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WAVES PROPAGATION FOR THE CONFORMABLE TIME-FRACTIONAL BENJAMIN-BONA-MAHONY-BURGERS EQUATION ARISING ON THE SURFACE OF WATER

Muhammad Zafarullah Baber[1,2,3], Tahir Shahzad[4], Muhammad Waqas Yasin[5], Tukur Abdulkadir Sulaiman[6,7], Mustafa Bayram[8], Taha Radwan[9,†] and Karim K. Ahmed[10]

Abstract This study investigates solitary wave solutions of the conformable time-fractional Benjamin-Bona-Mahony-Burgers (TFBBMB) equation. This is a special type of Sobolev-type equation that can describe long waves on the surface of water in a channel with a minimal amplitude. Three techniques are employed: The generalized projective Riccati equation (GPRE) approach, the modified auxiliary equation (MAE) technique, and the generalized exponential rational function (GERF) approach. We obtain formally generated expressions that are symbolically verified by direct substitution into the original equation. The solutions are obtained in the form of exponential functions, rational functions, hyperbolic functions, and trigonometric forms. Real nonsingular solutions are identified and presented with appropriate parameter restrictions, while physically admissible solutions are highlighted for their relevance to wave propagation phenomena. Moreover, linear stability analysis is discussed for the classical BBMB equation. Unique physical problems (initial and boundary conditions) are constructed for real-valued, nonsingular solutions that can serve as benchmark problems for numerical methods.

Keywords Sobolev-type equation, conformable TFBBMB equation, GPRE approach, GERF approach, fractional derivatives, MAE technique, linearize analysis, unique physical problems.

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[†]The corresponding author.
[1]Department of Mathematics, Shanghai University and Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, China
[2]Department of Mathematics and Statistics, The University of Lahore, Sargodha Campus, Pakistan
[3]Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan
[4]Department of Basic Sciences and Humanities, Narowal Campus, University of Engineering and Technology Lahore 54890, Pakistan
[5]Department of Mathematics, University of Narowal, Narowal, Pakistan
[6]Department of Mathematics, Science Faculty, Firat University, 23119 Elazig, Turkiye
[7]Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602105, Tamil Nadu, India
[8]Department of Computer Engineering, Biruni University, Istanbul, Turkiye
[9]Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia
[10]Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman
Email: zafarullah8883@gmail.com(M. Z. Baber), tahirshahzad@uet.edu.pk(T. Shahzad), waqas.yasin@uon.edu.pk(M. W. Yasin), tasulaiman@firat.edu.tr(T. A. Sulaiman), mbayram@biruni.edu.tr(M. Bayram), t.radwan@qu.edu.sa(T. Radwan), karim.kamal.502@gmail.com(K. K. Ahmed)

1. Introduction

Nonlinear equations arise in applied sciences and engineering because they contribute to the development of mathematical models as a result of physical systems, from fluid dynamics to astrophysics. Nonlinearity is important in these physical problems because each parameter is influenced by a number of factors. Solving these equations allows researchers to gain a better understanding of these processes and may result in the discovery of previously unknown facts through general observations. Wave phenomena are used in many areas of mathematics and physics, including elasticity, optical fibres, acoustic waves in crystals, fluid dynamics, plasma, and some other branches of engineering and applied sciences. Dissipation effects must be considered when describing the propagation of very low-amplitude long waves in nonlinear dispersive media for the model to accurately represent real physical systems. The form of the partial differential equation (PDE) is

$$w_t - \alpha \Delta w - \eta \Delta w_t = g(x, w, \nabla w), \tag{1.1}$$

where $w = w(x,t)$, $x \in \Omega \subset \mathbb{R}^n$, and $t > 0$. The unknown parameters α and η are non-negative, Δ is the Laplace operator. These types of nonlinear equations arise in many fields of mathematics and physics. The Eq. (1.1) is characterized by the mixed derivative of space and time, which together appear in the highest derivative and were first studied by Sobolev. In this general nonlinear Eq. (1.1). The time derivative is involved with the highest derivative term, which is the special case of the Sobolev-type equation, which is known as a pseudo-parabolic equation. These models arise in many fields, such as dispersive, unidirectional wave propagation and long waves. Where the unknown function w is the velocity, or maybe velocity, the space variable is the direction of propagation, and t is the time elapsed. The special case of the Sobolev-type equation is the Benjamin-Bona-Mahony-Burgers (BBMB) equation, which is written as

$$w_t + w_{xxt} - \alpha w_{xx} + \beta w_x + w w_x = 0, \tag{1.2}$$

here β, and α are nonnegative constants. Wave phenomena play a very important role in applied science and engineering because they contribute to the development of mathematical models as a result of physical systems. While a small-amplitude wave travels through dispersive media, the physical system is represented by the dissipation effect.

There are many analytical and numerical techniques proposed for the solutions of the Sobolev-type equations. The modified weak Gelarkin method is used by Gao and Wang to solve the Sobolev equation [11]. Korpusov et al. proved the stability of the 3D wave nonlinear equations of Sobolev type [22]. For the Sobolev equation, Zhang and Gao [37] proposed a fully discrete local discontinuous Galerkin technique. For the nonlinear Sobolev equation, He et al. [15] employed time discontinuous Galerkin and characteristic FEM. Haq et al. find the numerical solutions for the generalized BBMB equations by using the Haar wavelets technique [14], Zhang et al. proposed the linearly compact scheme for the 2-dimensional Sobolev equation [40] Akcagil et al. applied G'/G-expansion approach to find the exact traveling wave solutions for the Sobolev-type pseudo-parabolic equations [4], Gözükizil et al. used the tanh-coth method to find the exact solutions of some nonlinear pseudo-parabolic equations [12], the same author also used the tanh method to find the exact solutions of some BBMB-type nonlinear pseudo-parabolic equations [13], Zhang and Liu [38] investigated numerically the BBMB equation by using a newly developed three-point fourth-order compact operator; Zhang et al. [36] investigated the two-dimensional nonlinear Sobolev equations by using two classes of high-order

numerical schemes; Zhang et al. [39] investigated the BBMB equation by using two linearized finite difference schemes, and etc.

Peregrine [26] and Benjamin [9] proposed the Nonlinear BBMB equation, which can describe long waves on the surface of water in a channel with a very small amplitude. Fractional calculus is better than classical calculus in its application to real-world phenomena [29,32]. Many mathematical problems are solved by using the fractional-order derivatives [2,35]. Applications of fractional-order derivatives include acoustic wave transmission in rigid porous materials, magnetism, small-amplitude wave travel through dispersive media, ultrasonic wave propagation in human malignant bone, physical systems, and other applications in engineering and sciences. So, in this study, we consider the conformable TFBBMB equation as follows:

$$D_t^{\epsilon} w + D_t^{\epsilon}(D_{xx})w - \alpha D_{xx} w + \beta D_x w + w D_x = 0. \tag{1.3}$$

Numerous methods have been developed to identify various types of solitary wave solutions, including the G'/G-expansion method. Where ϵ is the fractional-order which is $\epsilon \in (0,1)$. Now, these days several approaches have been developed to identify multiple kinds of solitary wave solutions, such as G'/G-expansion approach [34], the new MEDA method [17,28], the improved modified extended tanh function method [16], the generalized Kudryashov method [5], the Hirota bilinear method [18,30], the Riccati equation mapping technique [7], F-expansion approach [1], the modified extended mapping method [3], etc.

We chose three methods to investigate the solitary wave solutions for the governing equation because they complement each other. The GPRE method yields physically meaningful soliton solutions to the propagation of waves in dispersive media. The GERF technique is able to generate solutions in exponential, hyperbolic, and trigonometric forms, which yields a wide range of solutions. The MAE approach provides a systematic procedure for obtaining rational, trigonometric, and hyperbolic solutions. Each of these methods has its own limitations: Parameter sensitivity in GPRE, algebraic complexity in GERF, and restrictive ansatz in MAE, but cross-verification, comprehensive solution space coverage, and comparison of the methods address these. These methods are widely accepted, fast, and reproducible.

The main contributions of our work are: (i) The first comprehensive comparative study applying three different analytical tools (GPRE, GERF, and MAE) simultaneously to the TFBBMB equation; (ii) the derivation of a variety of solitary wave solutions in both bright and dark form, as well as singular, kink, exponential, rational, and trigonometric forms; (iii) the construction of unique physical problems (initial and boundary conditions) with verified corner compatibility conditions for multiple solutions, which can serve as practical benchmarks for numerical verification of the solutions; and (iv) detailed linear stability analysis of the classical BBMB equation. Our work is different from the published studies [25] and [24] in that it uses three solution methods, engenders more complete physical problem sets, and offers more extensive 3D graphical representations and detailed physical explanations.

2. Fractional order derivative

We explain the conformable time-fractional derivative in this section.

Definition 2.1. Consider the variable unknown function $f : [0,\infty) \to \mathbb{R}$, of ϵ-th order "conformable fractional derivative", which is also defined as:

$$T_{\epsilon}(f)(t) = \lim_{\delta \to 0} \frac{-f(t) + f(t + \delta t^{1-\epsilon})}{\delta}, \tag{2.1}$$

$\epsilon \in (0,1)$ for every $t > 0$. Further information can be found in [8].

Definition 2.2. For a function f, the conformable integral is given for $\epsilon \geq 0$.

$$I_{\epsilon}^{h_1}(f)(r) = \int_{h_1}^{r} \frac{f(t)}{t^{1-\epsilon}} dt, \tag{2.2}$$

$\epsilon \in (0,1)$ for every $t > 0$. Further information can be found in [8].

Theorem 2.1. *Let $t > 0$, and let u and v be functions that are be functions that are ϵ-differentiable for $\epsilon \in (0,1]$. Then the following properties hold:*

- ***Linearity:*** $T_{\epsilon}(h_2 u + h_3 v) = h_2 T_{\epsilon}(u) + h_3 T_{\epsilon}(v)$ *for all* $h_2, h_3 \in \mathbb{R}$.
- ***Power Rule:*** $T_{\epsilon}(t^q) = q(t^{q-\epsilon})$ *for all* q.
- ***Product Rule:*** $T_{\epsilon}(u.v) = uT_{\epsilon}(v) + vT_{\epsilon}(u)$.
- ***Quotient Rule:*** $T_{\epsilon}(\frac{u}{v}) = \frac{vT_{\epsilon}(u) - uT_{\epsilon}(v)}{v^2}$.
- ***Constant Rule:*** $T_{\epsilon}(h_2) = 0$ *for every constant functions* $f(t) = h_2$.
- ***Differentiability:*** f *is differentiable, then* $T_{\epsilon}(u)(t) = t^{1-\epsilon} * \frac{h_3 f(t)}{dt}$.

Some other properties of the fractional order are discussed in [33].

3. Solitary wave structures

This section is divided into three subsections to explore the solitary wave structure for the conformable TFBBMB equation, which is a Sobolev-type equation. In the first subsection, we utilize the GPRE approach; the second subsection deals with the GERF approach; and in the third subsection, we apply the MAE technique. So, we consider the time-fractional wave transformation to replace the PDE with an ODE such as $w(x,t)=\Theta(\zeta)$, where $\zeta = \gamma x - c\frac{t^{\epsilon}}{\epsilon}$. So, we obtained the ODE as follows,

$$\gamma^2 c\Theta''(\zeta) + \alpha\gamma^2\Theta'(\zeta) - (\beta - c)\Theta(\zeta) - \frac{\gamma}{2}\Theta^2(\zeta) = 0, \tag{3.1}$$

where Θ is a polynomial and $' = \frac{d}{d\zeta}$.

3.1. GPRE approach

The GPRE approach [31] generates the polynomials in the following form by considering the general solution of Eq. (3.1).

$$\Theta(\zeta) = l_0 + \sum_{i=1}^{K} \Omega^{i-1}(\zeta)[l_i\Omega(\zeta) + k_i\Lambda(\zeta)]. \tag{3.2}$$

Where l_i and k_i are non-negative and can be solved later, and the functions $\Lambda(\zeta)$ and $\Omega(\zeta)$ satisfy the ODEs as given below.

$$\Omega'(\zeta) = \delta_1\Lambda(\zeta)\Omega(\zeta), \tag{3.3}$$

$$\Lambda'(\zeta) = A - \delta_2\Omega(\zeta) + \delta_1\Lambda(\zeta)^2, \quad \delta_1 = \pm 1, \tag{3.4}$$

such that

$$\Lambda(\zeta)^2 = -\delta_1 \left(\frac{\left(\delta_2^2 - 1\right) \Omega(\zeta)^2}{A} + A - 2\delta_2 \Omega(\zeta) \right), \tag{3.5}$$

here δ_2 and A are constants. The value of K is now the previous Eq. (3.1) applying the homogeneous balancing principle, and $K = 2$ is entered in Eq. (3.2).

$$\Theta(\zeta) = l_0 + l_1\Omega(\zeta) + l_2\Omega(\zeta)^2 + k_1\Lambda(\zeta) + k_2\Lambda(\zeta)\Omega(\zeta). \tag{3.6}$$

Utilizing the Eq. (3.3) & Eq. (3.4), Determine the derivatives of Eq. (3.6) and enter them in the Eq. (3.1). Following simplification, the coefficients of the same power of $\Theta(\zeta)$ are gathered, and they are then set to zero. So, we obtained the system of algebraic equations and solved these algebraic equations using Mathematica and obtained the solutions.

