

EXISTENCE OF THREE SOLUTIONS FOR DISCRETE BOUNDARY VALUE PROBLEMS WITH CONTINUOUS-TIME*

Junjie Mao¹, Wentao Huang^{1,2} and Feng Xiong^{1,2,†}

Abstract Using critical point theory, we establish sufficient conditions for the existence of at least three solutions to discrete boundary value problems with continuous-time. Under suitable assumptions on the nonlinearity, we obtain the existence of three positive solutions via the strong maximum principle. To the best of our knowledge, this is the first study of such discrete boundary value problems. Finally, three concrete examples illustrate the main results.

Keywords Three solutions, boundary value problems, continuous-time, critical point theory, strong maximum principle.

MSC(2010) 39A27, 39A12.

1. Introduction

In this paper, we use the standard notations $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ to represent the sets of real numbers, integers, and positive integers, respectively. For any integers a and b satisfying $a < b$, we denote the discrete interval by $\mathbb{Z}(a, b) = \{a, a + 1, \dots, b\}$.

We investigate the following boundary value problems:

$$\begin{cases} -u(s+1) + 2u(s) - u(s-1) = \lambda f(s, u(s)), & s \in [1, N), \\ u(s) = 0, & s \in [0, 1) \cup [N, N+1). \end{cases} \quad (1.1)$$

Here, $N \in \mathbb{N}$ with $N \geq 2$, and $[1, N)$ denotes the half-open interval of real numbers from 1 to N . The parameter $\lambda \in \mathbb{R}$, and $f(s, u)$ is continuous with respect to u for each $s \in [1, N)$.

In recent years, the study of nonlinear difference equations has attracted increasing attention, as many research fields are developing discrete models to facilitate investigation, such as fluid dynamics, petroleum engineering, biology, econometrics and computational mechanics [8, 12, 14, 18, 27].

Historically, difference equations were predominantly investigated via fixed point theory and the method of upper and lower solution techniques, yielding excellent results [7, 17, 20, 24, 36]. In 2003, Guo and Yu [15] investigated the following second-order superlinear difference equations

[†]The corresponding author.

¹College of Mathematics and Statistics, Guangxi Normal University, 541006 Guilin, China

²Center for Applied Mathematics of Guangxi, Guangxi Normal University, 541006 Guilin, China

*The authors were supported by the National Natural Science Foundation of China (No. 12561026), Guangxi basic ability promotion program for young and middle-aged teachers (No. 2025KY0092) and Green talent inclusive support policy research program (No. RZ2400004152).

Email: elmsley@stu.gxnu.edu.cn (J. Mao), huangwentao@163.com (W. Huang), xiongfengq1@163.com (F. Xiong)

by applying critical point theory:

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z}, \quad (1.2)$$

and obtained the existence of periodic and subharmonic solutions.

Since then, a growing number of researchers have employed critical point theory to study nonlinear difference equations, conducting extensive research and obtaining numerous results [11, 16, 19, 25, 26, 31–35, 38].

In particular, several researchers [1–5, 13, 21, 23, 37] investigated the following boundary value problems when $\alpha = \beta = 0$ and $\alpha = 0, \beta = 1$:

$$\begin{cases} -\Delta^2 x_{k-1} = \lambda f(k, x_k), \\ x_0 = \alpha x_1, \quad x_{T+1} = \beta x_T, \end{cases} \quad (1.3)$$

and obtained various existence results for problem (1.3). To establish the existence of solutions, they employed a wide range of analytical tools, including critical point theory, the method of upper and lower solutions, the Leray-Schauder nonlinear alternative, and various cone fixed point theorems. However, they did not consider the case involving continuous-time.

In 2018, Balanov et al. [6] proposed the following continuous-time difference equation:

$$-\Delta^2 x(t-1) = f(t, x(t)), \quad t \in \mathbb{R}, \quad (1.4)$$

as an open problem. Due to the challenges of establishing an appropriate variational framework in a suitable working space, there has been a noticeable lack of literature on discrete boundary value problems with continuous-time. Recently, Lin et al. [28] combined critical point theory with gluing arguments and established existence and multiplicity results for periodic solutions of problem (1.4). This opened a new avenue for studying continuous-time difference equations through critical point theory and therefore motivates us to investigate discrete boundary value problems with continuous-time. In many natural science disciplines, continuous-time difference equations are commonly employed to characterize observed evolutionary dynamics [29, 30]. They hold both considerable theoretical value and substantial practical relevance. The main contributions and innovative aspects of this work are outlined below:

- (1) Compared to the studies in [1–5, 13, 21, 23, 37], we establish the existence of three solutions for discrete BVPs with continuous-time.
- (2) Regarding problem (1.1) without continuous-time, the existence of three solutions has been established [10]. In contrast, as illustrated by the examples, the solutions to the continuous-time difference problem investigated here are not merely isolated points, but functions defined over continuous intervals.
- (3) Moreover, we construct several illustrative examples and provide graphical representations of their solutions to visually verify our theoretical findings.

This paper is structured as follows. In Section 2, we outline the necessary preliminaries. In Section 3, we state and prove our main theorems. In Section 4, we provide three illustrative examples to demonstrate the main results. In Section 5, we conclude the work with remarks and suggestions for future research.

