

CODIMENSION-TWO BIFURCATIONS AND CHAOS OF A DISCRETE PREDATOR–PREY SYSTEM WITH ALLEE EFFECT

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Abstract We investigate local codimension-two resonances of the coexistence fixed point in a forward-Euler discretization of a predator–prey model with a sigmoid functional response and an Allee effect. Earlier work established the fixed points and codimension-one period-doubling and Neimark–Sacker bifurcations. Here we derive the spectral locations of the 1:2, 1:3, and 1:4 resonance candidates, state the regular two-parameter unfolding and normal-form nondegeneracy requirements, and compute the corresponding normal-form coefficients. For a representative strong-Allee parameter set, the analytical resonance locations agree with MatContM continuation to the displayed precision. We also separate algebraic existence from biological feasibility under strong and weak Allee effects and clarify the local biological scope of the Euler map. The bifurcation and maximal-Lyapunov-exponent diagrams illustrate the dynamical complexity.

Keywords Allee effect, Neimark–Sacker bifurcation, strong resonance, chaos.

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1. Introduction

Discrete-time population models can display dynamical behavior that is not present in their continuous-time counterparts, and bifurcation theory for maps provides a systematic framework for studying such effects [12]. Discrete predator–prey models, epidemic models, information-diffusion models, and other biological or applied maps have therefore received considerable attention; representative studies include [2–9, 11, 14, 18, 19, 21, 23, 25, 26].

Common discretization approaches include forward Euler schemes, nonstandard finite-difference schemes, and direct discrete modeling. For predator–prey systems, these approaches have been used to study local stability, period-doubling and Neimark–Sacker bifurcations, strong resonances, periodic dynamics, and chaos [2, 3, 6, 8, 9, 19, 23]. Recent work has also emphasized multiparameter bifurcations and strong-resonance organization in discrete ecological maps [1, 13, 15–17].

A continuous predator–prey system with an Allee effect in the prey and a sigmoid functional response was studied in [20]:

$$\begin{cases} \frac{dx}{dt} = r \left(1 - \frac{x}{k}\right) (x - m)x - \frac{sx^2y}{x^2 + a^2}, \\ \frac{dy}{dt} = \frac{px^2y}{x^2 + a^2} - cy. \end{cases}$$

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Here $x(t)$ and $y(t)$ denote prey and predator abundance. We assume $r, k, s, a, p, c > 0$. The parameter m represents the Allee threshold: $0 < m < k$ corresponds to a strong Allee effect, whereas $m < 0$ corresponds to a weak Allee effect (within the parameter range considered in the cited model).

Applying the forward Euler method with step size $h > 0$ gives the map studied in [4]:

$$\begin{cases} x_{n+1} = x_n \left[1 + h \left(r \left(1 - \frac{x_n}{k} \right) (x_n - m) - \frac{s x_n y_n}{x_n^2 + a^2} \right) \right], \\ y_{n+1} = y_n \left[1 + h \left(\frac{p x_n^2}{x_n^2 + a^2} - c \right) \right]. \end{cases} \quad (1.1)$$

The fixed points and codimension-one period-doubling and Neimark–Sacker bifurcations of this map were investigated previously in [4]. The purpose of the present work is narrower and more precise: With (k, r) as unfolding parameters, we locate the 1:2, 1:3, and 1:4 strong-resonance candidates of the coexistence fixed point, verify regular two-parameter unfolding at the representative numerical points, compute normal-form coefficients, and compare the analytical locations with numerical continuation. This codimension-two analysis is the main addition beyond [4].

Biological domain of the Euler map

The positive quadrant is not automatically invariant under the forward-Euler discretization. Indeed, the predator equation can be written as

$$y_{n+1} = y_n \left[1 + h \left(\frac{p x_n^2}{x_n^2 + a^2} - c \right) \right]. \quad (1.2)$$

For $x_n \geq 0$, the multiplier in (1.2) is bounded below by $1 - hc$. Thus $0 < h \leq 1/c$ is a sufficient condition for nonnegativity of the predator update whenever $y_n \geq 0$. In contrast, the prey update can be written as

$$x_{n+1} = x_n \left[1 + h \left(r \left(1 - \frac{x_n}{k} \right) (x_n - m) - \frac{s x_n y_n}{x_n^2 + a^2} \right) \right], \quad (1.3)$$

and, for any fixed $x_n > 0$, the multiplier in (1.3) becomes negative for sufficiently large y_n . Hence there is no parameter-independent global invariance statement for the whole positive quadrant. In particular, for the numerical choice $h = 3$ and $c = 0.9$, the predator multiplier tends to $1 - hc = -1.7$ as $x_n \rightarrow 0$, so positive predator densities near the predator axis can be mapped to negative values. Consequently, the analytical bifurcation results below are local results in a neighborhood of a feasible coexistence fixed point. In all newly generated orbit computations, an iterate is classified as biologically inadmissible once either component becomes negative (or the orbit becomes nonfinite/divergent); such trajectories are not interpreted as biological population dynamics.

The paper is organized as follows. Section 2 separates algebraic fixed-point existence from biological feasibility, gives a sign-complete local stability criterion, and analyzes the three strong-resonance candidates together with unfolding requirements. Section 3 compares analytical and continuation results and plots two parameters bifurcation curves, the bifurcation diagrams and the corresponding Maximum Lyapunov Exponents curve. Section 4 discusses the main results of the paper.

2. Qualitative analysis of the system

Throughout the local analysis we assume

$$r, k, s, a, p, c, h > 0, \quad p > c,$$

and all displayed denominators are required to be nonzero. For the coexistence fixed point we additionally impose $k > x^*$ and $x^* > m$, as stated below. These assumptions imply, in particular, $kps(p - c) \neq 0$.

Solving the fixed-point equations of (1.1) gives the algebraic equilibria

$$E_0 = (0, 0), \quad E_1 = (m, 0), \quad E_2 = (k, 0), \quad E_3 = (x^*, y^*),$$

where

$$x^* = a\sqrt{\frac{c}{p-c}}, \quad (2.1)$$

$$y^* = \frac{r(1 - x^*/k)(x^* - m)((x^*)^2 + a^2)}{sx^*}. \quad (2.2)$$

Algebraic existence and biological feasibility must be distinguished. The origin E_0 is always feasible in $\mathbb{R}_+^2$; E_2 is feasible because $k > 0$; and E_1 is biologically feasible only when $m \geq 0$ (for $m = 0$ it coincides with E_0). In particular, under a weak Allee effect $m < 0$, E_1 remains an algebraic fixed point but is not biologically feasible. The coexistence equilibrium satisfies $x^* > 0$ when $p > c$ and is feasible precisely when $y^* > 0$, i.e. when $k > x^*$ and $x^* > m$. Therefore, under a strong Allee effect its feasibility condition is $0 < m < x^* < k$, whereas under a weak Allee effect $m < 0$ the condition $x^* > m$ is automatic and feasibility reduces to $p > c$ and $k > x^*$.

The boundary fixed points were treated in [4]. We therefore focus on the local stability and nonhyperbolic cases of the feasible coexistence fixed point E_3 .

Lemma 2.1. [14] *Let $F(\lambda) = \lambda^2 + P\lambda + Q$ with $F(1) > 0$, and let λ_1, λ_2 be its roots. Then: (i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$; (ii) exactly one multiplier is inside the unit circle if and only if $F(-1) < 0$; (iii) both multipliers have modulus greater than one if and only if $F(-1) > 0$ and $Q > 1$; (iv) $\lambda = -1$ is a simple unit multiplier with the other multiplier off the unit circle when $F(-1) = 0$ and $P \neq 0, 2$; and (v) a complex-conjugate pair lies on the unit circle when $Q = 1$ and $P^2 - 4Q < 0$. The equality cases are nonhyperbolic and must be treated separately.*

The Jacobian matrix of system (1.1) at E_3

$$J = \begin{bmatrix} -\frac{2\sqrt{(p-c)}cachr(k+m) + k(c-p)(2chmr - hmpr - p) - ca^2hr(2c+p)}{kp(c-p)} - \frac{chs}{p} & \\ \frac{2\left(\sqrt{(p-c)}ca + 2mc - 2mp\right)\left(\sqrt{(p-c)}ca + kc - kp\right)rh}{sk(c-p)} & 1 \end{bmatrix},$$

and the corresponding characteristic equation is

$$F(\lambda) = \lambda^2 + P\lambda + Q, \quad (2.3)$$

where

$$P = \frac{2achr(k+m)\sqrt{(p-c)c} + (c-p)(hmr(2c-p) - 2p)k - a^2chr(2c+p)}{kp(c-p)},$$

$$Q = \frac{1}{kp(c-p)} \left(2\sqrt{-(c-p)}cachr(k+m)(ch-hp-1) - 2cr(c-p)(a^2c - mk(c-p))h^2 \right. \\ \left. + r(c(2c+p)a^2 - mk(2c-p)(c-p))h + kp(c-p) \right).$$

For the characteristic polynomial (2.3), introduce the sign functions

$$\Phi_-(k, r) := F(-1) = 1 - P + Q, \quad (2.4)$$

$$\Phi_Q(k, r) := 1 - Q. \quad (2.5)$$

The following formulation avoids imposing an unproved global ordering on auxiliary roots such as k_1 and k_2 .