Family of solution: $\left\{ l_0 = \frac{2(c-\beta)}{\gamma}, \quad l_1 = 0, \quad k_1 = 0, \quad l_2 = -\frac{12c\gamma\delta_1^3\left(\delta_2^2-1\right)}{A}, \quad k_2 = 0 \right\}.$

Type 1: If $\delta_1 = -1$, $A > 0$, that will provide the travelling wave solution for the Eq. (1.3) such as

$$w_1(x,t) = \frac{2(c-\beta)}{\gamma} - \frac{12Ac\gamma\delta_1^3\left(\delta_2^2-1\right)\operatorname{sech}^2\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2\operatorname{sech}\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}, \tag{3.7}$$

$$w_2(x,t) = \frac{2(c-\beta)}{\gamma} - \frac{12Ac\gamma\delta_1^3\left(\delta_2^2-1\right)\operatorname{csch}^2\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2\operatorname{csch}\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}. \tag{3.8}$$

Type 2: If $\delta_1 = 1$, $A > 0$, that will provide the periodic wave solution for the Eq. (1.3) such as

$$w_3(x,t) = \frac{2(c-\beta)}{\gamma} - \frac{12Ac\gamma\delta_1^3\left(\delta_2^2-1\right)\sec^2\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2\sec\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}, \tag{3.9}$$

$$w_4(x,t) = \frac{2(c-\beta)}{\gamma} - \frac{12Ac\gamma\delta_1^3\left(\delta_2^2-1\right)\csc^2\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2\csc\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}. \tag{3.10}$$

Type 3: If $\delta_2 = A = 0$, that will provide the plane wave solution for the Eq. (1.3) such as

$$w_5(x,t) = \frac{2(c-\beta)}{\gamma} - \frac{12c\gamma c_1^2\delta_1^3\left(\delta_2^2-1\right)}{A\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)^2}. \tag{3.11}$$

Family of solution 2: $\left\{ l_0 = \frac{c-\beta}{\gamma}, l_1 = -\frac{12\alpha^2\gamma\left(\delta_1^4+\delta_1^2\right)\delta_2}{Ac\delta_1^2\left(2\delta_1^2-1\right)}, k_1 = \frac{3\left(\alpha\gamma\delta_1^2\right)}{\delta_1}, l_2 = -\frac{6c\gamma\delta_1^3\left(\delta_2^2-1\right)}{A}, k_2 = 0 \right\}.$

Type 1: If $\delta_1 = -1$, $A > 0$, that will provide the travelling wave solution for the Eq. (1.3) such as

$$w_6(x,t) = \frac{c-\beta}{\gamma} + \frac{3\alpha\sqrt{A}\gamma\delta_1\tanh\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)}{\delta_2\operatorname{sech}\left(\sqrt{A}\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)+1}$$

$$- \frac{12\alpha^2\gamma\left(\delta_1^4+\delta_1^2\right)\delta_2 \mathrm{sech}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{c\delta_1^2\left(2\delta_1^2-1\right)\left(\delta_2 \mathrm{sech}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)} - \frac{6Ac\gamma\left(\delta_2^2-1\right)\delta_1^3 \mathrm{sech}^2\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2 \mathrm{sech}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}, \tag{3.12}$$

$$w_7(x,t) = \frac{c-\beta}{\gamma} + \frac{3\alpha\sqrt{A}\gamma\delta_1 \coth\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\delta_2 \mathrm{csch}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1} - \frac{12\alpha^2\gamma\left(\delta_1^4+\delta_1^2\right)\delta_2 \mathrm{csch}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{c\delta_1^2\left(2\delta_1^2-1\right)\left(\delta_2 \mathrm{csch}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)} - \frac{6Ac\gamma\left(\delta_2^2-1\right)\delta_1^3 \mathrm{csch}^2\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2 \mathrm{csch}\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}. \tag{3.13}$$

Type 2: If $\delta_1 = 1$, $A > 0$, that will provide the periodic wave solution for the Eq. (1.3) such as

$$w_8(x,t) = \frac{c-\beta}{\gamma} + \frac{3\alpha\sqrt{A}\gamma\delta_1 \tan\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\delta_2 \sec\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1} - \frac{12\alpha^2\gamma\left(\delta_1^4+\delta_1^2\right)\delta_2 \sec\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{c\delta_1^2\left(2\delta_1^2-1\right)\left(\delta_2 \sec\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)} - \frac{6Ac\gamma\left(\delta_2^2-1\right)\delta_1^3 \sec^2\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2 \sec\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}, \tag{3.14}$$

$$w_9(x,t) = \frac{c-\beta}{\gamma} + \frac{3\alpha\sqrt{A}\gamma\delta_1 \cot\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\delta_2 \csc\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1} - \frac{12\alpha^2\gamma\left(\delta_1^4+\delta_1^2\right)\delta_2 \csc\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{c\delta_1^2\left(2\delta_1^2-1\right)\left(\delta_2 \csc\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)} - \frac{6Ac\gamma\left(\delta_2^2-1\right)\delta_1^3 \csc^2\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)}{\left(\delta_2 \csc\left(\sqrt{A}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)+1\right)^2}. \tag{3.15}$$

Type 3: If $\delta_2 = A = 0$, that will provide the plane wave solution for the Eq. (1.3) such as

$$w_{10}(x,t) = +\frac{c-\beta}{\gamma} + \frac{3\alpha\gamma\delta_1}{\delta_1\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)} - \frac{12\alpha^2\gamma c_1\left(\delta_1^4+\delta_1^2\right)\delta_2}{Ac\delta_1^2\left(2\delta_1^2-1\right)\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)} - \frac{6c\gamma c_1^2\left(\delta_2^2-1\right)\delta_1^3}{A\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)^2}. \tag{3.16}$$

3.2. GERF approach

The GERF approach [6, 19, 23] generates the polynomials in the following form by considering the general solution of Eq. (3.1).

$$\Theta(\zeta) = a_0 + \sum_{k=1}^{K} a_k \Lambda(\zeta)^k + \sum_{k=1}^{K} b_k \Lambda(\zeta)^{-k}, \tag{3.17}$$

where

$$\Lambda(\zeta) = \frac{B_1 e^{A_1\zeta} + B_2 e^{A_2\zeta}}{B_3 e^{A_3\zeta} + B_4 e^{A_4\zeta}}, \tag{3.18}$$

where a_0, a_k and b_k are the constants, and B_i, A_i $(i = 1, 2, 3, 4)$. We get $K = 2$ from the homogeneous balance principle. As a result, Eq. (3.17) can be expressed in the following polynomial:

$$\Theta(\zeta) = a_0 + a_1\Lambda(\zeta) + \frac{b_1}{\Lambda(\zeta)} + a_2\Lambda(\zeta)^2 + \frac{b_2}{\Lambda(\zeta)^2}. \tag{3.19}$$

Family of solution 1: If $B_1 = -1, B_2 = -1, B_3 = 1, B_4 = -1, A_1 = 1, A_2 = -1, A_3 = 1, A_4 = -1$, then Eq. (3.18) changes into,

$$\Lambda(\zeta) = -\frac{\cosh(\zeta)}{\sinh(\zeta)}. \tag{3.20}$$

Substitute Eq. (3.19) by the help of Eq. (3.20) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = \sqrt{\frac{27\alpha(-\alpha-10\beta)}{50}}, a_1 = \frac{\sqrt{6\alpha(-\alpha-10\beta)}}{5}, a_2 = -\sqrt{\frac{3\alpha(-\alpha-10\beta)}{50}}, b_1 = 0, b_2 = 0, c = -\frac{\alpha}{10}, \gamma = \frac{\sqrt{-\alpha-10\beta}}{2\sqrt{6}\sqrt{\alpha}}$, we gained the singular wave solution of Eq. (1.3) such as

$$\begin{aligned} w_{11}(x,t) = {} & \frac{3}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{-\alpha-10\beta} - \frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{-\alpha-10\beta}\coth^2\left(\frac{\alpha t^{\epsilon}}{10\epsilon} + \frac{x\sqrt{-\alpha-10\beta}}{2\sqrt{6}\sqrt{\alpha}}\right) \\ & - \frac{1}{5}\sqrt{6}\sqrt{\alpha}\sqrt{-\alpha-10\beta}\coth\left(\frac{\alpha t^{\epsilon}}{10\epsilon} + \frac{x\sqrt{-\alpha-10\beta}}{2\sqrt{6}\sqrt{\alpha}}\right). \end{aligned} \tag{3.21}$$

Set 2: When $a_0 = \frac{1}{5}(-3)\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{-\alpha-10\beta}, a_1 = 0, a_2 = 0, b_1 = -\frac{1}{5}\sqrt{6}\sqrt{\alpha}\sqrt{-\alpha-10\beta}, b_2 = \frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{-\alpha-10\beta}, c = -\frac{\alpha}{10}, \gamma = -\frac{\sqrt{-\alpha-10\beta}}{2\sqrt{6}\sqrt{\alpha}}$, we obtained the dark wave solution of Eq. (1.3) such as,

$$\begin{aligned} w_{12}(x,t) = {} & -\frac{3}{5}\sqrt{\frac{3\alpha}{2}}\sqrt{-\alpha-10\beta} + \frac{1}{5}\sqrt{\frac{3\alpha}{2}}\sqrt{-\alpha-10\beta}\tanh^2\left(\frac{x\sqrt{-\alpha-10\beta}}{2\sqrt{6\alpha}} - \frac{\alpha t^{\epsilon}}{10\epsilon}\right) \\ & - \frac{1}{5}\sqrt{6\alpha}\sqrt{-\alpha-10\beta}\tanh\left(\frac{x\sqrt{-\alpha-10\beta}}{2\sqrt{6\alpha}} - \frac{\alpha t^{\epsilon}}{10\epsilon}\right). \end{aligned} \tag{3.22}$$

Family of solution 2: If $B_1 = -i, B_2 = i, B_3 = 1, B_4 = 1, A_1 = i, A_2 = -i, A_3 = i, A_4 = -i$, then Eq. (3.18) changes into,

$$\Lambda(\zeta) = -\frac{\sin(\zeta)}{\cos(\zeta)}. \tag{3.23}$$

Substitute Eq. (3.19) by the help of (3.23) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = -i\sqrt{\frac{27\alpha(-10i\beta)}{50}}, a_1 = \frac{\sqrt{6\alpha(\alpha-10i\beta)}}{5}, a_2 = -i\sqrt{\frac{3\alpha(\alpha-10i\beta)}{50}}, b_1 = 0, b_2 = 0, c = -\frac{1}{10}(i\alpha), \gamma = \frac{\sqrt{\alpha-10i\beta}}{2\sqrt{6\alpha}}$ we gained the solitary wave solution (SWS) of Eq. (1.3) such as,

$$\begin{aligned} w_{13}(x,t) =& -\frac{3}{5}i\sqrt{\frac{3\alpha}{2}}\sqrt{\alpha-10i\beta} - \frac{1}{5}i\sqrt{\frac{3\alpha}{2}}\sqrt{\alpha-10i\beta}\tan^2\left(\frac{i\alpha t^{\epsilon}}{10\epsilon} + \frac{x\sqrt{\alpha-10i\beta}}{2\sqrt{6\alpha}}\right) \\ & - \frac{1}{5}\sqrt{6\alpha}\sqrt{\alpha-10i\beta}\tan\left(\frac{i\alpha t^{\epsilon}}{10\epsilon} + \frac{x\sqrt{\alpha-10i\beta}}{2\sqrt{6\alpha}}\right). \end{aligned} \tag{3.24}$$

Family of solution 3: If $B_1 = 1, B_2 = 0, B_3 = 1, B_4 = 1, A_1 = 1, A_2 = 0, A_3 = 1, A_4 = 0$ then Eq. (3.18) changes into,

$$\Lambda(\zeta) = \frac{\exp(\zeta)}{\exp(\zeta)+1}. \tag{3.25}$$

Substitute Eq. (3.19) by the help of (3.25) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = -\frac{2\gamma(\alpha+\beta)}{\gamma^2-1}, a_1 = \frac{2(\alpha\gamma+\beta\gamma)}{\gamma^2-1}, a_2 = -\frac{4(\alpha\gamma+\beta\gamma)}{\gamma^2-1}, b_1 = 0, b_2 = 0, c = \frac{-\alpha\gamma^2-\beta}{\gamma^2-1}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$\begin{aligned} w_{14}(x,t) =& -\frac{2\gamma(\alpha+\beta)}{\gamma^2-1} - \frac{4(\alpha\gamma+\beta\gamma)\exp\left(2\left(\gamma x - \frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{(\gamma^2-1)\epsilon}\right)\right)}{(\gamma^2-1)\left(1+e^{\gamma x - \frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{(\gamma^2-1)\epsilon}}\right)^2} \\ & + \frac{2(\alpha\gamma+\beta\gamma)e^{\gamma x - \frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{(\gamma^2-1)\epsilon}}}{(\gamma^2-1)\left(1+e^{\gamma x - \frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{(\gamma^2-1)\epsilon}}\right)}. \end{aligned} \tag{3.26}$$