2. Preliminaries

For a sequence x , denote $\Delta x(p) = x(p+1) - x(p)$, $\Delta^2 x(p-1) = \Delta x(p) - \Delta x(p-1)$. Let $X = \{x : \mathbb{Z}(0, N) \rightarrow \mathbb{R} : x(0) = x(N) = 0\}$ be the $(N-1)$ -dimensional Banach space. We equip this space with the norm

$$\|x\| := \left(\sum_{p=0}^{N-1} (\Delta x(p))^2 \right)^{\frac{1}{2}},$$

along with the maximum norm defined by

$$\|x\|_{\infty} := \max_{p \in \mathbb{Z}(1, N-1)} |x(p)|.$$

For each $t \in [0, 1)$, define

$$\Phi(x) = \frac{1}{2} \sum_{p=0}^{N-1} (\Delta x(p))^2, \quad \Psi(x) = \sum_{p=1}^{N-1} F(t+p, x(p)),$$

where $F(t+p, u) = \int_0^u f(t+p, s) ds$ for each $(p, u) \in \mathbb{Z}(1, N-1) \times \mathbb{R}$. Denote

$$J_t(x) = \Phi(x) - \lambda \Psi(x).$$

It is readily seen that $\Phi, \Psi \in C^1(X, \mathbb{R})$, which means they are continuously Gâteaux differentiable. Using the boundary conditions, we obtain

$$\begin{aligned} \Phi'(x)(y) &= \lim_{\delta \rightarrow 0} \frac{\Phi(x + \delta y) - \Phi(x)}{\delta} \\ &= \sum_{p=0}^{N-1} \Delta x(p) \Delta y(p) \\ &= \sum_{p=1}^{N-1} \Delta x(p-1) y(p) - \sum_{p=1}^{N-1} \Delta x(p) y(p) \\ &= \sum_{p=1}^{N-1} -\Delta^2 x(p-1) y(p), \end{aligned}$$

and

$$\Psi'(x)(y) = \lim_{\delta \rightarrow 0} \frac{\Psi(x + \delta y) - \Psi(x)}{\delta} = \sum_{p=1}^{N-1} f(t+p, x(p)) y(p),$$

for any $x, y \in X$.

For each fixed $t \in [0, 1)$, the parameter t selects one discrete section of the continuous-time problem (1.1). Since $y \in X$ is arbitrary on $\mathbb{Z}(1, N-1)$, the critical point condition $J'_t(x)(y) = 0$ for all $y \in X$ is equivalent to

$$-x(p+1) + 2x(p) - x(p-1) = \lambda f(t+p, x(p)), \quad p \in \mathbb{Z}(1, N-1).$$

This gives the following discrete Dirichlet boundary value problem:

$$\begin{cases} -x(p+1) + 2x(p) - x(p-1) = \lambda f(t+p, x(p)), & p \in \mathbb{Z}(1, N-1), \\ x(0) = x(N) = 0. \end{cases} \quad (2.1)$$

Thus, for each fixed $t \in [0, 1)$, the critical points of J_t are precisely the solutions of (2.1). Notice that (2.1) is not independent of (1.1); rather, it is obtained from (1.1) by writing $s = t + p$, where $t \in [0, 1)$ and $p \in \mathbb{Z}(1, N - 1)$. In this correspondence, the discrete boundary conditions $x(0) = x(N) = 0$ match exactly the continuous-time boundary conditions $u(s) = 0$ on $[0, 1) \cup [N, N + 1)$.

Conversely, suppose that for each $t \in [0, 1)$ we choose a solution $x_t \in X$ of (2.1). Define $u : [0, N + 1) \rightarrow \mathbb{R}$ by

$$u(s) = \begin{cases} x_t(p), & s = t + p, \quad t \in [0, 1), \quad p \in \mathbb{Z}(1, N - 1), \\ 0, & s \in [0, 1) \cup [N, N + 1). \end{cases}$$

For every $s \in [1, N)$, there are unique $t \in [0, 1)$ and $p \in \mathbb{Z}(1, N - 1)$ such that $s = t + p$. Using (2.1), we have

$$-u(s + 1) + 2u(s) - u(s - 1) = -x_t(p + 1) + 2x_t(p) - x_t(p - 1) = \lambda f(s, u(s)).$$

Together with $u(s) = 0$ on $[0, 1) \cup [N, N + 1)$, this shows that u is a solution of problem (1.1).

The results presented below are essential for our analysis.

Lemma 2.1 (Proposition 2.2, [10]). *Fix $x \in X$, such that either*

$$x(p) > 0 \quad \text{or} \quad -x(p + 1) + 2x(p) - x(p - 1) \geq 0,$$

for each $p \in \mathbb{Z}(1, N - 1)$. Then, either $x \equiv 0$ or $x(p) > 0$ for each $p \in \mathbb{Z}(1, N - 1)$.

Lemma 2.2 (Lemma 2.2, [22]). *For any $x \in X$, the following inequality holds:*

$$\|x\|_\infty \leq \frac{\sqrt{N}}{2} \|x\|.$$

3. Main results

Denote

$$F_q(t) = \sum_{p=1}^{N-1} F(t + p, q), \quad \forall q \in \mathbb{R}, \quad t \in [0, 1).$$

Theorem 3.1. *Let c and d be two positive constants subject to the inequality*

$$Nd^2 > 2c^2, \tag{3.1}$$

such that

(A₁) $f(s, u) \geq 0$ for each $(s, u) \in [1, N) \times [-c, c]$;

(A₂) $Nd^2 F_c < 2c^2 F_d$;

(A₃) Let $\beta_1 \geq 0$ be a constant satisfying

$$\beta_1 = \limsup_{|u| \rightarrow \infty} \sup_{\substack{t \in [0, 1) \\ p \in \mathbb{Z}(1, N-1)}} \frac{F(t + p, u)}{|u|^2}.$$

Moreover, it satisfies

$$\frac{d^2}{F_d} < \frac{2}{(N - 1)N\beta_1}.$$

Furthermore, put

$$\tau_1 = \frac{d^2}{F_d}, \quad \tau_2 = \min \left\{ \frac{2c^2}{NF_c}, \frac{2}{(N-1)N\beta_1} \right\}.$$

It follows that problem (1.1) possesses at least three different solutions provided that $\lambda \in \Lambda := (\tau_1, \tau_2)$, where $F_d = \inf_{t \in [0,1]} F_d(t)$, $F_c = \sup_{t \in [0,1]} F_c(t)$.