Lemma 2.2. *Assume that E_3 is feasible, i.e. $p > c$, $k > x^*$, and $x^* > m$. Then $F(1) > 0$. Moreover:*

1. E_3 is a sink if $\Phi_- > 0$ and $\Phi_Q > 0$;
2. E_3 is a saddle if $\Phi_- < 0$;
3. E_3 is a source if $\Phi_- > 0$ and $\Phi_Q < 0$;
4. $\Phi_- = 0$ gives a period-doubling boundary with a multiplier -1 whenever $P \neq 0, 2$; the case $P = 2$, $Q = 1$ is the double multiplier $(-1, -1)$ and is treated as the 1:2 codimension-two candidate;
5. $\Phi_Q = 0$ together with $P^2 - 4 < 0$ gives a complex unit-modulus pair and hence a Neimark–Sacker boundary; its intersections with the strong-resonance trace values are treated separately below.

No universal statement such as $x^* < k_1 < k_2$ is used unless the signs and ordering of the relevant roots have first been established in the parameter slice under consideration.

Proof. Using (2.1) and the feasibility assumptions, the expression for $F(1)$ simplifies to

$$F(1) = \frac{2ch^2r(p-c)(k-x^*)(x^*-m)}{kp} > 0. \quad (2.6)$$

The five assertions then follow directly from Lemma 2.1 by applying the signs of $F(-1) = \Phi_-$ and $1 - Q = \Phi_Q$. The equality cases are stated separately because the strict sign tests do not classify nonhyperbolic points. In particular, the double -1 case is the intersection $F(-1) = 0$, $Q = 1$, equivalently $P = 2$, $Q = 1$. $\square$

For $h = 3$, $a = 0.8$, $c = 0.9$, $m = 0.2$, $s = 1.5$, and $p = 1.2$, Figure 1 illustrates the sign-based classification of Lemma 2.2 as (k, r) varies. This numerical slice is illustrative; it is not used to assert a global ordering of symbolic threshold functions.

We now restrict attention to nonhyperbolic values at E_3 . Codimension-one period-doubling and Neimark–Sacker bifurcations were studied in [4]; the present analysis concerns codimension-two strong resonances.

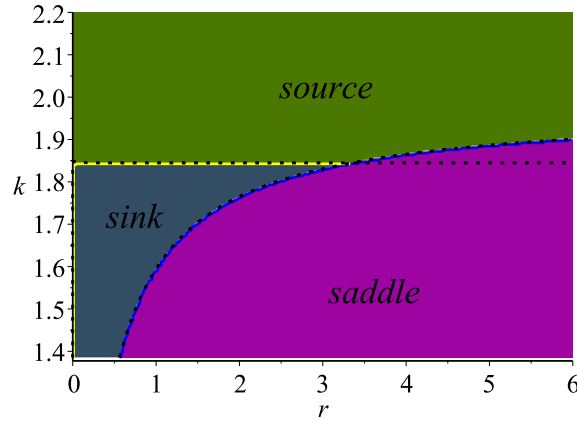


Figure 1. Numerical sign classification of the feasible fixed point E_3 in the (r, k) plane for the parameter set stated below. The regions are obtained from the signs in Lemma 2.2.

2.1. Codimension-two strong resonances at E_3

Following [12, 13], keep (m, s, a, p, c, h) fixed and use (k, r) as two independent unfolding parameters. Since the characteristic polynomial is $\lambda^2 + P\lambda + Q$, write

$$T(k, r) = \text{tr } J = -P(k, r), \quad D(k, r) = \det J = Q(k, r).$$

A spectral resonance point gives a regular two-parameter unfolding only if

$$\Delta_{kr} := \det \begin{pmatrix} T_k & T_r \\ D_k & D_r \end{pmatrix} \neq 0. \quad (2.7)$$

Thus the spectral equations in Theorems 2.1–2.3 locate resonance candidates; a generic codimension-two bifurcation additionally requires (2.7) and the relevant normal-form nondegeneracy conditions. These requirements are stated explicitly below and checked numerically for the representative parameter set in Section 3.

Let $u = x - x^*$, $v = y - y^*$, then system (1.1) becomes

$$\begin{cases} u \implies u + h \left(r \left(1 - \frac{u + x^*}{k} \right) (u + x^* - m) (u + x^*) - \frac{s(u + x^*)^2 (v + y^*)}{a^2 + (u + x^*)^2} \right), \\ v \implies v + h \left(\frac{p(u + x^*)^2 (v + y^*)}{a^2 + (u + x^*)^2} - c(v + y^*) \right), \end{cases} \quad (2.8)$$

which is equivalent to

$$U \mapsto JU + F(U), \quad U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad (2.9)$$

$$F(U) = \begin{bmatrix} a_{30}u^3 + a_{21}u^2v + a_{20}u^2 + a_{11}uv \\ b_{30}u^3 + b_{21}u^2v + b_{20}u^2 + b_{11}uv \end{bmatrix} + O(\|U\|^4),$$

where

$$\begin{aligned}
 a_{11} &= -\frac{2hcs(c-p)^2}{ap^2\sqrt{(p-c)c}}, \quad a_{21} = \frac{hs(4c-p)(c-p)^2}{p^3a^2}, \\
 a_{20} &= -\frac{hr}{ap^2k(c-p)c} \left(\sqrt{(p-c)c} \left(c(4c^2 - 5pc - 2p^2)a^2 - km(4c-p)(c-p)^2 \right) \right. \\
 &\quad \left. + ac^2(c-p)(4c-5p)(k+m) \right), \\
 a_{30} &= \frac{hr}{a^2p^3k} \left(4\sqrt{(p-c)c}a(2c-p)(c-p)(k+m) - (8c^3 - 12c^2p + 4cp^2 + p^3)a^2 \right. \\
 &\quad \left. + 4km(2c-p)(c-p)^2 \right), \\
 b_{11} &= 2\frac{c(c-p)^2h}{ap\sqrt{-(c-p)c}}, \quad b_{21} = -\frac{(4c-p)(c-p)^2h}{p^2a^2}, \\
 b_{20} &= \frac{r(4c-p)(c-p) \left(\sqrt{(p-c)c}ak + \sqrt{(p-c)c}am - a^2c + ckm - kmp \right) h}{apsk\sqrt{(p-c)c}}, \\
 b_{30} &= -\frac{4hr(2c-p)(c-p) \left(\sqrt{(p-c)c}ak + \sqrt{(p-c)c}am - a^2c + ckm - kmp \right)}{p^2ska^2}.
 \end{aligned}$$

Theorem 2.1. *If*

$$\begin{aligned}
 k &= -\frac{ac \left(2\sqrt{-(c-p)cm}(ch - hp - 1) - a(2c^2h - 2chp - 2c - p) \right)}{2\sqrt{(p-c)c}ac(ch - hp - 1) + m(c-p)(2c^2h - 2chp - 2c + p)}, \\
 r &= -\frac{2 \left(2m\sqrt{(p-c)c}(ch - hp - 1) - a(2c^2h - 2chp - 2c - p) \right)}{(c-p)(a^2c + cm^2 - m^2p)^2ch^2} \\
 &\quad \times \left(\sqrt{(p-c)c}(a^2c - cm^2 + m^2p) - 2acm(c-p) \right),
 \end{aligned}$$

the Jacobian at E_3 has the double multiplier $(-1, -1)$, so this is a spectral 1:2 resonance candidate. If, in addition, $\Delta_{kr} \neq 0$, $C_{R2} \neq 0$, and $D_{R2} + 3C_{R2} \neq 0$, the point is a generic codimension-two 1:2 resonance in the sense of [12, 13]. The normal form of system (1.1) takes the form