Family of solution 4: If $B_1 = 1-i, B_2 = 1+i, B_3 = 1, B_4 = 1, A_1 = i, A_2 = -i, A_3 = i, A_4 = -i$ then Eq. (3.18) changes into,

$$\Lambda(\zeta) = 1 + \tan(\zeta). \tag{3.27}$$

Substitute Eq. (3.19) by the help of (3.27) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = \frac{1}{2}i\left((-2+2i)a_1 - 4a_2 + (1+i)b_1 + b_2\right), c = \beta, \gamma = 0$, we gained the SWS of Eq. (1.3) such as,

$$\begin{aligned} w_{15}(x,t) =& \frac{1-3i}{20\alpha+80i\beta} + \frac{(-8i\alpha a_1 + 32a_1\beta + (2-i))\cos^2\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)}{((8+4i)\alpha - (16-32i)\beta)\left(\sin\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right) + \cos\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)\right)^2} \\ & + \frac{\left(\frac{3}{10} + \frac{i}{10}\right)((6+2i)\alpha a_1 - (8-24i)a_1\beta + i)\cos\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)}{(\alpha+4i\beta)\left(\sin\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right) + \cos\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)\right)} \\ & + \frac{(-8\alpha a_1 - 32ia_1\beta + i)\sec^2\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)\left(\sin\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right) + \cos\left(\gamma x - \frac{ct^{\epsilon}}{\epsilon}\right)\right)^2}{(32+16i)\alpha - (64-128i)\beta} \end{aligned}$$

$$+ a_1 \sec\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\left(\sin\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right) + \cos\left(\gamma x - \frac{ct^\epsilon}{\epsilon}\right)\right)$$
$$- \left(\frac{9}{5} + \frac{3i}{5}\right) a_1 - 8i\beta + 8ic. \tag{3.28}$$

Family of solution 5: If $B_1 = -3, B_2 = -1, B_3 = 1, B_4 = 1, A_1 = 1, A_2 = -1, A_3 = 1, A_4 = -1$ then Eq. (3.18) changes into,

$$\Lambda(\zeta) = -\tanh(\zeta) - 2. \tag{3.29}$$

Substitute Eq. (3.19) by the help of (3.29) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = \frac{4\gamma(\alpha-2\beta)+5a_1\left(4\gamma^2-1\right)+a_2\left(18-72\gamma^2\right)}{4\gamma^2-1}, b_1 = 7a_1 - 32a_2, b_2 = -3\left(5a_2 - a_1\right), c = \frac{2\alpha\gamma^2-\beta}{4\gamma^2-1}$ we obtained the travelling wave solution of Eq. (1.3) such as,

$$w_{16}(x,t) = \frac{4\gamma(\alpha-2\beta)+5a_1\left(4\gamma^2-1\right)+a_2\left(18-72\gamma^2\right)}{4\gamma^2-1}$$
$$- \frac{3\left(5a_2-a_1\right)\cosh^2\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{(4\gamma^2-1)\epsilon}\right)}{\left(-\sinh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon}\right) - 2\cosh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon}\right)\right)^2}$$
$$+ \frac{\left(7a_1-32a_2\right)\cosh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{(4\gamma^2-1)\epsilon}\right)}{-\sinh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon}\right) - 2\cosh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon}\right)}$$
$$+ a_2\mathrm{sech}^2\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right)\left(-\sinh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right)\right.$$
$$\left.-2\cosh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right)\right)^2 + a_1\mathrm{sech}\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right)$$
$$\times\left(-\sinh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right) - 2\cosh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2-\beta\right)}{\left(4\gamma^2-1\right)\epsilon}\right)\right). \tag{3.30}$$

Set 2: When $a_0 = 5a_1 - 18a_2, b_1 = 7a_1 - 32a_2, b_2 = -3\left(5a_2 - a_1\right), c = \frac{2\alpha\gamma^2+\beta}{4\gamma^2+1}$ we obtained the travelling wave solution of Eq. (1.3) such as,

$$w_{17}(x,t) = - \frac{3\left(5a_2-a_1\right)\cosh^2\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2+\beta\right)}{(4\gamma^2+1)\epsilon}\right)}{\left(-\sinh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2+\beta)}{(4\gamma^2+1)\epsilon}\right) - 2\cosh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2+\beta)}{(4\gamma^2+1)\epsilon}\right)\right)^2}$$
$$+ \frac{\left(7a_1-32a_2\right)\cosh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2+\beta\right)}{(4\gamma^2+1)\epsilon}\right)}{-\sinh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2+\beta)}{(4\gamma^2+1)\epsilon}\right) - 2\cosh\left(\gamma x - \frac{t^\epsilon(2\alpha\gamma^2+\beta)}{(4\gamma^2+1)\epsilon}\right)}$$
$$+ a_2\mathrm{sech}^2\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)\left(-\sinh\left(\gamma x - \frac{t^\epsilon\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)\right.$$

$$-2\cosh\left(\gamma x-\frac{t^{\epsilon}\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)\Bigg)^2+a_1\mathrm{sech}\left(\gamma x-\frac{t^{\epsilon}\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)$$
$$\times\left(-\sinh\left(\gamma x-\frac{t^{\epsilon}\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)-2\cosh\left(\gamma x-\frac{t^{\epsilon}\left(2\alpha\gamma^2+\beta\right)}{\left(4\gamma^2+1\right)\epsilon}\right)\right)+5a_1-18a_2. \tag{3.31}$$

Family of solution 6: If $B_1=-1,B_2=0,B_3=1,B_4=1,A_1=0,A_2=1,A_3=0,A_4=1$ then Eq. (3.18) changes into,

$$\Lambda(\eta)=-\frac{1}{\exp(\eta)+1}. \tag{3.32}$$

Substitute Eq. (3.19) by the help of (3.32) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0=-\frac{3\left(49\alpha\gamma^3-79\alpha\gamma-128\beta\gamma\right)}{128(\gamma^2+1)},a_1=-\frac{5\left(33\alpha\gamma^3-95\alpha\gamma-128\beta\gamma\right)}{64(\gamma^2+1)},b_1=\frac{3\alpha\gamma}{64},b_2=-\frac{1}{256}(9\alpha\gamma)$, $c=\frac{\beta-\alpha\gamma^2}{\gamma^2+1}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$w_{18}(x,t)=\frac{a_2}{\left(e^{\gamma x-\frac{t^{\epsilon}\left(\beta-\alpha\gamma^2\right)}{\left(\gamma^2+1\right)\epsilon}}+1\right)^2}+\frac{3}{64}\alpha\gamma\left(-e^{\gamma x-\frac{t^{\epsilon}\left(\beta-\alpha\gamma^2\right)}{\left(\gamma^2+1\right)\epsilon}}-1\right)$$
$$+\frac{5\left(33\alpha\gamma^3-95\alpha\gamma-128\beta\gamma\right)}{64\left(\gamma^2+1\right)\left(e^{\gamma x-\frac{t^{\epsilon}\left(\beta-\alpha\gamma^2\right)}{\left(\gamma^2+1\right)\epsilon}}+1\right)}-\frac{3\left(49\alpha\gamma^3-79\alpha\gamma-128\beta\gamma\right)}{128\left(\gamma^2+1\right)}$$
$$-\frac{1}{256}9\alpha\gamma\left(e^{\gamma x-\frac{t^{\epsilon}\left(\beta-\alpha\gamma^2\right)}{\left(\gamma^2+1\right)\epsilon}}+1\right)^2. \tag{3.33}$$

Set 2: When $a_0=\frac{a_1}{2},b_1=0,b_2=0,c=\frac{-\alpha\gamma^2-\beta}{\gamma^2-1}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$w_{19}(x,t)=\frac{a_1}{2}+\frac{a_2}{\left(e^{\gamma x-\frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{\left(\gamma^2-1\right)\epsilon}}+1\right)^2}-\frac{a_1}{e^{\gamma x-\frac{t^{\epsilon}\left(-\alpha\gamma^2-\beta\right)}{\left(\gamma^2-1\right)\epsilon}}+1}. \tag{3.34}$$

Family of solution 7: If $B_1=1,B_2=0,B_3=1,B_4=1,A_1=1,A_2=0,A_3=1,A_4=0$ then Eq. (3.18) changes into,

$$\Lambda(\zeta)=-\tanh(\zeta). \tag{3.35}$$

Substitute Eq. (3.19) by the help of (3.35) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0=\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}$, $a_1=-\frac{1}{5}\sqrt{6}\sqrt{\alpha}\sqrt{\alpha+10\beta}$, $a_2=\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}$, $b_1=0$, $b_2=0$, $c=-\frac{\alpha}{21}$, $\gamma=-\frac{\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}$, we obtained the travelling wave solution of Eq. (1.3) such as,

$$w_{20}(x,t)=\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}+\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}\tanh^2\left(\frac{x\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}-\frac{\alpha t^{\epsilon}}{10\epsilon}\right)$$

$$-\frac{1}{5}\sqrt{6}\sqrt{\alpha}\sqrt{\alpha+10\beta}\tanh\left(\frac{x\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}-\frac{\alpha t^{\epsilon}}{10\epsilon}\right). \tag{3.36}$$

Set 2: When $a_0 = -\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}$, $a_1 = 0$, $a_2 = 0$, $b_1 = \frac{1}{5}\sqrt{6\alpha(\alpha+10\beta)}$, $b_2 = -\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}$, $c = -\frac{\alpha}{10}$, $\gamma = \frac{\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}$ we obtained the travelling wave solution of Eq. (1.3) such as

$$\begin{aligned} w_{21}(x,t) =& -\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}-\frac{1}{5}\sqrt{\frac{3}{2}}\sqrt{\alpha}\sqrt{\alpha+10\beta}\coth^2\left(\frac{\alpha t^{\epsilon}}{10\epsilon}+\frac{x\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}\right)\\ &-\frac{1}{5}\sqrt{6}\sqrt{\alpha}\sqrt{\alpha+10\beta}\coth\left(\frac{\alpha t^{\epsilon}}{10\epsilon}+\frac{x\sqrt{\alpha+10\beta}}{2\sqrt{6}\sqrt{\alpha}}\right). \end{aligned} \tag{3.37}$$

Family of solution 8: If $B_1 = 2-i, B_2 = 2+i, B_3 = 1, B_4 = 1, A_1 = i, A_2 = -i, A_3 = i, A_4 = -i$ then Eq. (3.18) changes into,

$$\Lambda(\zeta) = 2 + \tan(\zeta). \tag{3.38}$$

Substitute Eq. (3.19) by the help of (3.38) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = -\frac{60\left(-\beta+8c\gamma^2+c\right)}{\gamma}$, $a_1 = \frac{(18-4i)\left(-\beta+8c\gamma^2+c\right)}{\gamma}$, $a_2 = -\frac{(2-i)\left(-\beta+8c\gamma^2+c\right)}{\gamma}$, $b_1 = \frac{(90+20i)\left(-\beta+8c\gamma^2+c\right)}{\gamma}$, $b_2 = -\frac{(50+25i)\left(-\beta+8c\gamma^2+c\right)}{\gamma}$, we obtained the SWS of Eq. (1.3) such as

$$\begin{aligned} w_{22}(x,t) =& -\frac{60\left(-\beta+8c\gamma^2+c\right)}{\gamma}+\frac{(90+20i)\left(-\beta+8c\gamma^2+c\right)\cos\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)}{\gamma\left(\sin\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)+2\cos\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\right)}\\ &+\frac{(18-4i)\left(-\beta+8c\gamma^2+c\right)\sec\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\left(\sin\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)+2\cos\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\right)}{\gamma}\\ &-\frac{(50+25i)\left(-\beta+8c\gamma^2+c\right)\cos^2\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)}{\gamma\left(\sin\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)+2\cos\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\right)^2}\\ &-\frac{(2-i)\left(-\beta+8c\gamma^2+c\right)\sec^2\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\left(\sin\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)+2\cos\left(\gamma x-\frac{ct^{\epsilon}}{\epsilon}\right)\right)^2}{\gamma}. \end{aligned} \tag{3.39}$$

Family of solution 9: If $B_1 = 1, B_2 = 2, B_3 = 1, B_4 = 1, A_1 = 1, A_2 = 0, A_3 = 1, A_4 = 0$ then Eq. (3.18) changes into,

$$\Lambda(\eta) = \frac{2+\exp(\eta)}{1+\exp(\eta)}. \tag{3.40}$$

Substitute Eq. (3.19) by the help of (3.40) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = \frac{1}{19}\left(-57a_1 - 114a_2 - 4\sqrt{19}\beta + 4\sqrt{19}c\right), b_1 = 3a_1 + 8a_2, b_2 = -a_1 - 3a_2, \gamma = \frac{\sqrt{19}}{2}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$w_{23}(x,t) = \frac{4(c-\beta)}{\sqrt{19}}+\frac{(a_1+4a_2)\,e^{\frac{3ct^{\epsilon}}{\epsilon}+\frac{\sqrt{19}x}{2}}+(a_1+5a_2)\,e^{\frac{4ct^{\epsilon}}{\epsilon}}}{\left(3e^{\frac{ct^{\epsilon}}{\epsilon}+\frac{\sqrt{19}x}{2}}+2e^{\frac{2ct^{\epsilon}}{\epsilon}}+e^{\sqrt{19}x}\right)^2}. \tag{3.41}$$

