i = \nu_i / \lambda_i$. The choice $\nu_1 = 0.6$, $\lambda_1 = 0.1389$, $\nu_2 = 0.6$, $\lambda_2 = 0.1611$ gives $m_1 = 4.3197$ and $m_2 = 3.7244$. One sample path of $r(t)$ in $\mathcal{M} = \{1, 2\}$ is drawn in Figure 1.

Throughout the simulations all parameters of system (4.1) except β_1 , β_2 and σ are held fixed as follows. Following [12] we take $a = 0.001$; the rates $b_1 = 0.25$, $b_2 = 0.8$, $\alpha = 0.3$, $c = 0.12$, $\eta = 0.25$, $\vartheta = 0.2$ and $\delta = 0.25$ are those of [1]; and $\Lambda = 0.025$, $\mu = 0.02$ and $\ell = 0.21$ are estimated from the data, the first two fixing the disease-free level $\Lambda/\mu = 1.25$. All rates are measured in day^{-1} , except a (in ind.^{-1}) and Λ (in ind. day^{-1}). The epidemic variables start from $(S, E, Q, I, R)(0) = (1.1, 0.4, 0.4, 0.2, 0.1)$ and the driving process from $r(0) = 1$, $\theta(0) = 0$, so that the current regime begins at a jump.

Example 4.1. Select $\beta_1 = 0.4$, $\beta_2 = 1.3$ and $\sigma = 0.25$. Then $\underline{\beta} = \min\{\beta_1, \beta_2\} = 0.4$, and since

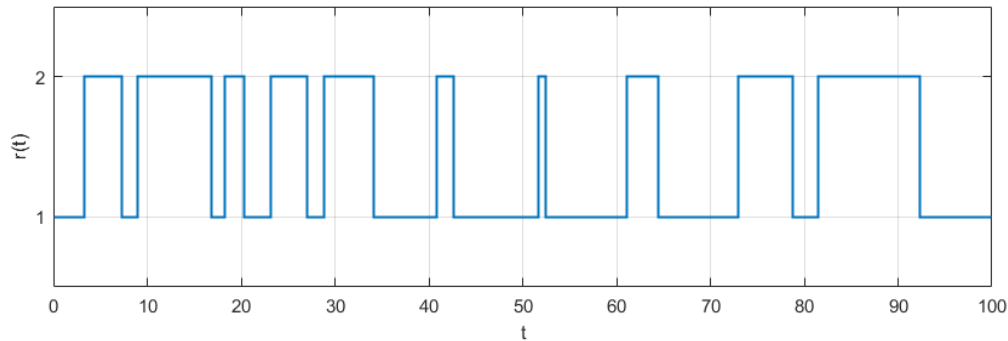


Figure 1. A sample path of the semi-Markov process $\{r(t)\}$ with state space $\mathcal{M} = \{1, 2\}$. The shape $\nu_i < 1$ makes the Gamma sojourns right-skewed, producing both short and long sojourns, unlike the memoryless exponential clock of a Markov chain.

$f'(0) = 1$ and $\Lambda/\mu = 0.025/0.02 = 1.25$ we have $\left(\frac{\Lambda}{\mu}\right)^2 f'(0) = 1.5625$, so that

$$\sqrt{\frac{\beta}{\left(\frac{\Lambda}{\mu}\right)^2 f'(0)}} = \sqrt{\frac{0.4}{1.5625}} = \sqrt{0.256} \approx 0.5060 > \sigma = 0.25,$$

which is precisely the first line of (3.7). Direct calculation from the expressions above gives $\mathcal{R}_0^1 = 0.4476 < 1$, $\mathcal{R}_0^2 = 1.4547 > 1$, and $\mathcal{R}_0^s = 0.9139 < 1$. The example is instructive and deserves a comment: Run on its own, the second regime would sustain the infection, since $\mathcal{R}_0^2 > 1$. The switched system nevertheless dies out, because the mean sojourn times $m_1 = 4.3197$ and $m_2 = 3.7244$ enter the threshold and pull its value below one. The switched system can therefore go extinct even though one of its subsystems is persistent, which shows that neither the most nor the least favourable regime, nor their average alone, decides the outcome. Theorem 3.1 predicts extinction, in agreement with Figure 2.