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \mapsto \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C_{R2}\xi_1^3 + D_{R2}\xi_1^2\xi_2 + O(\|\xi\|^4) \end{bmatrix}$$

where

$$\begin{aligned}
 C_{R2} &= \frac{L_1hs^2}{2(a^2c + m^2(c-p))^2a^2p^4}, \\
 L_1 &= 2\sqrt{(p-c)c}amp(3c^2h - 3chp - 6c + 2p)(2(c-p)c(a^2c + m^2(c-p))h \\
 &\quad - c(2c+p)a^2 - m^2(2c-p)(c-p)) - 6c^3(c-p)^3(a^2c + m^2(c-p))^2h^3 \\
 &\quad + 4c^2(c-p)^2(a^2c + m^2(c-p))(c(8c+p)a^2 + 2m^2(4c-p)(c-p))h^2 \\
 &\quad - c(c-p)(c^2(56c^2 + 2cp - 7p^2)a^4 + 2cm^2(c-p)(56c^2 - 16cp - 9p^2)a^2)
 \end{aligned}$$

$$\begin{aligned}
& +m^4(56c^2-34cp+p^2)(c-p)^2)h+2c(c^2(16c^2-6cp+p^2)a^4 \\
& +2m^2(c-p)(16c^3-12c^2p-3cp^2+2p^3)a^2+m^4(2c-p)(8c-5p)(c-p)^2), \\
D_{R2} &= \frac{L_2s^2}{2p^4a^2(c-p)(a^2c+m^2(c-p))^4} \left(4\sqrt{(p-c)}cam(a^2c+m^2(c-p)) \right. \\
& \left. +a^4c^2-6cm^2(c-p)a^2+m^4(c-p)^2 \right), \\
L_2 &= 2\sqrt{(p-c)}cam \left(28c^3(c-p)^4(a^2c+m^2(c-p))h^4-2c^2(c-p)^3(64a^2c^2p \right. \\
& +a^2c-64c^2m^2+73cm^2p-9m^2p^2)h^3+c(c-p)^2(3c(80c^2-8cp+5p^2)a^2 \\
& -m^2(c-p)(240c^2-96cp+7p^2))h^2-(c-p)(c(256c^3-62c^2p-35cp^2-6p^3)a^2 \\
& -m^2(c-p)(256c^3-178c^2p-11cp^2+2p^3))h+4(8c^2-4cp-p^2)((4c^2-cp)a^2 \\
& -m^2(4c-3p)(c-p))) -14c^3(c-p)^4(a^4c^2-6cm^2(c-p)a^2+m^4(c-p)^2)h^4 \\
& +2c^2(c-p)^3(c^2(32c+3p)a^4-12cm^2(16c-p)(c-p)a^2+m^4(32c-7p)(c-p)^2) \\
& \times h^3-2c(c-p)^2(c^2(60c^2+3cp+8p^2)a^4-10cm^2(c-p)(36c^2-9cp+p^2)a^2 \\
& +m^4(60c^2-33cp+4p^2)(c-p)^2)h^2+c(c-p)(c^2(128c^2-2cp-9p^2)a^4-2m^2 \\
& \times (c-p)(384c^3-180c^2p-49cp^2-4p^3)a^2+m^4(128c^2-118cp+15p^2)(c-p)^2)h \\
& -4c^3(4c-p)^2a^4+4m^2(c-p)(96c^4-96c^3p+6c^2p^2+8cp^3+p^4)a^2 \\
& \left. -4cm^4(4c-3p)^2(c-p)^2 \right).
\end{aligned}$$

Proof. The map (1.1) at E_3 has two multipliers -1 when $\det(J) - 1 = 0$ and $\text{tr}(J) + 2 = 0$, we can compute the expressions of k and r . Furthermore, by

$$\begin{aligned}
Jq_0 &= -q_0, & Jq_1 &= -q_1 + q_0, & J^T p_1 &= -p_1, & J^T p_0 &= -p_0 + p_1, \\
\langle q_0, p_0 \rangle &= \langle q_1, p_1 \rangle = 1, & \langle q_1, p_0 \rangle &= \langle q_0, p_1 \rangle = 0,
\end{aligned}$$

where

$$J = \begin{bmatrix} -3 & -\frac{chs}{p} \\ \frac{4p}{chs} & 1 \end{bmatrix}.$$

We can compute

$$q_0 = \begin{bmatrix} -\frac{chs}{2p} \\ 1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} \frac{chs}{4p} \\ 0 \end{bmatrix}, \quad p_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} \frac{4p}{chs} \\ 2 \end{bmatrix}.$$

Let $\begin{bmatrix} u \\ v \end{bmatrix} = y_1 q_0 + y_2 q_1$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The map (2.8) becomes

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} G(Y) \\ H(Y) \end{bmatrix},$$

$$G(Y) = \langle p_0, F(y_1 q_0 + y_2 q_1) \rangle, \quad H(Y) = \langle p_1, F(y_1 q_0 + y_2 q_1) \rangle.$$

Expand G and H into Taylor series with respect to Y :

$$G(Y) = \sum_{2 \leq i+j \leq 3} g_{ij} y_1^i y_2^j + O(\|Y\|^4), \quad H(Y) = \sum_{2 \leq i+j \leq 3} h_{ij} y_1^i y_2^j + O(\|Y\|^4).$$

Through some transformations, the normal form can be obtained and the coefficients C and D are as follows:

$$\begin{aligned} C_{R2} &= h_{30} + g_{20}h_{20} + \frac{1}{2}h_{20}^2 + \frac{1}{2}h_{20}h_{11}, \quad D_{R2} = h_{21} + 3g_{30} + \frac{1}{2}g_{20}h_{11} + \frac{5}{4}h_{20}h_{11} \\ &+ h_{20}h_{02} + 3g_{20}^2 + \frac{5}{2}g_{20}h_{20} + \frac{5}{2}g_{11}h_{20} + h_{20}^2 + \frac{1}{2}h_{11}^2. \end{aligned}$$

□

Theorem 2.2. *If*

$$\begin{aligned} k &= -\frac{ac \left(2\sqrt{(p-c)} cm (ch - hp - 1) - a (2c^2h - 2chp - 2c - p) \right)}{2\sqrt{(p-c)} cac (ch - hp - 1) + m(c-p)(2c^2h - 2chp - 2c + p)}, \\ r &= -\frac{3 \left(2\sqrt{(p-c)} cm (ch - hp - 1) - a (2c^2h - 2chp - 2c - p) \right)}{2(c-p)(a^2c + cm^2 - m^2p)^2 ch^2} \\ &\quad \times \left(\sqrt{(p-c)} c (a^2c - cm^2 + m^2p) - 2acm(c-p) \right), \end{aligned}$$

the multipliers at E_3 are $e^{\pm 2\pi i/3}$, so this is a spectral 1:3 resonance candidate. A generic 1:3 strong resonance further requires $\Delta_{kr} \neq 0$ and nonvanishing of the resonant quadratic coefficient B_{R3} together with the standard cubic nondegeneracy coefficient computed from the normal form [12, 13]. The normal form of system (2.8) is as follows

$$\zeta \mapsto e^{\frac{2\pi}{3}i} \zeta + B_{R3} \bar{\zeta}^2 + C_{R3} \zeta |\zeta|^2 + O(|\zeta|^4),$$

where

$$\begin{aligned} B_{R3} &= \frac{M_1 (-\sqrt{3} + 3i) \sqrt{(p-c)} cs}{24p^2 a (c-p) (a^2c + m^2(c-p))^2 c}, \\ M_1 &= 3(i + \sqrt{3}) \sqrt{(p-c)} c \left(2(c-p) c (a^2c + m^2(c-p)) h - c(2c+p) a^2 \right. \\ &\quad \left. - m^2(2c-p)(c-p) amp + \frac{1}{2}(i + \sqrt{3}) \left(-8c(c-p)^2 (a^2c + m^2(c-p))^2 h^2 \right. \right. \\ &\quad \left. \left. + 3 \left((i\sqrt{3}p + 8c + 3p) ca^2 + (c-p) (i\sqrt{3}p + 8c - p) m^2 \right) (a^2c + m^2(c-p)) (c-p) \right. \right. \\ &\quad \left. \left. \times h - 6c^2(4c-p) a^4 - 12m^2(c-p) (4c^2 - 2cp - p^2) a^2 - 6m^4(4c-3p)(c-p)^2 \right) c, \right. \\ C_{R3} &= \frac{(\sqrt{3} - 3i)(\sqrt{3} + 9i) s^2}{288p^4 ca^2 (c-p) (a^2c + m^2(c-p))^4} \\ &\quad \times \left(2\sqrt{3} \sqrt{(p-c)} camp \left(2(c-p) c (a^2c + m^2(c-p)) h - c(2c+p) a^2 \right. \right. \\ &\quad \left. \left. - m^2(2c-p)(c-p) \right) - 2 \left(\sqrt{3} + i \right) (a^2c + m^2(c-p))^2 (c-p)^2 c^2 h^2 - \frac{1}{13} (i + 2\sqrt{3}) \right) \end{aligned}$$