Family of solution 10: If $B_1 = 2, B_2 = 1, B_3 = 1, B_4 = 1, A_1 = 1, A_2 = 0, A_3 = 1, A_4 = 0$ then Eq. (3.18) changes into,

$$\Lambda(\eta) = \frac{\exp(\eta)}{\exp(\eta) + 1}. \tag{3.42}$$

Substitute Eq. (3.19) by the help of (3.42) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = -\frac{2\gamma(\alpha+\beta)}{\gamma^2-1}, a_1 = \frac{2(\alpha\gamma+\beta\gamma)}{\gamma^2-1}, a_2 = -\frac{4(\alpha\gamma+\beta\gamma)}{\gamma^2-1}, b_1 = 0, b_2 = 0, c = \frac{-\alpha\gamma^2-\beta}{\gamma^2-1}$, we obtained the exponential function solution of Eq. (1.3) such as,

$$\begin{aligned} w_{24}(x,t) = & \frac{12(c-\beta)\left(12\alpha\gamma^2 - 5\beta + 24c\gamma^2 + 5c\right)}{\gamma\left(2\alpha\gamma^2 - \beta + 2c\gamma^2 + c\right)} \\ & + \frac{16(c-\beta)\left(5\alpha\gamma^2 - 2\beta + 11c\gamma^2 + 2c\right)\left(e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)^2}{3\gamma\left(2\alpha\gamma^2 - \beta + 2c\gamma^2 + c\right)\left(2e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)^2} \\ & + \frac{4(c-\beta)\left(16\alpha\gamma^2 - 7\beta + 28c\gamma^2 + 7c\right)\left(2e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)^2}{3\gamma\left(2\alpha\gamma^2 - \beta + 2c\gamma^2 + c\right)\left(e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)^2} \\ & - \frac{4(c-\beta)\left(76\alpha\gamma^2 - 31\beta + 160c\gamma^2 + 31c\right)\left(e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)}{3\gamma\left(2\alpha\gamma^2 - \beta + 2c\gamma^2 + c\right)\left(2e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)} \\ & - \frac{4(c-\beta)\left(68\alpha\gamma^2 - 29\beta + 128c\gamma^2 + 29c\right)\left(2e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)}{3\gamma\left(2\alpha\gamma^2 - \beta + 2c\gamma^2 + c\right)\left(e^{\gamma x - \frac{ct^\epsilon}{\epsilon}} + 1\right)}. \end{aligned} \tag{3.43}$$

Family of solution 11: If $B_1 = 1, B_2 = 1, B_3 = 1, B_4 = 1, A_1 = 0, A_2 = 0, A_3 = 1, A_4 = -1$ then Eq. (3.18) changes into,

$$\Lambda(\eta) = \frac{2}{\exp(-\eta) + \exp(\eta)}. \tag{3.44}$$

Substitute Eq. (3.19) by the help of (3.44) in the Eq. (3.1) We get the unknown values of the constants.

Set 1: When $a_0 = -i(\alpha + 2\beta), a_1 = 0, a_2 = \frac{3}{2}i(\alpha + 2\beta), b_1 = 0, b_2 = 0, c = \frac{1}{4}(2\beta - \alpha), \gamma = -\frac{i}{2}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$w_{25}(x,t) = -i(2\beta + \alpha) + \frac{6i(2\beta + \alpha)}{\left(e^{-\frac{(2\beta-\alpha)t^\epsilon}{4\epsilon} - \frac{ix}{2}} + e^{\frac{(2\beta-\alpha)t^\epsilon}{4\epsilon} + \frac{ix}{2}}\right)^2}. \tag{3.45}$$

Set 2: When as,$a_0 = -\frac{4(\alpha\gamma+2\beta\gamma)}{4\gamma^2-1}, a_1 = 0, a_2 = \frac{6(\alpha\gamma+2\beta\gamma)}{4\gamma^2-1}, b_1 = 0, b_2 = 0, c = \frac{-2\alpha\gamma^2-\beta}{4\gamma^2-1}$ we obtained the exponential function solution of Eq. (1.3) such as,

$$\begin{aligned} w_{26}(x,t) = & -\frac{4(\alpha\gamma + 2\beta\gamma)}{4\gamma^2 - 1} \\ & + \frac{24(\alpha\gamma + 2\beta\gamma)}{(4\gamma^2 - 1)\left(\exp\left(\gamma x - \frac{t^\epsilon(-2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon}\right) + \exp\left(\frac{t^\epsilon(-2\alpha\gamma^2-\beta)}{(4\gamma^2-1)\epsilon} - \gamma x\right)\right)^2}. \end{aligned} \tag{3.46}$$

4. MAE technique

The polynomials are gained by taking the solution of Eq.(3.1) as follows [20, 21],

$$\Theta(\zeta) = a_0 + \sum_{k=1}^{K} \left[a_k \omega^{(z\zeta)k} + b_k \omega^{-(z\zeta)k} \right], \tag{4.1}$$

where the constants a_0, a_i and b_i $(i = 1, 2, 3, \cdots K)$ that are found later, here $\omega(\zeta)$ is the solution that is given below.

$$z'(\zeta) = \frac{1}{\ln(\omega)} \left(\beta_0 + \beta_1 \omega^{z\zeta} + \nu \omega^{-z\zeta} \right), \tag{4.2}$$

here, β_0, β_1, ν and z with $z > 0$ $z \neq 1$ are constant that are calculate later. Now, applying the homogeneous balancing principle to determine the value of K from the above equation. Utilizing the homogeneous balancing principle, then enter $K = 2$ in Eq. (4.1).

$$\Theta(\zeta) = a_0 + a_1 \omega^{z\zeta} + b_1 \omega^{-z\zeta} + a_2 \omega^{2z\zeta} + b_2 \omega^{-2z\zeta}. \tag{4.3}$$

Applying the Eq. (4.2), Calculate the derivatives of Eq. (4.3) and enter them in the Eq. (3.1). Following simplification, compute the coefficients of $\omega^{(z\zeta)i}$ and $\omega^{-(z\zeta)i}$ of the same power, then set them to zero. So, we obtained a system of algebraic equations and solved these algebraic equations by applying the software Mathematica and obtained the solutions.

Family of solutions 1:

$$a_0 = \frac{-\beta + \gamma^2 \left(\beta_0 \left(\alpha + \beta_0 c \right) + 2\beta_1 c\nu \right) + c}{\gamma}, a_1 = \frac{\gamma^4 \left(\beta_0 \left(\alpha + \beta_0 c \right) + 2\beta_1 c\nu \right)^2 - (c - \beta)^2}{2\beta_1 \gamma^3 \left(\alpha + \beta_0 c \right)}, b_1 = 0,$$
$$a_2 = 0, b_2 = 0.$$

Case-1. The SWSs occur when $\beta_0^2 - 4\beta_1\nu < 0$ and $\nu \neq 0$.

$$\begin{aligned} & w_{27}(x,t) \\ &= \frac{-\beta + \gamma^2 \left(\beta_0 \left(\beta_0 c + \alpha \right) + 2\beta_1 c\nu \right) + c}{\gamma} \\ &\quad + \frac{\left(\gamma^4 \left(\beta_0 \left(\alpha + \beta_0 c \right) + 2\beta_1 c\nu \right)^2 - (c - \beta)^2 \right) \left(\sqrt{4\beta_1\nu - \beta_0^2} \tan \left(\frac{1}{2} \sqrt{4\beta_1\nu - \beta_0^2} \left(\gamma x - \frac{ct^\epsilon}{\epsilon} \right) \right) - \beta_0 \right)}{4\beta_1 \gamma^3 \nu \left(\alpha + \beta_0 c \right)}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} & w_{28}(x,t) \\ &= \frac{-\beta + \gamma^2 \left(\beta_0 \left(\alpha + \beta_0 c \right) + 2\beta_1 c\nu \right) + c}{\gamma} \\ &\quad - \frac{\left(\gamma^4 \left(\beta_0 \left(\alpha + \beta_0 c \right) + 2\beta_1 c\nu \right)^2 - (c - \beta)^2 \right) \left(\beta_0 + \sqrt{4\beta_1\nu - \beta_0^2} \cot \left(\frac{1}{2} \sqrt{4\beta_1\nu - \beta_0^2} \left(\gamma x - \frac{ct^\epsilon}{\epsilon} \right) \right) \right)}{4\beta_1 \gamma^3 \nu \left(\alpha + \beta_0 c \right)}, \end{aligned} \tag{4.5}$$

Case-2. The hyperbolic solutions when $\beta_0^2 - 4\beta_1\nu > 0$ and $\nu \neq 0$,

$$\begin{aligned} & w_{29}(x,t) \\ &= \frac{-\beta + \gamma^2\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)+c}{\gamma} \\ &\quad - \frac{\left(\gamma^4\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)^2-(c-\beta)^2\right)\left(\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\tanh\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)\right)}{4\beta_1\gamma^3\nu\left(\alpha+\beta_0 c\right)}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} & w_{30}(x,t) \\ &= \frac{-\beta + \gamma^2\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)+c}{\gamma} \\ &\quad - \frac{\left(\gamma^4\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)^2-(c-\beta)^2\right)\left(\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\coth\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)\right)\right)}{4\beta_1\gamma^3\nu\left(\alpha+\beta_0 c\right)}. \end{aligned} \tag{4.7}$$

Case-3. The rational solutions when $\beta_0^2 - 4\beta_1\nu = 0$ and $\nu \neq 0$,

$$\begin{aligned} w_{31}(x,t) = & \frac{-\beta + \gamma^2\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)+c}{\gamma} \\ & - \frac{\left(\gamma^4\left(\beta_0\left(\alpha+\beta_0 c\right)+2\beta_1 c\nu\right)^2-(c-\beta)^2\right)\left(\beta_0\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)+2\right)}{4\beta_1\gamma^3\nu\left(\alpha+\beta_0 c\right)\left(\gamma x-\frac{ct^\epsilon}{\epsilon}\right)}. \end{aligned} \tag{4.8}$$

Family of solutions 2:

$$a_0 = -\left(\frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}}+\frac{\beta}{\gamma}\right), a_1 = 0, b_1 = -\frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}} - \frac{12}{5}\alpha\beta_1\gamma,$$

$$a_2 = 0, b_2 = -\frac{12\alpha\beta_1^2\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}, c = -\frac{\alpha}{5\sqrt{\beta_0^2-4\beta_1\nu}}.$$

Case-1. The SWSs occur when $\beta_0^2 - 4\beta_1\nu < 0$ and $\nu \neq 0$,

$$\begin{aligned} & w_{32}(x,t) \\ &= -\frac{\beta}{\gamma} - \frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}} \\ &\quad - \frac{48\alpha\beta_1^2\gamma\nu^2}{5\sqrt{\beta_0^2-4\beta_1\nu}\left(\sqrt{4\beta_1\nu-\beta_0^2}\tan\left(\frac{1}{2}\sqrt{4\beta_1\nu-\beta_0^2}\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)-\beta_0\right)^2} \\ &\quad + \frac{2\nu\left(\frac{1}{5}(-12)\alpha\beta_1\gamma - \frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}\right)}{\sqrt{4\beta_1\nu-\beta_0^2}\tan\left(\frac{1}{2}\sqrt{4\beta_1\nu-\beta_0^2}\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)-\beta_0}, \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
&w_{33}(x,t)\\
&=-\frac{\beta}{\gamma}-\frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}}\\
&-\frac{48\alpha\beta_1^2\gamma\nu^2}{5\sqrt{\beta_0^2-4\beta_1\nu}\left(\beta_0+\sqrt{4\beta_1\nu-\beta_0^2}\cot\left(\frac{1}{2}\sqrt{4\beta_1\nu-\beta_0^2}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)\right)^2}\\
&-\frac{2\nu\left(\frac{1}{5}(-12)\alpha\beta_1\gamma-\frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}\right)}{\beta_0+\sqrt{4\beta_1\nu-\beta_0^2}\cot\left(\frac{1}{2}\sqrt{4\beta_1\nu-\beta_0^2}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)}.
\end{aligned}
\tag{4.10}
$$

Case-2. The hyperbolic solutions when $\beta_0^2-4\beta_1\nu>0$ and $\nu\neq 0$,

$$
\begin{aligned}
&w_{34}(x,t)\\
&=-\frac{\beta}{\gamma}-\frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}}\\
&-\frac{48\alpha\beta_1^2\gamma\nu^2}{5\sqrt{\beta_0^2-4\beta_1\nu}\left(\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\tanh\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)\right)^2}\\
&-\frac{2\nu\left(\frac{1}{5}(-12)\alpha\beta_1\gamma-\frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}\right)}{\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\tanh\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)},
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&w_{35}(x,t)\\
&=-\frac{\beta}{\gamma}-\frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}}\\
&-\frac{48\alpha\beta_1^2\gamma\nu^2}{5\sqrt{\beta_0^2-4\beta_1\nu}\left(\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\coth\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)\right)^2}\\
&-\frac{2\nu\left(\frac{1}{5}(-12)\alpha\beta_1\gamma-\frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}\right)}{\beta_0+\sqrt{\beta_0^2-4\beta_1\nu}\coth\left(\frac{1}{2}\sqrt{\beta_0^2-4\beta_1\nu}\left(\frac{\alpha t^{\epsilon}}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)\right)}.
\end{aligned}
\tag{4.12}
$$

Case-3. The rational solutions when $\beta_0^2-4\beta_1\nu=0$ and $\nu\neq 0$,

$$
\begin{aligned}
&w_{36}(x,t)\\
&=-\frac{\beta}{\gamma}-\frac{6\alpha\gamma^2\left(\beta_0\sqrt{\beta_0^2-4\beta_1\nu}+2\beta_1\nu\right)+\alpha}{5\gamma\sqrt{\beta_0^2-4\beta_1\nu}}
\end{aligned}
$$

$$-\frac{48\alpha\beta_1^2\gamma\nu^2\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)^2}{5\sqrt{\beta_0^2-4\beta_1\nu}\left(\beta_0\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)+2\right)^2}$$
$$-\frac{2\nu\left(\frac{1}{5}(-12)\alpha\beta_1\gamma-\frac{12\alpha\beta_0\beta_1\gamma}{5\sqrt{\beta_0^2-4\beta_1\nu}}\right)\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)}{\beta_0\left(\frac{\alpha t^\epsilon}{5\epsilon\sqrt{\beta_0^2-4\beta_1\nu}}+\gamma x\right)+2}. \tag{4.13}$$