Proof. The proof relies on applying Theorem 4.1 of [10]. Retaining the definitions of the space X and the functionals Φ , Ψ , and J_t from Section 2, it is evident that Φ and Ψ are continuously Gâteaux differentiable, satisfying $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Fix $\lambda \in \Lambda$ and choose the constant r as

$$r = \frac{2c^2}{N}.$$

Recalling the definition $\Phi(x) = \frac{1}{2}\|x\|^2$, for any $x \in X$ satisfying $\Phi(x) \leq r$, it follows that $\|x\|^2 \leq \frac{4c^2}{N}$. By virtue of the inequality $\|x\|_\infty \leq \frac{\sqrt{N}}{2}\|x\|$, we deduce that $\|x\|_\infty \leq c$. Consequently,

$$\begin{aligned} \frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} &\leq \frac{\max_{\|x\|_\infty \leq c} \sum_{p=1}^{N-1} F(t+p, x(p))}{r} \\ &\leq \frac{N \sum_{p=1}^{N-1} \max_{|u| \leq c} F(t+p, u)}{2c^2} \\ &\leq \frac{NF_c}{2c^2}. \end{aligned}$$

Now, let $\bar{x} \in X$ be given by

$$\bar{x}(p) = \begin{cases} d, & p \in \mathbb{Z}(1, N-1), \\ 0, & p = 0 \text{ or } p = N. \end{cases}$$

We have

$$\Phi(\bar{x}) = \frac{1}{2}\|\bar{x}\|^2 = d^2 > r,$$

and

$$\frac{\Psi(\bar{x})}{\Phi(\bar{x})} = \frac{F_d(t)}{\frac{1}{2}\|\bar{x}\|^2} \geq \frac{F_d}{d^2}.$$

Subsequently, we establish that J_t is coercive, that is,

$$\lim_{\|x\| \rightarrow +\infty} J_t = +\infty.$$

Since $\beta_1 \geq 0$ and $\lambda < \frac{2}{(N-1)N\beta_1}$, there exists a real number $l \in (\beta_1, \frac{2}{(N-1)N\lambda})$. By means of the definition of β_1 , we get

$$F(t+p, u) \leq l|u|^2 + C(p),$$

where $C(p) \geq 0$ for all $p \in \mathbb{Z}(1, N-1)$. This yields

$$\Psi(x) = \sum_{p=1}^{N-1} F(t+p, x(p)) \leq l \sum_{p=1}^{N-1} |x(p)|^2 + C \leq l(N-1)\|x\|_\infty^2 + C,$$

where $C = \sum_{p=1}^{N-1} C(p)$.

Now, we obtain

$$\begin{aligned} J_t(x) &= \Phi(x) - \lambda\Psi(x) \\ &= \frac{1}{2}\|x\|^2 - \lambda\Psi(x) \\ &\geq \frac{2}{N}\|x\|_\infty^2 - \lambda l(N-1)\|x\|_\infty^2 - \lambda C \\ &= \left(\frac{2}{N} - \lambda l(N-1)\right)\|x\|_\infty^2 - \lambda C. \end{aligned}$$

For $l < \frac{2}{(N-1)N\lambda}$, we know $\frac{2}{N} - \lambda l(N-1) > 0$. It means $J_t(x) \rightarrow +\infty$ when $\|x\| \rightarrow +\infty$. From the above discussion, it follows that J_t is coercive.

For each fixed $t \in [0, 1)$, problem (2.1) admits at least three distinct solutions. Denote them by

$$x_t^{(1)}, \quad x_t^{(2)}, \quad x_t^{(3)}.$$

For $i = 1, 2, 3$, define $u_i : [0, N+1) \rightarrow \mathbb{R}$ by

$$u_i(s) = \begin{cases} x_t^{(i)}(p), & s = t + p, \quad t \in [0, 1), \quad p \in \mathbb{Z}(1, N-1), \\ 0, & s \in [0, 1) \cup [N, N+1). \end{cases}$$

Then each u_i satisfies problem (1.1). Moreover, since $x_t^{(1)}, x_t^{(2)}, x_t^{(3)}$ are distinct for each fixed $t \in [0, 1)$, the functions u_1, u_2, u_3 are distinct. Consequently, problem (1.1) admits at least three distinct solutions. $\square$

Remark 3.1. If $\beta_1 = 0$, that is, $\frac{2}{(N-1)N\beta_1} \rightarrow +\infty$, then $\tau_2 = \frac{2c^2}{NF_c}$.

For fixed $t \in [0, 1)$, let

$$F^+(t+p, u) = \int_0^u f(t+p, \xi^+) d\xi, \quad (p, u) \in \mathbb{Z}(1, N-1) \times \mathbb{R},$$

where $\xi^+ = \max\{\xi, 0\}$, and define

$$J_t^+(x) := \Phi(x) - \lambda\Psi^+(x), \quad \Psi^+(x) = \sum_{p=1}^{N-1} F^+(t+p, x(p)).$$

Here, we give a problem whose solution is associated with the critical point of J_t^+ :

$$\begin{cases} -x(p+1) + 2x(p) - x(p-1) = \lambda f(t+p, x^+(p)), & p \in \mathbb{Z}(1, N-1), \\ x(0) = x(N) = 0. \end{cases} \quad (3.2)$$

Corollary 3.1. Assume that condition (A_1) is strengthened to $f(s, u) > 0, (s, u) \in [1, N) \times \mathbb{R}$ while all the other assumptions remain unchanged. With each $\lambda \in \Lambda$, problem (1.1) admits at least three positive solutions.

Proof. Under the assumptions of Theorem 3.1, problem (3.2) possesses three distinct solutions. Since

$$-x(p+1) + 2x(p) - x(p-1) = \lambda f(t+p, x^+(p)) > 0,$$

it is clear that $x = 0$ is not a solution to problem (3.2). By Lemma 2.1, every non-zero solution of problem (3.2) is positive in $\mathbb{Z}(1, N-1)$, and hence it is a positive solution of (2.1).

Thus, for each fixed $t \in [0, 1)$, problem (2.1) admits at least three positive solutions. Applying the same construction used in the proof of Theorem 3.1, we obtain at least three positive solutions of problem (1.1). $\square$

Next, we present another theorem establishing the existence of three solutions.