$$\begin{aligned}
& \times (a^2c + m^2(c-p)) \left((5ip\sqrt{3} - 52c - 17p)ca^2 + (c-p) \left(ip\sqrt{3} - 52c + 7p \right) m^2 \right) \\
& \times (c-p) ch - 2\sqrt{3} (c^2(4c-p)a^4 + 2m^2(c-p)(4c^2 - 2cp - p^2)a^2 \\
& + m^4(4c-3p)(c-p)^2)c \left(-3(i-\sqrt{3})\sqrt{(p-c)c} (2(c-p)c(a^2c + m^2(c-p))h \right. \\
& \left. - c(2c+p)a^2 - m^2(2c-p)(c-p)) amp - 4(\sqrt{3}+i)(a^2c + m^2(c-p))^2 \right. \\
& \times (c-p)^2 c^2 h^2 - 2\sqrt{3}(a^2c + m^2(c-p)) \times ((2ip\sqrt{3} - 8c - p)ca^2 \\
& + (ip\sqrt{3} - 8c + 2p)(c-p)m^2)(c-p)ch - 3(-i+\sqrt{3})(c^2(4c-p)a^4 + 2m^2(c-p) \\
& \times (4c^2 - 2cp - p^2)a^2 + m^4(4c-3p)(c-p)^2)c \left. \right) + \frac{g_{21}}{2} + \frac{2|g_{11}|^2}{3+\sqrt{3}i}, \\
g_{11} = & \frac{i\sqrt{(p-c)}cs}{2acp^2(p-c)(a^2c + m^2(c-p))^2} \\
& \times \left(-2\sqrt{(p-c)}c\sqrt{3}amp(2(c-p)c(a^2c + m^2(c-p))h \right. \\
& \left. - c(2c+p)a^2 - m^2(2c-p)(c-p)) + 2(i+\sqrt{3})(a^2c + m^2(c-p))^2(c-p)^2c^2h^2 \right. \\
& + \frac{1}{13}(i+2\sqrt{3})(a^2c + m^2(c-p)) \left((5i\sqrt{3}p - 52c - 17p)ca^2 + (c-p) \right. \\
& \times (i\sqrt{3}p - 52c + 7p)m^2)(c-p)ch + 2\sqrt{3}(c^2(4c-p)a^4 + 2m^2(c-p) \\
& \times (4c^2 - 2cp - p^2)a^2 + m^4(4c-3p)(c-p)^2)c \left. \right), \\
\frac{g_{21}}{2} = & -\frac{hs^2(i\sqrt{3}-3)}{36p^4(a^2c + m^2(c-p))^2a^2} (9i\sqrt{(p-c)}c\sqrt{3}(2(c-p)c(a^2c + m^2(c-p))h \\
& - c(2c+p)a^2 - m^2(2c-p)(c-p))amp^2 \\
& + 2(2i\sqrt{3}-3)(4c-p)(a^2c + m^2(c-p))^2(c-p)^2c^2h^2 \\
& - \frac{(-6+9i\sqrt{3})(a^2c + m^2(c-p))(c-p)ch}{31} \left((9i\sqrt{3}cp + 9i\sqrt{3}p^2 + 248c^2 \right. \\
& \left. - 211cp - 25p^2)ca^2 + (c-p)(9i\sqrt{3}cp - 3i\sqrt{3}p^2 + 248c^2 - 211cp + 29p^2)m^2 \right) \\
& + 18i\sqrt{3}(c^2(4c^2 - 6cp + p^2)a^4 + m^2(8c^2 - 4cp - p^2)(c-p)^2a^2 \\
& + 2m^4(2c-p)(c-p)^3)c).
\end{aligned}$$

Proof. The map (2.8) at E_3 has a pair complex multiplier $e^{\pm \frac{2\pi}{3}i}$ if $\det(J) - 1 = 0$ and $\text{Trace}(J) + 1 = 0$. Solving the two equations, we obtain the expressions of k and r ,

$$J = \begin{bmatrix} -2 & -\frac{chs}{p} \\ 3\frac{p}{chs} & 1 \end{bmatrix}.$$

Solving

$$Jq_0 = e^{\frac{2\pi}{3}i}q, \quad J^T p_0 = e^{-\frac{2\pi}{3}i}p, \quad \langle p_0, q_0 \rangle = 1,$$

we have

$$q_0 = \begin{bmatrix} \frac{csh(i\sqrt{3}-3)}{6p} \\ 1 \end{bmatrix}, \quad p_0 = \begin{bmatrix} \frac{ip\sqrt{3}}{csh} \\ \frac{1+\sqrt{3}i}{2} \end{bmatrix}.$$

Let $U = zq_0 + \bar{z}\bar{q}_0$. Then

$$z \mapsto e^{\frac{2\pi}{3}i}z + g(z, \bar{z}), \quad g(z, \bar{z}) = \langle p, F(zq_0 + \bar{z}\bar{q}_0) \rangle = \sum_{i+j \geq 2} \frac{1}{i!j!} g_{ij} z^i \bar{z}^j.$$

Through some transformations, one can obtain the normal form, with the coefficients

$$B_{R3} = \frac{g_{02}}{2}, \quad C_{R3} = \frac{g_{20}g_{11}(2e^{\frac{2\pi}{3}i} + e^{-\frac{2\pi}{3}i} - 3)}{2(e^{-\frac{2\pi}{3}i} - 1)(e^{\frac{4\pi}{3}i} - e^{\frac{2\pi}{3}i})} + \frac{|g_{11}|^2}{1 - e^{-\frac{2\pi}{3}i}} + \frac{g_{21}}{2}.$$

□

Theorem 2.3. *If*

$$k = -\frac{ac \left(2\sqrt{(p-c)} cm (ch - hp - 1) - a (2c^2h - 2chp - 2c - p) \right)}{2\sqrt{(p-c)} cac (ch - hp - 1) + m (c - p) (2c^2h - 2chp - 2c + p)},$$

$$r = -\frac{\left(2m\sqrt{(p-c)} c (ch - hp - 1) - 2a (c - p) ch + a (2c + p) \right)}{(c - p) (a^2c + cm^2 - m^2p)^2 ch^2}$$

$$\times \left(\sqrt{(p-c)} c (a^2c - cm^2 + m^2p) - 2acm (c - p) \right),$$

the multipliers at E_3 are $\pm i$, so this is a spectral 1:4 resonance candidate. A generic 1:4 strong resonance further requires $\Delta_{kr} \neq 0$, $D_{R4} \neq 0$, and the corresponding normalized strong-resonance coefficient to avoid its degenerate boundary [12, 13]. The normal form of system (2.8) is as follows

$$\zeta \mapsto i\zeta + C_{R4}\zeta|\zeta|^2 + D_{R4}\bar{\zeta}^3 + O(|\zeta|^4),$$

where

$$C_{R4} = \frac{(-1 - 3i)s^2}{16p^4(a^2c + m^2(c-p))^4(c-p)ca^2} \left(2\sqrt{(p-c)} camp (2(c-p)c(a^2c + cm^2 - m^2p)h - c(2c+p)a^2 - m^2(2c-p)(c-p)) + c((c-p)(c(8c+3p)a^2 + (8c-p)(c-p)m^2)(a^2c + m^2(c-p))h - 2c^2(4c-p)a^4 - 4m^2(c-p)(4c^2 - 2cp - p^2)a^2 - 2m^4(4c-3p)(c-p)^2) - i(4c^2h - 4chp - 8c + 5p)(c-p)ch(a^2c + m^2(c-p))^2) \right. \\ \left. (2c^2(c-p)^2(a^2c + m^2(c-p))^2h^2 - c(4c-p)(c-p)(a^2c + m^2(c-p))^2h + i(-2c^2(c-p)^2(a^2c + m^2(c-p))^2 \times h^2 + (c-p)c(a^2c + m^2(c-p))(c(8c+3p)a^2 + (8c-p)(c-p)m^2)h - 2c(c^2(4c-p)a^4 + 2m^2(c-p)(4c^2 - 2cp - p^2)a^2) \right)$$

