

ANALYTICAL SOLUTIONS AND EFFICIENT COMPACT SCHEMES OF THE CAPUTO-HADAMARD TIME FRACTIONAL MODEL FOR NANOFLUID*

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Abstract In this paper, we consider the Caputo-Hadamard time fractional model for unsteady MHD mixed convection problem of Maxwell fractional nanofluid. Two kinds of boundary conditions are proposed, which are Dirichlet boundary conditions and Robin boundary conditions. The analytical solutions for the equation are derived by using the method of separation of variables and the generalized Laplace transform. The numerical schemes are developed based on the L1 and L1-2 approximations for the Caputo-Hadamard fractional derivatives and the compact finite difference scheme for spatial derivative. Stability and convergence of the obtained numerical scheme are analyzed theoretically. Numerical examples are constructed to demonstrate the accuracy of the numerical scheme as compared to theoretical analysis.

Keywords Caputo-Hadamard, analytical solution, separation of variables, generalized Laplace transform, compact scheme.

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1. Introduction

Fractional order partial differential equations have gained more and more attention during the last few decades, due to its demonstrated applications in many areas of science and engineering such as hydrology, physics, finance, biology, chemistry, fluid mechanics as well as bioengineering [20, 23]. This is because of the nonlocal nature of the fractional order derivatives which provide a powerful tool to capture the heterogeneity of the substances that exhibit anomalous diffusion. The most important advantage of the fractional order derivatives is they can describe the memory and hereditary properties of various substances [5].

In recent years, many papers [3, 7–10, 12, 14, 34, 36, 37, 39] have discussed the numerical computation and higher-order numerical schemes of fractional derivatives. Especially, with the increasing research interest in fractional differential equations, particularly in fluid dynamics and heat transfer, issues such as how to construct high-precision numerical schemes have be-

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come the focus of many scholars. To the authors' best knowledge, the detailed discussions on high-order numerical approximations for Caputo, Riemann-Liouville and Riesz derivatives have been conducted. Gao et al. gave an L1-2 interpolation approximation formula for the Caputo fractional derivative of order α ($0 < \alpha < 1$) in [14]. On this basis, Alikhanov discovered some superconvergent interpolation points and built L2-1 $_{\sigma}$ formula, which can reach the $(3 - \alpha)$ -th order convergence uniformly [3]. In [12], Gao et al. considered L2-1 $_{\sigma}$ interpolation approximation of multi-term Caputo fractional derivatives. With respect to space fractional operators, Tadjeran et al. [34] gave an asymptotic expansion for the error in the shifted Grünwald finite difference formula, which plays a vital role in construction of high order difference schemes. Another version of shifted formula was first proposed by Tuan and Gorenflo [37]. As a generalization of the symmetric difference operator, fractional shifted Grünwald formulae can approximate Riemann-Liouville fractional derivatives with the second-order accuracy. Tian et al. [36] introduced the weighted and shifted Grünwald difference (WSGD) operators for Riemann-Liouville fractional derivatives, then they numerically solved space fractional diffusion equations in one and two dimensions, the stability and convergence of the difference schemes are theoretically established. Relate to the Riesz derivative, the scholars in [7] used the central difference quotient formula to approximate the Riesz fractional derivative, then developed a Crank-Nicolson difference scheme of order two in both time and space to solve a one-dimensional space-fractional partial differential equation. A fourth-order numerical differentiation formula to approximate the weighted value of Riesz fractional derivatives at three points was proposed by Zhao et al. [39]. Ding et al. [9] derived a different higher-order numerical differentiation formula to approximate the Riesz fractional derivative and applied it to the numerical solution of the space-fractional differential equation in the Riesz derivative type. With respect to the Caputo-Hadamard fractional derivative, Fan et al. proposed the L1-2 formula, L2-1 $_{\sigma}$ formula and H2N2 formula in [10]. Among them, the numerical formulas L1-2 and L2-1 $_{\sigma}$ are for order $\alpha \in (0, 1)$ with $(3 - \alpha)$ -th order convergence, and H2N2 formula is for order $\alpha \in (1, 2)$ with $(3 - \alpha)$ -th order convergence too.

Many researchers have worked on fractional derivatives for viscous fluid. They have used fractional calculus to explain the viscoelastic behavior of material [2, 11, 35, 38]. At present, the utilization of nanofluids is a subject of bounteous reviews. Cao et al. [6] dealt with fractional Maxwell viscoelastic nanofluid over a moving plate. Pan et al. [29] derived a thermal transport equation with fractional calculus to investigate the effects of anomalous diffusion of nanoparticles on convective heat transfer of nanofluids through a porous medium in boundary layer flow. They solved the fractional order temperature equation with the Caputo derivative by the finite difference method and discussed in detail. Aman et al. [4] used Caputo time fractional derivatives to obtained the exact solutions for unsteady MHD mixed convection problem of Maxwell fractional nanofluid. They found that rate of heat transfer increases with increasing nanoparticles volume fraction and Caputo time fractional parameters. This latest area appealed to us to investigate a problem on nanofluids by using Caputo-Hadamard fractional derivative model.

Over the past few years, the Hadamard type derivatives are used in some practical problems related to mechanics and engineering, e.g., both planar and three-dimensional elasticities, or the fracture analysis [1] and the Lomnitz logarithmic creep law of special substances, e.g., igneous rock [15]. But the Hadamard fractional derivative seems not to be easily used in applied science and engineering [25]. Jarad et al. [21] modified the Hadamard fractional derivatives into a more suitable one having physically interpretable initial conditions similar to the ones in the Caputo setting. They defined the Caputo-type modifications of the Hadamard fractional derivatives and presented properties of such derivatives. The alternative definition is the Caputo-Hadamard fractional derivative.

Several work have been done for solving fractional equations with Caputo-Hadamard derivatives analytically and numerically. To mention a few, Li et al. [24] applied the amended Laplace transform and the amended Mellin transform to obtain the fundamental solution and its asymptotic estimates of the time-space fractional diffusion equation. Fan et al. [10] proposed three kinds of numerical formulas for approximating the Caputo-Hadamard fractional derivatives. Then the authors applied these three numerical formulas to long-term integration of fractional differential systems. Liu et al. [28] established an averaging principle for Caputo-Hadamard fractional stochastic differential equations. It is showed the solution of the Caputo-Hadamard fractional stochastic differential equation converges to the solution of the averaged equation when the time scale parameter tends to zero. Hammad et al. [19] studied the existence and uniqueness of solutions to the fractional integro-differential equation with the Caputo-Hadamard derivative under nonlocal boundary conditions. Li et al. [27] constructed an Euler-Maruyama (EM) scheme and a fast EM scheme based on the sum-of-exponentials approximation to decrease the computational cost of the EM scheme for the Caputo-Hadamard fractional stochastic differential equations.

While fractional models for non-Newtonian fluids have been extensively studied, the Caputo-Hadamard fractional derivative, with its logarithmic kernel, is particularly suited for problems exhibiting slow, anomalous dynamics across broad temporal scales, a feature often observed in complex nanofluid systems. However, efficient and theoretically-grounded numerical schemes for such models with mixed boundary conditions remain a challenge. This work aims to bridge this gap by developing a high-order compact numerical scheme with rigorous stability and convergence guarantees, which is crucial for reliable long-term simulations in industrial applications.

Aman et al. [4] considered Poiseuille flow of water based-nanofluid with graphene nanoparticles in a vertical channel with the following governing equations of momentum, energy and mass:

$$\begin{aligned}\rho_{nf} D_t^\alpha u &= -\frac{\partial p}{\partial t} + \mu_{nf} \frac{\partial^2 u}{\partial y^2} - (\sigma_{nf} B_0^2 + \frac{\mu_{nf}}{k_1})u + (\rho\beta_T)_{nf} g(T - T_0) \\ &\quad + (\rho\beta_C)_{nf} g(C - C_0) + H(t)(\lambda_0 + \lambda e^{i\omega t}), \\ (\rho c_p)_{nf} D_t^\alpha T &= k_{nf} \frac{\partial^2 T}{\partial y^2} + 4\alpha_0^2(T - T_0), \\ D_t^\alpha C &= D_{nf} \frac{\partial^2 C}{\partial y^2},\end{aligned}$$

where $u = u(y, t)$, $T = T(y, t)$, $C = C(y, t)$, $\rho_{nf}, \mu_{nf}, \sigma_{nf}, \beta_T, \beta_C, g, (\rho c_p)_{nf}, k_{nf}, \alpha_0, D_{nf}$ are respectively fluid velocity in the x direction, temperature, concentration, density, dynamic viscosity, electrical conductivity of the base fluid, volumetric thermal expansion co-efficient, gravitational acceleration, heat capacitance of nanofluids, thermal conductivity of nanofluid, radiation absorption and thermal diffusion coefficient.

Following the methodology presented in that article, we consider simplifying the above system of equations to the following form for solution:

$${}_{CH}D_{a,t}^\alpha u(x, t) = A \frac{\partial^2 u(x, t)}{\partial x^2} - Bu(x, t) + F(x, t), \quad (1.1)$$

where $x \in [0, L]$, $t \in [a, T]$ ($0 < a$) and ${}_{CH}D_t^\alpha$ denotes the Caputo-Hadamard fractional order derivative of α ($0 < \alpha < 1$) which is defined as [16, 17, 21]

$${}_{CH}D_{a,t}^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}, \quad 0 < a < t,$$

with $\delta^n y(s) = \left(s \frac{d}{ds}\right)^n y(s)$, $n - 1 < \alpha < n \in \mathbb{Z}^+$.

In this paper, we mainly consider Eq. (1.1) with the following two boundary value problems. The first case is subject to the initial and Dirichlet boundary conditions:

$$u(x, a) = u_a(x), \quad (1.2a)$$

$$u(0, t) = 0, \quad u(L, t) = 0. \quad (1.2b)$$

The second case is subject to the initial and Robin boundary conditions:

$$u(x, a) = u_a(x), \quad (1.3a)$$

$$-u_x(0, t) + u(0, t) = b_l(t), \quad u_x(L, t) + u(L, t) = b_r(t). \quad (1.3b)$$

The main goal of this article is to derive the analytic solutions and construct the effective finite difference schemes for the Caputo-Hadamard time fractional model for unsteady MHD mixed convection problem of Maxwell fractional nanofluid.

2. Analytical solutions

In this section the method of separation of variables [18] is employed to solve the above two boundary value problems.

2.1. Dirichlet boundary condition

Next, we first employ the method of separation of variables to solve for the eigenvalue functions of the corresponding homogeneous equation, i.e. $F(x, t) \equiv 0$. Suppose its solution exhibits a separable form in variables $u(x, t) = X(x)T(t)$, substitute it into the equation (1.1) and the conditions (1.2b), the boundary value problem of the following ordinary differential equation constructed from a spatial function $X(x)$ is obtained:

$$AX''(x) + \lambda X(x) = 0, \quad (2.1a)$$

$$X(0) = 0, \quad X(L) = 0 \quad (2.1b)$$

and a fractional order equation constructed from a temporal function $T(t)$:

$${}_CH D_{a,t}^\alpha T(t) + (\lambda + B)T(t) = 0,$$

where λ is the separated constant.