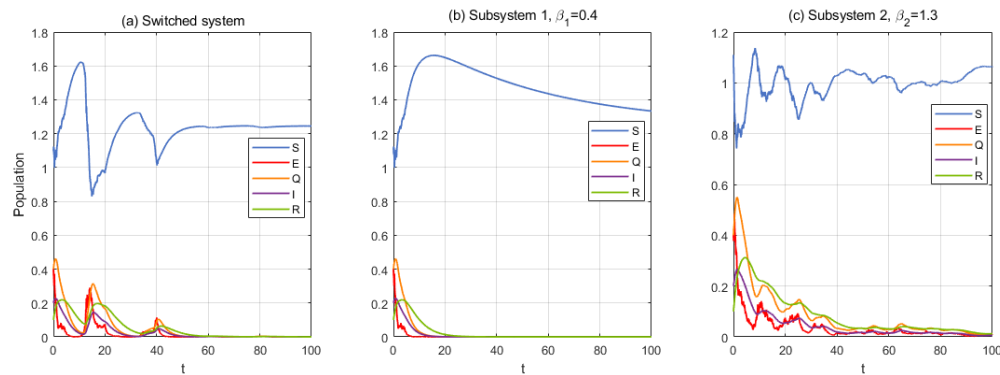


Figure 2. (a) Sample paths of $(S(t), E(t), Q(t), I(t), R(t))$ for system (4.1), with the parameters and initial data of Section 4, in Example 4.1, where the disease dies out. (b) Sample paths of the subsystem in state 1, where the disease dies out. (c) Sample paths of the subsystem in state 2, where the disease persists.

Example 4.2. Take $\beta_1 = 0.95$, $\beta_2 = 0.8$ and $\sigma = 0.6$. Now $\bar{\beta} = \max\{\beta_1, \beta_2\} = 0.95$ and

$b_2 + \alpha + \vartheta + \mu = 0.8 + 0.3 + 0.2 + 0.02 = 1.32$, whence

$$\sqrt{\frac{\bar{\beta}^2}{2(b_2 + \alpha + \vartheta + \mu)}} = \sqrt{\frac{0.95^2}{2 \times 1.32}} = \sqrt{0.3419} \approx 0.5847 < \sigma = 0.6,$$

so the second line of (3.7) is satisfied. By direct calculation, $\mathcal{R}_0^1 = 0.8437 < 1$, $\mathcal{R}_0^2 = 0.7104 < 1$, and $\mathcal{R}_0^s = 0.7820 < 1$. Theorem 3.1 predicts extinction of the disease in system (4.1), as confirmed by Figure 3. This example illustrates the content of Remark 3.1: The value $\mathcal{R}_0^s < 1$ is not an independent assumption here, it follows from $\sigma = 0.6$ exceeding the bound 0.5847. The corresponding noise-free threshold of Remark 3.3 is nevertheless larger than one, $\mathcal{R}_0 = \langle \beta \rangle x_1 / \kappa \approx 0.8806 \times 1.5625 / 1.32 \approx 1.0423$, so the extinction displayed in Figure 3 is driven by the size of the noise and not by a subcritical transmission rate.

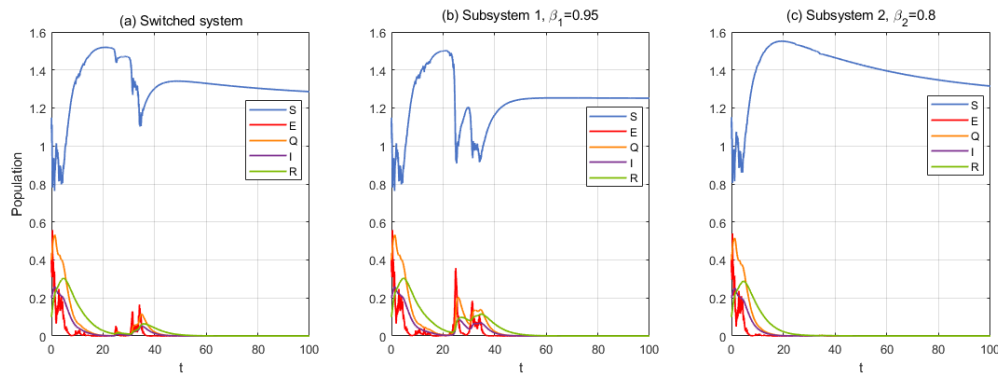


Figure 3. (a) Sample paths of $(S(t), E(t), Q(t), I(t), R(t))$ for system (4.1), with the parameters and initial data of Section 4, in Example 4.2, where the disease dies out. (b) Sample paths of the subsystem in state 1, where the disease dies out. (c) Sample paths of the subsystem in state 2, where the disease also dies out.