5. Linear stability analysis

In this part, we talked about the classical stability of the BBMB equation when we take $\alpha = 1$. We selected the transformation as [10, 27],

$$M = N + \epsilon_p w(x,t). \tag{5.1}$$

Here N is the constant, steady-state solution of Eq. (1.2) and entering the value we gain,

$$N\epsilon_p w_x + \epsilon_p w_t + \beta\epsilon_p w_x + 2w\epsilon_p^2 w_x - \alpha\epsilon_p w_{xx} + \epsilon_p w_{xxt} = 0. \tag{5.2}$$

Linearising in the form of ϵ_p,

$$N\epsilon_p w_x + \epsilon_p w_t + \beta\epsilon_p w_x - \alpha\epsilon_p w_{xx} + \epsilon_p w_{xxt} = 0. \tag{5.3}$$

Consider that the following is the solution to the above equation:

$$w(x,t) = \aleph e^{i(kx+\sigma t)}, \tag{5.4}$$

where σ represents the frequency and the normalized wave number k, respectively. Putting this transformation in Eq. (5.3),

$$\aleph\epsilon_p\left(k^2(\alpha - i\sigma) + ik(\beta + N) + i\sigma\right) = 0. \tag{5.5}$$

We get the value of σ after solving the above equation as,

$$\sigma = \frac{\beta k + kN - i\alpha k^2}{k^2 - 1}. \tag{5.6}$$

In a situation where all values of k except 1 in the above equation. Here, the wavenumber σ is an imaginary value. As a result, there is no solution superposition to decay, and the perturbation increases exponentially. This indicates the instability of the dispersion. If $k = 0$, then the dispersion is stable.

6. Physical representation

This section discusses the graphical behavior of some solutions that are successfully achieved by utilizing three different techniques. This section covers the graphical behavior of a few solutions that were successfully achieved utilizing three distinct methods. These solutions are constructed

in different forms, such as travelling waves and solitary waves. These wave solutions are very fruitful for the phenomena of waves on the surface of water in a channel with a very small amplitude. These waves show the physical behaviors of dispersive, unidirectional wave propagation and long waves. The acquired results are highly valuable for comparing computational and experimental solutions and understanding the nonlinear dynamics. This model also plays an important role in ocean engineering because it has applications on the surface of water. The 3D plot and corresponding contours are drawn with the aid of *Mathematica* software, which specifies the physical behavior of the state variable, and contours are created along with the 3D, plot. Future applied science investigation into nonlinear wave problems may benefit from the physics of our results.

Figure 1 shows the dark travelling wave behavior for the solution $w_1(x,t)$, while figure 2 gives us the bright wave for the solution $w_3(x,t)$, figure 3 provided us the dark-bright wave behavior clearly for the solution $w_6(x,t)$. Figures 4,5 show the solitary wave behavior for the solutions $w_{11}(x,t)$ and $w_{12}(x,t)$ by choosing the different values of parameters that are mentioned in the captions of the figures. Figures 6, 8, 9, and 10 are drawn for the exponential function solutions of $w_{14}(x,t)$, $w_{18}(x,t)$, $w_{19}(x,t)$. and $w_{26}(x,t)$. Figure 11 gives us the kink wave for the solution $w_{27}(x,t)$. Figures 7, 12,13, 14 and 15 also give us the solitary wave behavior for the solutions $w_{14}(x,t)$, $w_{18}(x,t)$, $w_{19}(x,t)$. and $w_{26}(x,t)$. Solutions of this equation are isolated waveforms which preserve their form and energy as they propagate with a given speed. These are the waves commonly defined by a hyperbolic secant (sech) function, which are phenomena such as shock waves or discontinuous perturbations which do not diffuse. Special waves, travelling waves, are a special solution that not only preserves its shape and velocity but also interacts with other solitons without change of form, even upon collision. When the fractional BBMB equation is solved, the effects of memory or anomalous diffusion on a system become more complicated since the time-fractional derivative is added to the solution. A more precise account of running waves in systems where the past contribution or non-locality effects are of importance, including in viscoelastic substances or in some fluid dynamics cases, is given by the solitary wave and travelling wave solutions of the time-fractional BBMB equation, which are typically presented as a sech-squared function. These solutions emphasize the more realistic, more complex behavior of waves in systems with the use of the fractional time derivatives.

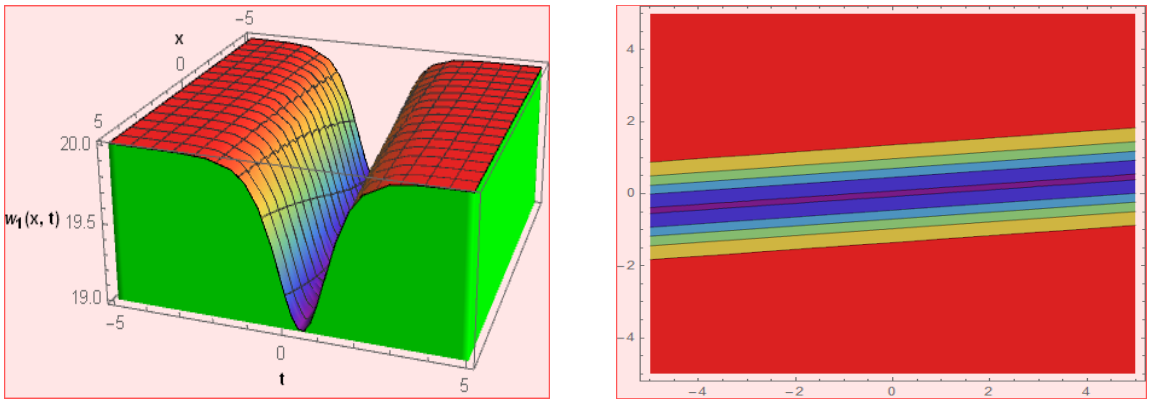


Figure 1. The physical representation for the solution $w_1(x,t)$ when constants are such as $\gamma = 0.2, \beta = 0.1, \delta_2 = 0.2, A = 0.3, c = 2.1, \delta_1 = -1$, and $\epsilon = 0.99$.

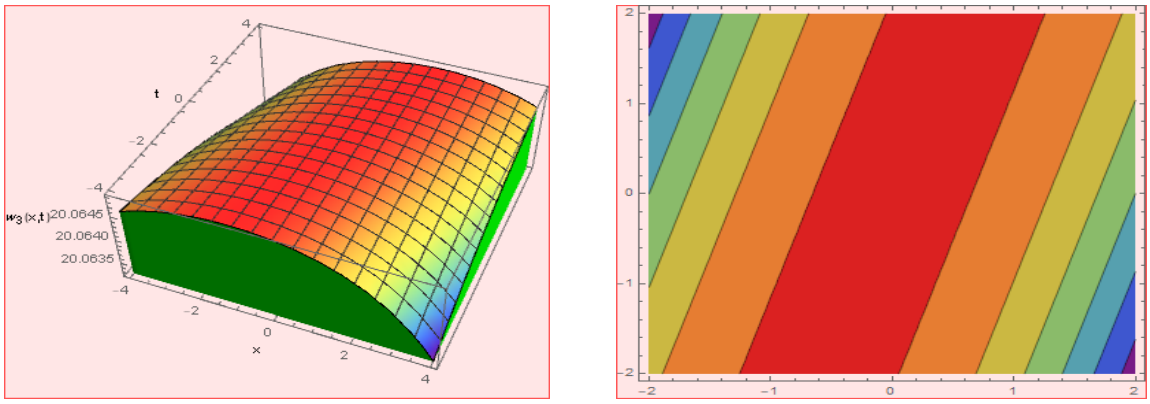


Figure 2. The physical representation for the solution $w_3(x,t)$ when constants are such as $\beta = 3.1, A = 0.3, \gamma = 0.3, c = .09, \delta_1 = .3, \delta_2 = 0.2$, and $\epsilon = 0.99$.

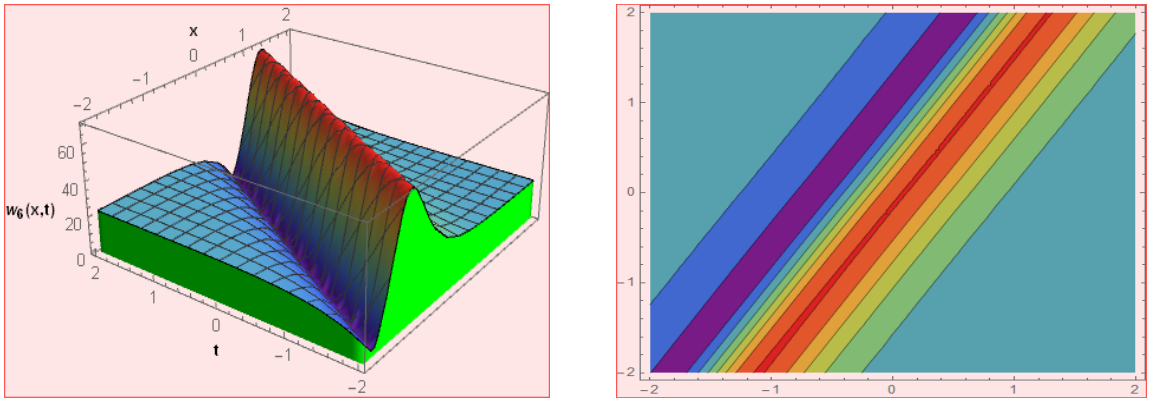


Figure 3. The physical representation for the solution $w_6(x,t)$ when constants are such as $\alpha = 2.7, A = 0.3, \delta_2 = 0.2, \gamma = 5.2, c = 3.1, \delta_1 = -1, \beta = .1$, and $\epsilon = 0.99$.

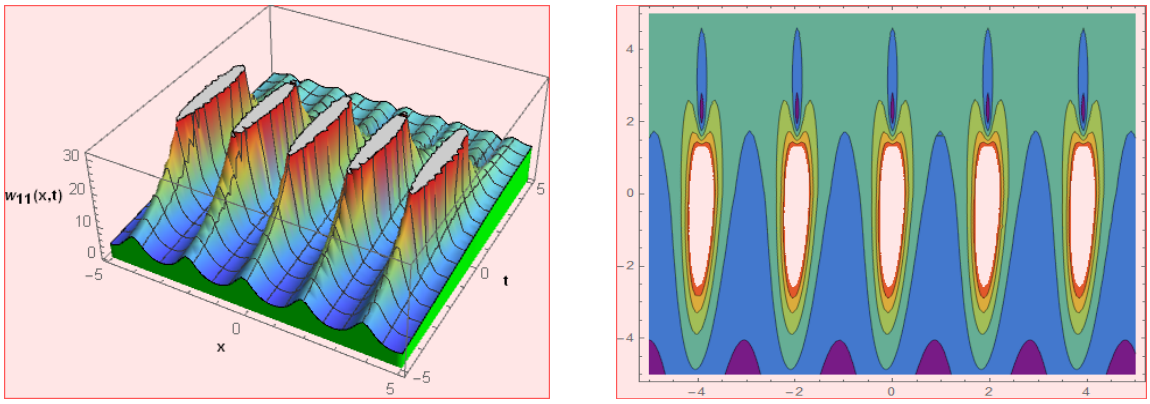


Figure 4. The physical representation for the solution $w_{11}(x,t)$ when constants are such as $\alpha = -1.5, \beta = -9.1$, and $\epsilon = 0.9$.

7. Selection of unique physical problem

In this section, we have chosen specific physical problems from the aforementioned solutions that are useful to investigators to gain approximate solutions. The TFBBMB equation is proposed to

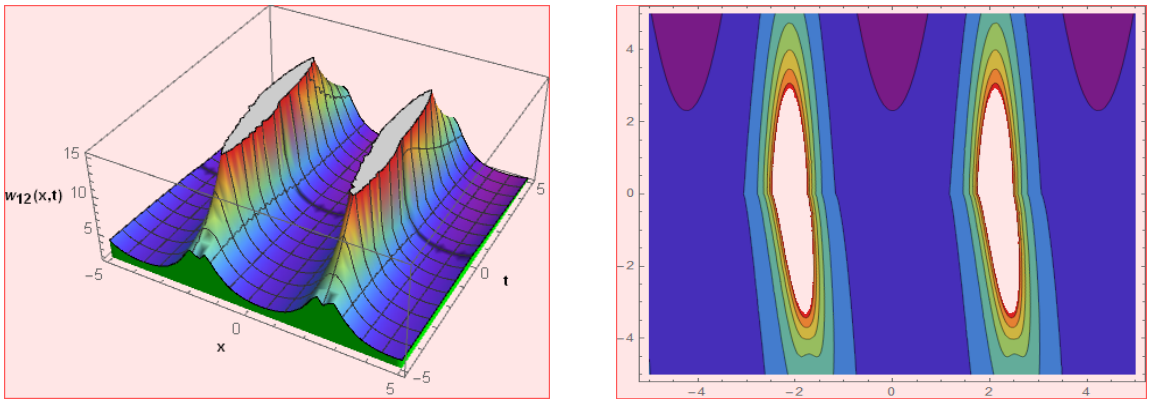


Figure 5. The physical representation for the solution $w_{12}(x,t)$ when constants are such as $\alpha = 0.9, \beta = 1.1$, and $\epsilon = 0.7$.