Theorem 3.2. *Let c_1, c_2 , and d be positive constants satisfying the inequalities*

$$2c_1^2 < Nd^2 < c_2^2, \quad (3.3)$$

such that

$$(B_1) \quad f(s, u) \geq 0, \quad (s, u) \in [1, N) \times \mathbb{R};$$

$$(B_2) \quad N \max \left\{ \frac{F_{c_1}}{2c_1^2}, \frac{F_{c_2}}{c_2^2} \right\} < \frac{F_d}{2d^2}.$$

It follows that problem (1.1) possesses at least three solutions provided that

$$\lambda \in \Lambda_1 := \left(\frac{2d^2}{F_d}, \min \left\{ \frac{2c_1^2}{NF_{c_1}}, \frac{c_2^2}{NF_{c_2}} \right\} \right),$$

where $F_d = \inf_{t \in [0, 1)} F_d(t)$, $F_{c_1} = \sup_{t \in [0, 1)} F_{c_1}(t)$, $F_{c_2} = \sup_{t \in [0, 1)} F_{c_2}(t)$.

Proof. Since $\Phi(x) = \frac{1}{2} \sum_{p=0}^{N-1} (\Delta x(p))^2 = \frac{1}{2} \|x\|^2$, we see that Φ is coercive.

Notice that

$$\frac{a^2 + b^2}{2} - \left(\frac{a+b}{2} \right)^2 = \left(\frac{a-b}{2} \right)^2 \geq 0.$$

Thus, it follows that

$$\begin{aligned} \Phi\left(\frac{x_1 + x_2}{2}\right) &= \frac{1}{2} \sum_{p=0}^{N-1} \left(\frac{\Delta x_1(p) + \Delta x_2(p)}{2} \right)^2 \\ &\leq \frac{1}{2} \sum_{p=0}^{N-1} \frac{(\Delta x_1(p))^2 + (\Delta x_2(p))^2}{2} \\ &= \frac{1}{2} (\Phi(x_1) + \Phi(x_2)). \end{aligned}$$

Consequently, Φ is a convex functional. Moreover, it is easy to see

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Thus, the assumptions of Corollary 3.1 of [9] regarding the properties of Φ and Ψ have been verified.

Let x_1 and x_2 be two local minima of J_t such that

$$\Psi(x_1) \geq 0, \quad \Psi(x_2) \geq 0.$$

Since local minima of J_t are critical points, x_1 and x_2 satisfy

$$-x_i(p+1) + 2x_i(p) - x_i(p-1) = \lambda f(t+p, x_i(p)), \quad i = 1, 2.$$

Then, by Lemma 2.1 and (B_1) , we have $x_1(p), x_2(p) \geq 0$, $p \in \mathbb{Z}(1, N-1)$. Hence, for every $\alpha \in [0, 1]$,

$$\alpha x_1 + (1 - \alpha)x_2 \geq 0.$$

Since $f(s, u) \geq 0$, it follows that

$$F(s, u) = \int_0^u f(s, \xi) d\xi \geq 0, \quad u \geq 0.$$

Therefore,

$$\Psi(\alpha x_1 + (1 - \alpha)x_2) \geq 0, \quad \alpha \in [0, 1],$$

and consequently

$$\inf_{\alpha \in [0, 1]} \Psi(\alpha x_1 + (1 - \alpha)x_2) \geq 0.$$

Thus condition (H_3) holds.

Let

$$\rho_1 = \frac{2c_1^2}{N}, \quad \rho_2 = \frac{2c_2^2}{N}.$$

When $\|x\| \leq \sqrt{2\rho_1}$, by Lemma 2.2, we have

$$\|x\|_\infty \leq \frac{\sqrt{N}}{2} \sqrt{2\rho_1} = c_1.$$

Similarly, it holds that

$$\|x\|_\infty \leq \frac{\sqrt{N}}{2} \sqrt{2\rho_2} = c_2,$$

with $\|x\| \leq \sqrt{2\rho_2}$. Similarly, we have

$$\begin{aligned} \frac{\sup_{x \in \Phi^{-1}(-\infty, \rho_1)} \Psi(x)}{\rho_1} &\leq \frac{\sup_{\|x\| \leq \sqrt{2\rho_1}} \sum_{p=1}^{N-1} F(t+p, x(p))}{\rho_1} \\ &\leq \frac{\sum_{p=1}^{N-1} F(t+p, c_1)}{\rho_1} \\ &= \frac{N}{2c_1^2} \sum_{p=1}^{N-1} F(t+p, c_1) \\ &\leq \frac{N}{2c_1^2} F_{c_1}, \end{aligned}$$

and

$$\begin{aligned} \frac{\sup_{x \in \Phi^{-1}(-\infty, \rho_2)} \Psi(x)}{\rho_2} &\leq \frac{\sup_{\|x\| \leq \sqrt{2\rho_2}} \sum_{p=1}^{N-1} F(t+p, x(p))}{\rho_2} \\ &\leq \frac{\sum_{p=1}^{N-1} F(t+p, c_2)}{\rho_2} \\ &= \frac{N}{2c_2^2} \sum_{p=1}^{N-1} F(t+p, c_2) \\ &\leq \frac{N}{2c_2^2} F_{c_2}. \end{aligned}$$

Let $\bar{v} \in X$ be given by

$$\bar{v}(p) = \begin{cases} d, & p \in \mathbb{Z}(1, N-1), \\ 0, & p = 0 \text{ or } p = N. \end{cases}$$

Clearly, $\Phi(\bar{v}) = \frac{1}{2} \sum_{p=0}^{N-1} (\Delta \bar{v}(p))^2 = d^2$. From (3.3), we have

$$\rho_1 < \Phi(\bar{v}) < \frac{\rho_2}{2}.$$

By Corollary 3.1 of [9], for any $\lambda \in \left(\frac{2d^2}{F_d}, \min\left\{ \frac{2c_1^2}{NF_{c_1}}, \frac{c_2^2}{NF_{c_2}} \right\} \right)$, problem (2.1) has at least three solutions.

Therefore, for each fixed $t \in [0, 1)$, problem (2.1) has at least three solutions. Applying the same construction used in the proof of Theorem 3.1, problem (1.1) admits at least three solutions. $\square$

Following the same proof as for Corollary 3.1, we obtain the following corollary.