2.1.1. Eigenfunctions

The discussion for the eigenvalue problem (2.1a) is divided into three cases.

- $\lambda = 0$, the general solution is $X(x) = C_1 + C_2x$. The result $C_1 = C_2 = 0$ follows the boundary conditions (2.1b) that leads to the trivial solution $u(x, t) \equiv 0$.
- $\lambda < 0$, the general solution is $X(x) = C_1 e^{rx} + C_2 e^{-rx}$, where $r = \sqrt{-\frac{\lambda}{A}}$. The boundary conditions (2.1b) leads to $C_1 = -C_2$ and $C_2(e^{-rL} - e^{rL}) = 0$, which result in the trivial solution too.

- $\lambda > 0$, it yields the general solution as $X(x) = C_1 \cos(kx) + C_2 \sin(kx)$, and $k = \sqrt{\frac{\lambda}{A}}$. The boundary conditions first leads to $C_1 = 0$. To ensure $u(x, t)$ is not the trivial solution, the following equation have to satisfy:

$$\sin kL = 0.$$

Then,

$$k = k_n = \frac{n\pi}{L} \quad (n = 1, 2, \dots)$$

it results a series of discrete eigenvalues and their associated eigenfunctions:

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2,$$

and

$$X_n(x) = C_n \sin(k_n x),$$

which satisfy orthogonality as basis functions of Fourier series.

2.1.2. General solution

To solve the following temporal Caputo-Hadamard fractional-order differential equation, we first introduce the generalized Laplace and inverse Laplace transform [25].

Definition 2.1. For a given function $\phi(t)$ defined on $[a, +\infty)$ ($a > 0$), the modified Laplace transform of $\phi(t)$ is defined as

$$\Phi(s) = \mathcal{L}\{\phi(t)\} = \int_a^\infty e^{-s \log \frac{t}{a}} \phi(t) \frac{dt}{t}, \quad s \in \mathbb{C}.$$

And the inverse modified Laplace transform of $G(s)$ is given by:

$$\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s \log \frac{t}{a}} \Phi(s) ds, \quad c > 0, \quad i^2 = -1.$$

Lemma 2.1. The following equality holds [25]:

$$\mathcal{L}\{ {}_{CH}D_{a,t}^\alpha \phi(t) \} = s^\alpha \Phi(s) - s^{\alpha-1} \phi(a).$$

By means of the formula [30]:

$$\int_0^\infty e^{-st} t^{\alpha k + \beta - 1} E_{\alpha, \beta}^k(\pm \lambda t^\alpha) dt = \frac{k! s^{\alpha - \beta}}{(s^\alpha \mp \lambda)^{k+1}}, \quad \Re(s) > |\lambda|^{\frac{1}{\alpha}},$$

where $E_{\alpha, \beta}(z)$ is the Mittag-Leffler function [22]:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha > 0, \beta > 0, |z| < \infty,$$

there exists the following lemma.

Lemma 2.2. The following equality holds [25]:

$$\int_a^\infty e^{-s \log \frac{\tau}{a}} \left(\log \frac{\tau}{a}\right)^{\alpha k + \beta - 1} E_{\alpha, \beta}^k(\pm \lambda (\log \frac{\tau}{a})^\alpha) \frac{d\tau}{\tau} = \frac{k! s^{\alpha - \beta}}{(s^\alpha \mp \lambda)^{k+1}}, \quad \Re(s) > |\lambda|^{\frac{1}{\alpha}}.$$

Expand the driving term in (1.1) in terms of the eigenfunction basis as follows:

$$F(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(k_n x), \quad (2.2)$$

where

$$f_n(t) = \frac{2}{L} \int_0^L F(x, t) \sin(k_n x) dx.$$

Letting $v(x, t) = \sum_{n=1}^{\infty} g_n(t) X_n(x) = \sum_{n=1}^{\infty} g_n(t) \sin(k_n x)$ and substituting it and the source term (2.2) into the equation (1.1) leads to the initial value problem for a fractional-order ordinary differential equation:

$$\begin{aligned} {}_{CH}D_{a,t}^{\alpha} g_n(t) + (Ak_n^2 + B)g_n(t) &= f_n(t), \\ g_n(a) &= \frac{2}{L} \int_0^L u_a(x) \sin(k_n x) dx, \end{aligned} \quad (2.3)$$

due to the orthogonality of eigenfunctions $X_n(x) = \sin(k_n x)$.

Applying the generalized Laplace transform to both sides of the equation (2.3):

$$s^{\alpha} G_n(s) - s^{\alpha-1} g_n(a) + (Ak_n^2 + B)G_n(s) = F_n(s)$$

i.e.

$$G_n(s) = \frac{s^{\alpha-1} g_n(a) + F_n(s)}{s^{\alpha} + Ak_n^2 + B},$$

then applying the inverse Laplace transform to both sides of the above equation, from Lemmas 2.1 and 2.2, one obtains [25]:

$$\begin{aligned} g_n(t) &= E_{\alpha,1}(-(Ak_n^2 + B)(\log \frac{t}{a})^{\alpha}) g_n(a) \\ &\quad + \int_a^t (\log \frac{t}{s})^{\alpha-1} E_{\alpha,\alpha}(-(Ak_n^2 + B)(\log \frac{t}{s})^{\alpha}) f_n(s) \frac{ds}{s}. \end{aligned} \quad (2.4)$$

Hence we have the general solution of the equation (1.1), (1.2a) and (1.2b):

$$u(x, t) = \sum_{n=1}^{\infty} g_n(t) \sin(k_n x). \quad (2.5)$$

2.2. Robin boundary condition

Since the method of separation of variables cannot handle a problem with non-homogeneous boundary conditions, we first employ a variable substitution, $u(x, t) = v(x, t) + \omega(x)$, $v(x, t)$ satisfying homogeneous boundary conditions and $\omega(x)$ satisfying non-homogeneous boundary conditions. It should be noted that this method is only applicable to special cases when b_l , b_r in the non-homogeneous boundary conditions (1.3b) are constants. It leads to the following two equations with respect to $\omega(x)$ and $v(x, t)$, respectively:

$$A\omega''(x) - B\omega(x) = 0, \quad (2.6a)$$

$$-\omega'(0) + \omega(0) = b_l, \quad \omega'(L) + \omega(L) = b_r, \quad (2.6b)$$

and

$${}_CHD_{a,t}^\alpha v(x,t) = A \frac{\partial^2 v(x,t)}{\partial x^2} - Bv(x,t) + F(x,t), \quad (2.7a)$$

$$v(x,a) = u_a(x) - \omega(x) := \psi(x), \quad (2.7b)$$

$$-v_x(0,t) + v(0,t) = 0, v_x(L,t) + v(L,t) = 0. \quad (2.7c)$$

For the second-order linear ordinary differential equation with constant coefficients (2.6a) and (2.6b), it is straightforward to obtain its solution as

$$\omega(x) = d_1 e^{cx} + d_2 e^{-cx}, \quad (2.8)$$

where $d_1 = \frac{b_l e^{-cL}(c-1) + b_r(c+1)}{e^{cL}(c+1)^2 - e^{-cL}(c-1)^2}$, $d_2 = \frac{b_r(c-1) + b_l e^{cL}(c+1)}{e^{cL}(c+1)^2 - e^{-cL}(c-1)^2}$, $c = \sqrt{\frac{B}{A}}$.

Next, the method of separation of variables is employed to solve the equation (2.7a) with initial condition (2.7b) and homogeneous boundary conditions (2.7c).

2.2.1. Eigenfunctions

Suppose the solution exhibits a separable form in variables $v(x,t) = X(x)T(t)$ for the corresponding homogeneous equation as $F(x,t) \equiv 0$, then substitute it into the equation (2.7a), (2.7b) and (2.7c) the boundary value problem of the following ordinary differential equation constructed from a spatial function $X(x)$ is obtained:

$$AX''(x) + \lambda X(x) = 0, \quad (2.9a)$$

$$-X'(0) + X(0) = 0, X'(L) + X(L) = 0. \quad (2.9b)$$

The discussion of the above eigenvalue problem is divided into three cases.

- $\lambda = 0$, it clearly leads to the trivial solution $v(x,t) \equiv 0$.
- $\lambda < 0$, the general solution is $X(x) = C_1 e^{rx} + C_2 e^{-rx}$, where $r = \sqrt{-\frac{\lambda}{A}}$. The boundary conditions (2.9b) leads to the trivial solution too.
- $\lambda > 0$, it yields the general solution as $X(x) = C_1 \cos(kx) + C_2 \sin(kx)$, and $k = \sqrt{\frac{\lambda}{A}}$. From the boundary conditions, it obtains that

$$C_1 = kC_2, C_1 \cos(kL) + C_2 \sin(kL) - C_1 k \sin(kL) + C_2 k \cos(kL) = 0.$$

To ensure the existence of non-trivial solutions, k must satisfy the following condition:

$$(1 - k^2) \sin(kL) + 2k \cos(kL) = 0.$$

Thus the eigenfunction is:

$$X_n(x) = C_n(k_n \cos(k_n x) + \sin(k_n x)),$$

where k_n is the solution of the following nonlinear transcendental equation:

$$\tan(kL) = \frac{2k}{k^2 - 1}, \quad (2.10)$$

and the eigenvalue $\lambda_n = Ak_n^2$, where k_n are arranged in ascending order, and $k_n > 0, n = 1, 2, \dots$.

2.2.2. Orthogonality of eigenfunctions

Let k_n be the n -th eigenvalue, and $X_n(x)$ is the corresponding eigenfunction. Discuss the orthogonality of eigenfunctions and evaluate the integral $\int_0^L X_m(x)X_n(x)dx$ ($m \neq n$).