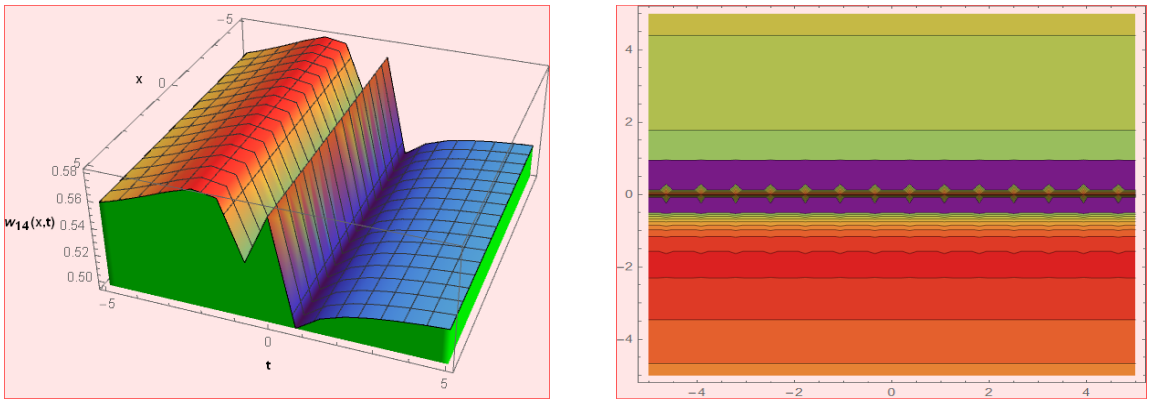


Figure 6. The physical representation for the solution $w_{14}(x,t)$ when constants are such as $\alpha = 0.1, \beta = 0.7, \gamma = 0.3, x = 1$, and $\epsilon = .25$.

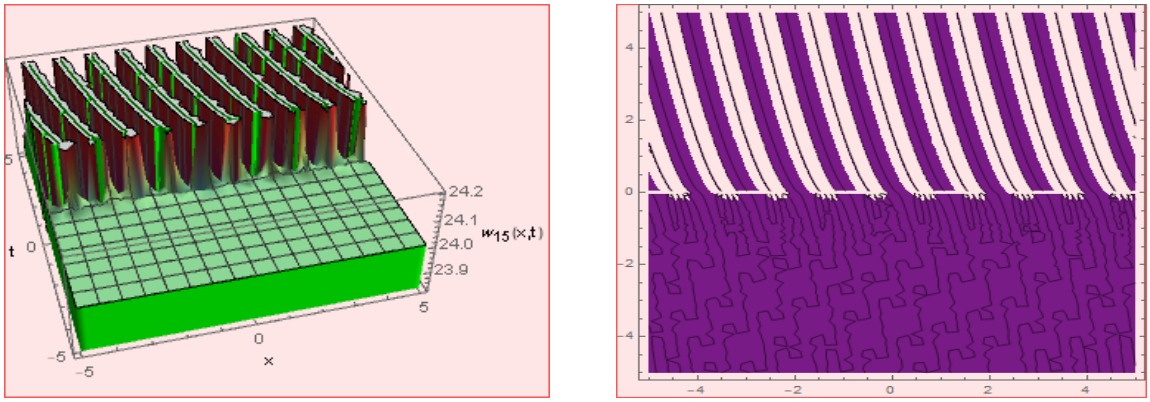


Figure 7. The physical representation for the solution $w_{15}(x,t)$ when constants are such as $\alpha = 4.1, a_1 = 0.3, \beta = 2.1, \gamma = 3.1, c = -0.9$, and $\epsilon = 0.7$.

model nonlinear wave propagation in media with dispersion and dissipation. Thus, the derived analytical solutions are useful for many engineering and physical applications. Bright and dark solitary wave solutions are solutions of the propagation of long, weakly nonlinear, surface waves in shallow water hydraulic channels (rivers, irrigation canals, shallow coastal waters), where the

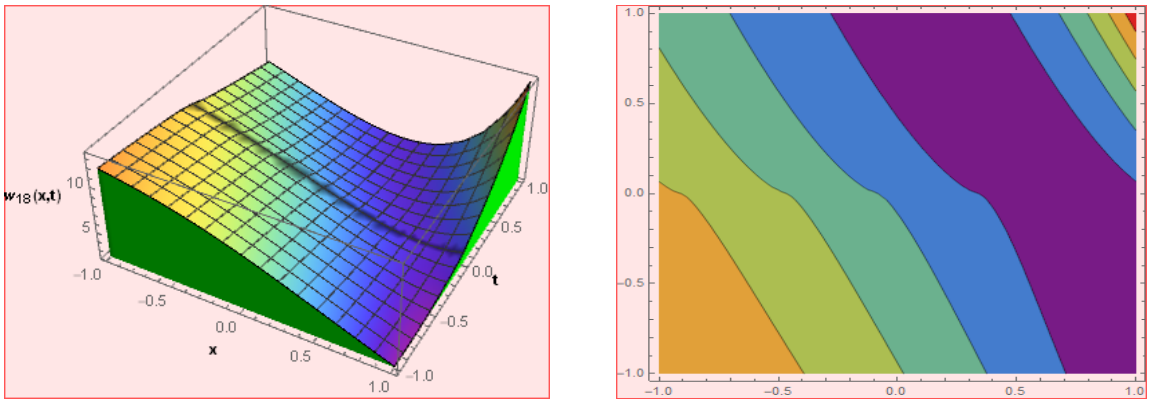


Figure 8. The physical representation for the solution $w_{18}(x,t)$ when constants are such as $\alpha = 2.5, a_2 = 0.2, \beta = 3.1, \gamma = 1.4$, and $\epsilon = 0.7$.

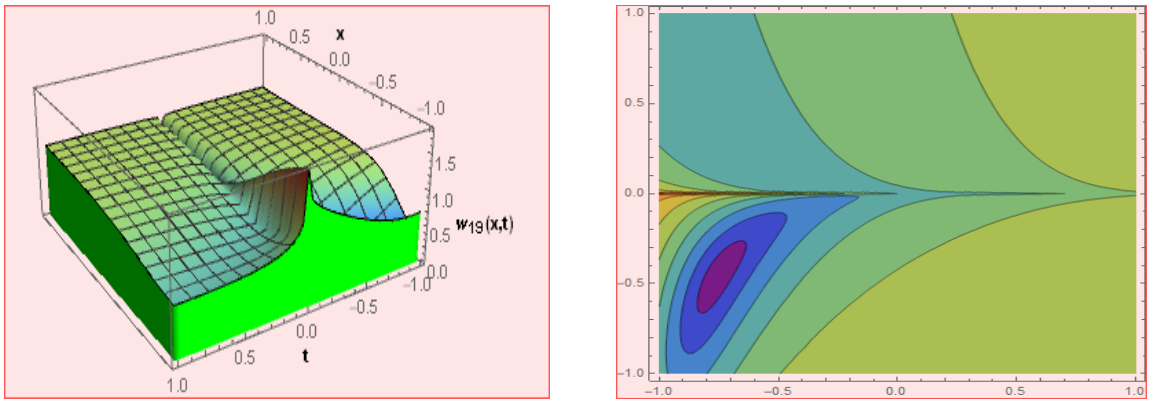


Figure 9. The physical representation for the solution $w_{19}(x,t)$ when constants are such as $\alpha = 0.5, a_1 = 2.1, a_2 = 3.2, \beta = 0.1, \gamma = 2.4$, and $\epsilon = 0.2$.

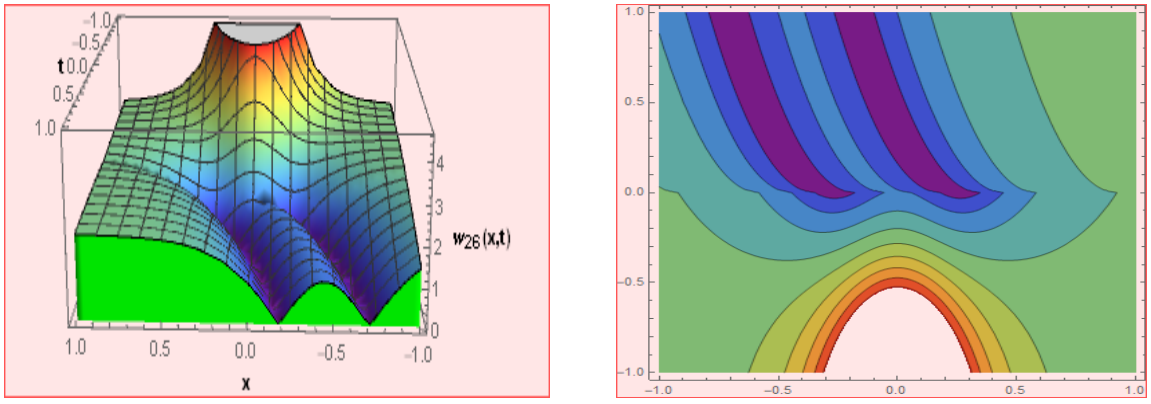


Figure 10. The physical representation for the solution $w_{26}(x,t)$ when constants are such as $\alpha = 0.9, \beta = 2.1, \gamma = 2.5$, and $\epsilon = 0.5$.

effects of viscous dissipation are not negligible. There are three types of solutions: The periodic wave solutions represent repeating wave trains that arise through oscillatory forcing, and the exponential and rational solutions represent wave attenuation, transition layers, and localized disturbances. These wave structures are also applicable to the propagation of nonlinear acous-

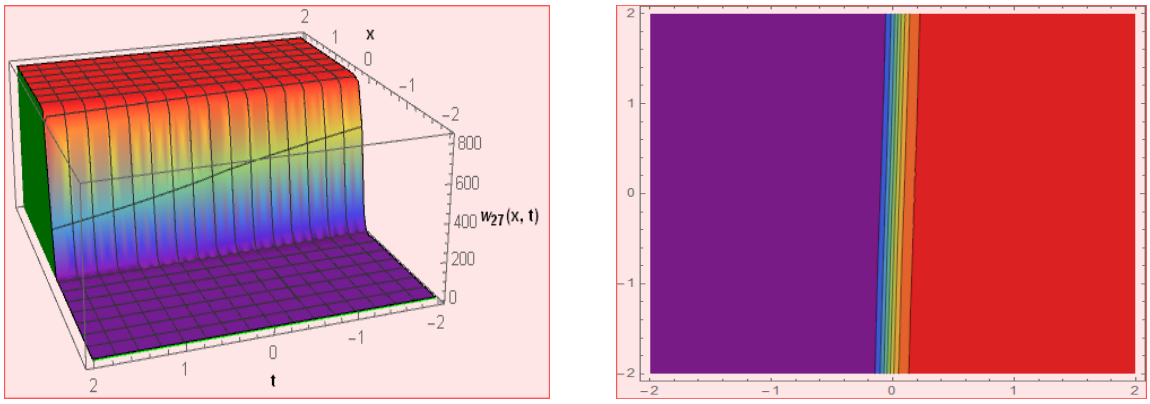


Figure 11. The physical representation for the solution $w_{27}(x,t)$ when constants are such as $\alpha = 0.7, \beta = 1.1, \beta_0 = 4.2, \beta_1 = 2.2, A = 0.3, \gamma = 4.5, c = 0.1, \nu = 0.1$, and $\epsilon = 0.9$.

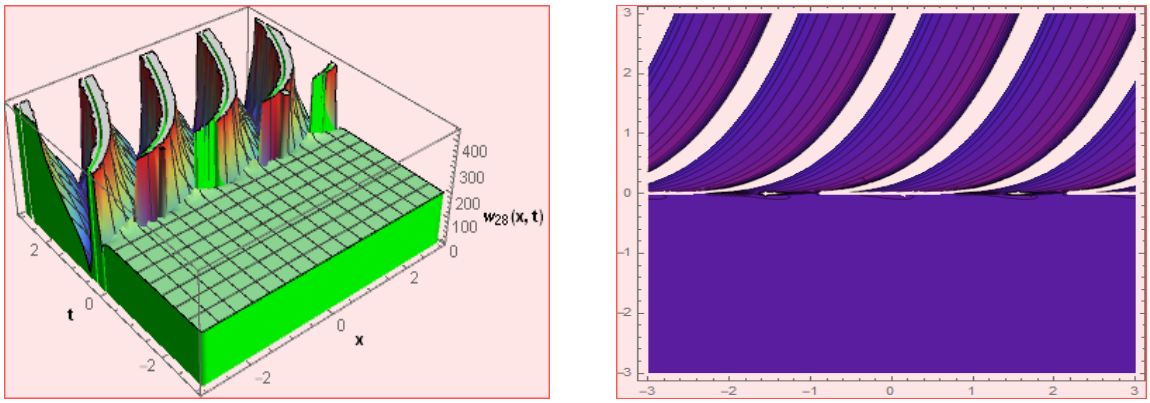


Figure 12. The physical representation for the solution $w_{28}(x,t)$ when constants are such as $\alpha = 0.7, \beta = 1.1, \beta_0 = 4.2, \beta_1 = 2.2, A = 0.3, \gamma = 4.5, c = 3.1, \nu = 2.1$, and $\epsilon = 0.3$.