Example 4.3. Let $\beta_1 = 4.3$, $\beta_2 = 4.8$ and $\sigma = 2.1$, and take $\varrho = 0.5$. This value is consistent with the level at which the susceptible population settles in Figure 4(a); the hypothesis (3.10) is therefore plausible for this realization (cf. Remark 3.5). With $\underline{\beta} = 4.3$, $\bar{\beta} = 4.8$, $(\Lambda/\mu)^2 f'(0) = 1.5625$ and $\varrho^{p-1} \Lambda f'(0)/\mu = 0.5 \times 1.25 = 0.625$, the two endpoints $L(0.5)$ and $R(0.5)$ of (3.11) are

$$L(0.5) = \sqrt{\frac{2\bar{\beta}}{\frac{\varrho\Lambda}{\mu} f'(0) + \left(\frac{\Lambda}{\mu}\right)^2 f'(0)}} = \sqrt{\frac{9.6}{0.625 + 1.5625}} \approx 2.0949,$$

$$R(0.5) = \sqrt{\frac{2\bar{\beta}\mu}{\varrho f'(0)\Lambda}} = \sqrt{\frac{0.192}{0.0125}} \approx 3.9192,$$

so that

$$L(0.5) \approx 2.0949 \leq \sigma = 2.1 < 3.9192 \approx R(0.5),$$

that is, the assumptions of Theorem 3.2 are satisfied. By direct calculation, $\mathcal{R}_0^1 = 1.0023 > 1$, $\mathcal{R}_0^2 = 1.1189 > 1$, and $\mathcal{R}_0^s = 1.0563 > 1$. Theorem 3.2 predicts persistence of the disease in system (4.1). Figure 4 confirms this conclusion.

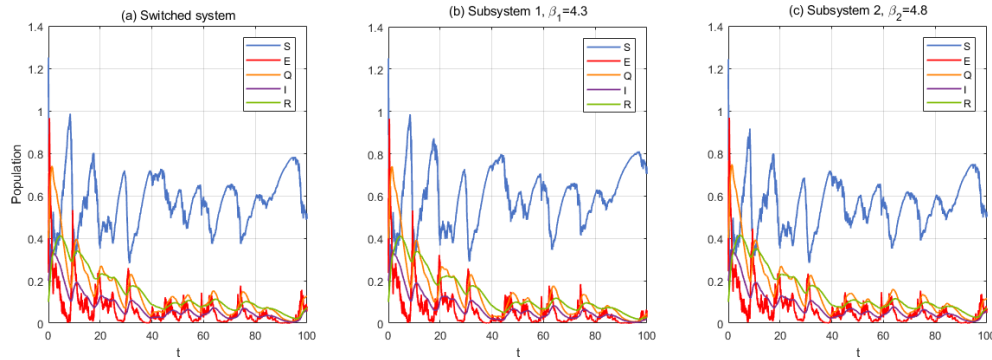


Figure 4. (a) Sample paths of $(S(t), E(t), Q(t), I(t), R(t))$ for system (4.1), with the parameters and initial data of Section 4, in Example 4.3, where the disease persists. (b) Sample paths of the subsystem in state 1, where the disease persists. (c) Sample paths of the subsystem in state 2, where the disease also persists.

Example 4.4. Keep $\beta_1 = 4.3$ and $\beta_2 = 4.8$ but reduce the noise intensity to $\sigma = 0.5$. This value does not meet the requirement of Theorem 3.2, and not merely for the particular choice made in Example 4.3: The left endpoint $L(\varrho)$ of (3.11) introduced in Remark 3.5 decreases in ϱ , so its smallest admissible value is attained at the largest value ϱ can take, namely $\varrho = \Lambda/\mu = 1.25$, for which