The eigenfunction equation (2.9a) leads to

$$-k_n^2 X_n = \frac{d}{dx}(X'_n), \quad -k_m^2 X_m = \frac{d}{dx}(X'_m).$$

Multiplying the first equation by X_m and the second one by X_n , subtracting the two equations and integrating them yields:

$$\begin{aligned} & (k_m^2 - k_n^2) \int_0^L X_m X_n dx \\ &= \int_0^L [X_m \frac{d}{dx}(X'_n) - X_n \frac{d}{dx}(X'_m)] dx \\ &= X_m X'_n|_0^L - \int_0^L X'_n X'_m dx - X_n X'_m|_0^L + \int_0^L X'_m X'_n dx \\ &= (X_m(L)X'_n(L) - X_m(0)X'_n(0)) - (X_n(L)X'_m(L) - X_n(0)X'_m(0)) \\ &= (X_n(0)X'_m(0) - X_m(0)X'_n(0)) - (X_n(L)X'_m(L) - X_m(L)X'_n(L)). \end{aligned}$$

From the boundary condition of the eigenfunction equation (2.9a), we have

$$X_m(0) = X'_m(0), \quad X_m(L) = -X'_m(L), \quad X_n(0) = X'_n(0), \quad X_n(L) = -X'_n(L),$$

therefore

$$\int_0^L X_m X_n dx = 0. \quad (2.11)$$

2.2.3. General solution

Similar to Section 2.1.2, expand the driving term in terms of the eigenfunction basis as follows:

$$F(x, t) = \sum_{n=1}^{\infty} f_n(t)(k_n \cos(k_n x) + \sin(k_n x)), \quad (2.12)$$

where

$$f_n(t) = \frac{2}{L} \int_0^L F(x, t)(k_n \cos(k_n x) + \sin(k_n x)) dx.$$

Letting $v(x, t) = \sum_{n=1}^{\infty} g_n(t)(k_n \cos(k_n x) + \sin(k_n x))$ and substituting it and the source term (2.12) into the equation (2.7a) leads to the initial value problem for a fractional-order ordinary differential equation:

$$\begin{aligned} {}_{CH}D_{a,t}^{\alpha} g_n(t) + (Ak_n^2 + B)g_n(t) &= f_n(t), \\ g_n(a) &= \frac{2}{L} \int_0^L \psi(x)(k_n \cos(k_n x) + \sin(k_n x)) dx. \end{aligned} \quad (2.13)$$

Applying the generalized Laplace transform and inverse transform to both sides of the equation (2.13), we arrive at its general solution similar to (2.4). Then we have the general solution of the equation (2.7a):

$$v(x, t) = \sum_{n=1}^{\infty} g_n(t)(k_n \cos(k_n x) + \sin(k_n x)),$$

where $g_n(t)$ is defined by Eq. (2.4). Then we arrive at the general solution of the equations (1.1), (1.3a) and (1.3b) :

$$u(x, t) = \sum_{n=1}^{\infty} g_n(t)(k_n \cos(k_n x) + \sin(k_n x)) + \omega(x),$$

where $\omega(x)$ is defined by Eq. (2.8).

Given that the analytical solutions in series form are computationally challenging and are also limited to constant non-homogeneous boundary conditions, compact numerical schemes are proposed to solve the equations in the following sections.

3. Compact finite difference schemes

To construct a compact finite difference scheme, we first discretize the interval $[0, L]$ with $x_i = ih$ ($0 \leq i \leq M$), and $t_k = a + k\tau$ ($0 \leq k \leq N$), where $h = L/M$ and $\tau = (T - a)/N$ are the spatial and temporal step sizes, respectively. Denote $t_{k+\frac{1}{2}} = (t_k + t_{k+1})/2$, $\Omega_h = \{x_i | 0 \leq i \leq M\}$, $\Omega_\tau = \{t_k | 0 \leq k \leq N\}$. Then the domain $[0, L] \times [a, T]$ is covered by $\Omega_h \times \Omega_\tau$. For any mesh function $u = \{u_i^k | 0 \leq i \leq M, 0 \leq k \leq N\}$ defined on $\Omega_h \times \Omega_\tau$, we introduce the following notations

$$\begin{aligned} \delta_x u_{i-\frac{1}{2}}^k &= \frac{1}{h} (u_i^k - u_{i-1}^k), \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \\ \delta_x^2 u_i^k &= \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} = \frac{1}{h} (\delta_x u_{i+\frac{1}{2}}^k - \delta_x u_{i-\frac{1}{2}}^k), \quad 1 \leq i \leq M-1, \quad 0 \leq k \leq N, \\ \tilde{\delta}_x^2 u_i^k &\equiv \begin{cases} \frac{2}{h} \delta_x u_{\frac{1}{2}}^k, & i = 0, \quad 0 \leq k \leq N, \\ \frac{1}{h} (\delta_x u_{i+\frac{1}{2}}^k - \delta_x u_{i-\frac{1}{2}}^k), & 1 \leq i \leq M-1, \quad 0 \leq k \leq N, \\ -\frac{2}{h} \delta_x u_{M-\frac{1}{2}}^k, & i = M, \quad 0 \leq k \leq N. \end{cases} \end{aligned}$$

In addition, we introduce the average operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ for Dirichlet boundary condition and Robin boundary condition, respectively. And the L1 discrete Caputo-Hadamard fractional derivative operator D_τ^α , as follows [16]:

$$\mathcal{A}u_i \equiv \begin{cases} \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \leq i \leq M-1, \\ u_i, & i = 0, M, \end{cases} \quad (3.1)$$

$$\tilde{\mathcal{A}}u_i \equiv \begin{cases} \frac{1}{6}(5u_0 + u_1), & i = 0, \\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \leq i \leq M-1, \\ \frac{1}{6}(u_{M-1} + 5u_M), & i = M, \end{cases} \quad (3.2)$$

$$D_\tau^\alpha u_i^k = \sum_{j=0}^k b_{k-j,k} u_i^j, \quad (3.3)$$

where

$$b_{k-j,k} = \begin{cases} -a_{1,k}, & j = 0, \\ a_{j,k} - a_{j+1,k}, & j = 1, 2, \dots, k-1, \\ a_{k,k}, & j = k, \end{cases}$$

and

$$a_{j,k} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\log \frac{t_j}{t_{j-1}}} \left(\left(\log \frac{t_k}{t_{j-1}} \right)^{1-\alpha} - \left(\log \frac{t_k}{t_j} \right)^{1-\alpha} \right).$$

Lemma 3.1. [31, 32]. *If $f(x) \in C^6[x_0, x_M]$, then it holds that*

$$\begin{aligned} & \frac{5}{6}f''(x_0) + \frac{1}{6}f''(x_1) \\ &= \frac{2}{h} \left[\frac{f(x_1) - f(x_0)}{h} - f'(x_0) \right] - \frac{h}{6}f'''(x_0) + O(h^3), \end{aligned} \quad (3.4a)$$

$$\begin{aligned} & \frac{1}{12}[f''(x_{i-1}) + 10f''(x_i) + f''(x_{i+1})] \\ &= \frac{1}{h^2}[f(x_{i-1}) - 2f(x_i) + f(x_{i+1})] + O(h^4), \end{aligned} \quad (3.4b)$$

$$\begin{aligned} & \frac{1}{6}f''(x_{M-1}) + \frac{5}{6}f''(x_M) \\ &= \frac{2}{h} \left[f'(x_M) - \frac{f(x_M) - f(x_{M-1}))}{h} \right] + \frac{h}{6}f'''(x_M) + O(h^3). \end{aligned} \quad (3.4c)$$

Lemma 3.2. [16]. *Suppose $0 < \alpha < 1$, $u \in C^2[a, t_k]$, and denote*

$${}_{CH}D_{a,t}^\alpha u^k = D_\tau^\alpha u^k + R_t^k = a_{k,k}u^k - \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})u^j - a_{1,k}u^0 + R_t^k, \quad (3.5)$$

where $R_t^k \leq C\tau^{2-\alpha}$.

3.1. Dirichlet boundary condition

We now construct a compact difference scheme for solving Eq. (1.1) with the initial and Dirichlet boundary conditions Eqs. (1.2a) - (1.2b). Suppose $u(x, t) \in C_{x,t}^{6,2}([0, L] \times [a, T])$. Considering Eq. (1.1) at the point (x_i, t_k) , and then acting the average operator $\mathcal{A}$ on both sides of Eq. (1.1), we have

$$\mathcal{A} {}_{CH}D_{a,t}^\alpha u(x_i, t_k) = \mathcal{A} \mathcal{A} \frac{\partial^2 u(x_i, t_k)}{\partial x^2} - \mathcal{B} \mathcal{A} u(x_i, t_k) + \mathcal{A} F(x_i, t_k).$$

Denote the grid function $U_i^k = u(x_i, t_k)$, $0 \leq i \leq M$, $0 \leq k \leq N$. Using the L1 discrete Caputo-Hadamard fractional derivative operator D_τ^α in Eq. (3.3) to approach the term ${}_{CH}D_{a,t}^\alpha u(x_i, t_k)$ and the compact scheme (3.4b) in Lemma 3.1 to approach the term $\mathcal{A} \frac{\partial^2 u(x_i, t_k)}{\partial x^2}$, we have

$$\mathcal{A} D_\tau^\alpha U_i^k = A \delta_x^2 U_i^k - \mathcal{B} \mathcal{A} U_i^k + \mathcal{A} F_i^k + R_i^k, \quad (3.6)$$

where $|R_i^k| \leq C_0(\tau^{2-\alpha} + h^4)$, $1 \leq i \leq M-1$, $1 \leq k \leq N$, C_0 is a constant.

Omitting small terms R_i^k in Eq. (3.6), and replacing the function U_i^k with its numerical approximation u_i^k , as well as noticing the initial condition and Dirichlet boundary condition in Eqs. (1.2a)-(1.2b), we can obtain the following compact finite difference scheme

$$\mathcal{A}D_\tau^\alpha u_i^k = A\delta_x^2 u_i^k - B\mathcal{A}u_i^k + \mathcal{A}F_i^k, \quad 1 \leq i \leq M-1, 1 \leq k \leq N, \quad (3.7a)$$

$$u_i^0 = u_a(x_0), \quad 0 \leq i \leq M, \quad (3.7b)$$

$$u_0^k = u_M^k = 0, \quad 1 \leq k \leq N. \quad (3.7c)$$

3.2. Robin boundary condition

We now construct a compact difference scheme for solving Eq. (1.1) with the initial and Robin boundary conditions Eqs. (1.3a)-(1.3b). Suppose $u(x, t) \in C_{x,t}^{6,2}([0, L] \times [a, T])$. Denote the grid function $U_i^k = u(x_i, t_k)$, $0 \leq i \leq M$, $0 \leq k \leq N$. Considering Eqs. (1.1) at the point (x_i, t_k) , and acting the average operator $\tilde{\mathcal{A}}$ on both sides of Eq. (1.1), then using the L1 discrete Caputo-Hadamard fractional derivative operator D_τ^α in Eq. (3.3) to approach the term ${}_{CH}D_{a,t}^\alpha u(x_i, t_k)$ and the compact scheme in Eq. (3.2) to approach the term $\tilde{\mathcal{A}}\frac{\partial^2 u(x_i, t_k)}{\partial x^2}$.