$$\begin{aligned}
& + m^4 (4c - 3p) (c - p)^2) + 2\sqrt{-(c - p)} camp (2(c - p) c (a^2 c + m^2(c - p)) h \\
& - c(2c + p) a^2 - m^2(2c - p) (c - p))) + \frac{(-1 + i) s^2}{8p^4 (a^2 c + m^2(c - p))^4 (c - p) ca^2} \\
& \times \left(\left(2c^2 (c - p)^2 (a^2 c + m^2(c - p))^2 h^2 - c(4c - p) (c - p) (a^2 c + m^2(c - p))^2 h \right)^2 \right. \\
& + \left(-2c^2 (c - p)^2 (a^2 c + m^2(c - p))^2 h^2 + (c - p) c (a^2 c + m^2(c - p)) (c(8c + 3p) a^2 \right. \\
& + (8c - p) (c - p) m^2) h - 2c (c^2 (4c - p) a^4 + 2m^2 (c - p) (4c^2 - 2cp - p^2) a^2 \\
& + m^4 (4c - 3p) (c - p)^2) + 2\sqrt{(p - c)} camp (2(c - p) c (a^2 c + m^2(c - p)) h \\
& - c(2c + p) a^2 - m^2(2c - p) (c - p)))^2 \Big) + \frac{(1 + i)}{16p^4 (a^2 c + m^2(c - p))^4 (c - p) ca^2} \\
& \times \left(16 \left(\frac{1}{4} sc (4c (c - p)^2 (a^2 c + m^2(c - p))^2 h^2 - (c - p) (c(8c + 3p) a^2 + (8c - p) \right. \right. \\
& \times (c - p) m^2) (a^2 c + m^2(c - p)) h + 2c^2 (4c - p) a^4 + 4m^2 (c - p) (4c^2 - 2cp - p^2) a^2 \\
& + 2m^4 (4c - 3p) (c - p)^2) - \frac{1}{2} samp \sqrt{(p - c)} c (2(c - p) c (a^2 c + m^2(c - p)) h \\
& - c(2c + p) a^2 - m^2(2c - p) (c - p)))^2 + 9c^2 s^2 h^2 p^2 (c - p)^2 (a^2 c + m^2(c - p))^4 \Big) \\
& + \frac{s^2 h}{4p^4 (a^2 c + m^2(c - p))^2 a^2} \left(3(1 + i) \sqrt{(p - c)} c (2(c - p) c (a^2 c + m^2(c - p)) h \right. \\
& - c(2c + p) a^2 - m^2(2c - p) (c - p)) amp^2 - \frac{1 + 5i}{13} (4c - p) (a^2 c + m^2(c - p))^2 \\
& \times (c - p)^2 c^2 h^2 - \frac{1 + 5i}{13} ((9ip(c + p) + 104c^2 - 85cp - 7p^2) ca^2 \\
& + (c - p) (9icp - 3ip^2 + 104c^2 - 85cp + 11p^2) m^2) (a^2 c + m^2(c - p)) (c - p) ch \\
& + 6(1 + i) (c^2 (4c^2 - 6cp + p^2) a^4 + m^2 (8c^2 - 4cp - p^2) (c - p)^2 a^2 \\
& + 2m^4 (2c - p) (c - p)^3) c \Big), \\
D_{R4} = & \frac{(1 - i) s}{4p^4 (a^2 c + m^2(c - p))^4 (c - p) ca^2} \left(2c^2 (c - p)^2 (a^2 c + m^2(c - p))^2 h^2 \right. \\
& - c(4c - p) (c - p) (a^2 c + cm^2 - m^2 p)^2 h + i \left(-2c^2 (c - p)^2 (a^2 c + m^2(c - p))^2 h^2 \right. \\
& + (c - p) c (a^2 c + cm^2 - m^2 p) (c(8c + 3p) a^2 + (8c - p) (c - p) m^2) h \\
& - 2c \left(c^2 (4c - p) a^4 + 2m^2 (c - p) (4c^2 - 2cp - p^2) a^2 + m^4 (4c - 3p) (c - p)^2 \right) \\
& + 2\sqrt{(p - c)} camp (2(c - p) c (a^2 c + cm^2 - m^2 p) h - c(2c + p) a^2 \\
& - m^2(2c - p) (c - p))) \left(\frac{1}{4} sc (4c (c - p)^2 (a^2 c + cm^2 - m^2 p)^2 h^2 \right. \\
& - (c - p) (c(8c + 3p) a^2 + (8c - p) (c - p) m^2) (a^2 c + cm^2 - m^2 p) h + 2c^2 (4c - p) a^4 \\
& + 4m^2 (c - p) (4c^2 - 2cp - p^2) a^2 + 2m^4 (4c - 3p) (c - p)^2 \Big) \\
& - \frac{1}{2} samp \sqrt{(p - c)} c (2(c - p) c (a^2 c + m^2(c - p)) h - c(2c + p) a^2
\end{aligned}$$

$$\begin{aligned}
& -m^2(2c-p)(c-p)) - \frac{3}{4}icshp(c-p)(a^2c+cm^2-m^2p)^2) \\
& + \frac{(1+i)s}{4p^4(a^2c+cm^2-m^2p)^4(c-p)ca^2} \left(\frac{1}{4}sc(4c(c-p)^2(a^2c+cm^2-m^2p)^2h^2 \right. \\
& - (c-p)(c(8c+3p)a^2+(8c-p)(c-p)m^2)(a^2c+cm^2-m^2p)h+2c^2(4c-p)a^4 \\
& + 4m^2(c-p)(4c^2-2cp-p^2)a^2+2m^4(4c-3p)(c-p)^2) - \frac{1}{2}samp\sqrt{(p-c)c} \\
& \times (2(c-p)c(a^2c+cm^2-m^2p)h-c(2c+p)a^2-m^2(2c-p)(c-p)) \\
& - \frac{3}{4}icshp(c-p)(a^2c+cm^2-m^2p)^2) \left(2\sqrt{(p-c)c}amp(2(c-p)c(a^2c+m^2(c-p))h \right. \\
& - c(2c+p)a^2-m^2(2c-p)(c-p)) + c((c-p)(c(8c+3p)a^2+(8c-p)(c-p)m^2) \\
& \times (a^2c+cm^2-m^2p)h-2c^2(4c-p)a^4-4m^2(c-p)(4c^2-2cp-p^2)a^2 \\
& - 2m^4(4c-3p)(c-p)^2) + i(4c^2h-4chp-8c+5p)(c-p)ch(a^2c+cm^2-m^2p)^2) \\
& - \frac{s^2h}{4p^4(a^2c+cm^2-m^2p)^2a^2} \left((1+i)\sqrt{(p-c)c}(2(c-p)c(a^2c+cm^2-m^2p)h \right. \\
& - c(2c+p)a^2-m^2(2c-p)(c-p))amp^2 + (1+i)(4c-p)(a^2c+m^2(c-p))^2 \\
& \times (c-p)^2c^2h^2 - (1+i)((3icp-ip^2+8c^2-7cp-p^2)ca^2 \\
& + (c-p)(3icp-ip^2+8c^2-7cp+p^2)m^2)(a^2c+cm^2-m^2p)(c-p)ch+ \\
& + 2(1+i)(c^2(4c^2-6cp+p^2)a^4+m^2(8c^2-4cp-p^2)(c-p)^2a^2 \\
& + 2m^4(2c-p)(c-p)^3)c) \Big).
\end{aligned}$$

Proof. The map (2.8) has a fixed point E_3 with a pair complex multiplier $\pm i$ if $\det(J) - 1 = 0$ and $\text{tr}(J) = 0$, which can be computed the expressions of k and r . Let

$$Jq_0 = iq_0, \quad J^T p_0 = -ip_0, \quad \langle p_0, q_0 \rangle = 1, \quad J = \begin{bmatrix} -1 & -\frac{chs}{p} \\ 2\frac{p}{chs} & 1 \end{bmatrix}.$$

Then we have

$$q_0 = \begin{bmatrix} \frac{(-1+i)chs}{2p} \\ 1 \end{bmatrix}, \quad p_0 = \begin{bmatrix} \frac{ip}{chs} \\ \frac{1+i}{2} \end{bmatrix}.$$

Let $U = zq_0 + \bar{z}\bar{q}_0$. Then

$$z \mapsto e^{\frac{\pi}{2}i}z + g(z, \bar{z}), \quad g(z, \bar{z}) = \langle p, F(zq_0 + \bar{z}\bar{q}_0) \rangle = \sum_{i+j \geq 2} \frac{1}{i!j!} g_{ij} z^i \bar{z}^j.$$

Through some transformations, one can obtain the normal form, with the coefficients

$$C_{R4} = \frac{1+3i}{4}g_{20}g_{11} + \frac{1-i}{2}|g_{11}|^2 - \frac{1+i}{4}|g_{02}|^2 + \frac{1}{2}g_{21},$$

$$D_{R4} = \frac{i-1}{4}g_{11}g_{02} - \frac{1+i}{4}g_{02}|g_{20}| + \frac{1}{6}g_{03}.$$

□

3. Numerical simulations

The numerical continuation is used here as an independent check of the analytical resonance locations rather than as a substitute for the genericity conditions. In the MatContM output for continuation of the Neimark–Sacker curve with free parameters (r, k) , we read a reported vector in the order (x, y, r, k, ρ) , where ρ is the real part of the critical unit multiplier used by the resonance test function; it is not an additional continuation parameter. We use the following abbreviations: NS = Neimark–Sacker, PD = period-doubling, BP = branch point, $Rj = 1:j$ resonance candidate, LPPD = limit-point/period-doubling (fold–flip) candidate, GPD = generalized period-doubling, CH = Chenciner, CP = cusp point, and LP = limit point (fold). Software labels are reported as candidates unless their defining genericity conditions have been checked.