Corollary 3.2. *If we assume that $f(s, u) > 0$ for any $(s, u) \in [1, N) \times \mathbb{R}$ while all the other assumptions of Theorem 3.2 remain unchanged, problem (1.1) admits at least three positive solutions.*

Proof. We omit the proof, as it is identical to that of Corollary 3.1. $\square$

To simplify the conditions of Corollary 3.2, we impose assumptions on the nonlinear term. Let

$$f(s, u) = g(s)h(u),$$

for all $(s, u) \in [1, N) \times \mathbb{R}$. We consider the following boundary value problems:

$$\begin{cases} -\Delta^2 u(s-1) = \lambda g(s)h(u(s)), & s \in [1, N), \\ u(s) = 0, & s \in [0, 1) \cup [N, N+1), \end{cases} \quad (3.4)$$

where g, h are positive continuous functions. This leads to the following result.

Corollary 3.3. *Let c_1, c_2 , and d be positive constants satisfying (3.3). Define*

$$H(q) = \int_0^q h(\xi) d\xi, \quad q > 0,$$

and

$$G_* = \inf_{t \in [0, 1)} \sum_{p=1}^{N-1} g(t+p), \quad G^* = \sup_{t \in [0, 1)} \sum_{p=1}^{N-1} g(t+p).$$

Assume that

$$(B'_1) \quad NG^* \max \left\{ \frac{H(c_1)}{2c_1^2}, \frac{H(c_2)}{c_2^2} \right\} < \frac{G_* H(d)}{2d^2}.$$

Then, for each

$$\lambda \in \Lambda'_1 := \left(\frac{2d^2}{G_* H(d)}, \frac{1}{NG^*} \min \left\{ \frac{2c_1^2}{H(c_1)}, \frac{c_2^2}{H(c_2)} \right\} \right),$$

problem (3.4) admits at least three positive solutions.

4. Examples

To illustrate our main results, we present three examples.

Example 4.1. Let $N = 5$. Define

$$h(u) = \begin{cases} \frac{1}{1000}, & u \leq \frac{1}{10}, \\ \frac{111}{100}u - \frac{11}{100}, & \frac{1}{10} < u \leq 1, \\ u, & 1 < u \leq 6, \\ 24u - 138, & 6 < u \leq 7, \\ 30, & u > 7. \end{cases}$$

For $(s, u) \in [1, 5] \times \mathbb{R}$, define

$$f(s, u) = \begin{cases} \frac{1}{2}h(u), & s \in [1, 2) \cup [4, 5], \\ \frac{1}{3}h(u), & s \in [2, 4]. \end{cases}$$

Then $f(s, u) > 0$ for all $(s, u) \in [1, 5] \times \mathbb{R}$, and f is continuous with respect to u .

Let

$$H(u) = \int_0^u h(\xi) d\xi.$$

Since h is bounded as $|u| \rightarrow \infty$, we have $\beta_1 = 0$. Take $c = 1/10, d = 7$. A direct calculation gives

$$H(c) = 10^{-4}, \quad H(d) = 35.95055.$$

Hence

$$F_c = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \right) H(c) = \frac{1}{6000}, \quad F_d = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \right) H(d) \approx 59.9176.$$

Moreover,

$$Nd^2 = 245 > \frac{1}{50} = 2c^2,$$

and

$$Nd^2 F_c \approx 0.0408 < 1.1984 \approx 2c^2 F_d.$$

Thus condition (A_2) holds. Since $\beta_1 = 0$,

$$\tau_1 = \frac{d^2}{F_d} \approx 0.8178, \quad \tau_2 = \frac{2c^2}{NF_c} = 24.$$

Therefore, for each

$$\lambda \in (0.8178, 24),$$

problem (1.1) admits at least three positive solutions by Corollary 3.1.

In particular, choose $\lambda = 1$. For $k = 1, 2, 3$, define

$$u_k(s) = \begin{cases} 0, & s \in [0, 1) \cup [5, 6), \\ k[1 + (s-1)(2-s)], & s \in [1, 2), \\ \frac{3}{2}k[1 + (s-2)(3-s)], & s \in [2, 3), \\ \frac{3}{2}k[1 + (s-3)(4-s)], & s \in [3, 4), \\ k[1 + (s-4)(5-s)], & s \in [4, 5). \end{cases}$$

It is easy to see that

$$1 \leq u_k(s) \leq \frac{45}{8} < 6, \quad s \in [1, 5), \quad k = 1, 2, 3.$$

Thus

$$f(s, u_k(s)) = \begin{cases} \frac{1}{2}u_k(s), & s \in [1, 2) \cup [4, 5), \\ \frac{1}{3}u_k(s), & s \in [2, 4). \end{cases}$$

on $s \in [1, 5)$. Moreover, by writing $s = t + p$, $t \in [0, 1)$, $p = 1, 2, 3, 4$, a direct calculation gives

$$-u_k(t + p + 1) + 2u_k(t + p) - u_k(t + p - 1) = f(t + p, u_k(t + p)).$$

Therefore,

$$-u_k(s + 1) + 2u_k(s) - u_k(s - 1) = \lambda f(s, u_k(s)), \quad s \in [1, 5).$$

Together with the boundary condition $u_k(s) = 0, s \in [0, 1) \cup [5, 6)$, we conclude that u_1, u_2, u_3 are three positive solutions of problem (1.1).

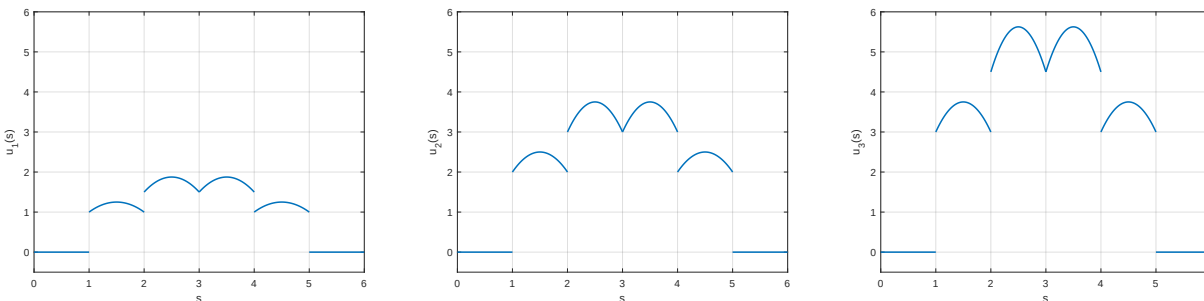


Figure 1. Three solutions in Example 4.1.