For $i = 0$, we have

$$\begin{aligned} & \tilde{\mathcal{A}}D_\tau^\alpha U_0^k \\ &= A \left[\frac{5}{6} \frac{\partial^2 u(x_0, t_k)}{\partial x^2} + \frac{1}{6} \frac{\partial^2 u(x_1, t_k)}{\partial x^2} \right] - B\tilde{\mathcal{A}}U_0^k + \tilde{\mathcal{A}}F_0^k + \tilde{\mathcal{A}}(R_t^k(u(x_0, t_k))) \\ &= A \left\{ \frac{2}{h} \left[\frac{u(x_1, t_k) - u(x_0, t_k)}{h} - \frac{\partial u(x_0, t_k)}{\partial x} \right] - \frac{h}{6} \frac{\partial^3 u(x_0, t_k)}{\partial x^3} \right\} \\ & \quad - B\tilde{\mathcal{A}}U_0^k + \tilde{\mathcal{A}}F_0^k + R_0^k \\ &= A \left\{ \frac{2}{h} \frac{\partial^2 u(x_0, t_k)}{\partial x^2} - \frac{2}{h} \frac{\partial u(x_0, t_k)}{\partial x} - \frac{h}{6} \frac{\partial^3 u(x_0, t_k)}{\partial x^3} \right\} - B\tilde{\mathcal{A}}U_0^k + \tilde{\mathcal{A}}F_0^k + R_0^k, \end{aligned} \quad (3.8)$$

where $R_0^k = \tilde{\mathcal{A}}(R_t^k(u(x_0, t_k))) + O(h^3)$.

Differentiating Eq. (1.1) with respect to x and letting $x \rightarrow x_0^+$, and using the boundary condition in Eq. (1.3b), we obtain

$$\begin{aligned} A \frac{\partial^3 u(x_0, t_k)}{\partial x^3} &= {}_{CH}D_{a,t}^\alpha \left(\frac{\partial u(x_0, t_k)}{\partial x} \right) + B \frac{\partial u(x_0, t_k)}{\partial x} - \frac{\partial F(x_0, t_k)}{\partial x} \\ &= {}_{CH}D_{a,t}^\alpha (U_0^k - b_l(t_k)) + B(U_0^k - b_l(t_k)) - \frac{\partial F_0^k}{\partial x}. \end{aligned} \quad (3.9)$$

Substituting the boundary condition in Eq. (1.3b) and Eq. (3.9) into Eq. (3.8), we have

$$\begin{aligned} & \tilde{\mathcal{A}}D_\tau^\alpha U_0^k = A\tilde{\delta}_x^2 U_0^k - \frac{2A}{h} [U_0^k - b_l(t_k)] - \frac{h}{6} {}_{CH}D_{a,t}^\alpha (U_0^k - b_l(t_k)) \\ & \quad - \frac{Bh}{6} [U_0^k - b_l(t_k)] + \frac{h}{6} \frac{\partial F_0^k}{\partial x} - B\tilde{\mathcal{A}}U_0^k + \tilde{\mathcal{A}}F_0^k + R_0^k \\ &= A\tilde{\delta}_x^2 U_0^k - \left(\frac{2A}{h} + \frac{Bh}{6} \right) U_0^k - \frac{h}{6} {}_{CH}D_{a,t}^\alpha U_0^k - B\tilde{\mathcal{A}}U_0^k \\ & \quad + \left(\frac{2A}{h} + \frac{Bh}{6} \right) b_l(t_k) + \frac{h}{6} {}_{CH}D_{a,t}^\alpha b_l(t_k) + \frac{h}{6} \frac{\partial F_0^k}{\partial x} \\ & \quad + \tilde{\mathcal{A}}F_0^k + R_0^k. \end{aligned} \quad (3.10)$$

Using the L1 fractional derivative operator D_τ^α to approach the term $\frac{h}{6} {}_{CH}D_{a,t}^\alpha U_0^k$ in the above equation, we obtain

$$\frac{h}{6} {}_{CH}D_{a,t}^\alpha U_0^k = \frac{h}{6} D_\tau^\alpha U_0^k + O(h\tau^{2-\alpha}).$$

Thus, Eq. (3.10) becomes

$$\begin{aligned} \tilde{\mathcal{A}}D_\tau^\alpha U_0^k &= A\tilde{\delta}_x^2 U_0^k - \left(\frac{2A}{h} + \frac{Bh}{6}\right) U_0^k - \frac{h}{6} D_\tau^\alpha U_0^k - B\tilde{\mathcal{A}}U_0^k \\ &\quad + \left(\frac{2A}{h} + \frac{Bh}{6}\right) b_l(t_k) + \frac{h}{6} {}_{CH}D_{a,t}^\alpha b_l(t_k) + \frac{h}{6} \frac{\partial F_0^k}{\partial x} \\ &\quad + \tilde{\mathcal{A}}F_0^k + \overline{R_0^k}, \end{aligned} \quad (3.11)$$

where $\overline{R_0^k} = O(h\tau^{2-\alpha}) + R_0^k$, and hence $|\overline{R_0^k}| \leq C_1(\tau^{2-\alpha} + h^3)$, $1 \leq k \leq N$, C_1 is a constant.

Similarly, for $i = M$, we can obtain

$$\begin{aligned} \tilde{\mathcal{A}}D_\tau^\alpha U_M^k &= A\tilde{\delta}_x^2 U_M^k - \left(\frac{2A}{h} + \frac{Bh}{6}\right) U_M^k - \frac{h}{6} D_\tau^\alpha U_M^k - B\tilde{\mathcal{A}}U_M^k \\ &\quad + \left(\frac{2A}{h} + \frac{Bh}{6}\right) b_r(t_k) + \frac{h}{6} {}_{CH}D_{a,t}^\alpha b_r(t_k) - \frac{h}{6} \frac{\partial F_M^k}{\partial x} \\ &\quad + \tilde{\mathcal{A}}F_M^k + \overline{R_M^k}, \end{aligned} \quad (3.12)$$

where $|\overline{R_M^k}| \leq C_2(\tau^{2-\alpha} + h^3)$, $1 \leq k \leq N$, C_2 is a constant.

For $1 \leq i \leq M-1$, utilizing Lemmas 3.1 and 3.2, we obtain

$$\tilde{\mathcal{A}}D_\tau^\alpha U_i^k = A\tilde{\delta}_x^2 U_i^k - B\tilde{\mathcal{A}}U_i^k + \tilde{\mathcal{A}}F_i^k + \overline{R_i^k}, \quad (3.13)$$

where $|\overline{R_i^k}| \leq C_3(\tau^{2-\alpha} + h^4)$, $1 \leq i \leq M-1$, $1 \leq k \leq N$, C_3 is a constant.

Omitting small terms $\overline{R_i^k}$ in Eqs. (3.11)-(3.13), and then replacing the function U_i^k with its numerical approximation u_i^k , as well as noticing the initial conditions in Eq. (1.3a), we can obtain the following compact finite difference scheme

$$\begin{aligned} \tilde{\mathcal{A}}D_\tau^\alpha u_0^k &= A\tilde{\delta}_x^2 u_0^k - \left(\frac{2A}{h} + \frac{Bh}{6}\right) u_0^k - \frac{h}{6} D_\tau^\alpha u_0^k - B\tilde{\mathcal{A}}u_0^k \\ &\quad + \left(\frac{2A}{h} + \frac{Bh}{6}\right) b_l(t_k) + \frac{h}{6} {}_{CH}D_{a,t}^\alpha b_l(t_k) + \frac{h}{6} \frac{\partial F_0^k}{\partial x} + \tilde{\mathcal{A}}F_0^k, \end{aligned} \quad (3.14a)$$

$$\tilde{\mathcal{A}}D_\tau^\alpha u_i^k = A\tilde{\delta}_x^2 u_i^k - B\tilde{\mathcal{A}}u_i^k + \tilde{\mathcal{A}}F_i^k, \quad 1 \leq i \leq M-1, \quad (3.14b)$$

$$\begin{aligned} \tilde{\mathcal{A}}D_\tau^\alpha u_M^k &= A\tilde{\delta}_x^2 u_M^k - \left(\frac{2A}{h} + \frac{Bh}{6}\right) u_M^k - \frac{h}{6} D_\tau^\alpha u_M^k - B\tilde{\mathcal{A}}u_M^k \\ &\quad + \left(\frac{2A}{h} + \frac{Bh}{6}\right) b_r(t_k) + \frac{h}{6} {}_{CH}D_{a,t}^\alpha b_r(t_k) - \frac{h}{6} \frac{\partial F_M^k}{\partial x} + \tilde{\mathcal{A}}F_M^k, \end{aligned} \quad (3.14c)$$

$$u_i^0 = u_a(x_0), \quad 0 \leq i \leq M, \quad (3.14d)$$

where $1 \leq k \leq N$.

4. Stability of the two schemes

Lemma 4.1. [16] For $0 < \alpha < 1$, the coefficients $a_{j,k}$ ($1 \leq j \leq k, 1 \leq k \leq N$) defined in Lemma 3.2 satisfy:

$$a_{k,k} > a_{k-1,k} > \cdots > a_{j,k} > a_{j-1,k} > \cdots > a_{1,k} > 0.$$

4.1. Dirichlet boundary condition

We now introduce some notations about the discrete inner product and the corresponding norm. Let $\mathcal{V}_D = \{v | v = (v_0, v_1, \dots, v_{M-1}, v_M), v_0 = v_M = 0\}$. For any $u, v \in \mathcal{V}_D$, we define the discrete inner products L_2 norm, H_1 semi-norm and the maximum norm as follows:

$$\begin{aligned} (u, v)_D &= h \sum_{i=1}^{M-1} u_i v_i, \quad \|u\|_D = \sqrt{(u, u)_D}, \\ \langle \delta_x u, \delta_x v \rangle_D &= h \sum_{i=0}^{M-1} (\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}), \quad \|\delta_x u\|_D = \sqrt{\langle \delta_x u, \delta_x u \rangle_D}, \\ \|u\|_\infty &= \max_{0 \leq i \leq M} |u_i|. \end{aligned}$$

Lemma 4.2. [16, 26] For any mesh functions $u, v \in \mathcal{V}_D$, it holds that

$$-(\delta_x^2 u, v)_D = \langle \delta_x u, \delta_x v \rangle_D.$$

Lemma 4.3. [33] For any mesh functions $u, v \in \mathcal{V}_D$, let

$$I(u, v) = (\mathcal{A}u, -\delta_x^2 v)_D = -h \sum_{i=0}^{M-1} (\mathcal{A}u_i) \delta_x^2 v_i,$$

then $I(u, v)$ is an inner product which is define on $\mathcal{V}_D$. Subsequently, denote $(u, v)_A = I(u, v)$ and $\|u\|_A = \sqrt{(u, u)_A}$, and it holds

$$\frac{2}{3} \|\delta_x u\|_D^2 \leq \|u\|_A^2 \leq \|\delta_x u\|_D^2.$$

Lemma 4.4. [33] For any mesh functions $u, v \in \mathcal{V}_D$, it holds that

$$\|u\|_\infty \leq \frac{\sqrt{L}}{2} \|\delta_x u\|_D.$$

Subsequently, the following theorem is proven by introducing a new inner-product norm, utilizing the monotonicity of the coefficients, and employing mathematical induction.