$$L(1.25) = \sqrt{\frac{9.6}{1.5625 + 1.5625}} \approx 1.7527 > 0.5.$$

In other words, no choice of $\varrho \in (0, \Lambda/\mu]$ makes $\sigma = 0.5$ admissible. For instance, for the particular choice $\varrho = \Lambda/\mu$ every $\sigma \in [1.7527, 2.4787)$ is admissible. This interval is the one attached to that choice of ϱ alone, since the admissible set depends on ϱ , and smaller values of ϱ extend it to the right while raising its left endpoint. Hence 1.7527 is the smallest noise intensity that can be admitted at all. We have kept the example because it isolates the effect of the noise intensity: Direct calculation still gives $\mathcal{R}_0^s = 4.3567 > 1$, and Figure 5 shows persistence for both $\sigma = 0.5$ and $\sigma = 2.1$, the fluctuations about the endemic level being markedly stronger for the latter. Comparing, in each row, the solid curve with the dashed one therefore illustrates the qualitative conclusion that stronger noise means larger fluctuations and more uncertainty in the course of the epidemic, outside the range covered by the theorem.

Comparing, for every compartment, the two curves drawn in Figure 5, which correspond to Examples 4.3 and 4.4, shows that the larger noise intensity produces noticeably wider fluctuations around the endemic level and hence more uncertainty about the actual course of the transmission. The disease itself persists in both cases.

5. Conclusion

We have studied a stochastic SEQIRS model in which the contact parameter is switched by a semi-Markov process with arbitrary sojourn-time distributions. After establishing the existence of a unique global positive solution, we defined a threshold $\mathcal{R}_0^s$ through the stationary distribution of the embedded chain and the mean sojourn times, and we showed that it determines the long-time behavior of the model: The disease dies out almost surely under either of the two conditions listed in (3.7). The first condition (C1) couples the threshold requirement $\mathcal{R}_0^s < 1$ with an upper