Next, let us discuss another example.

Example 4.2. Let $N = 3$ and define

$$f(t, u) = h(u), \quad (t, u) \in [1, 3) \times \mathbb{R},$$

where

$$h(u) = \begin{cases} 0, & u \leq \frac{1}{2}, \\ 2u - 1, & \frac{1}{2} < u < 1, \\ u, & 1 \leq u \leq 4, \\ 56u - 220, & 4 < u < 5, \\ -60u + 360, & 5 \leq u < 6, \\ 0, & u \geq 6. \end{cases}$$

It is clear that f is continuous and

$$f(t, u) \geq 0, \quad (t, u) \in [1, 3] \times \mathbb{R}.$$

Let

$$F(u) = \int_0^u h(s) ds.$$

Take $c_1 = 3/5, d = 5, c_2 = 80$. Then

$$2c_1^2 = \frac{18}{25} < Nd^2 = 75 < c_2^2 = 6400.$$

Since f is independent of t , for any $q \in \mathbb{R}$ we have

$$F_q(t) = \sum_{p=1}^{N-1} F(t+p, q) = 2F(q).$$

Hence

$$F_{c_1} = 2F\left(\frac{3}{5}\right) = \frac{1}{50}, \quad F_d = 2F(5) = \frac{159}{2}, \quad F_{c_2} = 2F(80) = 2F(6) = \frac{279}{2}.$$

Thus,

$$N \max \left\{ \frac{F_{c_1}}{2c_1^2}, \frac{F_{c_2}}{c_2^2} \right\} = 3 \max \left\{ \frac{1}{36}, \frac{279}{12800} \right\} = \frac{1}{12} < \frac{F_d}{2d^2} = \frac{159}{100},$$

which verifies condition (B_2) .

By Theorem 3.2, problem (1.1) admits at least three solutions for

$$\lambda \in \Lambda_1 = \left(\frac{2d^2}{F_d}, \min \left\{ \frac{2c_1^2}{NF_{c_1}}, \frac{c_2^2}{NF_{c_2}} \right\} \right).$$

Moreover,

$$\frac{2d^2}{F_d} = \frac{100}{159} \approx 0.6289, \quad \frac{2c_1^2}{NF_{c_1}} = 12, \quad \frac{c_2^2}{NF_{c_2}} = \frac{12800}{837} \approx 15.2927.$$

Consequently,

$$\Lambda_1 = (0.6289, 12).$$

Now we choose $\lambda = 1 \in \Lambda_1$. Let

$$a(s) = 1 + \frac{1}{5} \sin^2(\pi s).$$

For $k = 1, 2, 3$, define

$$u_k(s) = \begin{cases} k a(s), & s \in [1, 3], \\ 0, & s \in [0, 1) \cup [3, 4]. \end{cases}$$

Since

$$a(s+1) = a(s),$$

and

$$1 \leq k a(s) \leq \frac{6k}{5} \leq \frac{18}{5} < 4, \quad k = 1, 2, 3,$$

we have

$$h(ka(s)) = ka(s), \quad k = 1, 2, 3.$$

Using $a(s+1) = a(s)$ and the boundary condition $u_k(s) = 0$ on $[0, 1) \cup [3, 4)$, we obtain

$$-u_k(s+1) + 2u_k(s) - u_k(s-1) = ka(s) = h(ka(s)) = f(s, u_k(s)), \quad s \in [1, 3].$$

Since $\lambda = 1$, it follows that

$$-u_k(s+1) + 2u_k(s) - u_k(s-1) = \lambda f(s, u_k(s)), \quad s \in [1, 3].$$

Together with

$$u_k(s) = 0, \quad s \in [0, 1) \cup [3, 4),$$

we conclude that u_1, u_2, u_3 are three solutions of problem (1.1).

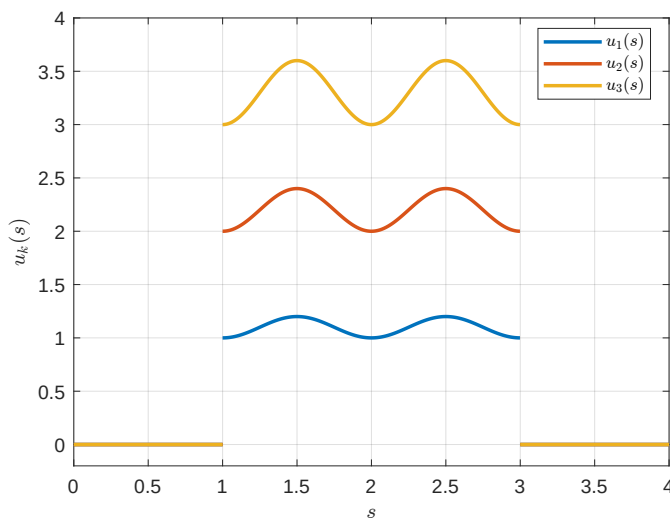


Figure 2. Three solutions in Example 4.2.

Finally, we give an example.