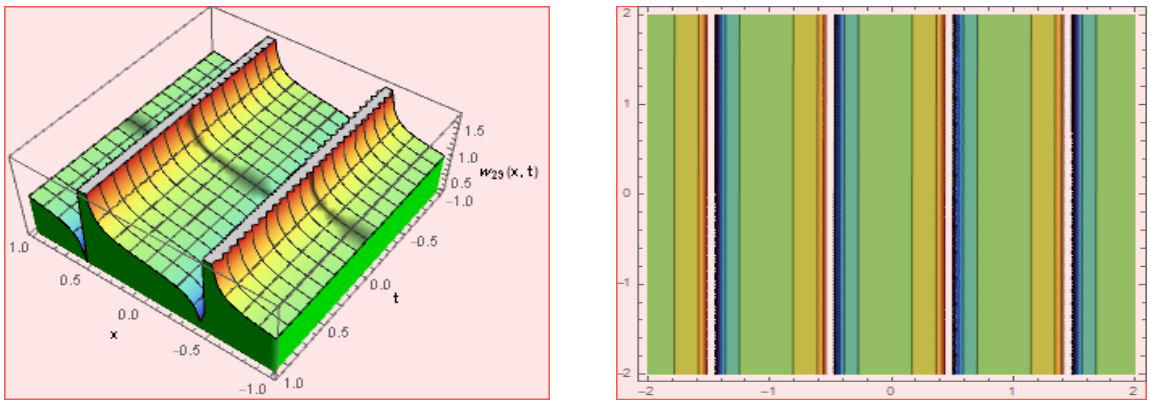


Figure 13. The physical representation for the solution $w_{29}(x,t)$ when constants are such as $\alpha = 0.7, \beta = 1.1, \beta_0 = -0.2, \beta_1 = 2.2, A = 0.3, \gamma = 1.5, c = 0.001, \nu = 2.1$, and $\epsilon = 0.5$.

tic waves, plasma waves, elastic waves in viscoelastic media, and the propagation of signals in nonlinear dispersive transmission lines. Furthermore, the model incorporates the conformable fractional derivative, thereby being able to account for memory and hereditary effects that are often observed in real materials and complex fluids, which are more appropriate to real physi-

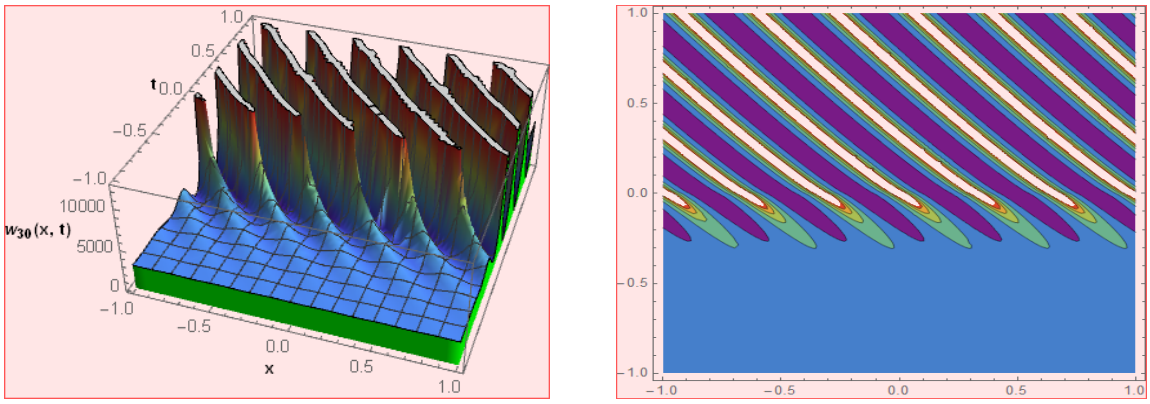


Figure 14. The physical representation for the solution $w_{30}(x,t)$ when constants are such as $\alpha = -0.7, \beta_0 = 0.1, \beta_1 = 2.2, \beta = 0.1, A = 0.3, \gamma = -3.5, c = 2.9, \nu = 3.4$, and $\epsilon = 0.9$.

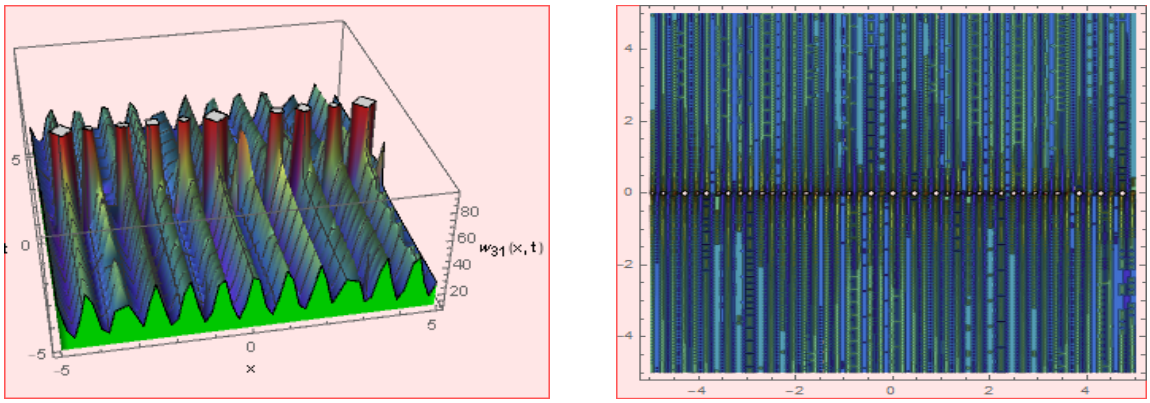


Figure 15. The physical representation for the solution $w_{31}(x,t)$ when constants are such as $\alpha = 0.9, \beta = 1.1, \beta_0 = 0.2, \beta_1 = 2.2, A = 0.3, \gamma = 6.5, c = 0.9, \nu = 2.1$, and $\epsilon = 0.1$.

cal systems than classical fractional derivative models. The analytically obtained solutions are related to physically meaningful initial-boundary value problems which are representative of realistic wave propagation situations. It is assumed that the computational domain is a one-dimensional propagation channel of one dimension, and the initial condition is directly given as the exact solution of the solitary wave at t=0, and the Dirichlet boundary conditions are applied with the analytical expressions at the left and right ends of the propagation domain. This configuration is typical of those of disturbances evolving in a finite dispersive medium without external forcing. These benchmark problems meet the necessary compatibility conditions and serve as good validation examples for finite difference, finite element, and spectral and mesh-free numerical schemes. In addition, the solution families are all associated with different physical configurations: Bright solitons are localized peaks above the equilibrium water level, dark solitons are localized dips below the equilibrium water level, periodic solutions are continuous wave trains, and exponential solutions are dissipative wave decay as they propagate. On an engineering level, the exact solutions gathered in this work can be used as reference solutions for testing of numerical algorithms, numerical accuracy of these solutions, and nonlinear analysis of interactions between waves in fractional dispersive media. The proposed physical arrangements and the specified initial and boundary conditions allow for a direct comparison of analytical and numerical solutions, as well as the establishment of a reliable computational model for wave

propagation problems such as shallow-water hydraulics, coastal engineering, nonlinear acoustics, plasma dynamics, and others related to Sobolev equations.

For the selected benchmark problem, let the computational domain be

$$a \leq x \leq b, \qquad 0 \leq t \leq T,$$

where a and b denote the left and right spatial boundaries, respectively, and T is the final simulation time. The initial condition is prescribed from the exact analytical solution as

$$w(x,0) = w_0(x),$$

while the Dirichlet boundary conditions are defined by

$$w(a,t) = g_L(t), \qquad w(b,t) = g_R(t).$$

To ensure that the initial-boundary value problem is well posed, the prescribed initial and boundary data must satisfy the corner compatibility conditions at $(a,0)$ and $(b,0)$. These conditions are given by

$$w_0(a) = g_L(0),$$

and

$$w_0(b) = g_R(0).$$

Using the analytical travelling-wave solution, we obtain

$$w_0(a) = w(a,0) = g_L(0),$$

and

$$w_0(b) = w(b,0) = g_R(0).$$

Hence, the initial condition and the boundary conditions are mutually consistent at the space-time corner points of the computational domain. Therefore, the benchmark problem satisfies the required corner compatibility conditions, ensuring that the prescribed initial-boundary data are mathematically consistent and suitable for validating numerical methods.

Example 7.1. We choose the solution Eq. (3.9) to construct the unique physical problems such as
The IC is constructed as

$$w_3\,(x,\,0) = 3.07692 + \frac{0.618659\sec^2((0.712039\ +0.i)x)}{(1-0.2\sec((0.712039\ +0.i)x))^2}. \tag{7.1}$$

The BCs are constructed as,

$$w_3\,(-1,t) = \frac{0.618659\sec^2(0.547723(0.\ -5.1t))}{(1-0.2\sec(0.547723(0.\ -5.1t)))^2} + 3.07692, \tag{7.2}$$

$$w_3\,(1,t) = \frac{0.618659\sec^2(0.547723(2.6\ -5.1t))}{(1-0.2\sec(0.547723(2.6\ -5.1t)))^2} + 3.07692. \tag{7.3}$$

The physical behavior of these conditions is shown in figure 16 in the form of 2-dimension.

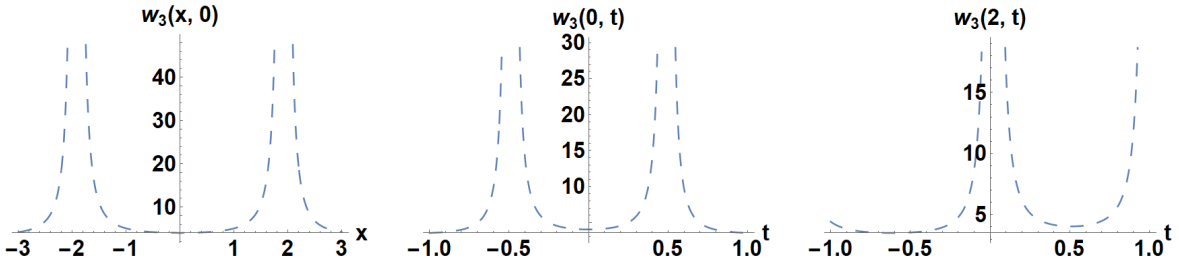


Figure 16. The 2-dimension behavior of IC and BCs for $w_3(x,t)$.

Example 7.2. We choose the solution Eq. (3.10) to construct the unique physical problem such as
The IC is constructed as

$$w_4\left(x,0\right)=0.669433\ -\frac{3.04328\coth^4(x+2.)}{0.13241\coth^2(x+2.)-1.3241}. \tag{7.4}$$

The BCs are constructed as,

$$w_4\left(0,t\right)=0.669433\ -\frac{3.04328\coth^4\left(1.5t^{0.6}+2\right)}{0.13241\coth^2\left(1.5t^{0.6}+2\right)-1.3241}, \tag{7.5}$$

$$w_4\left(2,t\right)=0.669433\ -\frac{3.04328\coth^4\left(1.5t^{0.6}+4\right)}{0.13241\coth^2\left(1.5t^{0.6}+4\right)-1.3241}. \tag{7.6}$$

The physical behavior of these conditions is shown in figure 17 in the form of a 2-dimension.

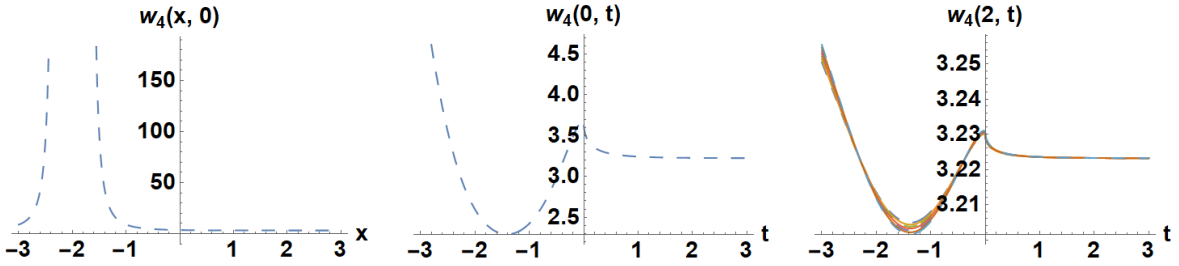


Figure 17. The 2-dimension behavior of IC and BCs for $w_4(x,t)$.

Example 7.3. We choose the solution Eq. (3.11) to construct the unique physical problem such as
The IC is constructed as

$$w_5\left(x,0\right)=0.047619\ -\frac{15.3629}{(4.2x+0.)^2}. \tag{7.7}$$

The BCs are constructed as,

$$w_5\left(0,t\right)=0.047619\ -\frac{15.3629}{(0.\ -0.333333t^{0.6})^2}, \tag{7.8}$$

$$w_5\left(1,t\right)=0.047619\ -\frac{15.3629}{(8.4\ -0.333333t^{0.6})^2}. \tag{7.9}$$

The physical behavior of these conditions is shown in figure 18 in the form of a 2-dimension.