Theorem 4.1. Let $\{u_i^k | 0 \leq i \leq M, 0 \leq k \leq N\}$ be the solutions of the following finite difference system

$$\mathcal{A}D_\tau^\alpha u_i^k = A\delta_x^2 u_i^k - B\mathcal{A}u_i^k + g_i^k, \quad 1 \leq i \leq M-1, 1 \leq k \leq N, \quad (4.1a)$$

$$u_i^0 = u_a(x_0), \quad 0 \leq i \leq M, \quad (4.1b)$$

$$u_0^k = u_M^k = 0, \quad 1 \leq k \leq N. \quad (4.1c)$$

Then, it holds that

$$\|u^k\|_\infty^2 \leq \frac{L}{4} \left[\frac{3}{2} \|u^0\|_A^2 + \frac{3\Gamma(1-\alpha)}{4A} \left(\log \frac{T}{a} \right)^\alpha \max_{1 \leq l \leq N} \|g^l\|_D^2 \right].$$

Proof. Multiplying Eq. (4.1a) by $-h\delta_x^2 u_i^k$, then summing up from 1 to $M-1$, we obtain

$$(\mathcal{A}D_\tau^\alpha u^k, -\delta_x^2 u^k)_D = A(\delta_x^2 u^k, -\delta_x^2 u^k)_D - B(\mathcal{A}u^k, -\delta_x^2 u^k)_D + (g^k, -\delta_x^2 u^k)_D. \quad (4.2)$$

By Lemma 4.3, we obtain $B(\mathcal{A}u^k, -\delta_x^2 u^k)_D = B\|u\|_A^2$ and $(\mathcal{A}D_\tau^\alpha u^k, -\delta_x^2 u^k)_D = a_{k,k}\|u^k\|_A^2 - \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})(u^j, u^k)_A - a_{1,k}(u^0, u^k)_A$. Substituting them into (4.2) gives

$$\begin{aligned} & a_{k,k}\|u^k\|_A^2 + A\|\delta_x^2 u^k\|_D^2 + B\|u\|_A^2 \\ &= \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})(u^j, u^k)_A + a_{1,k}(u^0, u^k)_A + (g^k, -\delta_x^2 u^k)_D. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, and noticing that $a_{j+1,k} - a_{j,k}$ and $a_{1,k}$ are positive, we obtain

$$\begin{aligned} & a_{k,k}\|u^k\|_A^2 + A\|\delta_x^2 u^k\|_D^2 + B\|u\|_A^2 \\ &\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \frac{\|u^j\|_A^2 + \|u^k\|_A^2}{2} + a_{1,k} \frac{\|u^0\|_A^2 + \|u^k\|_A^2}{2} \\ &\quad + A\|\delta_x^2 u^k\|_D^2 + \frac{1}{4A}\|g^k\|_D^2. \end{aligned}$$

Simplifying the above inequality, we have

$$\begin{aligned} a_{k,k}\|u^k\|_A^2 &\leq \frac{1}{2}a_{k,k}\|u^k\|_A^2 + \frac{1}{2}\sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\|u^j\|_A^2 \\ &\quad + \frac{a_{1,k}}{2}\|u^0\|_A^2 + \frac{1}{4A}\|g^k\|_D^2. \end{aligned} \quad (4.3)$$

Multiplying Eq. (4.3) by 2, we obtain

$$a_{k,k}\|u^k\|_A^2 \leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\|u^j\|_A^2 + a_{1,k}\|u^0\|_A^2 + \frac{1}{2A}\|g^k\|_D^2.$$

Noticing that $\frac{1}{a_{1,k}} \leq \Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha$ (see [16]), we have

$$\begin{aligned} & a_{k,k}\|u^k\|_A^2 \\ &\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\|u^j\|_A^2 + a_{1,k} \left[\|u^0\|_A^2 + \frac{\Gamma(1-\alpha)}{2A} \left(\log \frac{T}{a}\right)^\alpha \|g^k\|_D^2 \right] \\ &\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\|u^j\|_A^2 + a_{1,k}H, \end{aligned}$$

where

$$H = \|u^0\|_A^2 + \frac{\Gamma(1-\alpha)}{2A} \left(\log \frac{T}{a}\right)^\alpha \max_{1 \leq l \leq N} \|g^l\|_D^2.$$

We now prove by the mathematical induction that

$$\|u^k\|_A^2 \leq H, \quad 1 \leq k \leq N. \quad (4.4)$$

Indeed, Eq. (4.4) is obviously true for $k = 1$. Suppose that Eq. (4.4) is valid for $k = 2, 3, \dots, l-1$. Then we see that

$$\begin{aligned} a_{l,l} \|u^k\|_A^2 &\leq \sum_{j=1}^{l-1} (a_{j+1,l} - a_{j,l}) \|u^j\|_A^2 + a_{1,l} H \\ &\leq \sum_{j=1}^{l-1} (a_{j+1,l} - a_{j,l}) H + a_{1,l} H \\ &\leq a_{l,l} H. \end{aligned}$$

This concludes that Eq. (4.4) is true and using Lemma 4.3 and Lemma 4.4, it is easy to come to the conclusion. $\square$

Based on Theorem 4.1, we can conclude that the compact finite difference scheme in Eqs. (3.7a)-(3.7c) is unconditionally stable with respect to the initial values u^0 and the source term F .

4.2. Robin boundary condition

Let $\mathcal{V}_R = \{v | v = (v_0, v_1, \dots, v_{M-1}, v_M)\}$. For any $u, v \in \mathcal{V}_R$, we define the discrete inner products L_2 norm, H_1 semi-norm and the maximum norm as follows:

$$\begin{aligned} (u, v)_R &= h \left(\frac{1}{2} u_0 v_0 + \sum_{i=1}^{M-1} u_i v_i + \frac{1}{2} u_M v_M \right), \quad \|u\|_R = \sqrt{(u, u)_R} \\ \langle \delta_x u, \delta_x v \rangle_R &= h \sum_{i=0}^{M-1} (\delta_x u_{i+\frac{1}{2}}) (\delta_x v_{i+\frac{1}{2}}), \quad \|\delta_x u\|_R = \sqrt{\langle \delta_x u, \delta_x u \rangle_R}, \\ \langle u, v \rangle_{\tilde{A}} &= (u, v)_R - \frac{h^2}{12} \langle \delta_x u, \delta_x v \rangle_R, \quad \|u\|_{\tilde{A}} = \sqrt{\langle u, u \rangle_{\tilde{A}}}, \\ (u, v)_0 &= h \sum_{i=1}^{M-1} u_i v_i, \quad \|u\|_0 = \sqrt{(u, u)_0}, \quad \|u\|_\infty = \max_{0 \leq i \leq M} |u_i|. \end{aligned}$$

Lemma 4.5. [31, 32] For any mesh functions $u, v \in \mathcal{V}_R$, it holds

$$-(\tilde{\delta}_x^2 u, v)_R = \langle \delta_x u, \delta_x v \rangle_R.$$

Lemma 4.6. [13, 32] For any mesh functions u, v defined on Ω_R , it holds

$$(\tilde{\mathcal{A}}u, v)_R = \langle u, v \rangle_{\tilde{A}}, \quad \frac{2}{3} \|u\|_R^2 \leq \|u\|_{\tilde{A}}^2 \leq \|u\|_R^2.$$

Lemma 4.7. For any sequence $y = \{y^0, y^1, y^2, y^3, \dots\}$, it holds

$$y^k \cdot D_\tau^\alpha y^k \geq \frac{1}{2} \left\{ a_{k,k} (y^k)^2 - \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) (y^j)^2 - a_{1,k} (y^0)^2 \right\},$$

where $0 < \alpha < 1$, $a_{j,k}$ and D_τ^α are given in Lemma 3.2.

Proof. Since $a_{j+1,k} - a_{j,k} > 0$ and $a_{j,k} > 0$, we have

$$\begin{aligned} -y^k \cdot D_\tau^\alpha y^k &= -y^k \left[a_{k,k} y^k - \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) y^j - a_{1,k} y^0 \right] \\ &\leq -a_{k,k} (y^k)^2 + \frac{1}{2} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) [(y^k)^2 + (y^j)^2] \\ &\quad + \frac{1}{2} a_{1,k} [(y^k)^2 + (y^0)^2] \\ &= \frac{1}{2} \left\{ (-a_{k,k}) (y^k)^2 + \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) (y^j)^2 + a_{1,k} (y^0)^2 \right\}, \end{aligned}$$

which completes the proof. $\square$

Lemma 4.8. [31, 32] For any mesh function $u \in \mathcal{V}_R$ and any positive constant ε , it holds

$$\|u\|_\infty^2 \leq \varepsilon \|\delta_x u\|_R^2 + \left(\frac{1}{\varepsilon} + 1 \right) \|u\|_R^2.$$

We now proceed to prove the following theorem by constructing a novel inner-product norm and leveraging the monotonicity of the coefficients, combined with the principle of mathematical induction.

Theorem 4.2. . Let $\{u_i^k | 0 \leq i \leq M, 0 \leq k \leq N\}$ be the solutions of the following finite difference system

$$\tilde{A} D_\tau^\alpha u_0^k = A \tilde{\delta}_x^2 u_0^k - \left(\frac{2A}{h} + \frac{Bh}{6} \right) u_0^k - \frac{h}{6} D_\tau^\alpha u_0^k - B \tilde{A} u_0^k + g_0^0, \quad 1 \leq k \leq N, \quad (4.5a)$$

$$\tilde{A} D_\tau^\alpha u_i^k = A \tilde{\delta}_x^2 u_i^k - B \tilde{A} u_i^k + g_i^k, \quad 1 \leq i \leq M-1, 1 \leq k \leq N, \quad (4.5b)$$

$$\tilde{A} D_\tau^\alpha u_M^k = A \tilde{\delta}_x^2 u_M^k - \left(\frac{2A}{h} + \frac{Bh}{6} \right) u_M^k - \frac{h}{6} D_\tau^\alpha u_M^k - B \tilde{A} u_M^k + g_M^k, \quad 1 \leq k \leq N, \quad (4.5c)$$

$$u_i^0 = u_a(x_0), \quad 0 \leq i \leq M. \quad (4.5d)$$

Then, it holds

$$\begin{aligned} &\|u^k\|_{\tilde{A}}^2 + \frac{h^2}{12} [(u_0^k)^2 + (u_M^k)^2] \\ &\leq 2\|u^0\|_{\tilde{A}}^2 + \frac{h^2}{12} [(u_0^0)^2 + (u_M^0)^2] + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a} \right)^\alpha \right]^2 \max_{1 \leq l \leq N} \|g^l\|_0^2 \\ &\quad + \frac{\Gamma(1-\alpha) \left(\log \frac{T}{a} \right)^\alpha}{4A\varepsilon} h^2 \left(\max_{1 \leq l \leq N} |g_0^l|^2 + \max_{1 \leq l \leq N} |g_M^l|^2 \right), \end{aligned}$$

where $\varepsilon = \min\{1, \frac{(\log \frac{T}{a})^{-\alpha}}{48A\Gamma(1-\alpha)}\}$.