Example 4.3. Let $N = 3$, $g(s) \equiv 1$ and define

$$h(u) = \begin{cases} 10^{-3}, & u \leq 1, \\ 10^{-3} + 1.999(u-1) + 100 \sin^2[\pi(u-1)], & 1 < u \leq 2, \\ u, & 2 < u \leq 20, \\ 20e^{-(u-20)}, & u > 20. \end{cases}$$

For $(s, u) \in [1, 3] \times \mathbb{R}$, let

$$f(s, u) = g(s)h(u) = h(u).$$

Then g and h are positive continuous functions. Let

$$H(u) = \int_0^u h(\xi) d\xi.$$

Choose $c_1 = 1/2$, $d = 2$, $c_2 = 100$. Clearly,

$$2c_1^2 = \frac{1}{2} < Nd^2 = 12 < c_2^2 = 10000, \quad \sum_{p=1}^{N-1} g(t+p) = 2.$$

Moreover,

$$H\left(\frac{1}{2}\right) = 0.0005, \quad H(2) = 51.0015, \quad H(100) = 269.0015 - 20e^{-80} \approx 269.0015.$$

Hence

$$3 \cdot 2 \max \left\{ \frac{H(c_1)}{2c_1^2}, \frac{H(c_2)}{c_2^2} \right\} \approx 0.1614 < 2 \cdot \frac{H(d)}{2d^2} \approx 12.7504.$$

Thus condition (B'_1) is satisfied. Furthermore,

$$\Lambda'_1 = \left(\frac{2d^2}{2H(d)}, \frac{1}{3 \cdot 2} \min \left\{ \frac{2c_1^2}{H(c_1)}, \frac{c_2^2}{H(c_2)} \right\} \right) \approx (0.0784, 6.1957).$$

Taking $\lambda = 1 \in \Lambda'_1$, Corollary 3.3 implies that problem (3.4) admits at least three positive solutions.

We now give three explicit positive solutions. Let

$$\omega(s) = 1 + \frac{1}{4} \sin(2\pi s), \quad s \in \mathbb{R},$$

and take $A_1 = 3$, $A_2 = 7$, $A_3 = 12$. For $k = 1, 2, 3$, define

$$u_k(s) = \begin{cases} A_k \omega(s), & s \in [1, 3], \\ 0, & s \in [0, 1) \cup [3, 4]. \end{cases}$$

Since $3/4 \leq \omega(s) \leq 5/4$, we have

$$2 < u_k(s) < 20, \quad s \in [1, 3], \quad k = 1, 2, 3.$$

Therefore $h(u_k(s)) = u_k(s)$ on $[1, 3]$. Since ω is 1-periodic,

$$-u_k(s+1) + 2u_k(s) - u_k(s-1) = u_k(s) = \lambda g(s)h(u_k(s)), \quad s \in [1, 3].$$

Together with

$$u_k(s) = 0, \quad s \in [0, 1) \cup [3, 4],$$

we conclude that u_1, u_2, u_3 are three positive solutions of problem (3.4).

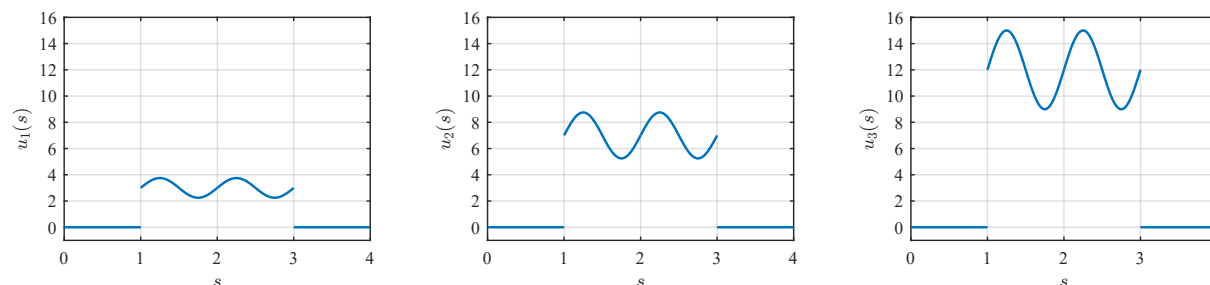


Figure 3. Three solutions in Example 4.3.

5. Conclusion

In this paper, we studied discrete boundary value problems with continuous-time. Different from previous works [1–5, 13, 21, 23, 37], the continuous-time setting has been taken into consideration for the first time. By constructing a suitable variational framework and applying critical point theory, we established sufficient conditions for the existence of at least three solutions, as detailed in Theorems 3.1 and 3.2.

Building upon the existence results, we further refined our analysis to guarantee the positivity of the solutions. By invoking the strong maximum principle, we derived Corollaries 3.1 and 3.2, which provide assurance that the obtained solutions are positive. Additionally, for the specific case where the nonlinear term involves separated variables, we provided Corollary 3.3, offering simplified and easily verifiable algebraic conditions.

To validate our theoretical findings, three illustrative examples were discussed. The examples verified the applicability of the theoretical conditions, and in particular, the graphical example illustrated that the obtained solutions are not isolated points. Future work may focus on investigating problems with additional parameters, exploring different boundary conditions, or extending the current analysis to higher-order difference equations with continuous-time.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and suggestions, which helped improve the manuscript.

References

- [1] R. P. Agarwal and D. O'Regan, *Boundary value problems for discrete equations*, Appl. Math. Lett., 1997, 10(4), 83–89.
- [2] R. P. Agarwal and D. O'Regan, *A fixed-point approach for nonlinear discrete boundary value problems*, Comput. Math. Appl., 1998, 36(10–12), 115–121.
- [3] R. P. Agarwal and D. O'Regan, *Nonpositone discrete boundary value problems*, Nonlinear Anal., 2000, 39(2), 207–215.
- [4] R. P. Agarwal, K. Perera and D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal., 2004, 58(1–2), 69–73.