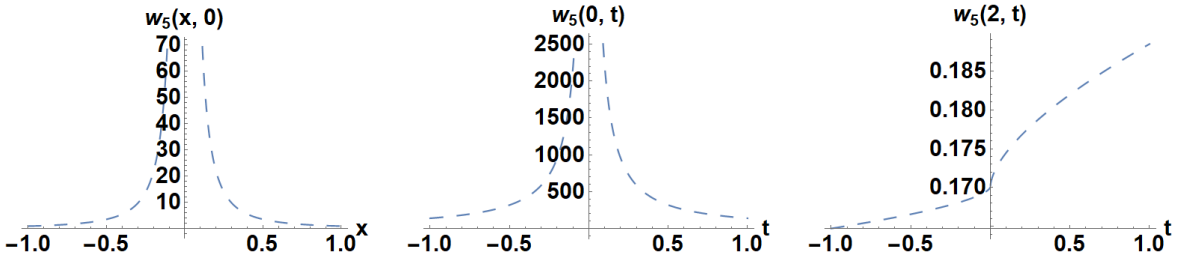


Figure 18. The 2-dimension behavior of IC and BCs for $w_5(x,t)$.

Example 7.4. We choose the solution Eq. (3.33) to construct the unique physical problem such as
The IC is constructed as

$$w_{18}(x,0) = 3.92641 - 0.123047\left(e^{1.4x+0.}+1\right)^2 + 0.164063\left(-e^{1.4x+0.}-1\right) + \frac{0.2}{\left(e^{1.4x+0.}+1\right)^2} - \frac{17.463}{e^{1.4x+0.}+1}. \tag{7.10}$$

The BCs are constructed as,

$$w_{18}(0,t) = 3.92641 - 0.123047\left(e^{0.868726t^{0.7}+0.}+1\right)^2 + 0.164063\left(-e^{0.868726t^{0.7}+0.}-1\right) + \frac{0.2}{\left(e^{0.868726t^{0.7}+0.}+1\right)^2} - \frac{17.463}{e^{0.868726t^{0.7}+0.}+1}, \tag{7.11}$$

$$w_{18}(2,t) = 3.92641 - 0.123047\left(e^{0.868726t^{0.7}+2.8}+1\right)^2 + 0.164063\left(-e^{0.868726t^{0.7}+2.8}-1\right) + \frac{0.2}{\left(e^{0.868726t^{0.7}+2.8}+1\right)^2} - \frac{17.463}{e^{0.868726t^{0.7}+2.8}+1}. \tag{7.12}$$

The physical behavior of these conditions is shown in figure 19 in the form of a 2-dimension.

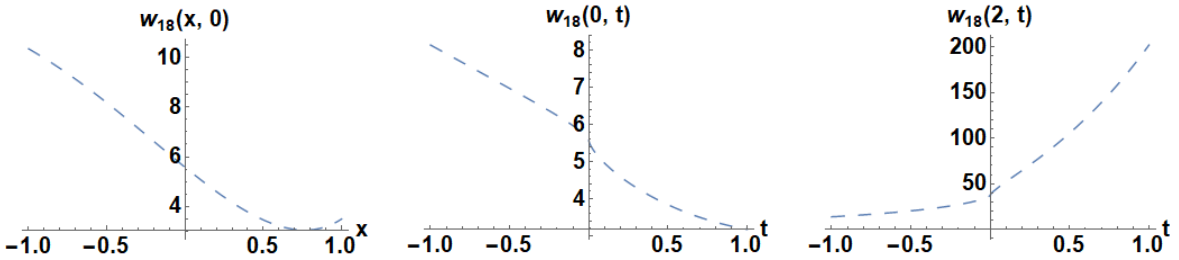


Figure 19. The 2-dimension behavior of IC and BCs for $w_{18}(x,t)$.

Example 7.5. We choose the solution Eq. (3.34) to construct the unique physical problem such as
The IC is constructed as

$$w_{19}(x,0) = 1.05 + \frac{3.2}{\left(e^{2.4x+0.}+1\right)^2} - \frac{2.1}{e^{2.4x+0.}+1}. \tag{7.13}$$

The BCs are constructed as,

$$w_{19}\left(0,t\right)=1.05+\frac{3.2}{\left(e^{3.13025t^{0.2}+0.}+1\right)^{2}}-\frac{2.1}{e^{3.13025t^{0.2}+0.}+1}, \tag{7.14}$$

$$w_{19}\left(1,t\right)=1.05+\frac{3.2}{\left(e^{3.13025t^{0.2}+4.8}+1\right)^{2}}-\frac{2.1}{e^{3.13025t^{0.2}+4.8}+1}. \tag{7.15}$$

The physical behavior of these conditions is shown in figure 20 in the form of a 2-dimension.

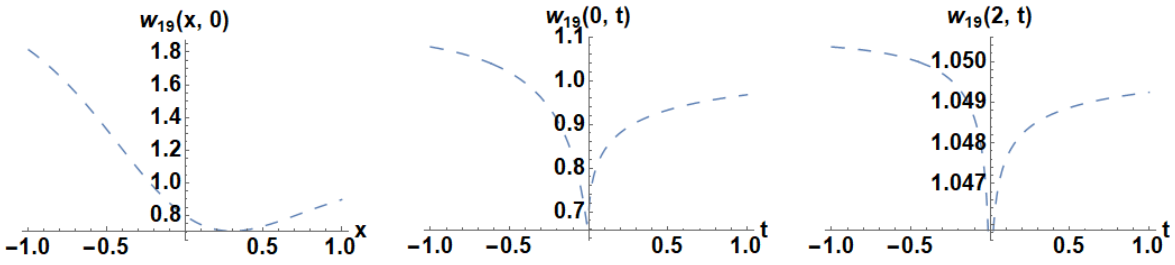


Figure 20. The 2-dimension behavior of IC and BCs for $w_{19}(x,t)$.

Example 7.6. We choose the solution Eq. (4.7) to construct the unique physical problem such as
The IC is constructed as

$$w_{30}\left(x,0\right)=-152.501-535.98(0.1+(5.469+0.i)\cot((2.7345+0.i)(0.-3.5x))). \tag{7.16}$$

The BCs are constructed as,

$$w_{30}\left(0,t\right)=-152.501-535.98\left(0.1+(5.469+0.i)\cot\left((2.7345+0.i)\left(0.-3.22222t^{0.9}\right)\right)\right), \tag{7.17}$$

$$w_{30}\left(2,t\right)=-152.501-535.98\left(0.1+(5.469+0.i)\cot\left((2.7345+0.i)\left(-3.22222t^{0.9}-7.\right)\right)\right). \tag{7.18}$$

The physical behavior of these conditions is shown in figure 21 in the form of a 2-dimension.

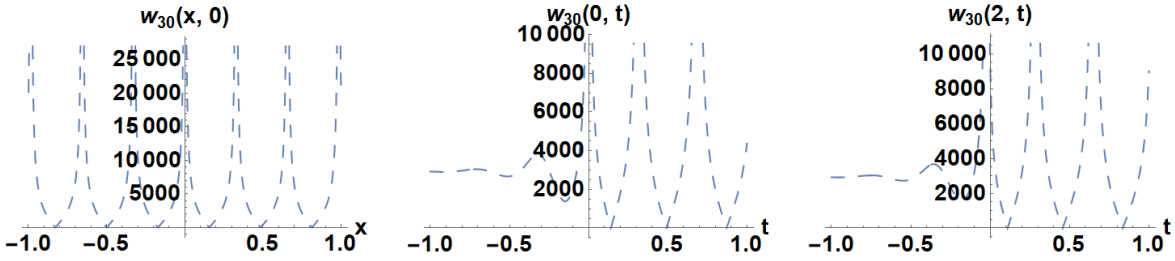


Figure 21. The 2-dimension behavior of IC and BCs for $w_{30}(x,t)$.

Example 7.7. We choose the solution Eq. (4.8) to construct the unique physical problem such as
The IC is constructed as

$$w_{31}\left(x,0\right)=\frac{103.405(3.1(0.-3.5x)+2)}{0.-3.5x}-81.0598. \tag{7.19}$$

The BCs are constructed as,

$$w_{31}(0,t) = \frac{103.405\left(3.1\left(0.-2.71429t^{0.7}\right)+2\right)}{0.-2.71429t^{0.7}} - 81.0598, \tag{7.20}$$

$$w_{31}(2,t) = \frac{103.405\left(3.1\left(-2.71429t^{0.7}-7.\right)+2\right)}{-2.71429t^{0.7}-7.} - 81.0598. \tag{7.21}$$

The physical behavior of these conditions is shown in figure 22 in the form of a 2-dimension.

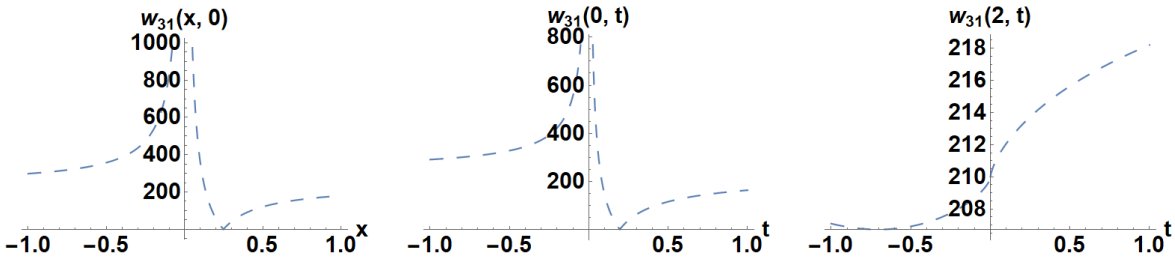


Figure 22. The 2-dimension behavior of IC and BCs for $w_{31}(x,t)$.

Example 7.8. We choose the solution Eq. (4.10) to construct the unique physical problem such as
The IC is constructed as

$$w_{33}(x,0) = 0. - \frac{7.46496\mathrm{csch}^2(0.547723(0.2x+0.))}{(2.2\mathrm{csch}(0.547723(0.2x+0.))+1)^2}. \tag{7.22}$$

The BCs are selected as,

$$w_{33}(0,t) = 0. - \frac{7.46496\mathrm{csch}^2\left(0.547723\left(0.-50.t^{0.002}\right)\right)}{\left(2.2\mathrm{csch}\left(0.547723\left(0.-50.t^{0.002}\right)\right)+1\right)^2}, \tag{7.23}$$

$$w_{33}(2,t) = 0. - \frac{7.46496\mathrm{csch}^2\left(0.547723\left(0.4-50.t^{0.002}\right)\right)}{\left(2.2\mathrm{csch}\left(0.547723\left(0.4-50.t^{0.002}\right)\right)+1\right)^2}. \tag{7.24}$$

The physical behavior of these conditions is shown in figure 23 in the form of a 2-dimension.

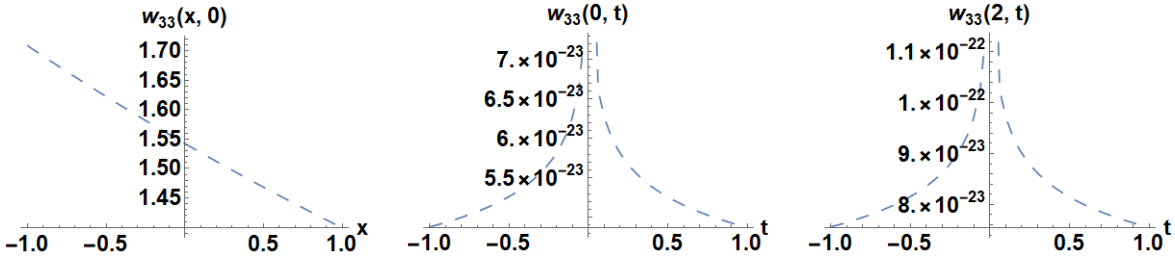


Figure 23. The 2-dimension behavior of IC and BCs for $w_{33}(x,t)$.

8. Conclusions

In this study, we investigate the solitary wave structures of the conformable TFBBMB equation. Three techniques (GPRE, GERF, and MAE) are applied, producing formally generated

expressions for a wide class of solutions. These expressions are symbolically verified by direct substitution. Among the obtained solutions, we identify real nonsingular solutions that are physically admissible for water wave applications. The physically admissible solutions include bright solitary waves, dark solitary waves, and kink waves that satisfy boundedness conditions on the chosen domains. Formal solutions with complex parameters or singularities are also presented for completeness of the mathematical analysis, but their physical interpretation requires additional constraints. Unique physical problems (initial and boundary conditions) are constructed for real-valued, nonsingular solutions, with corner compatibility conditions verified for each case. The current research can be extended in several useful directions in the future. The analytical solutions obtained in the present work can be used to solve higher-dimensional versions of the fractional BBMB equation, which are more realistic for modeling wave propagation in ocean basins and coastal zones. The alternative definitions of fractional derivatives, such as Caputo, Riemann-Liouville and Atangana-Baleanu operators, are worth investigating to elucidate the effect of various memory mechanisms on solitary wave dynamics. The exact solutions derived are used as benchmark solutions for testing and comparing the performance of numerical methods, such as high-order compact difference schemes, adaptive finite element methods, and physics-informed neural networks (PINNs). The applications should be related to propagation of tsunamis, internal wave dynamics, wave-structure interactions, and wave energy conversion, and validated with laboratory experiments and field measurements. Further studies of multi-soliton interactions, wave breaking, rogue wave generation, and wave turbulence studies would connect the mathematical solutions and the observed oceanographic phenomena. Incorporating stochastic forcing, machine learning methods, and climate change effects, especially the sea-level rise effects on wave characteristics, is a potential way to improve wave forecasting and coastal flooding modelling for operational use. These research directions all build on our current contributions, both in the fundamental mathematical and in the engineering aspects.

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