Proof. Multiplying Eq. (4.5a) by $\frac{1}{2} h u_0^k$, multiplying Eq. (4.5b) by $h u_i^k$, multiplying Eq. (4.5c)

by $\frac{1}{2}hu_M^k$, respectively, then summing up from 0 to M , we obtain

$$\begin{aligned} \left(\tilde{A}D_\tau^\alpha u^k, u^k\right)_R &= A\left(\tilde{\delta}_x^2 u^k, u^k\right)_R - B\left(\tilde{A}u^k, u^k\right)_R + (g^k, u^k)_R \\ &\quad - \left(\frac{2A}{h} + \frac{Bh}{6}\right) \cdot \frac{h}{2} \left[(u_0^k)^2 + (u_M^k)^2\right] \\ &\quad - \frac{h}{6} \cdot \frac{h}{2} \left(D_\tau^\alpha u_0^k \cdot u_0^k + D_\tau^\alpha u_M^k \cdot u_M^k\right). \end{aligned} \quad (4.6)$$

By Lemma 4.5 and Lemma 4.6, we obtain $A\left(\tilde{\delta}_x^2 u^k, u^k\right)_R = -A\|\delta_x u^k\|_R^2$ and $B\left(\tilde{A}u^k, u^k\right)_R = B\|u\|_{\tilde{A}}^2$, respectively. And noticing that $\left(\tilde{A}D_\tau^\alpha u^k, u^k\right)_R = a_{k,k}\|u^k\|_{\tilde{A}}^2 - \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\langle u^j, u^k \rangle_{\tilde{A}} - a_{1,k}\langle u^0, u^k \rangle_{\tilde{A}}$. Substituting them into (4.6) gives

$$\begin{aligned} &a_{k,k}\|u^k\|_{\tilde{A}}^2 + A\|\delta_x u^k\|_R^2 + B\|u\|_{\tilde{A}}^2 + \left(A + \frac{Bh^2}{12}\right) [(u_0^k)^2 + (u_M^k)^2] \\ &\quad + \frac{h^2}{12} \left(D_\tau^\alpha u_0^k \cdot u_0^k + D_\tau^\alpha u_M^k \cdot u_M^k\right) \\ &= \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\langle u^j, u^k \rangle_{\tilde{A}} + a_{1,k}\langle u^0, u^k \rangle_{\tilde{A}} + (g^k, u^k)_R. \end{aligned} \quad (4.7)$$

Applying Lemma 4.7 to the last term on the left-hand side of (4.7), and using the Cauchy-Schwarz inequality, noticing that $a_{j+1,k} - a_{j,k}$ and $a_{1,k}$ are positive, we obtain

$$\begin{aligned} &a_{k,k}\|u^k\|_{\tilde{A}}^2 + A\|\delta_x u^k\|_R^2 + B\|u\|_{\tilde{A}}^2 + \left(A + \frac{Bh^2}{12} + \frac{h^2}{24}a_{k,k}\right) [(u_0^k)^2 + (u_M^k)^2] \\ &\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k})\langle u^j, u^k \rangle_{\tilde{A}} + a_{1,k}\langle u^0, u^k \rangle_{\tilde{A}} + \frac{h^2}{24}a_{1,k} [(u_0^0)^2 + (u_M^0)^2] \\ &\quad + \frac{h^2}{24} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) [(u_0^j)^2 + (u_M^j)^2] + (g^k, u^k)_R \\ &\leq \frac{1}{2} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \left(\|u^j\|_{\tilde{A}}^2 + \|u^k\|_{\tilde{A}}^2\right) + a_{1,k} \left(\|u^0\|_{\tilde{A}}^2 + \frac{1}{4}\|u^k\|_{\tilde{A}}^2\right) \\ &\quad + \frac{h^2}{24} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) [(u_0^j)^2 + (u_M^j)^2] + \frac{h^2}{24}a_{1,k} [(u_0^0)^2 + (u_M^0)^2] \\ &\quad + (g^k, u^k)_R. \end{aligned} \quad (4.8)$$

We now evaluate the last term of (4.8), using the definition of inner product, and we have

$$(g^k, u^k)_R = \frac{h}{2}g_0^k \cdot u_0^k + h \sum_{i=1}^{M-1} g_i^k u_i^k + \frac{h}{2}g_M^k \cdot u_M^k. \quad (4.9)$$

Based on the Cauchy-Schwarz inequality with $\varepsilon > 0$, the definition of norm, and the result of Lemma 4.8 with $\varepsilon = 1$, we obtain

$$\frac{h}{2}g_0^k \cdot u_0^k \leq \frac{1}{2} \cdot A\varepsilon(u_0^k)^2 + \frac{1}{2} \cdot \frac{1}{4A\varepsilon}(hg_0^k)^2$$

$$\leq \frac{A\varepsilon}{2} \|\delta_x u^k\|_R^2 + A\varepsilon \|u^k\|_R^2 + \frac{1}{8A\varepsilon} (hg_0^k)^2, \quad (4.10a)$$

$$\frac{h}{2} g_M^k \cdot u_M^k \leq \frac{A\varepsilon}{2} \|\delta_x u^k\|_R^2 + A\varepsilon \|u^k\|_R^2 + \frac{1}{8A\varepsilon} (hg_M^k)^2, \quad (4.10b)$$

$$\begin{aligned} h \sum_{i=1}^{M-1} g_i^k u_i^k &= (g^k, u^k)_0 \\ &\leq \frac{a_{1,k}}{8} \|u^k\|_R^2 + \frac{2}{a_{1,k}} \|g^k\|_0^2. \end{aligned} \quad (4.10c)$$

Substituting Eqs. (4.10a)-(4.10c) into Eq. (4.9), choosing $\varepsilon \leq 1$, and applying Lemma 4.6, we obtain

$$\begin{aligned} (g^k, u^k)_R &\leq A\varepsilon \|\delta_x u^k\|_R^2 + \left(2A\varepsilon + \frac{a_{1,k}}{8}\right) \|u^k\|_R^2 \\ &\quad + \frac{1}{8A\varepsilon} \left[(hg_0^k)^2 + (hg_M^k)^2\right] + \frac{2}{a_{1,k}} \|g^k\|_0^2 \\ &\leq A \|\delta_x u^k\|_R^2 + \frac{3}{2} \left(2A\varepsilon + \frac{a_{1,k}}{8}\right) \|u^k\|_{\tilde{A}}^2 \\ &\quad + \frac{1}{8A\varepsilon} \left[(hg_0^k)^2 + (hg_M^k)^2\right] + \frac{2}{a_{1,k}} \|g^k\|_0^2. \end{aligned} \quad (4.11)$$

Substituting Eq. (4.11) into Eq. (4.8), we obtain

$$\begin{aligned} &a_{k,k} \|u^k\|_{\tilde{A}}^2 + B \|u\|_{\tilde{A}}^2 + \left(A + \frac{Bh^2}{12} + \frac{h^2}{24} a_{k,k}\right) [(u_0^k)^2 + (u_M^k)^2] \\ &\leq \frac{1}{2} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \|u^j\|_{\tilde{A}}^2 + \left(\frac{a_{k,k}}{2} + 3A\varepsilon - \frac{a_{1,k}}{16}\right) \|u^k\|_{\tilde{A}}^2 + a_{1,k} \|u^0\|_{\tilde{A}}^2 \\ &\quad + \frac{h^2}{24} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \left[(u_0^j)^2 + (u_M^j)^2\right] + \frac{h^2}{24} a_{1,k} [(u_0^0)^2 + (u_M^0)^2] \\ &\quad + \frac{1}{8A\varepsilon} \left[(hg_0^k)^2 + (hg_M^k)^2\right] + \frac{2}{a_{1,k}} \|g^k\|_0^2. \end{aligned} \quad (4.12)$$

Noticing that $a_{1,k} \geq \frac{1}{\Gamma(1-\alpha)} (\log \frac{T}{a})^{-\alpha}$ (see [16]), choosing $\varepsilon = \min\{1, \frac{(\log \frac{T}{a})^{-\alpha}}{48A\Gamma(1-\alpha)}\}$, we obtain $\left(\frac{a_{k,k}}{2} + 3A\varepsilon - \frac{a_{1,k}}{16}\right) \leq \frac{a_{k,k}}{2}$. Inserting it into the right-hand-side of Eq. (4.12), and then multiplying the inequality by 2, we simplify Eq. (4.12) to

$$\begin{aligned} &a_{k,k} \|u^k\|_{\tilde{A}}^2 + \left(2A + \frac{Bh^2}{6} + \frac{h^2}{12} a_{k,k}\right) [(u_0^k)^2 + (u_M^k)^2] \\ &\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \|u^j\|_{\tilde{A}}^2 + 2a_{1,k} \|u^0\|_{\tilde{A}}^2 + \frac{h^2}{12} a_{1,k} [(u_0^0)^2 + (u_M^0)^2] \\ &\quad + \frac{h^2}{12} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \left[(u_0^j)^2 + (u_M^j)^2\right] + \frac{1}{4A\varepsilon} \left[(hg_0^k)^2 + (hg_M^k)^2\right] \\ &\quad + \frac{4}{a_{1,k}} \|g^k\|_0^2. \end{aligned}$$

Hence, we obtain

$$a_{k,k} \|u^k\|_{\tilde{A}}^2 + \left(2A + \frac{Bh^2}{6}\right) [(u_0^k)^2 + (u_M^k)^2] + \frac{h^2}{12} a_{k,k} [(u_0^k)^2 + (u_M^k)^2]$$

$$\begin{aligned}
&= \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \|u^j\|_A^2 + \frac{h^2}{12} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \left[(u_0^j)^2 + (u_M^j)^2 \right] \\
&\quad + a_{1,k} \left\{ 2\|u^0\|_A^2 + \frac{h^2}{12} [(u_0^0)^2 + (u_M^0)^2] + \frac{1}{4A\varepsilon a_{1,k}} [(hg_0^k)^2 + (hg_M^k)^2] \right\} \\
&\quad + \frac{4}{a_{1,k}} \|g^k\|_0^2.
\end{aligned}$$