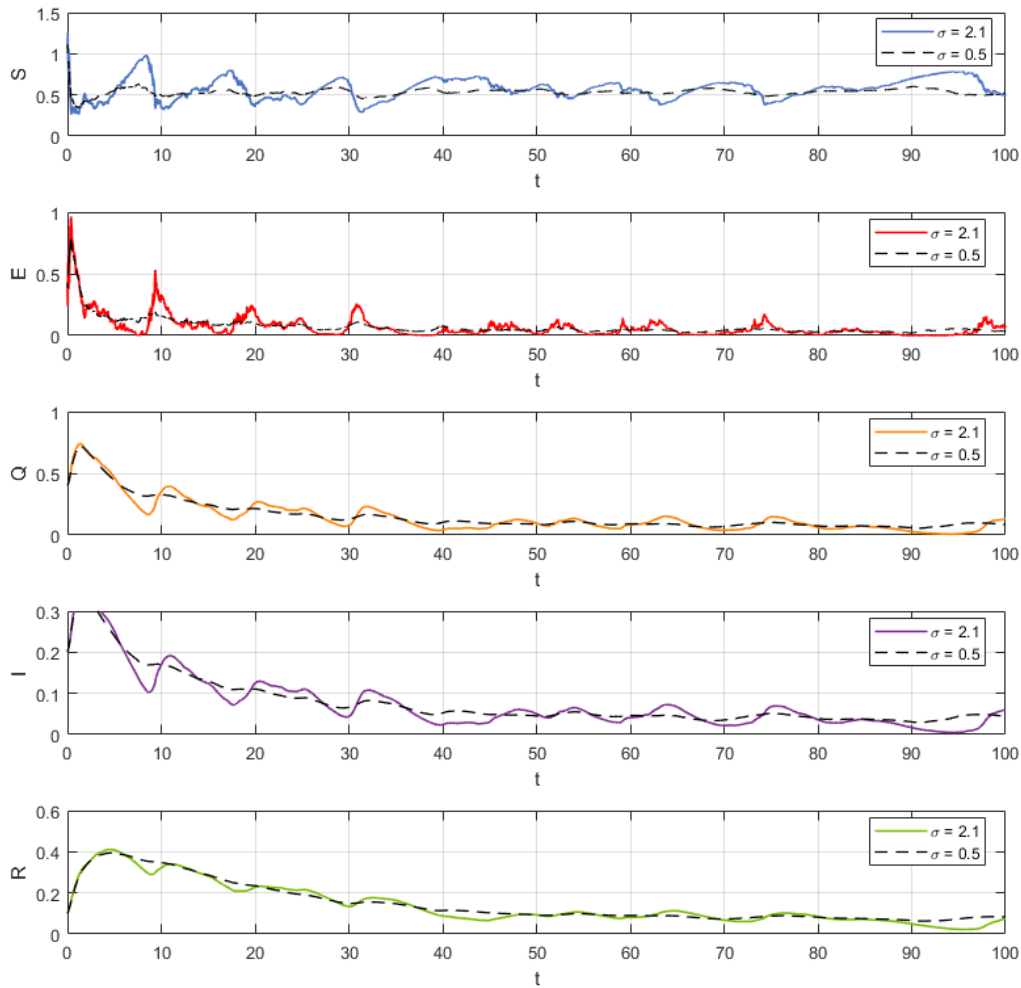


Figure 5. Simulations of $S(t)$, $E(t)$, $Q(t)$, $I(t)$, $R(t)$ for system (4.1) with the parameter values given above and $\beta_1 = 4.3$, $\beta_2 = 4.8$. Each of the five rows corresponds to one compartment and contains two sample paths: The solid curve is obtained with the larger noise intensity $\sigma = 2.1$ and the dashed one with $\sigma = 0.5$.

bound on the noise intensity, whereas the second condition (C2) imposes only a sufficiently large lower bound on the noise intensity and requires no separate threshold assumption, since this lower bound automatically forces $\mathcal{R}_0^s < 1$. The disease persists when $\mathcal{R}_0^s > 1$ together with the restriction (3.11) on the noise intensity (the latter carrying the auxiliary lower bound (3.10), which is needed only for $p \geq 2$). The threshold converges to the basic reproduction number of the corresponding deterministic system as the noise vanishes, and it is never larger than it. The numerical examples illustrate that the sojourn-time distributions can reverse the outcome predicted by any single subsystem, confirming that $\mathcal{R}_0^s$ captures an effect that $\mathcal{R}_0^k$ alone misses.

A word is in order on the role of the exponent p . Let $\kappa = b_2 + \alpha + \vartheta + \mu$ and $\langle \beta \rangle = \sum_{k \in \mathcal{M}} v_k m_k \beta_k / \sum_{k \in \mathcal{M}} v_k m_k$ be as in Remark 3.1, and put $u := (\Lambda/\mu)^p$, so that $x_1 = f'(0)u$ in the notation of (3.5). Since u is strictly increasing in $p \in \mathbb{N}$, the monotonicity of $\mathcal{R}_0^s$ in p follows

from that in u . In terms of u , (3.2) reads

$$\mathcal{R}_0^s = \frac{\langle \beta \rangle x_1}{\kappa + \frac{1}{2} \sigma^2 x_1^2} = \frac{\langle \beta \rangle f'(0) u}{\kappa + \frac{1}{2} \sigma^2 (f'(0))^2 u^2},$$

whose derivative with respect to u has the sign of $\kappa - \frac{1}{2} \sigma^2 (f'(0))^2 u^2$. Consequently $\mathcal{R}_0^s$ grows with p as long as $(\Lambda/\mu)^p < \sqrt{2\kappa}/(\sigma f'(0))$ and decreases afterwards: A super-linear incidence does not systematically help or hinder the epidemic, it amplifies whichever of the two competing effects dominates, the growth of the susceptible pool fed by recruitment on the one hand and the damping caused by the noise on the other. Only in the noise-free case $\sigma = 0$ is the threshold proportional to $(\Lambda/\mu)^p$; as soon as $\sigma > 0$ the quadratic term in the denominator destroys that proportionality, whatever the value of p . This dependence on p is invisible in models with a bilinear incidence and is one more reason for keeping p general. We note, however, that $p = 1$ remains special in one respect: As shown in Remark 3.5, the auxiliary lower bound (3.10) is vacuous for $p = 1$ but becomes a genuine requirement for $p \geq 2$.

Three questions are left open: Whether the sufficient condition (3.11) on σ can be weakened, whether the auxiliary lower bound (3.10) can be dispensed with for $p \geq 2$ and how the threshold changes when the immunity period is itself governed by a semi-Markov process. In the present model the loss of immunity is exponentially distributed at rate $\ell > 0$, but in reality the duration of immunity may depend on elapsed time; letting it follow a semi-Markov law would require extending the state space of the driving process.

Acknowledgements. The authors are grateful to the anonymous referees for their careful reading and for suggestions that improved the presentation of the paper.