For the strong-Allee parameter set

$$h = 3, \quad a = 0.8, \quad c = 0.9, \quad m = 0.2, \quad s = 1.5, \quad p = 1.2.$$

Theorems 2.1–2.3 and MatContM agree to the displayed precision. Numerical differentiation of (T, D) in (2.7) at the analytical points gives nonzero unfolding determinants.

Table 1. Analytical resonance locations and MatContM checks for the representative strong-Allee parameter set.

Point	Theorem (k, r)	MatContM (k, r)	Multipliers	Reported NF data	Δ_{kr}
$R4$	1.844662, 1.673809	1.844662, 1.673809	$\pm i$	$A = -1.234020 + 0.795969i$	11.6414
$R3$	1.844662, 2.510713	1.844662, 2.510713	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	$\text{Re}(c_1) = -0.555729$	17.4621
$R2$	1.844662, 3.347618	1.844662, 3.347618	$-1, -1$	$(C, D) =$ $(212.6088, -749.4324)$	23.2828

For $R2$, $D + 3C = -111.6060 \neq 0$ in the MatContM normalization. For $R3$, the reported cubic quantity is nonzero, and for $R4$, $|A| \approx 1.4685 \neq 1$; these numerical values support nondegeneracy in the software normalization. They should be interpreted together with the analytical normal forms, not as a replacement for them.

Case (1). In Figures 2 and 3, we consider the strong Allee effect. In Figure 2, let $s = 1.5$, $a = 0.8$, $c = 0.9$, $p = 1.2$, $m = 0.2$, $r = 0.75$ and $k = 2$, then system (1.1) has the fixed points $(0, 0)$, $(2, 0)$, $(0.2, 0)$, $(1.385640646, 0.3364379004)$. We use MatContM to simulate the multiple bifurcations. Take initial value $(1.3856, 0.3364)$ and $h = 3$, then vary k as a free parameter, the MatContM report is

```
label=NS, x=(1.385641 0.272539 1.844662),
normal form coefficient of NS=-1.561158e+01,
label=PD, x=(1.385641 0.081984 1.497753),
normal form coefficient of PD=-1.206156E+01,
label=BP, x=(1.385641 0 1.385641).
```

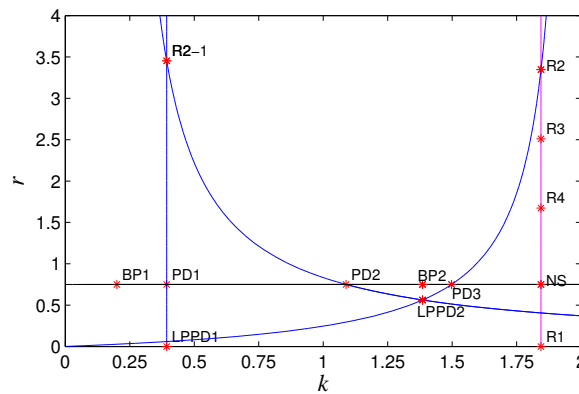


Figure 2. The bifurcation curves of k and r with $m = 0.2$.

Start from NS point and take k and r as the bifurcation parameters, the MatContM report is

```
label=R4, x=(1.385641 0.608237 1.673809 1.844662 0),
normal form coefficient of R4: A=-1.234020e+00+7.959693e-01i,
label=R3, x=(1.385641 0.912356 2.510713 1.844662 -0.5),
normal form coefficient of R3: Re(c-1)=-5.557292e-01,
label=R2, x=(1.385641 1.216475 3.347618 1.844662 -1),
normal form coefficient of R2: [c,d]=2.126088e+02,-7.494324e+02,
label=R1, x=(1.385641, 0, 0, 1.844662 1),
normal form coefficient of R1: s=0.
```

The label $R1$ is a MatContM test-function label for a possible 1:1 resonance. In the first (k, r) continuation above it occurs with the reported coefficient $s = 0$, so it is a nongeneric candidate and is not claimed here as an established 1:1 bifurcation. The labels $R2$, $R3$, and $R4$ denote 1:2, 1:3, and 1:4 resonance candidates, respectively.

From point BP2, take k as the bifurcation parameter, compute backward, there are point PD2, PD1, BP1, the MatContM output is

```
label=PD, x=(1.088889 0 1.088889),
normal form coefficient of PD=1.463507e+01,
label=PD, x=(0.393043 0 0.393043),
normal form coefficient of PD=-1.772429e+01,
label=BP, x=(0.2 0 0.2).
```

Take PD2 as the initial value and k and r as bifurcation parameters, compute forward there is a LPPD2 point, the MatContM output:

```
label=LPPD, x=(1.385641 0 0.562284 1.385641),
normal form coefficient for LPPD:[a/e,be]=-1.495888e-01,3.123485e-22,
first Lyapunov coefficient for second iterate=3.123485e-22.
```

Compute backward, there is a point R2-1 (another 1:2 resonance bifurcation point): The MatContM output is

```
label=R2, x=(0.393043 0 3.453463 0.393043),
normal form coefficient for R2: [c,d]=-3.497097e-05,-1.239380e+03.
```

From PD1 take k and r as bifurcation parameters, compute backward, there are two points R2-1 and LPPD1, the MatContM output is

```
label=LPPD, x=(0.393043 0 0 0.393043),
normal form coefficient for LPPD: [a/e,be]=-1.893160e-17,-5.101866e+00.
```

In the following, we take the same values of the parameters as above, but we choose k and h as two bifurcation parameters, the bifurcation diagram is shown in Figure 3. The concrete

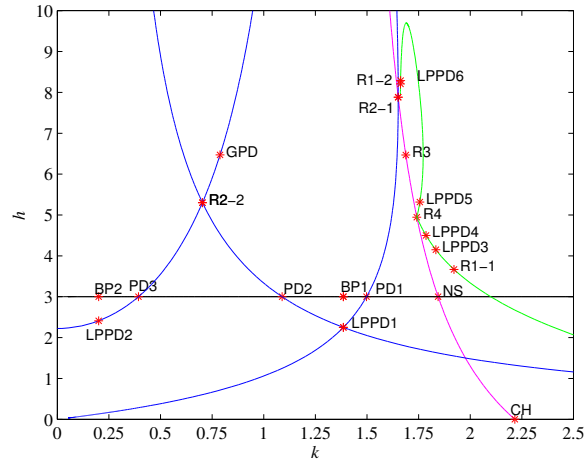


Figure 3. The bifurcation curves of k and h with $m = 0.2$.

plot steps and MatContM reports are as follows: We take $(1.3856, 0.3364)$ as initial value, first, we choose k as the bifurcation parameter, then compute forward and backward, the MatContM output is

```
label=NS, x=(1.385641 0.272539 1.844662),
normal form coefficient of NS=-1.561158e+01,
label=PD, x=(1.385641 0.081984 1.497753),
normal form coefficient of PD=-1.206156e+01,
label=BP, x=(1.385641 0 1.385641).
```

Take NS point as initial value and choose k and h as the bifurcation parameters, compute forward, the MatContM report is

```
label=CH, x=(1.385641 0.410408 0 2.216021 1.00),
normal form coefficient of CH=1.212983e-01.
```

MatContM reports a CH test-function candidate at $h = 0$; since the Euler map reduces to the identity map at $h = 0$, this point is highly degenerate and is not interpreted here as a generic Chenciner bifurcation.

Compute backward, the MatContM output is

```
label=R4, x=(1.385641 0.223777 4.945949 1.741447 0),
normal form coefficient of R4: A=-1.479569e+00+7.566413e-01i,
label=R3, x=(1.385641 0.196317 6.467324 1.688250 -0.5),
normal form coefficient of R3:Re(c-1)=-7.083264e-01,
label=R2, x=(1.385641 0.176203 7.882537 1.651301 -1),
normal form coefficient of R2:[c,d]=4.299310e+02,-1.222160e+03.
```

Then take PD1 as initial value and choose k and h as the bifurcation parameters, compute forward, the curve goes up through R2-1 point, compute backward, a point LPPD1 is present, the MatContM output is:

```
label=LPPD, x=(1.385641 0 2.249136 1.385641),
```

normal form coefficient for LPPD: [a/e, be] = -1.165107e-01, -1.451839e-08.