Since $\frac{1}{a_{1,k}} \leq \Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha$, we have

$$\begin{aligned}
&a_{k,k} \|u^k\|_A^2 + \left(2A + \frac{Bh^2}{6}\right) [(u_0^k)^2 + (u_M^k)^2] + \frac{h^2}{12} a_{k,k} [(u_0^k)^2 + (u_M^k)^2] \\
&\leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \|u^j\|_A^2 + \frac{h^2}{12} \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) \left[(u_0^j)^2 + (u_M^j)^2 \right] \\
&\quad + a_{1,k} \left\{ 2\|u^0\|_A^2 + \frac{h^2}{12} [(u_0^0)^2 + (u_M^0)^2] + \frac{\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha h^2}{4A\varepsilon} \right. \\
&\quad \times \left. \left(\max_{1 \leq l \leq N} |g_0^l|^2 + \max_{1 \leq l \leq N} |g_M^l|^2 \right) + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha \right]^2 \max_{1 \leq l \leq N} \|g^l\|_0^2 \right\}. \tag{4.13}
\end{aligned}$$

Denoting

$$\begin{aligned}
P^k &\equiv \|u^k\|_A^2 + \frac{h^2}{12} [(u_0^k)^2 + (u_M^k)^2], \\
I &\equiv 2\|u^0\|_A^2 + \frac{h^2}{12} [(u_0^0)^2 + (u_M^0)^2] + \frac{\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha h^2}{4A\varepsilon} \\
&\quad \times \left(\max_{1 \leq l \leq N} |g_0^l|^2 + \max_{1 \leq l \leq N} |g_M^l|^2 \right) + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha \right]^2 \max_{1 \leq l \leq N} \|g^l\|_0^2,
\end{aligned}$$

Eq. (4.13) can be simplified to

$$a_{k,k} P^k \leq \sum_{j=1}^{k-1} (a_{j+1,k} - a_{j,k}) P^j + a_{1,k} I, \quad 1 \leq k \leq N.$$

We now prove by the mathematical induction that

$$P^k \leq I, \quad 1 \leq k \leq N. \tag{4.14}$$

Indeed, Eq. (4.14) is obviously true for $k = 1$. Suppose that Eq. (4.14) is valid for $k = 2, 3, \dots, l-1$. Then we see

$$a_{l,l} P^l \leq \sum_{j=1}^{l-1} (a_{j+1,l} - a_{j,l}) P^j + a_{1,l} I \leq \sum_{j=1}^{l-1} (a_{j+1,l} - a_{j,l}) I + a_{1,l} I \leq a_{l,l} I.$$

This concludes that Eq. (4.14) is true and hence we complete the proof of Theorem 4.2. $\square$

Based on Theorem 4.2, we can conclude that the compact finite difference scheme in Eqs. (3.14a)-(3.14d) is unconditionally stable with respect to the initial values u^0 and the source term F .

5. Convergence of the two schemes

The following two theorems are readily derived from the Theorems 4.1 and 4.2 in the previous section.

5.1. Dirichlet boundary condition

Theorem 5.1. Assume that $u(x, t) \in C_{x,t}^{6,2}([0, L] \times [a, T])$ is the solution of Eqs. (1.1), (1.2a) and (1.2b), and $\{u_i^k, 0 \leq i \leq M, 0 \leq k \leq N\}$ are the solutions of the compact finite difference scheme (3.7a)-(3.7c). Let $e_i^k = U_i^k - u_i^k$ ($0 \leq i \leq M, 0 \leq k \leq N$), where $U_i^k = u(x_i, t_k)$. Then there exists a positive constant C such that

$$\|e^k\|_\infty \leq C(\tau^{2-\alpha} + h^4), \quad 0 \leq k \leq N,$$

where $C = \sqrt{\frac{3L\Gamma(1-\alpha)}{16A}} \left(\log \frac{T}{a}\right)^\alpha C_0^2$.

Proof. It follows from Theorem 4.1 that

$$\begin{aligned} \|e^k\|_\infty^2 &\leq \frac{L}{4} \left[\frac{3\Gamma(1-\alpha)}{4A} \left(\log \frac{T}{a}\right)^\alpha \max_{1 \leq l \leq N} \|R^l\|_D^2 \right] \\ &\leq \frac{3L\Gamma(1-\alpha)}{16A} \left(\log \frac{T}{a}\right)^\alpha C_0^2 (\tau^{2-\alpha} + h^4)^2 \\ &\leq C^2 (\tau^{2-\alpha} + h^4)^2, \end{aligned}$$

where $C = \sqrt{\frac{3L\Gamma(1-\alpha)}{16A}} \left(\log \frac{T}{a}\right)^\alpha C_0^2$. Hence, the proof is shown. $\square$

5.2. Robin boundary condition

Theorem 5.2. Assume that $u(x, t) \in C_{x,t}^{6,2}([0, L] \times [a, T])$ is the solution of Eqs. (1.1), (1.3a) and (1.3b), and $\{u_i^k, 0 \leq i \leq M, 0 \leq k \leq N\}$ are the solutions of the compact finite difference scheme (3.14a)-(3.14d). Let $e_i^k = U_i^k - u_i^k$ ($0 \leq i \leq M, 0 \leq k \leq N$), where $U_i^k = u(x_i, t_k)$. Then there exists a positive constant C such that

$$\|e^k\|_{\tilde{A}} \leq C(\tau^{2-\alpha} + h^4), \quad 0 \leq k \leq N,$$

where $C = \sqrt{\frac{\Gamma(1-\alpha)(\log \frac{T}{a})^\alpha}{4A\epsilon}} (C_1^2 + C_2^2) + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha \right]^2 C_3^2$.

Proof. It follows from Theorem 4.2 that

$$\begin{aligned} \|e^k\|_{\tilde{A}}^2 &\leq \|e^k\|_{\tilde{A}}^2 + \frac{h^2}{12} \left[(e_0^k)^2 + (e_M^k)^2 \right] \\ &\leq \frac{\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha}{4A\epsilon} h^2 \left(\max_{1 \leq l \leq N} |\overline{R_0^l}|^2 + \max_{1 \leq l \leq N} |\overline{R_M^l}|^2 \right) \\ &\quad + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha \right]^2 \max_{1 \leq l \leq N} \|\overline{R^l}\|_0^2 \\ &\leq \frac{\Gamma(1-\alpha) \left(\log \frac{T}{a}\right)^\alpha}{4A\epsilon} h^2 (C_1^2 + C_2^2) (\tau^{2-\alpha} + h^3)^2 \end{aligned}$$

$$\begin{aligned}
& + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a} \right)^\alpha \right]^2 \cdot C_3^2 (\tau^{2-\alpha} + h^4)^2 \\
& \leq C^2 (\tau^{2-\alpha} + h^4)^2,
\end{aligned}$$

where $C = \sqrt{\frac{\Gamma(1-\alpha)(\log \frac{T}{a})^\alpha}{4A\varepsilon} (C_1^2 + C_2^2) + 4 \left[\Gamma(1-\alpha) \left(\log \frac{T}{a} \right)^\alpha \right]^2 C_3^2}$. The proof is thus finished. $\square$

Next, we attempt to enhance the temporal accuracy order.

6. Numerical examples

In this section, we test the accuracy of the analytical solution and the accuracy and efficiency of the proposed numerical scheme by considering the following examples. In our computation, the L_∞ -norm error and convergence rate were calculated based on $e_\infty(\tau, h) = \|u^k - U^k\|_\infty$, $Rate(\tau) = \log_{\tau_1/\tau_2}(e_\infty(\tau_1, h)/e_\infty(\tau_2, h))$ when h is small and $Rate(h) = \log_{h_1/h_2}(e_\infty(\tau, h_1)/e_\infty(\tau, h_2))$ when τ is small.

Example 6.1. Consider the following fractional partial differential equation subject to the Dirichlet boundary conditions:

$$\begin{cases} {}_{CH}D_{a,t}^\alpha u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) + F(x, t), & 1 \leq t \leq T, \quad 0 < x < 1, \\ u(x, 1) = 0, \\ u(0, t) = 0, \quad u(1, t) = 0, \end{cases}$$

where $0 < \alpha < 1$ and the source term is given as $F(x, t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)}(\log t)^{3-\alpha} \cdot \sin^2(\pi x) + 2\pi^2[\sin^2(\pi x) - \cos^2(\pi x)] \cdot (\log t)^3 + (\log t)^3 \cdot \sin^2(\pi x)$. It can be seen that the exact solution is $u(x, t) = (\log t)^3 \cdot \sin^2(\pi x)$.

Figure 1 shows the analytical solution and the exact solution when we take $T = 100$ and truncate the solution's series expansion in Eq. (2.5) at the 50-th term.

Table 1 shows the maximum errors and the convergence rates in time when we take $h = 1/200$ for various time steps and different values of α at $T = 2$ for Example 6.1. Table 2 shows the maximum errors and the convergence rates in space when we took $\tau = 1/20000$ for various space steps and different values of α at $T = 2$ for Example 6.1. From Tables 1 to 2, one may see that the convergence rate in time is round about $2 - \alpha$, and the convergence rate in space in all cases is close to 4.0, which coincides with the theoretical analysis in Theorem 4.1.

Example 6.2. Consider the following fractional partial differential equation subject to the Robin boundary conditions:

$$\begin{cases} {}_{CH}D_{a,t}^\alpha u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) + F(x, t), & 1 \leq t \leq T, \quad 0 < x < 1, \\ u(x, 1) = 0, \\ -u_x(0, t) + u(0, t) = 0, \quad u_x(1, t) + u(1, t) = -(\log t)^2, \end{cases}$$

where $0 < \alpha < 1$ and the source term is given as $F(x, t) = \left[\frac{2}{\Gamma(3-\alpha)}(\log t)^{2-\alpha} + (\log t)^2 \right] \cdot x^5(1-x) - (20x^3 - 30x^4)(\log t)^2$. It can be seen that the exact solution is $u(x, t) = (\log t)^2 \cdot x^5(1-x)$.

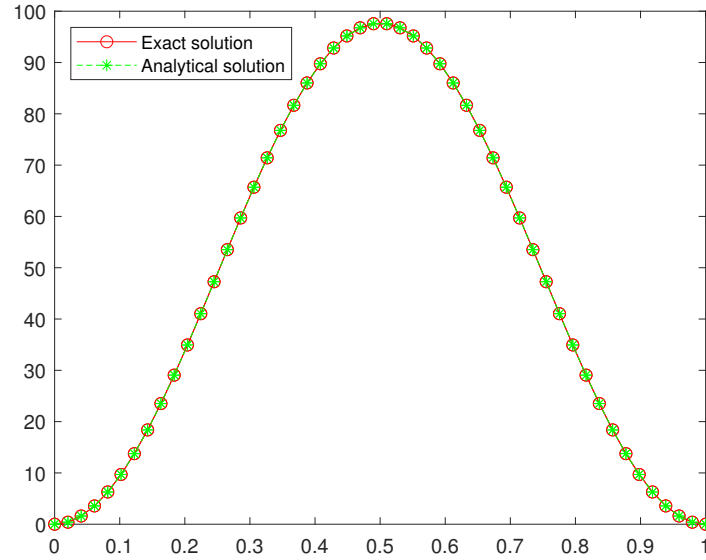


Figure 1. The analytical and exact solutions for Example 6.1.