- [5] F. M. Atici, *Existence of positive solutions of nonlinear discrete Sturm-Liouville problems*, Math. Comput. Model., 2000, 32(5–6), 599–607.
- [6] Z. Balanov, C. García-Azpeitia and W. Krawcewicz, *On variational and topological methods in nonlinear difference equations*, Commun. Pure Appl. Anal., 2018, 17(6), 2813–2844.
- [7] C. Bereanu and J. Mawhin, *Boundary value problems for second-order nonlinear difference equations with discrete ϕ -Laplacian and singular ϕ* , J. Differ. Equ. Appl., 2008, 14(10–11), 1099–1118.
- [8] J. E. Bolander, J. Eliáš, G. Cusatis and K. Nagai, *Discrete mechanical models of concrete fracture*, Eng. Fract. Mech., 2021, 257, 108030.
- [9] G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differ. Equations, 2008, 244(12), 3031–3059.
- [10] G. Bonanno, P. Candito and G. D’Agui, *Variational methods on finite dimensional Banach spaces and discrete problems*, Adv. Nonlinear Stud., 2014, 14(4), 915–939.
- [11] Y. Chen and Z. Zhou, *Homoclinic solutions for a discrete periodic Hamiltonian system with perturbed terms*, J. Geom. Anal., 2024, 34(12), 361.
- [12] Z. Chen and W. Yu, *A discrete model for pressure transient analysis in discretely fractured reservoirs*, SPE J., 2022, 27(3), 1708–1728.
- [13] J. R. Graef and J. Henderson, *Double solutions of boundary value problems for 2mth-order differential equations and difference equations*, Comput. Math. Appl., 2003, 45(6–9), 873–885.
- [14] T. E. Guerrero, C. A. Guevara, E. Cherchi and J. de D. Ortúzar, *Characterizing the impact of discrete indicators to correct for endogeneity in discrete choice models*, J. Choice Model., 2022, 42, 100342.
- [15] Z. Guo and J. Yu, *Existence of periodic and subharmonic solutions for second-order super-linear difference equations*, Sci. China Ser. A Math., 2003, 46(4), 506–515.
- [16] O. Hammouti, S. Taarabti and R. P. Agarwal, *Anisotropic discrete boundary value problems*, Appl. Anal. Discrete Math., 2023, 17(1), 232–248.
- [17] J. Henderson and H. B. Thompson, *Existence of multiple solutions for second-order discrete boundary value problems*, Comput. Math. Appl., 2002, 43(10–11), 1239–1248.
- [18] M. Holzer, Z. Richey, W. Rush and S. Schmidgall, *Locked fronts in a discrete time discrete space population model*, J. Math. Biol., 2022, 85(4), 39.
- [19] Z. Huang and Z. Zhou, *Positive solutions to the discrete boundary value problem involving the singular ϕ -Laplacian*, AIMS Math., 2025, 10(8), 17922–17939.
- [20] T. Jankowski, *First-order functional difference equations with nonlinear boundary value problems*, Comput. Math. Appl., 2010, 59(6), 1937–1943.
- [21] L. Jiang and Z. Zhou, *Existence of nontrivial solutions for discrete nonlinear two point boundary value problems*, Appl. Math. Comput., 2006, 180(1), 318–329.
- [22] L. Jiang and Z. Zhou, *Three solutions to Dirichlet boundary value problems for p -Laplacian difference equations*, Adv. Differ. Equ., 2008, 2008, 345916.
- [23] L. Jiang and Z. Zhou, *Multiple nontrivial solutions for a class of higher dimensional discrete boundary value problems*, Appl. Math. Comput., 2008, 203(1), 30–38.

- [24] E. Karapinar, *A short survey on the recent fixed point results on b-metric spaces*, Constr. Math. Anal., 2018, 1(1), 15–44.
- [25] J. Kuang and S. Cai, *Periodic solutions with prescribed minimal period for distributed delay differential equations*, Discrete Contin. Dyn. Syst. Ser. S, 2026, 21, 148–170.
- [26] J. Kuang and Z. Guo, *Periodic solutions with prescribed minimal period for second-order Hamiltonian systems with non-symmetric potentials*, Appl. Math. Lett., 2024, 155, 109123.
- [27] C. Lin, X. Sun, X. Su, H. Lai and X. Fang, *A discrete Boltzmann model with symmetric velocity discretization for compressible flow*, Chin. Phys. B, 2023, 32(11), 110503.
- [28] G. Lin, Z. Zhou, Z. Shen and J. Yu, *Existence of uncountably many periodic solutions for second-order superlinear difference equations with continuous time*, Bull. Math. Sci., 2025, 15(1), 2450010.
- [29] R. Sahadevan and N. Kannagi, *Continuous nonpoint symmetries of ordinary difference equations*, J. Math. Anal. Appl., 2006, 324(1), 199–215.
- [30] A. N. Sharkovsky, Y. L. Maistrenko and E. Y. Romanenko, *Difference Equations and their Applications*, Kluwer Academic Publishers, Dordrecht, 1993.
- [31] S. Wang and Z. Zhou, *Heteroclinic solutions for a difference equation involving the mean curvature operator*, Appl. Math. Lett., 2024, 147, 108827.
- [32] F. Xiong, *Infinitely many solutions for partial discrete Kirchhoff type problems involving p -Laplacian*, Mathematics, 2023, 11(15), 3288.
- [33] F. Xiong, W. Huang and Y. Xia, *Positive solutions of partial discrete Kirchhoff type problems*, Alex. Eng. J., 2024, 91, 503–509.
- [34] F. Xiong, Y. Xia and J. Cai, *Three solutions for a partial discrete BVP of the Kirchhoff type*, Adv. Contin. Discrete Models, 2025, 2025(1), 158.
- [35] F. Xiong and Z. Zhou, *Three solutions to Dirichlet problems for second-order self-adjoint difference equations involving p -Laplacian*, Adv. Differ. Equ., 2021, 2021(1), 192.
- [36] B. Zhang, L. Kong, Y. Sun and X. Deng, *Existence of positive solutions for BVPs of fourth-order difference equations*, Appl. Math. Comput., 2002, 131(2–3), 583–591.
- [37] G. Zhang and Z. Yang, *Existence of 2^n nontrivial solutions for discrete two-point boundary value problems*, Nonlinear Anal., 2004, 59(7), 1181–1187.
- [38] Z. Zhou, Y. Chen and J. Yu, *Positive solutions for a discrete Dirichlet problem involving the mean curvature operator*, J. Differ. Equ. Appl., 2025, 31(6), 825–840.

Received April 2026; Accepted October 2026; Available online October 2026.