At BP1, compute forward and backward, there are PD2, PD3, BP2 point, the MatContM output is

```
label=PD, x=(1.088889 0 1.088889),
normal form coefficient of PD=1.463507e+01,
label=PD, x=(0.393043 0 0.393043),
normal form coefficient of PD=-1.772429e+01,
label=BP, x=(0.2 0 0.2).
```

At PD2, choose k and h as the bifurcation parameters and keep other parameters fixed, compute forward and backward, the MatContM report is

```
label=LPPD, x=(1.385641 0 2.249136 1.385641),
normal form coefficient for LPPD: [a/e, be] = -1.165107e-01, 3.523123e-16,
first Lyapunov coefficient for second iterate=3.523123e-16,
label=R2, x=(0.702921 0 5.302361 0.702921),
normal form coefficient for R2: [c, d] = 5.293528e-05, -2.300404e+02.
```

At PD3, compute forward: The points R2-2 and GPD points are present, compute backward, the LPPD2 point is present, the MatContM report is

```
label=GPD, x=(0.787986 0 6.470854 0.787986),
normal form coefficient of GPD=-1.888828e+02,
label=LPPD, x=(0.2 0 2.411348 0.2),
normal form coefficient for LPPD: [a/e, be] = -1.128906e+00, -3.986011e-01.
```

At R4, choose k and h as the bifurcation parameters, compute forward, the points LPPD4, LPPD3, R1-1 are present, compute backward, the points LPPD5, PLLD6, R1-2 are present, the MatContM output is

```
label=LPPD, x=(1.540525 0.223191 4.5008 1.785915),
normal form coefficient for LPPD: [a/e, be] = -4.333846e+00, -7.926345e+01,
label=LPPD, x=(1.643292 0.226369 4.148629 1.833595),
normal form coefficient for LPPD: [a/e, be] = -7.367291e-01, -8.117302e+03,
label=R1, x=(1.811886 0.229891 3.668803 1.921243),
normal form coefficient of R1=-1;
label=LPPD, x=(1.326425 0.222725 5.316029 1.755914),
normal form coefficient for LPPD: [a/e, be] = 6.090110e-01, -2.214816e+02,
label=LPPD, x=(1.300420 0.195623 8.290160 1.662789),
normal form coefficient for LPPD: [a/e, be] = 1.137093e-01, -7.728823e+04,
label=R1, x=(1.305114, 0.194830 8.206949 1.662499),
normal form coefficient of R1=1.
```

Case (2). When considering the effect of the weak Allee effect on the system, we take $k = 2$, $m = -0.2$, $r = 0.75$, $s = 1.5$, $a = 0.8$, $c = 0.9$, $p = 1.2$. By choosing $h = 2$ and $h = 3$, the Figures 4 and 5 are plotted, respectively. Several bifurcation points are presented. For these two Figures, we omit the corresponding MatContM reports.

Figures 2–5 numerically support the spectral locations of the 1:2, 1:3, and 1:4 resonance candidates in Theorems 2.1–2.3. Generic bifurcation claims are made only where the unfolding and normal-form nondegeneracy conditions are also satisfied.

Case (3). We now use the reproducible orbit/Lyapunov protocol stated above. For a strong Allee effect, Figure 6 varies $r \in (0.1, 0.6)$ with $k = 2$, $m = 0.2$, $s = 1.5$, $a = 0.8$, $c = 0.9$, $p = 1.2$, $h = 3$, and initial condition $(1.385, 0.044)$. Figure 7 varies $h \in (0.5, 3.5)$ with $r = 0.5$ and initial

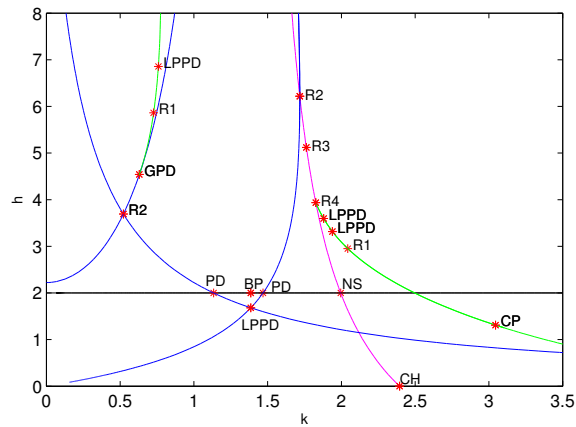


Figure 4. The bifurcation curves of k and h with $m = -0.2$.

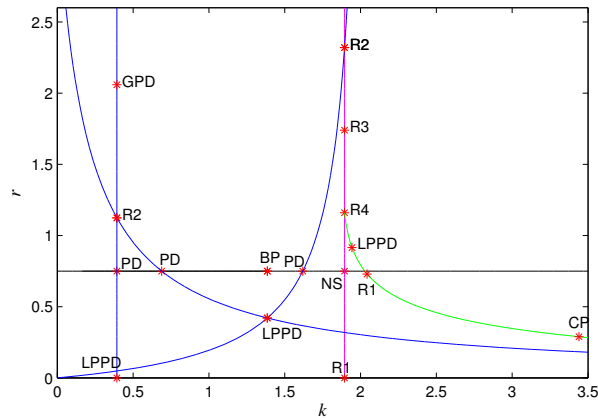


Figure 5. The bifurcation curves of k and r with $m = -0.2$ and $h = 3$.

condition $(1.385, 0.224)$. For a weak Allee effect, Figure 8 uses $m = -0.2$, $r = 0.5$ and varies $h \in (0.5, 3.5)$ from initial condition $(1.385, 0.299)$. Figure 9 varies $m \in (-0.5, 0.4)$ with $r = 0.5$, $k = 2$, $h = 2$ and initial condition $(1.385, 0.356)$.

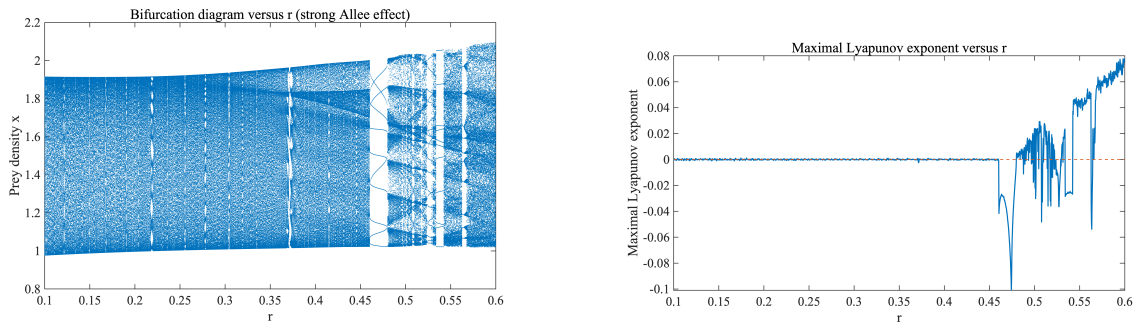


Figure 6. Strong Allee effect: Prey bifurcation diagram (left) and maximal Lyapunov exponent (right) versus r .

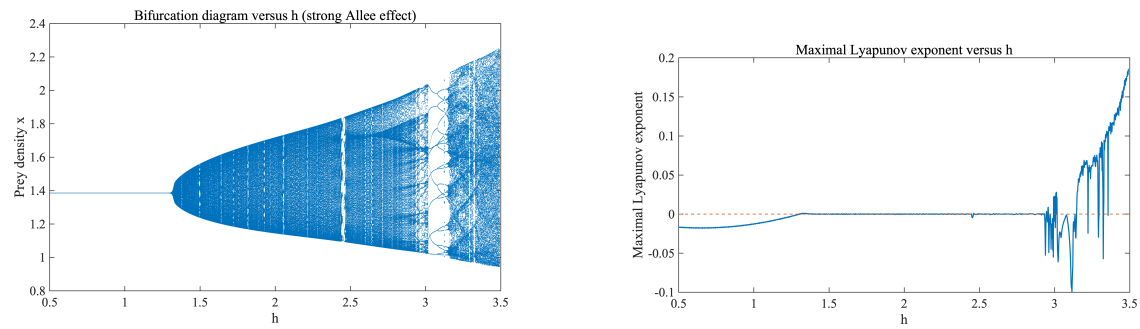


Figure 7. Strong Allee effect: Prey bifurcation diagram (left) and maximal Lyapunov exponent (right) versus h .