Table 1. Maximum errors and convergence rates in time at $T = 2$ for different α when $h = 1/200$ for Example 6.1.

τ	$e_{\infty}(\alpha = 0.3)$	Rate(τ)	$e_{\infty}(\alpha = 0.5)$	Rate(τ)	$e_{\infty}(\alpha = 0.8)$	Rate(τ)
$\frac{1}{8}$	3.531604e-4	-	1.051972e-3	-	4.176158e-3	-
$\frac{1}{16}$	1.139242e-4	1.632	3.801908e-4	1.468	1.832687e-3	1.188
$\frac{1}{32}$	3.635786e-5	1.648	1.363932e-4	1.479	8.008161e-4	1.194
$\frac{1}{64}$	1.151272e-5	1.660	4.870412e-5	1.486	3.492447e-4	1.197
$\frac{1}{128}$	3.624376e-6	1.667	1.733861e-5	1.490	1.521667e-4	1.199

Table 3 shows the maximum errors and the convergence rates in time when we took $h = 1/200$ for various time steps and different values of α at $T = 2$ for Example 6.2. Table 4 shows the maximum errors and the convergence rates in space when we took $\tau = 1/20000$ for various space steps and different values of α at $T = 2$ for Example 6.2. From Tables 3 to 4, one may see that the convergence rate in time is round about $2 - \alpha$, and the convergence rate in space in all cases is close to 4.0, which coincides with the theoretical analysis in Theorem 4.2.

Example 6.3. Consider Example 6.1 and Example 6.2 by using the L1-2 method, respectively. Table 5 shows the maximum errors and the convergence rates in time when we took $h = 1/200$ for various time steps and different values of α at $T = 2$ for Example 6.1 by using L1-2 method.

From the data of Table 5, we can see that the computational errors using the L1-2 method

Table 2. Maximum errors and convergence rates in space at $T = 2$ for different α when $\tau = 1/20000$ for Example 6.1.

h	$e_{\infty}(\alpha = 0.3)$	Rate(h)	$e_{\infty}(\alpha = 0.5)$	Rate(h)	$e_{\infty}(\alpha = 0.8)$	Rate(h)
$\frac{1}{2}$	1.688453e-1	-	1.611086e-1	-	1.462612e-1	-
$\frac{1}{4}$	7.634901e-3	4.467	7.342149e-3	4.456	6.778730e-3	4.431
$\frac{1}{8}$	4.425328e-4	4.109	4.257209e-4	4.108	3.936743e-4	4.106
$\frac{1}{16}$	2.715254e-5	4.027	2.612934e-5	4.026	2.448885e-5	4.007
$\frac{1}{32}$	1.689930e-6	4.006	1.634044e-6	3.999	1.856430e-6	3.722

Table 3. Maximum errors and convergence rates in time at $T = 2$ for different α when $h = 1/200$ for Example 6.2.

τ	$e_{\infty}(\alpha = 0.3)$	Rate(τ)	$e_{\infty}(\alpha = 0.5)$	Rate(τ)	$e_{\infty}(\alpha = 0.8)$	Rate(τ)
$\frac{1}{8}$	2.501244e-5	-	7.440248e-5	-	3.040447e-4	-
$\frac{1}{16}$	7.801524e-6	1.681	2.623018e-5	1.504	1.323598e-4	1.200
$\frac{1}{32}$	2.428498e-6	1.684	9.254086e-6	1.503	5.755046e-5	1.202
$\frac{1}{64}$	7.544676e-7	1.687	3.266914e-6	1.502	2.502262e-5	1.202
$\frac{1}{128}$	2.339552e-7	1.689	1.153793e-6	1.502	1.088257e-5	1.201

Table 4. Maximum errors and convergence rates in space at $T = 2$ for different α when $\tau = 1/20000$ for Example 6.2.

h	$e_{\infty}(\alpha = 0.3)$	Rate(h)	$e_{\infty}(\alpha = 0.5)$	Rate(h)	$e_{\infty}(\alpha = 0.8)$	Rate(h)
$\frac{1}{2}$	2.087418e-2	-	2.021543e-2	-	1.908102e-2	-
$\frac{1}{4}$	1.396329e-3	3.902	1.363380e-3	3.890	1.306734e-3	3.868
$\frac{1}{8}$	8.879076e-5	3.975	8.687974e-5	3.972	8.357721e-5	3.967
$\frac{1}{16}$	5.573507e-6	3.994	5.456035e-6	3.993	5.233651e-6	3.997
$\frac{1}{32}$	3.486856e-7	3.999	3.409329e-7	4.000	3.067495e-7	4.093

Table 5. Maximum errors and convergence rates in time at $T = 2$ for different α when $h = 1/200$ for Example 6.1 by using L1-2 method.

τ	$e_{\infty}(\alpha = 0.3)$	Rate(τ)	$e_{\infty}(\alpha = 0.5)$	Rate(τ)	$e_{\infty}(\alpha = 0.8)$	Rate(τ)
$\frac{1}{8}$	2.725947e-5	-	7.998476e-5	-	3.385164e-4	-
$\frac{1}{16}$	4.017418e-6	1.199	1.331903e-5	2.586	6.944427e-5	2.286
$\frac{1}{32}$	6.036339e-7	2.735	2.273617e-6	2.550	1.465840e-5	2.244
$\frac{1}{64}$	9.239722e-8	2.708	3.942477e-7	2.528	3.139833e-6	2.223
$\frac{1}{128}$	1.501402e-8	2.622	6.963855e-8	2.501	6.782631e-7	2.211

Table 6. Maximum errors and convergence rates in space at $T = 2$ for different α when $\tau = 1/20000$ for Example 6.1 by using L1-2 method.

h	$e_{\infty}(\alpha = 0.3)$	Rate(h)	$e_{\infty}(\alpha = 0.5)$	Rate(h)	$e_{\infty}(\alpha = 0.8)$	Rate(h)
$\frac{1}{2}$	1.688453e-1	-	1.611086e-1	-	1.462606e-1	-
$\frac{1}{4}$	7.634900e-3	4.467	7.342140e-3	4.456	6.778364e-3	4.431
$\frac{1}{8}$	4.425321e-4	4.109	4.257119e-4	4.108	3.933188e-4	4.107
$\frac{1}{16}$	2.715181e-5	4.027	2.612033e-5	4.027	2.413385e-5	4.027
$\frac{1}{32}$	1.689205e-6	4.007	1.625035e-6	4.007	1.501462e-6	4.007

Table 7. Maximum errors and convergence rates in time at $T = 2$ for different α when $h = 1/200$ for Example 6.2 by using L1-2 method.

τ	$e_{\infty}(\alpha = 0.3)$	Rate(τ)	$e_{\infty}(\alpha = 0.5)$	Rate(τ)	$e_{\infty}(\alpha = 0.8)$	Rate(τ)
$\frac{1}{8}$	1.723894e-6	-	5.103680e-6	-	2.784016e-5	-
$\frac{1}{16}$	2.497807e-7	2.787	8.527941e-7	2.581	5.810455e-6	2.260
$\frac{1}{32}$	3.606070e-8	2.792	1.443457e-7	2.563	1.245018e-6	2.223
$\frac{1}{64}$	5.177543e-9	2.800	2.467975e-8	2.548	2.692851e-7	2.209
$\frac{1}{128}$	7.321618e-10	2.822	4.244406e-9	2.540	5.846059e-8	2.204

Table 8. Maximum errors and convergence rates in space at $T = 2$ for different α when $\tau = 1/20000$ for Example 6.2 by using L1-2 method.

h	$e_{\infty}(\alpha = 0.3)$	Rate(h)	$e_{\infty}(\alpha = 0.5)$	Rate(h)	$e_{\infty}(\alpha = 0.8)$	Rate(h)
$\frac{1}{2}$	2.087418e-2	-	2.021543e-2	-	1.908101e-2	-
$\frac{1}{4}$	1.396329e-3	3.902	1.363380e-3	3.890	1.306750e-3	3.868
$\frac{1}{8}$	8.879080e-5	3.975	8.688022e-5	3.972	8.359798e-5	3.966
$\frac{1}{16}$	5.573546e-6	3.994	5.456546e-6	3.993	5.255567e-6	3.992
$\frac{1}{32}$	3.487250e-7	3.998	3.414505e-7	3.998	3.289542e-7	3.998

are much smaller than that of Table 1, and the results show that the convergence rates of errors in temporal direction tends to $(3 - \alpha)$ order, which is superior to the result of $(2 - \alpha)$ order in Table 1. Table 6 shows the maximum errors and the convergence rates in space when we took $\tau = 1/20000$ for various space steps and different values of α at $T = 2$ for Example 6.1 by using L1-2 method. The results achieve a convergence rate of approximately fourth order in space as shown in Table 2. Table 7 shows the maximum errors and the convergence rates in time when we took $h = 1/200$ for various time steps and different values of α at $T = 2$ for Example 6.2 by using L1-2 method. From the data of this table, we can see that the computational errors using the L1-2 method are much smaller than that of Table 3, and the results show that the convergence rates of errors in temporal direction tends to $(3 - \alpha)$ order, which is superior to the result of $(2 - \alpha)$ order in Table 3. Table 8 shows the maximum errors and the convergence rates in space when we took $\tau = 1/20000$ for various space steps and different values of α at $T = 2$ for Example 6.2 by using L1-2 method, while the theoretical analysis of this part is left for future work.

7. Conclusion

In this paper, we proposed the method of separation of variables and the generalized Laplace transform for unsteady MHD mixed convection problem of Maxwell fractional nanofluid. And based on the L1 and L1-2 approximations for the Caputo-Hadamard fractional derivatives we constructed the compact finite difference schemes for the spatial derivative. Finally, in the numerical results, we verified the accuracy of the analytical solution for the Dirichlet boundary condition and homogeneous Robin boundary condition. And for the problem with non-homogeneous Robin boundary condition, we verified the efficiency of the proposed numerical schemes.

The proposed framework is not limited to Maxwell nanofluids but can be extended to other viscoelastic fluid models with fractional operators. Its robustness under both Dirichlet and Robin boundary conditions makes it suitable for simulating a wide range of multiphysics problems in computational mechanics. Future work includes establishing a rigorous theoretical analysis for the L1-2 scheme, extending the model to handle variable coefficients and nonlinear terms, and exploring practical applications such as simulations with realistic nanofluid parameters and the implementation of parallel algorithms.

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