References

- [1] B. Boukanjime, T. Caraballo, M. El Fatini and M. El Khalifi, *Dynamics of a stochastic coronavirus (COVID-19) epidemic model with Markovian switching*, Chaos Solitons Fractals, 2020, 141, 110361.
- [2] L. Boulaasair, H. Bouzahir, R. Sai and M. A. Alaqeel, *Analysis of a hybrid SIR epidemic model driven by dual white noises and Markovian switching*, Results Phys., 2025, 76, 108385.
- [3] X. Cao, *Semi-Markov decision problems and performance sensitivity analysis*, IEEE Trans. Automat. Control, 2003, 48(5), 758–769.
- [4] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes, II*, Springer-Verlag, New York, 1975.
- [5] A. Lahrouz and L. Omari, *Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence*, Stat. Probab. Lett., 2013, 83(4), 960–968.
- [6] D. Li, J. Cui, M. Liu and S. Q. Liu, *The evolutionary dynamics of stochastic epidemic model with nonlinear incidence rate*, Bull. Math. Biol., 2015, 77(9), 1705–1743.
- [7] D. Li, S. Q. Liu and J. A. Cui, *Threshold dynamics and ergodicity of an SIRS epidemic model with semi-Markov switching*, J. Differential Equations, 2019, 266(7), 3973–4017.
- [8] J. Q. Li and Z. E. Ma, *Global analysis of SIS epidemic models with variable total population size*, Math. Comput. Modelling, 2004, 39(11–12), 1231–1242.
- [9] Y. Li, J. R. Chu and X. Y. Liao, *On persistence for a stochastic epidemic model with nonlinear incidence*, J. Univ. South China (Sci. Technol.), 2023, 37(1), 71–77.

- [10] N. Limnios and G. Oprisan, *Semi-Markov Processes and Reliability*, Birkhauser, Boston, 2001.
- [11] X. Z. Liu and P. Stechlinski, *Infectious Disease Modeling: A Hybrid System Approach*, Springer, Cham, 2017.
- [12] M. Mandal, S. Jana, S. K. Nandi, A. Khatua, S. Adak and T. K. Kar, *A model based study on the dynamics of COVID-19: Prediction and control*, Chaos Solitons Fractals, 2020, 136, 109889.
- [13] X. Mao, *Stability of stochastic differential equations with Markovian switching*, Stoch. Process. Appl., 1999, 79(1), 45–67.
- [14] M. A. Bin Masud and M. Ahmed, *Optimal control for system of stochastic differential equations with Lévy jumps*, Results Control Optim., 2025, 20, 100600.
- [15] T. H. Nguyen, D. H. Nguyen, N. N. Nguyen and T. D. Tuong, *Hybrid stochastic SIS epidemic models with vaccination: Stability of the disease-free state and applications*, Nonlinear Anal. Hybrid Syst., 2024, 53, 101492.
- [16] S. Ottaviano, *Threshold dynamics of SAIRS epidemic model with semi-Markov switching*, Math. Methods Appl. Sci., 2023, 46(9), 10289–10310.
- [17] M. J. Small and D. J. Morgan, *The relationship between a continuous-time renewal model and a discrete Markov chain model of precipitation occurrence*, Water Resour. Res., 1986, 22(10), 1422–1430.
- [18] Q. Q. Sun, D. J. Tan and S. W. Zhang, *Epidemic threshold of a COVID-19 model with Gaussian white noise and semi-Markov switching*, J. Franklin Inst., 2023, 360(10), 6846–6861.
- [19] X. Zhang, D. Jiang, A. Alsaedi and T. Hayat, *Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching*, Appl. Math. Lett., 2016, 59, 87–93.
- [20] X. Zhao, T. Feng, L. Wang and Z. P. Qiu, *Threshold dynamics and sensitivity analysis of a stochastic semi-Markov switched SIRS epidemic model with nonlinear incidence and vaccination*, Discrete Contin. Dyn. Syst. Ser. B, 2021, 26(12), 6131–6154.
- [21] L. Zu, D. Jiang and D. O'Regan, *Conditions for persistence and ergodicity of a stochastic Lotka–Volterra predator-prey model with regime switching*, Commun. Nonlinear Sci. Numer. Simul., 2015, 29(1–3), 1–11.

Received October 2025; Accepted October 2026; Available online October 2026.