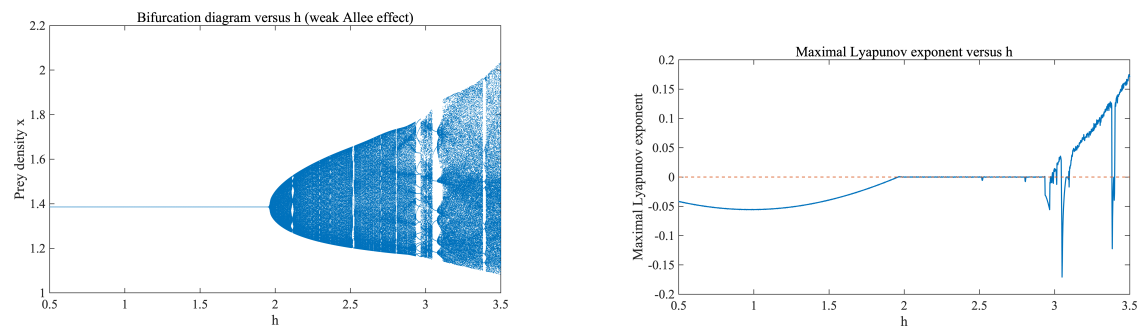


Figure 8. Weak Allee effect: Prey bifurcation diagram (left) and maximal Lyapunov exponent (right) versus h .

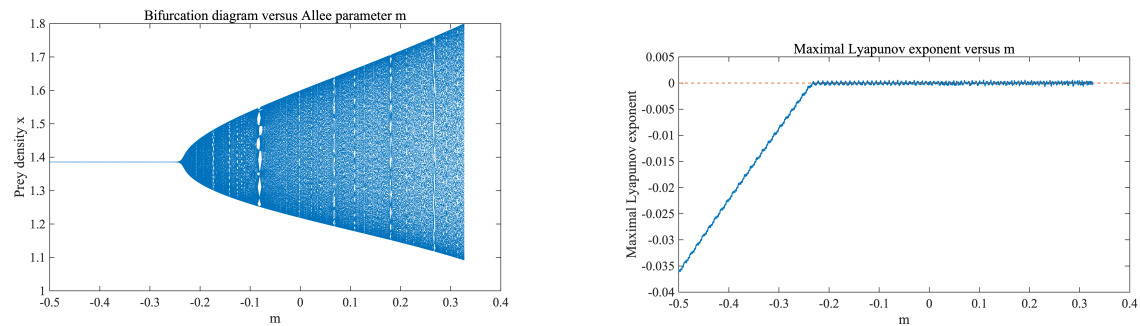


Figure 9. Bifurcation diagram (left) and maximal Lyapunov exponent (right) as the Allee parameter m varies from weak to strong regimes.

4. Discussion

The main result of this work is the local codimension-two analysis of the coexistence fixed point of the Euler map, beyond the codimension-one results reported in [4]. We have (i) separated algebraic fixed-point existence from biological feasibility for strong and weak Allee effects, (ii) reformulated the stability classification using the complete signs of $F(-1)$ and $1 - Q$ rather than assuming an ordering of symbolic thresholds, (iii) located the spectral 1:2, 1:3, and 1:4 strong-resonance candidates, (iv) added an explicit regular two-parameter unfolding test and normal-form nondegeneracy requirements, and (v) compared the analytical resonance locations with MatContM output, including multipliers and normal-form data.

References

- [1] M. Ahmidi, K. Mokni and M. Ch-Chaoui, *Influence of prey harvesting on the dynamics of a prey–predator system: Multi-parameter bifurcation analysis*, Nonlinear Dyn., 2025, 113, 31841–31870.
- [2] S. Akhtar, R. Ahmed, M. Batool, et al., *Stability, bifurcation and chaos control of a discretized Leslie prey-predator model*, Chaos, Solitons and Fractals, 2021, 152, 111345.
- [3] H. Benali, et. al., *Unveiling complexity: A discrete-time prey-predator model with immigration effects*, Iranian Journal of Science, 2025, 49, 449–462.
- [4] P. Chakraborty, et al., *Stability and bifurcation analysis of a discrete prey-predator model with sigmoid functional response and Allee effect*, Rendiconti del Circolo Matematico di Palermo Series 2, 2021, 70, 253–273.
- [5] Q. Chen, Z. Teng and F. Wang, *Fold-flip and strong resonance bifurcations of a discrete-time mosquito model*, Chaos, Solitons and Fractals, 2021, 144, 110704.
- [6] L. Cheng and H. Cao, *Bifurcation analysis of a discrete-time ratio-dependent predator-prey model with Allee Effect*, Commun. Nonlinear Sci. Numer. Simulat., 2016, 38, 288–302.
- [7] S. Das, A. N. Chattarjee and S. Rana, *Complex dynamics of a discrete time one prey two predator system with prey refuge*, Int. J. Appl. Comput. Math., 2025, 11, 86.
- [8] Q. Din, *Complexity and chaos control in a discrete-time prey-predator model*, Commun. Nonlinear Sci. Numer. Simulat., 2017, 49, 113–134.
- [9] D. Hu and H. Cao, *Bifurcation and chaos in a discrete-time predator-prey system of Holling and Leslie type*, Commun. Nonlinear Sci. Numer. Simulat., 2015, 22, 702–715.
- [10] C. Arancibia-Ibarra and J. Flores, *Dynamics of a Leslie-Gower predator-prey model with Holling type II functional response, Allee effect and a generalist predator*, Mathematics and Computers in Simulation, 2021, 188, 1–22.
- [11] M. A. Khan and Q. Din, *Codimension-two bifurcation in a class of a discrete-time predator-prey interaction with cannibalism*, Qualitative Theory of Dynamical Systems, 2025, 24, 67.
- [12] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, vol. 112, 3rd edn, Springer, New York, 2004.
- [13] Y. A. Kuznetsov and H. G. E. Meijer, *Numerical normal forms for codim 2 bifurcations of fixed points with at most two critical eigenvalues*, SIAM J. Sci. Comput., 2005, 26(6), 1932–1954.
- [14] X. Liu and D. Xiao, *Complex dynamic behaviors of discrete-time predator-prey system*, Chaos, Solitons and Fractals, 2007, 32, 80–94.
- [15] K. Mokni, H. Ben Ali, B. Ghosh and M. Ch-Chaoui, *Nonlinear dynamics of a Darwinian Ricker system with strong Allee effect and immigration*, Math. Comput. Simul., 2025, 229, 789–813.
- [16] K. Mokni and M. Ch-Chaoui, *Resonant bifurcations and complex dynamics in a coupled logistic prey–predator map with proportional immigration*, Chaos, 2026, 36, 063118.
- [17] K. Mokni and M. Ch-Chaoui, *Dynamics and bifurcation analysis of a discrete-time prey-predator model with ratio-dependent Holling type III response and strong Allee effect*, International Journal of Biomathematics, 2026, 2650061.

- [18] H. Mouhsine, et al., *Dynamicsunveiled: Investigating bifurcations and the strong Allee effect in a discrete-time prey-predator system*, International Journal of Dynamics and Control, 2025, 13, 324.
- [19] S. Al-Nassir, *Dynamic analysis of a harvested fractional-order biological system with its discretization*, Chaos, Solitons and Fractals, 2021, 152, 111308.
- [20] A. Rojas-Palma and E. Gonzalez-Olivares, *Optimal harvesting in a predator-prey model with Allee efect and sigmoid functional response*, Appl. Math. Model., 2012, 36(5), 1864–1874.
- [21] H. Rouah, *Bifurcation analysis of a nonlinear discrete prey-predator system*, Rendiconti del Circolo Matematico di Palermo Series 2, 2026, 75, 94.
- [22] P. K. Santra and G. S. Mahapatra, *Dynamical study of discrete prey-predator system incorporating proportional prey refuge with interval parameters*, Appl. Math. J. Chinese Univ., 2025, 40(2), 276–296.
- [23] A. Singh and P. Deolia, *Dynamical analysis and chaos control in discrete-time prey-predator model*, Commun. Nonlinear Sci. Numer. Simulat., 2020, 90, 105313.
- [24] K. Vishwakarma and M. Sen, *Role of Allee effect in prey and hunting cooperation in a generalist predator*, Mathematics and Computers in Simulation, 2021, 190, 622–640.
- [25] L. Zhang, et al., *Dynamic complexities in a discrete predator-prey system with lower critical point for the prey*, Mathematics and Computers in Simulation, 2014, 105, 119–131.
- [26] L. Zhang, C. Zhang and Z. He, *Codimension-one and codimension-two bifurcations of a discrete predator-prey system with strong Allee effect*, Mathematics and Computers in Simulation, 2019, 162, 155–178.

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